UMBRAL MOONSHINE

some papers in a Sammlung

arXiv:2606.10806v1 [cs.AI] 9 Jun 2026

Moonshine: An Autonomous Mathematical Research Agent Centered on Conjecture Generation

Xiaoyang Chen, Xiang Jiang
School of Mathematical Science, Tongji University
Moonshine Technology

Project homepage and source code: https://github.com/DeepMathLLM/Moonshine

Abstract

Moonshine is an autonomous agent whose central objective is to generate mathematical conjectures. Its core capability is to extract structure from classical problems, distill new concepts, and formulate conjectures of mathematical significance. Rather than treating the solution of a single proposition as its endpoint, Moonshine builds an extensible theoretical framework through conjecture generation, bridge building, and obstacle identification. This article uses Moonshine's exploration of the Jacobian conjecture as an example. It shows how the central logic of whether local nondegeneracy can force global injectivity is transferred to one-hidden-layer affine-ridge sigmoid networks. This leads to the formulation of the *Neural Jacobian Conjecture* (NJC): if such a network has strictly positive Jacobian determinant on the whole space, then it must be globally injective. By invoking GPT-5.5-pro and DeepSeek-V4-pro separately, Moonshine obtained independent complete proofs for the case $N = n+1$. In addition, with the assistance of ChatGPT through interactive use of its web interface with GPT-5.5-pro, a geometric-topological proof was developed. These results provide preliminary evidence for the plausibility of the conjecture. The general higher-width case $N \geq n+2$, however, remains unresolved and is left for further investigation. This work illustrates Moonshine's ability to autonomously generate meaningful mathematical problems and make rigorous progress on them.

1 Moonshine as an Autonomous Mathematical Research Agent Centered on Conjecture Generation

Moonshine is an autonomous mathematical research agent framework. It differs from question-answering systems or numerical computation tools in that its design goal is to **autonomously generate valuable mathematical conjectures** and to verify or refute them through structured exploration. Moonshine's behavior revolves around the following components:

- **Structural recognition and conjecture distillation.** It identifies core structural features in classical problems or mathematical objects, distills new concepts, and formulates precise and testable conjectures based on them.
- **Deep exploration and bridge building.** It connects conjectures with existing theories, explores interactions with other mathematical areas, and derives conditional conclusions.
- **Obstacle identification and boundary delineation.** Through proofs and counterexample construction, it clarifies sufficient conditions under which a conjecture holds and

identifies barriers that cannot be crossed, thereby determining the true range of applicability.

- **Research logs and memory.** It maintains long-term structured logs recording the evolution of conjectures, proof attempts, failed paths, and open subproblems, thereby forming an extensible theoretical framework.

Moonshine is organized around a runtime home directory (by default `~/.moonshine`), which contains configuration files, project folders, session logs, knowledge bases, skills, tools, and MCP server definitions. The agent operates in a research mode, in which it can iterate autonomously, search its historical memory, invoke verification tools, and gradually enrich its understanding of a given conjecture.

Moonshine's exploration of the Jacobian conjecture and the formulation of the Neural Jacobian Conjecture. Inspired by the classical complex Jacobian conjecture, Moonshine did not attempt to prove or disprove the original conjecture directly. Instead, it extracted the core logic: whether local nondegeneracy, expressed by a nonzero Jacobian determinant, can force global injectivity. This logic was then transferred to a restricted but structurally transparent function class, namely one-hidden-layer affine-ridge sigmoid networks. By analyzing the special algebraic and geometric structure of this network class, Moonshine distilled a new conjecture, the **Neural Jacobian Conjecture** (NJC). The conjecture asserts that if the Jacobian determinant of such a network is everywhere positive, then the network must be globally injective. This conjecture is both an analogy with the classical problem and an independently meaningful statement, because it attributes the rigidity from local diffeomorphism to global injectivity to the special affine-ridge structure.

The following sections present Moonshine's exploration of the NJC.

2 The Neural Jacobian Conjecture (NJC)

2.1 Function class and notation

Let $\sigma : \mathbb{R} \to (0, 1)$ be the logistic sigmoid function

$$\sigma(t) = \frac{1}{1 + e^{-t}}.$$

It is strictly increasing and satisfies $\sigma'(t) = \sigma(t)(1 - \sigma(t)) > 0$ for all $t \in \mathbb{R}$.

Definition 2.1 (Affine-ridge sigmoid networks)**.** For $n, N \geq 1$, a map $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a one-hidden-layer affine-ridge sigmoid network of width N if

$$F(x) = W^{(2)}\sigma\big(W^{(1)}x + b^{(1)}\big) + b^{(2)},$$

where $W^{(1)} \in \mathbb{R}^{N\times n}$, $W^{(2)} \in \mathbb{R}^{n\times N}$, $b^{(1)} \in \mathbb{R}^N$, $b^{(2)} \in \mathbb{R}^n$, and σ acts componentwise. This class is denoted by $\mathcal{N}_{n,N}$. When injectivity is unaffected by the notation, we write

$$F(x) = A\sigma(Bx + c).$$

Definition 2.2 (Positive-Jacobian subclass)**.** Define

$$\mathcal{N}^{+}_{n,N} = \{F \in \mathcal{N}_{n,N} : \det DF(x) > 0 \text{ for all } x \in \mathbb{R}^n\}.$$

2.2 The conjecture proposed by Moonshine

Conjecture 2.3 (Neural Jacobian Conjecture)**.** *For every $F \in \mathcal{N}^{+}_{n,N}$, the map F is globally injective.*

The motivation behind the conjecture is that a local diffeomorphism can in general form multiple sheets, whereas the special structure of affine-ridge sigmoid networks - in particular, the interaction between the kernel of the output weight matrix and the image of the hidden layer - may enforce uniqueness. If true, the NJC would provide a rigidity theorem in the neural-network setting, showing that local nondegeneracy implies global injectivity under this special structural constraint, and it would form an interesting parallel with the classical Jacobian conjecture.

2.3 A geometric reformulation

Let $h(x) = \sigma(Bx + c)$, and let

$$X_1 = h(\mathbb{R}^n) \subset (0,1)^N, \qquad X_2 = \ker A \subset \mathbb{R}^N.$$

Here X_1 is the hidden-layer submanifold and X_2 is the output kernel. Then F is injective if and only if

$$(p + X_2) \cap X_1 = \{p\}, \qquad \forall p \in X_1.$$

The positive-Jacobian condition $\det DF > 0$ is equivalent to the transversal intersection of X_1 and $p + X_2$ at every point, with local intersection index $+1$. Hence the NJC can be reformulated as follows: do transversality and positive local index force every affine fiber to intersect X_1 uniquely?

3 Partial Verification of the NJC in Low Width

Moonshine does not claim to have proved the NJC in full. It first analyzes the most accessible cases, namely when the hidden-layer width N equals the input dimension n, or is larger by one. These cases provide initial evidence for the conjecture.

3.1 The case $N = n$

Proposition 3.1. *If $N = n$ and $F \in \mathcal{N}^{+}_{n,n}$, then F is injective.*

Proof. When $N = n$, the matrices $A, B \in \mathbb{R}^{n\times n}$. Define

$$D(x) = \operatorname{diag}(\sigma'((Bx + c)_1), \ldots, \sigma'((Bx + c)_n)).$$

Since

$$\det DF(x) = \det A \cdot \det D(x) \cdot \det B > 0$$

for all x, and $\det D(x) > 0$, both $\det A$ and $\det B$ are nonzero. Thus A and B are invertible. The map $F(x) = A\sigma(Bx+c)$ is a composition of three injective maps: the invertible affine map $x \mapsto Bx + c$, the componentwise strictly increasing map $z \mapsto \sigma(z)$, and the invertible linear map $w \mapsto Aw$. Hence F is injective. $\square$

3.2 The case $N = n + 1$

This is the smallest width in which A has a nontrivial kernel. The main result of this section is the following.

Theorem 3.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a one-hidden-layer affine-ridge sigmoid network with $N = n + 1$ hidden units. If*

$$\det DF(x) > 0, \qquad \forall x \in \mathbb{R}^n,$$

then F is injective.

The first proof below was obtained by Moonshine using GPT-5.5-pro.

3.2.1 First algebraic proof: an injectivity lemma on convex sets

As a supplement to Theorem 3.2, we give a proof whose core is the following injectivity lemma for a linear projection applied to the graph of a scalar function.

Lemma 3.3 (Injectivity lemma on convex sets)**.** *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open convex set, let $h \in C^1(\Omega)$, and let $L : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be linear. Define*

$$T(y) = L(y, h(y)).$$

If $\det DT(y) \neq 0$ for all $y \in \Omega$, then T is injective.

Proof. Since $\det DT(y) \neq 0$, we have $\operatorname{rank} DT(y) = n$. Hence $\operatorname{rank} L = n$, because the image of $DT(y)$ is contained in the image of L. Thus $\dim \ker L = 1$. Let $k = (p, r) \in \mathbb{R}^n \times \mathbb{R}$ be a nonzero vector spanning $\ker L$. There are two cases.

Case 1: $r \neq 0$. Rescale k so that $r = -1$, hence $k = (p, -1)$. Define $Q(y, z) = y + pz$. Then $\ker Q = \mathbb{R}(p, -1) = \ker L$. Since both L and Q are full-rank maps from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, there exists $A_0 \in GL_n(\mathbb{R})$ such that $A_0 L = Q$. Acting on the target by A_0 does not affect injectivity, so we may assume

$$T(y) = y + ph(y).$$

Suppose $T(y_1) = T(y_2)$. Then

$$y_2 - y_1 = p\big(h(y_1) - h(y_2)\big).$$

Thus $y_2 - y_1$ is parallel to p. If $p = 0$, then $T(y) = y$, and there is nothing to prove. Assume $p \neq 0$. Then $y_2 = y_1 + sp$ for some $s \neq 0$. By convexity of Ω, the segment $\{y_1 + tp : 0 \leq t \leq s\}$ is contained in Ω. The equality above gives

$$s + h(y_1 + sp) - h(y_1) = 0.$$

Set $\psi(t) = t + h(y_1 + tp)$. Then $\psi(s) = \psi(0)$. Moreover,

$$\psi'(t) = 1 + \nabla h(y_1 + tp) \cdot p.$$

By the matrix determinant lemma,

$$\det DT(y_1 + tp) = \det\big(I + p(\nabla h)^T\big) = 1 + \nabla h(y_1 + tp) \cdot p.$$

Hence $\psi'(t) = \det DT(y_1 + tp)$. By assumption this never vanishes; since it is continuous, its sign is constant, and ψ is strictly monotone. This contradicts $\psi(s) = \psi(0)$ unless $s = 0$. Therefore $y_1 = y_2$.

Case 2: $r = 0$. Then $\ker L = \mathbb{R}(p, 0)$ with $p \neq 0$. Choose a linear isomorphism

$$P : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^n$$

such that $P(0, 1) = p$. Let

$$\widetilde{\Omega} = P^{-1}(\Omega), \qquad \widetilde{h}(u, s) = h(P(u, s)).$$

In coordinates $(u, s, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, define

$$\widetilde{L}(u, s, z) = L(P(u, s), z).$$

Then $\ker \widetilde{L} = \mathbb{R}(0, 1, 0)$. Consider

$$H = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R} \cong \mathbb{R}^n.$$

The restriction $\widetilde{L}|_H : H \to \mathbb{R}^n$ is a linear isomorphism. Let

$$J : H \to \mathbb{R}^n, \qquad J(u, 0, z) = (u, z),$$

and define

$$M = J \circ (\widetilde{L}|_H)^{-1} \in GL_n(\mathbb{R}).$$

Since $(0, s, 0) \in \ker \widetilde{L}$, we have

$$M\widetilde{L}(u, s, z) = J(u, 0, z) = (u, z).$$

After the source coordinate change $y = P(u, s)$ and the target linear change M, the map becomes

$$\widetilde{T}(u, s) = MT(P(u, s)) = M\widetilde{L}(u, s, \widetilde{h}(u, s)) = (u, \widetilde{h}(u, s)).$$

These transformations are invertible, so they do not affect injectivity; the Jacobian determinant is only multiplied by a nonzero constant.

The Jacobian of $\widetilde{T}$ is

$$D\widetilde{T}(u, s) = \begin{pmatrix} I_{n-1} & 0 \\ \partial_u \widetilde{h}(u, s) & \partial_s \widetilde{h}(u, s) \end{pmatrix},$$

and hence $\det D\widetilde{T}(u, s) = \partial_s \widetilde{h}(u, s)$. Therefore

$$\partial_s \widetilde{h}(u, s) \neq 0, \qquad (u, s) \in \widetilde{\Omega}.$$

For fixed u, the fiber

$$I_u = \{s \in \mathbb{R} : (u, s) \in \widetilde{\Omega}\}$$

is an interval, since $\widetilde{\Omega}$ is convex. The function $s \mapsto \widetilde{h}(u, s)$ has a continuous derivative which never vanishes on I_u, hence it is strictly monotone. If $\widetilde{T}(u_1, s_1) = \widetilde{T}(u_2, s_2)$, the first $n-1$ components give $u_1 = u_2$, and the last component gives $\widetilde{h}(u_1, s_1) = \widetilde{h}(u_1, s_2)$. Strict monotonicity then yields $s_1 = s_2$. Hence $\widetilde{T}$, and therefore T, is injective. $\square$

Proof of Theorem 3.2, first algebraic form. Write $F(x) = A\sigma(Bx + c)$, where $A \in \mathbb{R}^{n\times(n+1)}$, $B \in \mathbb{R}^{(n+1)\times n}$, and $c \in \mathbb{R}^{n+1}$. Since $\operatorname{rank} B = n$, some n rows of B are linearly independent. After reordering the hidden units, assume that the first n rows are independent. By an invertible affine change of input variables, the first n preactivations can be normalized to $x_1, \dots, x_n$. Thus the network can be written as

$$F(x) = C\sigma(x) + w\sigma(a^T x + \beta),$$

where $C \in \mathbb{R}^{n\times n}$, $w \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$, and $b \in \mathbb{R}^n$. Equivalently,

$$B = \begin{pmatrix} I_n \\ a^T \end{pmatrix}, \qquad A = \begin{pmatrix} C & w \end{pmatrix}.$$

Let $\Phi : \mathbb{R}^n \to (0,1)^n$ be $\Phi(x) = \sigma(x)$, and set

$$h(y) = \sigma(a^T \Phi^{-1}(y) + \beta), \qquad y \in (0,1)^n.$$

Then

$$F(x) = Cy + wh(y) = T(y), \qquad y = \Phi(x).$$

Since $\det DF(x) > 0$ and Φ is a diffeomorphism, $\det DT(y) \neq 0$ for all $y \in (0,1)^n$. The domain $(0,1)^n$ is open and convex, so Lemma 3.3 implies that T is injective. Hence F is injective. □

This proof highlights the fundamental role of convexity and a one-dimensional kernel in the low-width case of the NJC.

3.2.2 Second algebraic proof: one-dimensional monotonicity along the kernel direction

The following second proof was obtained by Moonshine using DeepSeek-V4-pro.

Proof. Again write $F(x) = A\sigma(Bx + c)$, where $A \in \mathbb{R}^{n\times(n+1)}$, $B \in \mathbb{R}^{(n+1)\times n}$, and $c \in \mathbb{R}^{n+1}$. As above, after normalization the network can be written as

$$F(x) = C\sigma(x) + w\sigma(a^T x + \beta).$$

Its Jacobian is

$$DF(x) = CS(x) + s_{n+1}(x) w a^T,$$

where

$$S(x) = \operatorname{diag}(\sigma'(x_1), \dots, \sigma'(x_n)), \qquad s_{n+1}(x) = \sigma'(a^T x + \beta).$$

All entries of $S(x)$ and $s_{n+1}(x)$ are positive.

Suppose, for contradiction, that F is not injective. Then there exist $p \neq q$ such that $F(p) = F(q)$. Let

$$u = \sigma(p) \in (0,1)^n, \qquad v = \sigma(q) \in (0,1)^n.$$

Define

$$G(u) = \sigma(a^T \sigma^{-1}(u) + \beta),$$

where $\sigma^{-1}(s) = \log(s/(1-s))$ acts componentwise. Then

$$F(x) = Cu + wG(u), \qquad u = \sigma(x).$$

Since $A = [C, w]$ has rank n, its kernel is one-dimensional. Choose

$$k = \begin{pmatrix} \widehat{k} \\ k_{n+1} \end{pmatrix} \in \ker A \setminus \{0\}.$$

Then

$$C\widehat{k} + k_{n+1} w = 0. \tag{1}$$

The equality $F(p) = F(q)$ implies that the difference of the hidden-layer outputs lies in $\ker A$. Thus for some $\lambda \neq 0$,

$$\begin{pmatrix} v \\ G(v) \end{pmatrix} - \begin{pmatrix} u \\ G(u) \end{pmatrix} = \lambda k.$$

In particular,

$$v = u + \lambda \widehat{k}, \qquad G(v) - G(u) = \lambda k_{n+1}. \tag{2}$$

Define

$$f(t) = G(u + t\widehat{k}) - G(u) - t k_{n+1}$$

for those t for which $u + t\widehat{k} \in (0,1)^n$. By (2), $f(0) = f(\lambda) = 0$, with $\lambda \neq 0$. Let

$$x_t = \sigma^{-1}(u + t\widehat{k}).$$

Convexity of $(0,1)^n$ ensures that x_t is well-defined for t between 0 and λ. Differentiating gives

$$f'(t) = \nabla G(u + t\widehat{k}) \cdot \widehat{k} - k_{n+1}.$$

A direct computation yields

$$\nabla G(u + t\widehat{k}) = s_{n+1}(x_t) a^T S(x_t)^{-1},$$

and hence

$$f'(t) = s_{n+1}(x_t) a^T S(x_t)^{-1} \widehat{k} - k_{n+1}. \tag{3}$$

We relate this expression to $\det DF(x_t)$.

Case 1: $k_{n+1} \neq 0$. Rescale k so that $k_{n+1} = -1$. Then (1) gives $w = C\widehat{k}$. Therefore

$$DF(x_t) = C\big(S(x_t) + s_{n+1}(x_t)\widehat{k} a^T\big).$$

Since $A = [C, w]$ has rank n, and $w = C\widehat{k}$, the matrix C is invertible. The matrix determinant lemma gives

$$\det DF(x_t) = \det C \det S(x_t)\big(1 + s_{n+1}(x_t) a^T S(x_t)^{-1} \widehat{k}\big).$$

By (3),

$$f'(t) = 1 + s_{n+1}(x_t) a^T S(x_t)^{-1} \widehat{k} = \frac{\det DF(x_t)}{\det C \det S(x_t)}.$$

Since $\det DF(x_t) > 0$, $\det S(x_t) > 0$, and $\det C \neq 0$ is constant, f' has a fixed nonzero sign on the interval between 0 and λ. Hence f is strictly monotone there.

Case 2: $k_{n+1} = 0$. Then $C\widehat{k} = 0$, with $\widehat{k} \neq 0$. Thus $\operatorname{rank} C \leq n - 1$. Since $[C, w]$ has rank n, we have $\operatorname{rank} C = n - 1$, and $w \notin \operatorname{Im} C$. Choose $v_0 \neq 0$ spanning the left kernel $\ker C^T$. Then $v_0^T w \neq 0$. The adjugate matrix has rank one and can be written as

$$\operatorname{adj}(C) = \gamma \widehat{k} v_0^T, \qquad \gamma \neq 0. \tag{4}$$

Writing

$$DF(x_t) = (C + s_{n+1}(x_t) w a^T S(x_t)^{-1}) S(x_t),$$

and using the rank-one perturbation formula for a rank $n-1$ matrix,

$$\det(C + uv^T) = v^T \operatorname{adj}(C) u,$$

we obtain

$$\det DF(x_t) = \det S(x_t)\, s_{n+1}(x_t) a^T S(x_t)^{-1} \widehat{k}\, \gamma(v_0^T w).$$

Since $k_{n+1} = 0$, (3) becomes

$$f'(t) = s_{n+1}(x_t) a^T S(x_t)^{-1} \widehat{k}.$$

Therefore

$$f'(t) = \frac{\det DF(x_t)}{\det S(x_t) \gamma(v_0^T w)}.$$

Again $\det DF(x_t) > 0$, $\det S(x_t) > 0$, and $\gamma(v_0^T w) \neq 0$, so f' has a fixed nonzero sign. Thus f is strictly monotone.

In both cases, f is strictly monotone on the closed interval with endpoints 0 and λ, but $f(0) = f(\lambda) = 0$ and $\lambda \neq 0$, a contradiction. Hence no distinct p, q with $F(p) = F(q)$ exist, and F is injective. □

3.2.3 Geometric-topological proof: the hidden-layer submanifold and one-dimensional fibers

The following geometric-topological proof was developed with the assistance of ChatGPT, using GPT-5.5-pro through interactive use of its web interface.

Use the notation

$$F(x) = A\sigma(Bx + c),$$

where

$$A \in \mathbb{R}^{n \times (n+1)}, \qquad B \in \mathbb{R}^{(n+1) \times n}.$$

Assume

$$\operatorname{rank} A = \operatorname{rank} B = n, \qquad \det DF(x) > 0 \quad \forall x \in \mathbb{R}^n.$$

Let

$$h(x) = \sigma(Bx + c),$$

and define

$$X_1 = h(\mathbb{R}^n) = \sigma(c + \operatorname{Im} B) \subset (0,1)^{n+1}, \qquad X_2 = \ker A.$$

Since B has full column rank, $c + \operatorname{Im} B$ is an affine n-plane in $\mathbb{R}^{n+1}$. Since the componentwise sigmoid is a diffeomorphism from $\mathbb{R}^{n+1}$ to $(0,1)^{n+1}$, X_1 is a smoothly embedded n-dimensional submanifold of $(0,1)^{n+1}$. Moreover $\dim X_2 = 1$.

By the hidden-layer intersection formulation, F is injective if and only if

$$(p + X_2) \cap X_1 = \{p\}, \qquad \forall p \in X_1.$$

It is therefore enough to prove that every affine line with direction X_2 meets X_1 at most once.

Fix $p \in X_1$, and choose $k \in X_2 \setminus \{0\}$ with $X_2 = \mathbb{R}k$. Define

$$I_p = \{t \in \mathbb{R} : p + tk \in (0,1)^{n+1}\}.$$

This is an open interval containing 0.

We describe X_1 in logit coordinates. Let

$$\operatorname{logit}(z) = \left(\log \frac{z_1}{1-z_1}, \dots, \log \frac{z_{n+1}}{1-z_{n+1}}\right).$$

Then

$$z \in X_1 \iff \operatorname{logit}(z) - c \in \operatorname{Im} B.$$

Choose $0 \neq \lambda \in (\operatorname{Im} B)^\perp$, and define

$$\Phi : (0,1)^{n+1} \to \mathbb{R}, \qquad \Phi(z) = \lambda \cdot (\operatorname{logit}(z) - c).$$

Then

$$X_1 = \{z \in (0,1)^{n+1} : \Phi(z) = 0\}.$$

Restrict Φ to the affine line $p + X_2$, and define

$$g : I_p \to \mathbb{R}, \qquad g(t) = \Phi(p + tk).$$

Then

$$p + tk \in X_1 \iff g(t) = 0.$$

Thus intersections of $p + X_2$ with X_1 correspond exactly to zeros of g, and $g(0) = 0$.

We first prove that if $t_0 \in I_p$ is a zero of g, then

$$g'(t_0) \neq 0.$$

This assertion concerns the derivative at zeros only; it does not say that g' is nonzero on all of I_p.

Let $g(t_0) = 0$, and set $z_0 = p + t_0 k$. Then $z_0 \in X_1$, so $z_0 = h(x_0) = \sigma(Bx_0 + c)$ for some x_0. Define

$$D(x_0) = \operatorname{diag}(\sigma'((Bx_0 + c)_1), \dots, \sigma'((Bx_0 + c)_{n+1})).$$

Then

$$T_{z_0} X_1 = D(x_0) \operatorname{Im} B.$$

Lemma 3.4. *At every zero t_0 of g, one has*

$$k \notin T_{z_0} X_1.$$

Proof. If $k \in T_{z_0} X_1$, then $k = D(x_0)Bv$ for some $v \in \mathbb{R}^n$. Since $k \in X_2 = \ker A$,

$$0 = Ak = AD(x_0)Bv.$$

But $AD(x_0)B = DF(x_0)$, and $\det DF(x_0) > 0$, so $AD(x_0)B$ is invertible. Hence $v = 0$, and therefore $k = 0$, a contradiction. □

Since X_1 is the zero level set of Φ, and on X_1

$$T_{z_0} X_1 = \ker d\Phi_{z_0},$$

Lemma 3.4 implies

$$d\Phi_{z_0}(k) \neq 0.$$

By the chain rule, because $g(t) = \Phi(p + tk)$,

$$g'(t_0) = d\Phi_{z_0}(k).$$

Thus every zero of g is nondegenerate, in the sense that its derivative is nonzero.

We next show that the signs of g' at all zeros are the same. For this purpose we use a separate linear-algebra lemma.

Lemma 3.5. *Let*

$$A : \mathbb{R}^{n+1} \to \mathbb{R}^n, \qquad B : \mathbb{R}^n \to \mathbb{R}^{n+1}$$

be linear maps with $\operatorname{rank} A = \operatorname{rank} B = n$. *Suppose*

$$\ker A = \mathbb{R}k, \qquad 0 \neq \lambda \in (\operatorname{Im} B)^{\perp}.$$

Then there exists a nonzero constant $C = C(A, B, k, \lambda)$ *such that for every* $L \in GL_{n+1}(\mathbb{R})$,

$$\det(ALB) = C \det L \, \lambda^T L^{-1} k.$$

In particular, if $\det L > 0$, *then there is a fixed sign* $\varepsilon \in \{\pm 1\}$, *depending only on* A, B, k, λ, *such that*

$$\operatorname{sgn} \det(ALB) = \varepsilon \operatorname{sgn}(\lambda^T L^{-1} k).$$

Proof. First, for any $M = [m_1, \ldots, m_n] \in \mathbb{R}^{(n+1)\times n}$, since A has rank n and $\ker A = \mathbb{R}k$, there is a nonzero constant c_A, depending only on A and k, such that

$$\det(AM) = c_A \det[M, k].$$

Indeed, choose a basis $u_1, \ldots, u_n$ of a complement to $\mathbb{R}k$, and set $U = [u_1, \ldots, u_n]$. Then $[U, k]$ is invertible and AU is invertible. Every M can be written uniquely as $M = UY + k\alpha^T$. Then

$$\det(AM) = \det(AU) \det Y, \qquad \det[M, k] = \det[U, k] \det Y,$$

which gives the desired proportionality.

Second, since $\operatorname{rank} B = n$, the image $\operatorname{Im} B$ is an n-dimensional hyperplane in $\mathbb{R}^{n+1}$. The map $q \mapsto \det[B, q]$ is a linear functional vanishing on $\operatorname{Im} B$, as is $q \mapsto \lambda^T q$. Since the space of such functionals is one-dimensional, there exists $c_B \neq 0$ such that

$$\det[B, q] = c_B \lambda^T q, \qquad \forall q \in \mathbb{R}^{n+1}.$$

Now take $M = LB$. Then

$$\det(ALB) = c_A \det[LB, k].$$

Since $[LB, k] = L[B, L^{-1}k]$,

$$\det[LB, k] = \det L \det[B, L^{-1}k] = \det L \, c_B \lambda^T L^{-1} k.$$

Thus

$$\det(ALB) = c_A c_B \det L \, \lambda^T L^{-1} k.$$

Set $C = c_A c_B \neq 0$. The sign statement follows immediately. $\square$

Return to the geometric proof. At a zero t_0, write again $z_0 = p + t_0 k = h(x_0)$. Since $z_0 = \sigma(Bx_0 + c)$,

$$D(x_0) = \operatorname{diag}\big(z_{0,1}(1 - z_{0,1}), \dots, z_{0,n+1}(1 - z_{0,n+1})\big),$$

which is a positive diagonal matrix. Moreover,

$$d\Phi_{z_0}(w) = \lambda^T D(x_0)^{-1} w, \qquad \text{so} \qquad d\Phi_{z_0}(k) = \lambda^T D(x_0)^{-1} k.$$

Applying Lemma 3.5 with $L = D(x_0)$, we get

$$\det(AD(x_0)B) = C \det D(x_0)\, d\Phi_{z_0}(k).$$

Since $\det D(x_0) > 0$, there exists a fixed sign ε, independent of the intersection point, such that

$$\operatorname{sgn}\det(AD(x_0)B) = \varepsilon \operatorname{sgn} d\Phi_{z_0}(k).$$

But $\det(AD(x_0)B) = \det DF(x_0) > 0$, so all zeros t_0 have the same sign of $d\Phi_{z_0}(k)$, and hence the same sign of $g'(t_0)$.

Finally, we use the following elementary one-dimensional fact: if $g : I \to \mathbb{R}$ is C^1, and $t_1 < t_2$ are two adjacent zeros with $g'(t_1) \neq 0$ and $g'(t_2) \neq 0$, then

$$\operatorname{sgn} g'(t_1) = -\operatorname{sgn} g'(t_2).$$

Indeed, if $g'(t_1) > 0$, then g is positive just to the right of t_1. Since there are no zeros in (t_1, t_2), it remains positive on that interval. Since $g(t_2) = 0$ and $g'(t_2) \neq 0$, it follows that $g'(t_2) < 0$. The other case is analogous.

If there were another intersection, then for some $\lambda_0 \in I_p \setminus \{0\}$, $g(\lambda_0) = 0$. On the compact interval with endpoints 0 and λ_0, the zeros are isolated and hence finite. Thus two adjacent zeros can be chosen. The one-dimensional fact forces the signs of g' at these two zeros to be opposite, whereas the argument above shows that all such signs are equal. This contradiction proves

$$g^{-1}(0) = \{0\}.$$

Equivalently,

$$(p + X_2) \cap X_1 = \{p\}.$$

Since $p \in X_1$ was arbitrary, F is globally injective.

This proof reveals the essence of the case $N = n + 1$: the one-dimensional output kernel reduces the intersection problem to a one-variable zero problem; nondegenerate zeros in one dimension have alternating derivative signs, while the positive Jacobian condition forces all local signs to agree. This mechanism no longer holds automatically for higher-dimensional fibers, which explains the difficulty of the case $N \geq n + 2$.

3.3 The higher-width case $N \geq n + 2$ remains open

For $N \geq n + 2$, the dimension of the kernel X_2 is at least two. Local positive index no longer excludes multiple intersections through one-dimensional sign alternation. Higher-dimensional maps can have multiple regular zeros, all with local index $+1$, while still failing to be injective. Thus the Neural Jacobian Conjecture remains open in this higher-width regime.

4 Conclusion and Outlook

By reflecting on the classical Jacobian conjecture, Moonshine extracted the core principle "local nondegeneracy implies global injectivity" and transferred it to one-hidden-layer affine-ridge sigmoid networks, thereby proposing the Neural Jacobian Conjecture. If fully true, the conjecture would reveal an intrinsic rigidity of a special class of neural networks. Even if it is eventually disproved, its exploration helps clarify the boundary between local diffeomorphism and global injectivity.

Moonshine has proved the NJC in the lowest nontrivial widths $N = n$ and $N = n + 1$, providing initial evidence for its plausibility. For the general higher-width case $N \geq n + 2$, the conjecture is neither proved nor disproved and remains an active open problem. This exemplifies Moonshine's working mode as a conjecture-generating mathematical agent: it formulates precise conjectures, establishes rigorous partial results, and identifies the unresolved boundary that guides subsequent research.

The complete source code, research logs, and intermediate verification records are available in the project repository: https://github.com/DeepMathLLM/Moonshine.

A Supplementary Remarks

Remark A.1 (The Jacobian determinant cannot be a nonzero constant). For $N > n$, the Jacobian determinant $\det DF(x)$ of a network $F \in \mathcal{N}_{n,N}$ can never be a nonzero constant. By the Cauchy-Binet formula,

$$\det(AD(x)B) = \sum_{|I|=n} \det(A_I)\det(B_I) \prod_{i \in I} \sigma'(b_i \cdot x + c_i),$$

where I ranges over all n-element subsets of $\{1, \ldots, N\}$, A_I is the $n \times n$ submatrix of A with columns in I, B_I is the $n \times n$ submatrix of B with rows in I, and the product consists of positive diagonal entries.

Choose a direction $u \in \mathbb{R}^n$ such that $b_i \cdot u \neq 0$ for all nonzero rows b_i. Along the ray $x = tu$, as $t \to +\infty$, each preactivation tends to $\pm\infty$, and hence $\sigma'(b_i \cdot (tu) + c_i) \to 0$. Thus every summand tends to zero, and $\det DF(tu) \to 0$. If $\det DF$ were a constant c, this limit would force $c = 0$. Hence the determinant cannot be a nonzero constant.

Remark A.2 (Necessity of full-rank conditions). If $\operatorname{rank} A < n$ or $\operatorname{rank} B < n$, then $AD(x)B$ has rank at most $\min(\operatorname{rank} A, \operatorname{rank} B) < n$, and therefore $\det DF(x) = 0$ for all x. Thus any $F \in \mathcal{N}^+_{n,N}$ automatically satisfies $\operatorname{rank} A = \operatorname{rank} B = n$, which in particular requires $N \geq n$.

Remark A.3 (Other activation functions). The proofs use only the facts that $\sigma'(t) > 0$ and that σ has a smooth inverse on its image. Therefore the same result holds for any strictly increasing C^1 activation function mapping $\mathbb{R}$ diffeomorphically onto an open interval, such as $(0, 1)$. The logistic sigmoid is merely a convenient example.

Remark A.4 (The complex case is not a direct analogue of the real case). The NJC considered in this paper is a real-domain statement. If sigmoid networks are complexified directly, the analogous Jacobian-type conclusion generally fails.

Consider the complex logistic sigmoid

$$\sigma(z) = \frac{1}{1 + e^{-z}}.$$

It is meromorphic, with poles at

$$z = (2k+1)\pi i, \qquad k \in \mathbb{Z}.$$

For a complex shallow sigmoid network

$$F(z) = W\sigma(Az+b) + c, \qquad z \in \mathbb{C}^n,$$

where $A \in \mathbb{C}^{N\times n}$, $W \in \mathbb{C}^{n\times N}$, and $b \in \mathbb{C}^N$, let the j-th preactivation be

$$\ell_j(z) = a_j^T z + b_j, \qquad j = 1, \dots, N.$$

The natural holomorphic domain is

$$\Omega = \mathbb{C}^n \setminus \bigcup_{j=1}^{N} \bigcup_{k\in\mathbb{Z}} \{z \in \mathbb{C}^n : \ell_j(z) = (2k+1)\pi i\}.$$

On this domain the complex sigmoid satisfies

$$\sigma(\zeta + 2\pi i) = \sigma(\zeta).$$

Thus one may define the period lattice of the input matrix A by

$$L(A) = \{v \in \mathbb{C}^n : Av \in (2\pi i\mathbb{Z})^N\}.$$

If $0 \neq v \in L(A)$, then $\sigma(A(z+v)+b) = \sigma(Az+b)$, and hence $F(z+v) = F(z)$. Such a period vector gives non-injectivity.

In the square case $N = n$, if $A, W \in GL_n(\mathbb{C})$, then

$$DF(z) = W \operatorname{diag}(\sigma'(Az+b))A.$$

On Ω, σ' has no zeros, so $\det DF(z) \neq 0$. However, for any nonzero $m \in \mathbb{Z}^n$,

$$v = A^{-1}(2\pi i m)$$

is a nonzero period vector, and therefore $F(z+v) = F(z)$. Thus the implication

$$\det DF(z) \neq 0 \text{ on } \Omega \quad \Longrightarrow \quad F \text{ is injective on } \Omega$$

is false in the complex setting.

For $N > n$, a second mechanism appears: cancellation by the output kernel. For example, with $n = 2$ and $N = 3$, define

$$F(z_1, z_2) = \begin{pmatrix} \sigma(z_1 + \sqrt{2}z_2) - \sigma(z_1 + z_2) \\ \sigma(\sqrt{3}z_1 + z_2) - \sigma(z_1 + z_2) \end{pmatrix}.$$

This corresponds to

$$A = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{3} & 1 \\ 1 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Take

$$z = \left(\frac{2\pi i}{\sqrt{3}-1}, \frac{2\pi i}{\sqrt{2}-1} \right).$$

Then

$$z_1 + \sqrt{2}z_2 = (z_1 + z_2) + 2\pi i, \qquad \sqrt{3}z_1 + z_2 = (z_1 + z_2) + 2\pi i.$$

By periodicity, $F(z) = 0 = F(0)$, with $z \neq 0$. Hence this network is not injective.

References

[1] Bass, H., Connell, E. H., and Wright, D. The Jacobian conjecture: reduction of degree and formal expansion of the inverse. *Bulletin of the American Mathematical Society (New Series)*, 7(2), 287–330, 1982.

[2] Gale, D., and Nikaido, H. The Jacobian matrix and global univalence of mappings. *Mathematische Annalen*, 159, 81–93, 1965.

[3] Guillemin, V., and Pollack, A. *Differential Topology*. Prentice-Hall, Englewood Cliffs, 1974.

[4] Hirsch, M. W. *Differential Topology*. Graduate Texts in Mathematics, Vol. 33. Springer, New York, 1976.

[5] Horn, R. A., and Johnson, C. R. *Matrix Analysis*. 2nd ed. Cambridge University Press, Cambridge, 2012.

[6] Keller, O.-H. Ganze Cremona-Transformationen. *Monatshefte fuer Mathematik und Physik*, 47, 299–306, 1939.

[7] Pinchuk, S. A counterexample to the strong real Jacobian conjecture. *Mathematische Zeitschrift*, 217, 1–4, 1994.

[8] Smale, S. Mathematical problems for the next century. *The Mathematical Intelligencer*, 20(2), 7–15, 1998.

[9] van den Essen, A. *Polynomial Automorphisms and the Jacobian Conjecture*. Progress in Mathematics, Vol. 190. Birkhaeuser, Basel, 2000.

CLASSICAL AND UMBRAL MOONSHINE: CONNECTIONS AND p-ADIC PROPERTIES

KEN ONO, LARRY ROLEN, SARAH TREBAT-LEDER

Abstract. The classical theory of *monstrous moonshine* describes the unexpected connection between the representation theory of the monster group M, the largest of the simple sporadic groups, and certain modular functions, called Hauptmoduln. In particular, the n-th Fourier coefficient of Klein's j-function is the dimension of the grade n part of a special infinite dimensional representation $V^\natural$ of the monster group. More generally the coefficients of Hauptmoduln are graded traces T_g of $g \in M$ acting on $V^\natural$. Similar phenomena have been shown to hold for the Mathieu group M_{24}, but instead of modular functions, *mock modular forms* must be used. This has been conjecturally generalized even further, to *umbral moonshine*, which associates to each of the 23 Niemeier lattices a finite group, infinite dimensional representation, and mock modular form. We use *generalized Borcherds products* to relate monstrous moonshine and umbral moonshine. Namely, we use mock modular forms from umbral moonshine to construct via generalized Borcherds products rational functions of the Hauptmoduln T_g from monstrous moonshine. This allows us to associate to each pure A-type Niemeier lattice a conjugacy class g of the monster group, and gives rise to identities relating dimensions of representations from umbral moonshine to values of T_g. We also show that the logarithmic derivatives of the Borcherds products are p-adic modular forms for certain primes p and describe some of the resulting properties of their coefficients modulo p.

1. Introduction

Monstrous moonshine begins with the surprising connection between the coefficients of the modular function

$$J(\tau) := j(\tau) - 744 = \frac{(1 + 240\sum_{n=1}^{\infty}\sum_{d|n} d^3q^n)^3}{\prod_{n=1}^{\infty}(1-q^n)^{24}q} - 744 = \frac{1}{q} + 196884q + 21493760q^2 + \dots$$

and the representation theory of the monster group M, which is the largest of the simple sporadic groups. Here $q := e^{2\pi i\tau}$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} \colon \Im z > 0\}$. McKay noticed that 196884, the q^1 coefficient of $J(\tau)$, can expressed as a linear combination of dimensions of irreducible

2010 *Mathematics Subject Classification.* 11F30, 11F33, 11F37.

Key words and phrases. Borcherds products, umbral moonshine, monstrous moonshine, mock modular forms, harmonic Maass forms.

The first author thanks the NSF and the Asa Griggs Candler Fund for their generous support. The second author thanks the University of Cologne and the DFG for his generous support via the University of Cologne postdoc grant DFG Grant D-72133-G-403-151001011. The third author thanks the NSF for its support. The authors began jointly discussing this work at the mock modular forms, moonshine, and string theory conference in Stony Brook, August 2013 and are grateful for the good hospitality and excellent conference. The authors are also grateful to John Duncan for useful comments which improved the quality of exposition.

representations of the monster group M. Namely,

$$196884 = 196883 + 1.$$

McKay saw that the same was true for other Fourier coefficients of $J(\tau)$. For example,

$$21493760 = 21296876 + 196883 + 1.$$

In [28], McKay and Thompson conjectured that the n-th Fourier coefficient of $J(\tau)$is the dimension of the grade n part of a special infinite-dimensional graded representation $V^\natural$ of M.

This was later expanded into the full monstrous moonshine conjecture by Thompson, Conway, and Norton [11, 27]. Since the graded dimension is just the graded trace of the identity element, they looked at the graded traces $T_g(\tau)$ of nontrivial elements g of M acting on $V^\natural$ and conjectured that they were all expansions of principal moduli, or Hauptmoduln, for certain genus zero congruence groups Γ_g commensurable with $\mathrm{SL}_2(\mathbb{Z})$. Note that these T_g are constant on each of the 194 conjugacy classes of M, and therefore are class functions, which automatically have coefficients which are $\mathbb{C}$-linear combinations of irreducible characters of M. Part of the task of proving monstrous moonshine was showing that they were in fact $\mathbb{Z}_{\geq 0}$-linear combinations.

By way of computer calculation, Atkin, Fong, and Smith [26] verified the existence of a virtual representation of M. Then using vertex-operator theory, Frenkel, Lepowsky, and Meurman [16] finally constructed a representation $V^\natural$ of M thereby providing a beautiful algebraic explanation for the original numerical observations of McKay and Thompson. Borcherds [1] further developed the theory of vertex-operator algebras, which he then used in [2] to prove the full conjectures as given by Conway and Norton.

Monstrous moonshine provides an example of coefficients of modular functions enjoying distinguished properties. Moreover, their values at Heegner points have also been considered important. A *Heegner point* τ of discrimant $d < 0$ is a complex number of the form $\tau = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ with $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, and $d = b^2 - 4ac$. The values of principal moduli at such points are called *singular moduli*. As an example of their importance, it is a classical fact that the singular moduli of $j(\tau)$ generate Hilbert class fields of imaginary quadratic fields. Moreover, the other McKay-Thompson series arising in monstrous moonshine satisfy analogous properties [7]. It is natural to ask what other interesting properties the values of the Hauptmoduln $T_g(\tau)$ could possess." We show that some of these values are related to another kind of moonshine, called *umbral moonshine*.

Recently, it was shown that phenomena similar to monstrous moonshine occur for other q-series and groups. In particular, the Mathieu group M_{24} exhibits moonshine [15, 19], with the role of the j-invariant played by a *mock modular form* of weight 1/2, denoted $H^{(2)}(\tau)$. A mock modular form is the holomorphic part of a *harmonic weak Maass form*. Cheng, Duncan, and Harvey conjecture in [10] that this is a special case of a more general phenomenon, which they call *umbral moonshine*. For each of the 23 Niemeier lattices X they associate a vector-valued mock modular form $H^X(\tau)$, a group G^X, and an infinite-dimensional graded representation K^X of G^X such that the Fourier coefficients of H^X encode the dimensions of the graded components of K^X.

In particular, if $c^X(n, h)$ is the n-th Fourier coefficient of the h-th component of H^X, then

$$c^X(n,h) = \begin{cases} a^X \dim_{K^X_{h,-D/4m}} & \text{if } n = -D/4m \text{ where } D \in \mathbb{Z}, D = h^2 \pmod{4m}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where $a^X \in \{1, 1/3\}$ and

$$K^X = \bigoplus_{h \pmod{2m}} \bigoplus_{\substack{D \in \mathbb{Z} \\ D = h^2 \pmod{4m}}} K^X_{h,-D/4m}.$$

For more information on umbral moonshine see Section 2 and for a definition of H^X see Section 4.

Using *generalized Borcherds products* (see [5]), we describe a connection between the mock modular forms $H^X(\tau)$ of umbral moonshine and the McKay-Thompson series $T_g(\tau)$ of monstrous moonshine. *Generalized Borcherds products* are a method to produce modular functions as infinite products of rational functions whose exponents come from the coefficients of mock modular forms, and they can be viewed as generalizations of the automorphic products in Theorem 13.3 of [3].

We focus on the Niemeier lattices X whose root systems are of pure A-type according to the ADE classification. They are listed in Table 1, along with their Coxeter numbers $m(X)$ and the notation we will use for the mock modular form H^X.

TABLE 1. Pure A-type root systems

Root System X	Coxeter Number $m(X)$	mock modular Form H^X
A_1^{24}	2	$H^{(2)}(\tau)$
A_2^{12}	3	$H^{(3)}(\tau)$
A_3^{8}	4	$H^{(4)}(\tau)$
A_4^{6}	5	$H^{(5)}(\tau)$
A_6^{4}	7	$H^{(7)}(\tau)$
A_8^{3}	9	$H^{(9)}(\tau)$
A_{12}^{2}	13	$H^{(13)}(\tau)$
A_{24}^{1}	25	$H^{(25)}(\tau)$

Table 2 gives the monstrous moonshine dictionary for the conjugacy classes g which correspond to pure A-type cases of umbral moonshine[1]. Note that $\eta(\tau)$ is the *Dedekind eta function*, defined by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

All of our Hauptmoduln are normalized so that they have the form $q^{-1} + O(q)$, which is why all of the η-quotients in the table have a constant added to them.

[1]The case $X = A_{24}$ corresponds to $g(X) = (25Z)$, which is what Conway and Norton call a "ghost element". This means that $\Gamma_0(25)$ is the only genus zero $\Gamma_0(N)$ that does not correspond to a conjugacy class of the monster group. The parentheses are used to indicate a ghost element.

TABLE 2. The dictionary of monstrous moonshine

Monster Conjugacy Class g	Congruence Subgroup Γ_g	McKay-Thomspon Series $T_g(\tau)$
$2B$	$\Gamma_0(2)$	$\eta(\tau)^{24}/\eta(2\tau)^{24}+24$
$3B$	$\Gamma_0(3)$	$\eta(\tau)^{12}/\eta(3\tau)^{12}+12$
$4C$	$\Gamma_0(4)$	$\eta(\tau)^{8}/\eta(4\tau)^{8}+8$
$5B$	$\Gamma_0(5)$	$\eta(\tau)^{6}/\eta(5\tau)^{6}+6$
$7B$	$\Gamma_0(7)$	$\eta(\tau)^{4}/\eta(7\tau)^{4}+4$
$9B$	$\Gamma_0(9)$	$\eta(\tau)^{3}/\eta(9\tau)^{3}+3$
$13B$	$\Gamma_0(13)$	$\eta(\tau)^{2}/\eta(13\tau)^{2}+2$
$(25Z)$	$\Gamma_0(25)$	$\eta(\tau)/\eta(25\tau)+1$

There is an evident correspondence between the pure A-type lattices X in Table 1 and the conjugacy classes g in Table 2. We give this correspondence in Table 3.

We show that for a pure A-type Niemeier lattice X and its corresponding conjugacy class $g := g(X)$, the "Galois (twisted) traces" of the CM values of the McKay-Thompson series $T_g(\tau)$ are the coefficients of the mock modular form H^X. To more precisely state this, we set up the following notation.

Let X be a pure A-type Niemeier lattice with Coxeter number $m := m(X)$ and corresponding conjugacy class $g := g(X)$. We call a pair (Δ, r) *admissible* if Δ is a negative fundamental discriminant and $r^2 \equiv \Delta \pmod{4m}$.

Theorem 1.1. *Let $c^+(n,h)$ be the n-th Fourier coefficient of the h-th component of H^X. Let (Δ, r) be an admissible pair for X. Then the twisted generalized Borcherds product*

$$\Psi_{\Delta,r}(\tau, H^X) := \prod_{n=1}^{\infty} P_\Delta(q^n)^{c^+\left(\frac{|\Delta|n^2}{4m}, \frac{rn}{2m}\right)},$$

where

$$P_\Delta(x) := \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} [1 - e(b/\Delta)x]^{\left(\frac{\Delta}{b}\right)}$$

is a rational function in $T_g(\tau)$ with a discriminant Δ Heegner divisor.

Remark 1. We consider only the pure A-type cases, because these are the ones for which the harmonic Maass form transforms under the Weil representation. See Section 4 for more information.

The next result gives a precise description of the rational functions in Theorem 1.1. In particular, it gives a "twisted" trace function for the values of T_g at points in the divisor and

the coefficients c^+ of the mock modular forms H^X. It is often the case that coefficients of automorphic forms can be expressed in terms of singular moduli (see e.g., [4, 6, 13, 29]).

TABLE 3. Correspondence Between Umbral and Monstrous Moonshine

Root System X	Conjugacy Class $g(X)$
A_1^{24}	$2B$
A_2^{12}	$3B$
A_3^{8}	$4C$
A_4^{6}	$5B$
A_6^{4}	$7B$
A_8^{3}	$9B$
A_{12}^{2}	$13B$
A_{24}^{1}	$(25Z)$

Corollary 1.2. *In the notation of Theorem 1.1, write*

$$\Psi_{\Delta,r}(\tau, H^X) = \prod_i \left(T_g(\tau) - T_g(\alpha_i)\right)^{\gamma_i}.$$

Then we have that

$$c^+\left(\frac{|\Delta|}{4m}, \frac{r}{2m}\right) = \frac{1}{\epsilon_\Delta}\sum_i \gamma_i \cdot T_g(\alpha_i),$$

where

$$\epsilon_\Delta = \sum_{b\in\mathbb{Z}/|\Delta|\mathbb{Z}} e(b/\Delta)\cdot\left(\frac{\Delta}{b}\right).$$

Remark 2. Assuming the umbral moonshine conjecture, the previous corollary implies the following "degree" formula in traces of singular moduli for classical moonshine functions:

$$\frac{1}{\epsilon_\Delta}\sum_i \gamma_i \cdot T_g(\alpha_i) = c^+\left(\frac{|\Delta|}{4m}, \frac{r}{2m}\right) = a^X \dim_{K^X_{r,|\Delta|/4m}}. \tag{1.2}$$

In the case where $m = 2$, the relationship between the coefficients of the mock-modular form and the dimensions of the graded components of the representation has been proven by Gannon[19], and so our work implies the following:

$$\frac{1}{\epsilon_\Delta}\sum_i \gamma_i \cdot T_{2B}(\alpha_i) = c^+\left(\frac{|\Delta|}{8}, \frac{r}{4}\right) = \dim_{K^{(2)}_{r,|\Delta|/8}}. \tag{1.3}$$

Example. Let $X = A_1^{24}$, so $m(X) = 2$ and $g(X) = 2B$. Then the corresponding McKay-Thompson series is

$$T_g(\tau) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24 = \frac{1}{q} + 276q + \dots.$$

We pick the admissible pair $(\Delta, r) = (-7, 1)$. In Section 5, we will show that

$$\begin{aligned}\Psi_{\Delta,r}(\tau, H^X) &= \frac{(T_g(\tau) - T_g(\alpha_1))^2}{(T_g(\tau) - T_g(\alpha_2))^2} = \frac{\left(T_g(\tau) - \frac{1-45\sqrt{-7}}{2}\right)^2}{\left(T_g(\tau) - \frac{1+45\sqrt{-7}}{2}\right)^2} \\ &= 1 + 90\sqrt{-7}q + (28350 + 45\sqrt{-7})q^2 + \dots,\end{aligned}$$

where $\alpha_1 := \frac{-1+\sqrt{-7}}{4}$ and $\alpha_2 := \frac{1+\sqrt{-7}}{4}$. Note that $T_g(\alpha_1)$ and $T_g(\alpha_2)$ are algebraic integers of degree 2 which form a full set of conjugates. Their twisted trace is

$$2[T_g(\alpha_1) - T_g(\alpha_2)] = -90\sqrt{-7},$$

which matches the q^1 Fourier coefficient above. To check Corollary 1.2, we note that

$$\epsilon_\Delta = \sum_{b \in \mathbb{Z}/7\mathbb{Z}} e(-b/7) \cdot \left(\frac{-7}{b}\right) = -\sqrt{-7}$$

and

$$\frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 90 = c^+(7/8, 1/4) = \dim_{K^{(2)}_{1,7/8}}.$$

Example. As a second example, again consider $X = A_1^{24}$, so $m(X) = 2$ and $g(X) = 2B$. We pick the admissible pair $(\Delta, r) = (-15, 1)$. Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the roots of

$$x^4 - 47x^3 + 192489x^2 - 9012848x + 122529840,$$

with γ_1, γ_2 having positive imaginary parts. Then

$$\Psi_{-15,1} = \frac{(T_g(\tau) - \gamma_1)^2 (T_g(\tau) - \gamma_2)^2}{(T_g(\tau) - \gamma_3)^2 (T_g(\tau) - \gamma_4)^2}.$$

We get that

$$\epsilon_{-15} = \sqrt{-15},$$

and

$$\frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 462 = c^+(15/8, 1/4) = \dim_{K^{(2)}_{1,15/8}}.$$

In view of this correspondence, it is clear that the mock modular forms of umbral moonshine have important properties. The congruence properties of their coefficients have just begun to be studied. For example, [12] examines the parity of the coefficients of the McKay-Thompson series for Mathieu moonshine in relation to a certain conjecture in [9], which in our case corresponds to $X = A_1^{24}$. Congruences modulo higher primes were also considered in [22].

Let $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Given the product expansion of a generalized Borcherds product, it is natural to consider its logarithmic derivative. It turns out that this logarithmic derivative has nice arithmetic properties. This idea was also used in [5] and [23].

Theorem 1.3. *Fix a pure A-type Niemeier lattice X with Coxeter number m. Let (Δ, r) be an admissible pair. Consider the logarithmic derivative*

$$f_{\Delta,r}(\tau) = \sqrt{\Delta} \sum a_{\Delta,r}(n) q^n := \sqrt{\Delta} \sum_n \sum_{ij=n} i c^+ \left(\frac{|\Delta| i^2}{4m}, \frac{ri}{2m} \right) \left(\frac{\Delta}{j} \right) q^n$$

of $\Psi_{\Delta,r}(\tau) = \Psi_{\Delta,r}(\tau, H^X)$. Then $f_{\Delta,r}(\tau)$ is a meromorphic weight 2 modular form.

When p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$, it turns out that $f_{\Delta,r}(\tau)$ is more than just a meromorphic modular form; it is a p-adic modular form. Essentially, a p-adic modular form is a q-series which is congruent modulo any power of p to a holomorphic modular form; we refer the reader to Section 6.1 for the definition.

Theorem 1.4. *Let X be a pure A-type Niemeier lattice with Coxeter number m. Let (Δ, r) be admissible and suppose p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$. Then $f_{\Delta,r}$ is a p-adic modular form of weight 2.*

We will use this result to study the p-divisibility of the coefficients $a_{\Delta,r}(n)$.

Corollary 1.5. *Let X, Δ, r, p be as above. Then for all $k \geq 1$ there exists $\alpha_k > 0$ such that*

$$\#\{n \leq x : a_{\Delta,r}(n) \not\equiv 0 \pmod{p^k}\} = O\left(\frac{x}{(\log x)^{\alpha_k}} \right).$$

In particular, if we let

$$\pi_{\Delta,r}(x; p^k) := \#\{n \leq x : a_{\Delta,r}(n) \equiv 0 \pmod{p^k}\},$$

then

$$\lim_{x \to \infty} \frac{\pi_{\Delta,r}(x; p^k)}{x} = 1.$$

Remark 3. Corollary 1.5 also applies to any constant multiple of $f_{\Delta,r}$ with integral coefficients. In the example below, we consider the coefficients of

$$\frac{f_{-7,1}(\tau)}{90\sqrt{-7}} = q + O(q^2).$$

However, it is not always the case that the analogous normalization has integral coefficients.

Example. We illustrate Corollary 1.5 for $X = A_1^{24}$, $\Delta = -7$, $r = 1$. Note that this is the same case considered in Example 1. The first few coefficients of the normalized logarithmic derivative are given by

$$\frac{f_{-7,1}(\tau)}{90\sqrt{-7}} =: \sum_{n \geq 1} a_{-7,1}(n) q^n = q + q^2 - 4371q^3 + q^4 + 17773755q^5 + \dots$$

The prime $p = 2$ is split in $\mathbb{Q}(\sqrt{-7})$, and so Theorem 1.4 and Corollary 1.5 do not apply. Therefore, we expect the coefficients $a_{-7,1}(n)$ to be equally distributed modulo 2, but cannot prove anything about them. The prime $p = 3$ is inert, so Corollary 1.5 tell us that, asymptotically, 100% of the coefficients $a_{-7,-1}(n)$ are divisible by 3. We illustrate this behavior in Table 1.

TABLE 4. Divisibility of $a_{-7,1}(n)$ by $p = 2, 3$

x	$\pi_2(x)/x$	$\pi_3(x)/x$
50	0.38	0.64
100	0.45	0.68
150	0.47	0.69
200	0.49	0.71
250	0.48	0.71
300	0.49	0.72
$\vdots$	$\vdots$	$\vdots$
∞	.5?	1

2. UMBRAL MOONSHINE

In this section, we summarize the main objects and conjectures of umbral moonshine. However, we first briefly describe Mathieu moonshine, which umbral moonshine generalized.

2.1. **Mathieu Moonshine.** In 2010, the study of a new form of moonshine commenced, called Mathieu moonshine. Let $\mu(z, \tau) := \mu(z, z, \tau)$ be Zwegers' famous function from his thesis [30], which is defined in the appendix. Let $H^{(2)}(\tau)$ be the q-series

$$H^{(2)}(\tau) := -8 \sum_{\omega \in \{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}} \mu(\omega, \tau) = 2q^{-1/8}(-1 + 45q + 231q^2 + \dots),$$

which occurs in the decomposition of the elliptic genus of a K3 surface into irreducible characters of the $N = 4$ superconformal algebra. This is a mock-modular form, and plays the role of $J(\tau)$ in Mathieu moonshine. Eguchi, Ooguri, and Tachikawa conjectured that the Fourier coefficients encode dimensions of irreducible representations of the Mathieu group M_{24} [15]. This was extended to the full Mathieu moonshine conjecture by [8, 14, 17, 18], which included providing mock modular forms $H_g^{(2)}$ for every $g \in M_{24}$. The existence of an infinite dimensional M_{24} module underlying the mock modular forms was shown by Gannon in 2012 [19].

2.2. **The Objects of Umbral Moonshine.** Cheng, Duncan, and Harvey generalized even further - conjecturing that Mathieu moonshine is but one example of a more general phenomenon which they call umbral moonshine [10].

For each of the 23 Niemeier root systems X, which are unions of irreducible simply-laced root systems with the same Coxeter number, they associate many objects, including a group G^X (playing the role of M), a mock modular form $H^X(\tau)$ (playing the role of $j(\tau)$), and an infinite dimensional graded G^X module K^X (playing the role of the M-module $V^\natural$) Table 2.2 gives a more complete list of the associated objects.

The ADE classification of simply laced Dynkin diagrams allows us to classify the irreducible components of the Niemeier root systems X. We will focus on the simplest cases - the root systems of pure A-type, i.e. $X = A_{m-1}^{24/(m-1)}$, where $(m-1) \mid 24$. In these cases, the lambency ℓ is an integer and equals m, and $\Gamma^X = \Gamma_0(m)$. The case $X = A_1^{24}$ corresponds to Mathieu moonshine, with $G^X = M_{24}$ and $H^X = H^{(2)}$, as defined above. We will generally refer to H^X ,

TABLE 5. This table gives the objects associated to a Niemeier root system X

L^X	The Niemeier lattice corresponding to X
m	The Coxeter number of all irreducible components of X
W^X	The Weyl group of X
$G^X := \mathrm{Aut}(L^X)/W^X$	The umbral group corresponding to X
π^X	The (formal) product of Frame shapes of Coxeter elements of irreducible components of X
Γ^X	The genus zero subgroup attached to X
T^X	The normalized Hauptmodul of Γ^X, whose eta-product expansion corresponds to π^X
ℓ	The lambency. A symbol that encodes the genus zero group Γ^X. Sometimes used instead of X to denote which case of umbral moonshine is being considered.
ψ^X	The unique meromorphic Jacobi form of weight 1 and index m satisfying certain conditions.
H^X	The vector-valued mock modular form of weight $1/2$ whose $2m$ components furnish the theta expansion of the finite part of ψ^X. Called the umbral mock modular form.
S^X	The vector-valued cusp form of weight $3/2$ which is the shadow of H^X. Called the umbral shadow.
H_g^X	The umbral McKay-Thompson series attached to $g \in G^X$. It is a vector-valued mock modular form of weight $1/2$, and equals H^X when g is the identity.
S_g^X	The vector-valued cusp form *conjectured* to be the shadow of H_g^X.
K^X	The *conjectural* infinite dimensional graded G^X-module whose graded super-dimension is encoded by H^X.

S^X, ψ^X, and T^X as $H^{(m)}$, $S^{(m)}$, $\psi^{(m)}$, and j_m respectively. These are the main quantities from Table 2.2 that we will work with, and we will only define them for pure A-type. This is done in Section 4.

2.3. **The Conjectures of Umbral Moonshine.** The main conjectures of umbral moonshine are as follows:

(1) The mock modular form H^X encodes the graded super-dimension of a certain infinite-dimensional, $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Q}$-graded G^X-module K^X.
(2) The graded super-characters H_g^X arising from the action of G^X on K^X are vector-valued mock modular forms with concretely specified shadows S_g^X.
(3) The umbral McKay-Thompson series H_g^X are uniquely determined by an optimal growth property which is directly analogous to the genus zero property of monstrous moonshine.

3. Vector-valued Modular Forms

In this section, we follow [5] in giving the needed background on vector-valued modular forms, though we state results in less generality.

3.1. **A Lattice related to $\Gamma_0(m)$.** We will define a lattice L and a dual lattice L' related to $\Gamma_0(m)$ such that the components of our vector-valued modular forms are labeled by the elements of L'/L.

We consider the quadratic space

$$V := \{X \in \mathrm{Mat}_2(\mathbb{Q}) : \mathrm{tr}(X) = 0\}$$

with the quadratic form $P(X) := m \det(X)$.[2] The corresponding bilinear form is then $(X, Y) := -m\,\mathrm{tr}(XY)$. Let L be the lattice

$$L := \left\{ \begin{pmatrix} b & -a/m \\ c & -b \end{pmatrix}; \quad a, b, c \in \mathbb{Z} \right\}.$$

The dual lattice is then given by

$$L' := \left\{ \begin{pmatrix} b/2m & -a/m \\ c & -b/2m \end{pmatrix}; \quad a, b, c \in \mathbb{Z} \right\}.$$

We will switch between viewing elements of L' as matrices and as quadratic forms, with the matrix

$$X = \begin{pmatrix} b/2m & -a/m \\ c & -b/2m \end{pmatrix}$$

corresponding to the integral binary quadratic form

$$Q = [mc, b, a] = mcx^2 + bcy + cy^2.$$

Note that then $P(X) = -\mathrm{Disc}(Q)/4m$.

We identify L'/L with $(\frac{1}{2m}\mathbb{Z})/\mathbb{Z}$, and the quadratic form P with the quadratic form $\frac{h}{2m} \mapsto \frac{-h^2}{4m}$ on $\mathbb{Q}/\mathbb{Z}$. We will also occasionally identify $\frac{h}{2m} \in \mathbb{Q}/\mathbb{Z}$ with $h \in \mathbb{Z}/2m\mathbb{Z}$.

For a fundamental discriminant D and $r/2m \in L'/L$ with $r^2 \equiv D \pmod{4N}$, let

$$Q_{D,r} := \{Q = [mc, b, a] : a, b, c \in \mathbb{Z}, \mathrm{Disc}(Q) = D, b \equiv r \pmod{2m}\}. \tag{3.1}$$

The action of $\Gamma_0(m)$ on this set is given by the usual action of congruence subgroups on binary quadratic forms. We will later be working with $Q_{D,r}/\Gamma_0(m)$.

[2]Note that this corrects a typo in [5].

3.2. **The Weil representation.** By $\mathrm{Mp}_2(\mathbb{Z})$ we denote the integral metaplectic group. It consists of pairs (γ, ϕ), where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi^2(\tau) = c\tau + d$. The group $\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$ is generated by $S := (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau})$ and $T := (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$.

We consider the *Weil representation* ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ corresponding to the discriminant form L'/L. Let $e(a) := e^{2\pi i a}$. We denote the standard basis elements of $\mathbb{C}[L'/L]$ by $\mathfrak{e}_h$, $h/2m \in L'/L$. Then the Weil representation ρ_L associated with the discriminant form L'/L is the unitary representation of $\widetilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)\mathfrak{e}_h = e(h^2/4m)\mathfrak{e}_h,$$

and

$$\rho_L(S)\mathfrak{e}_h = \frac{e(-1/8)}{\sqrt{2m}} \sum_{h' \in \mathbb{Z}/2m\mathbb{Z}} e(hh'/2m)\mathfrak{e}_{h'}.$$

3.3. **Harmonic weak Maass forms.** If $f \colon \mathbb{H} \to \mathbb{C}[L'/L]$ is a function, we write

$$f = \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} f_h \mathfrak{e}_h$$

for its decomposition into components. For $k \in \frac{1}{2}\mathbb{Z}$, let $M^!_{k,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$ valued weakly holomorphic modular forms of weight k and type ρ_L for the group $\widetilde{\Gamma}$. The subspaces of holomorphic modular forms (resp. cusp forms) are denoted by M_{k,ρ_L} (resp. S_{k,ρ_L}). Now, assume that $k \leq 1$. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form* (of weight k with respect to $\widetilde{\Gamma}$ and ρ_L) if it satisfies:

(1) $f(M\tau) = \phi(\tau)^{2k}\rho_L(M, \phi)f(\tau)$ for all $(M, \phi) \in \widetilde{\Gamma}$;
(2) $\Delta_k f = 0$;
(3) There is a polynomial

$$P_f(\tau) = \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} \sum_{\substack{n \in \mathbb{Z} - \frac{h^2}{4m}, \\ -\infty < < n \leq 0}} c^+(n, h)e(n\tau)\mathfrak{e}_h$$

such that

$$f(\tau) - P_f = O(e^{-\epsilon v})$$

for some $\epsilon > 0$ as $v \to +\infty$.

Note here that

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

is the usual weight k hyperbolic Laplace operator, and that $\tau = u + iv$. We denote the vector space of these harmonic weak Maass forms by $\mathcal{H}_{k,\rho_L}$. The Fourier expansion of any $f \in \mathcal{H}_{k,\rho_L}$

gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h\in\mathbb{Z}/2m\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}-\frac{h^2}{4m},\\ -\infty<<n}} c^+(n,h)e(n\tau)\mathfrak{e}_h, \tag{3.2}$$

$$f^-(\tau) = \sum_{h\in L'/L} \sum_{\substack{n\in\mathbb{Q},\\ n<0}} c^-(n,h)W(2\pi nv)e(n\tau)\mathfrak{e}_h, \tag{3.3}$$

and $W(x) := \int_{-2x}^{\infty} e^{-t}t^{-k}dt = \Gamma(1-k, 2|x|)$ for $x < 0$. Then f^+ is called the *holomorphic part* and f^- the *nonholomorphic part* of f. The polynomial P_f is also uniquely determined by f and is called its *principal part*. We define a *mock modular form* of weight k to be the holomorphic part f^+ of a harmonic weak Maass form f of weight k which has $f^- \neq 0$. Its weight is just the weight of the harmonic weak Maass form.

Recall that there is an antilinear differential operator defined by

$$\xi_k : \mathcal{H}_{k,\overline{\rho}_L} \to S_{2-k,\rho_L}, \quad f(\tau) \mapsto \xi_k(f)(\tau) := 2iy^k\overline{\frac{\partial}{\partial\overline{\tau}}},$$

where $\overline{\rho}_L$ is the complex conjugate representation. The Fourier expansion of $\xi_k(f)$ is given by

$$\xi_k(f) = -\sum_{h\in\mathbb{Z}/2m\mathbb{Z}} \sum_{n\in\mathbb{Q},n>0} (4\pi n)^{1-k}\overline{c^-(-n,h)}q^n\mathfrak{e}_h.$$

The kernel of ξ_k is equal to $M^!_{k,\overline{\rho}_L}$, and we have the following exact sequence:

$$0 \to M^!_{k,\overline{\rho}_L} \to \mathcal{H}_{k,\overline{\rho}_L} \to S_{2-k,\rho_L} \to 0.$$

We call $\xi_k(f)$ the *shadow* of f. Note that $\xi_k(f)$ uniquely determines f^-, but the f^+ is only determined up to the addition of a weakly holomorphic modular form.

4. Defining the Umbral mock modular Forms

In this section we define the mock modular forms $H^{(m)}$ from umbral moonshine, as well as their shadows $S^{(m)}$ and non-holomorphic parts. Note that we only give definitions for the pure A-type cases - see [10] for a more detailed and general definition. We also refer the reader to the appendix for definitions of $\varphi_1^{(m)}, \mu_{1,0}(\tau,z), \theta_{m,r}(\tau,z),$ and $R(u;\tau)$.

For each lambency $m \in \{2,3,4,5,7,9,13,25\}$, which correspond to the pure A-type cases, define the Jacobi form $\psi^{(m)}$ by

$$\psi^{(m)}(\tau,z) := c_m\varphi_1^{(m)}(\tau,z)\mu_{1,0}(\tau,z),$$

where $c_m = 2$ for $m = 2,3,4,5,7,13$ and $c_m = 1$ for $m = 9,25$. We can break up $\psi^{(m)}$ into a finite part $\psi_F^{(m)}$ and a polar part $\psi_P^{(m)}$. The polar part is given by

$$\psi_P^{(m)}(\tau,z) = \frac{24}{m-1}\mu_{m,0}(\tau,z).$$

Then the mock modular form $H^{(m)}$ is defined by

$$\psi_F^{(m)}(\tau,z) = \psi^{(m)}(\tau,z) - \psi_P^{(m)}(\tau,z) = \sum_{h\in\mathbb{Z}/2m\mathbb{Z}} H_h^{(m)}(\tau)\theta_{m,h}(\tau,z), \tag{4.1}$$

where

$$\theta_{m,h}(\tau, z) := \sum_{n \equiv h \pmod{2m}} q^{n^2/4m} y^k.$$

Note that $\psi^{(m)}$ satisfies an optimal growth condition, which is that

$$q^{1/4m} H_h^X(\tau) = O(1) \tag{4.2}$$

as $\tau \to i\infty$ for all $h \in \mathbb{Z}/2m\mathbb{Z}$.

We also define the following, which will respectively be the shadow, non-holomorphic part, and harmonic weak Maass form corresponding to the mock modular form $H^{(m)}$:

$$S_h^{(m)}(\tau) := \sum_{n \equiv h \pmod{2m}} n q^{n^2/4m}, \tag{4.3}$$

$$\begin{aligned} F_h^{(m)}(\tau) &:= \int_{-\overline{\tau}}^{i\infty} \frac{S_h^{(m)}(z)}{\sqrt{-i(z+\tau)}} dz \\ &= -2m q^{-(h-m)^2/4m} R\left(\frac{h-m}{2m}(2m\tau) + \frac{1}{2}; 2m\tau\right), \text{ and} \end{aligned} \tag{4.4}$$

$$\widehat{H}_h^{(m)}(\tau) := H_h^{(m)}(\tau) + F_h^{(m)}(\tau) \tag{4.5}$$

Note that by definition, $S_h^{(m)}(\tau) = -S_{-h}^{(m)}(\tau)$. Therefore, $S_0^{(m)} = S_m^{(m)} = 0$. The same is true of $H_h^{(m)}$. We can write this in terms of Shimura's theta functions as $S_h^{(m)}(\tau) = \theta(\tau; h, 2m, 2m, x)$ [25]. Then using the transformation laws for his θ-functions, we get that $S^{(m)}$ transforms as follows:

$$\begin{aligned} S_h^{(m)}(\tau+1) &= e(h^2/4m) S_h^{(m)}(\tau), \text{ and} \\ S_h^{(m)}(-1/\tau) &= \tau^{3/2} \frac{e(-1/8)}{\sqrt{2m}} \sum_{k \pmod{2m}} e(kh/2m) S_k^{(m)}(\tau). \end{aligned}$$

Thus, we have

$$\begin{aligned} S^{(m)}(\tau+1) &= \rho_L(T) S^{(m)}(\tau), \text{ and} \\ S^{(m)}(-1/\tau) &= \tau^{3/2} \rho_L(S) S^{(m)}(\tau). \end{aligned}$$

From these transformations, we see that $S^{(m)}(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$ is a weight $3/2$ vector-valued modular form transforming under the Weil representation ρ_L, i.e. an element of the space $M_{3/2,\rho_L}$. From [10], we know that $H^{(m)}$ is a mock modular form with shadow $S^{(m)}$.[3] This gives us the following theorem.

Theorem 4.1. *We have that $\widehat{H}^{(m)}(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$ is a weight 1/2 vector-valued harmonic weak Maass form transforming under the Weil representation $\overline{\rho}_L$, i.e., it is an element of $\mathcal{H}_{1/2,\overline{\rho}_L}$. Moreover, it has shadow $S^{(m)}(\tau)$, non-holomorphic part $F^{(m)}$, and principal part $P(\tau) = -2q^{-1/4m}(\mathfrak{e}_1 - \mathfrak{e}_{2m-1})$.*

[3] In fact, it is the only vector-valued mock modular form with shadow $S^{(m)}$ satisfying the optimal growth condition in 4.2.

5. Relating umbral and monstrous moonshine

In this section, we explain the relationship between the mock modular forms $H^{(m)}$ from umbral moonshine and the Hauptmoduln T_g from monstrous moonshine.

5.1. Twisted Generalized Borcherds Products.

We begin by giving the theorem of Bruinier and Ono we will use.

Let $c^+(n,h)$ be the n-th Fourier coefficient of $H_h^{(m)}$. Let (Δ, r) be an admissible pair, so that Δ is a negative fundamental discriminant and $r^2 \equiv \Delta \pmod{4m}$. Let $\Psi_{\Delta,r}(\tau, \widehat{H}^m)$ be the twisted generalized Borcherds product defined in Theorem 1.1.

Theorem 5.1. *(Theorem 6.1 in [5]) We have that $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ is a weight 0 meromorphic modular function on $\Gamma_0(m)$ with divisor $Z_{\Delta,r}(\widehat{H}^{(m)})$.*

For this theorem to make sense, we need to define the twisted Heegner divisor $Z_{\Delta,r}(\widehat{H}^{(m)})$ associated to $\widehat{H}^{(m)}$. It is defined by

$$Z_{\Delta,r}(\widehat{H}^{(m)}) := \sum_{h \in \mathbb{Z}/2m\mathbb{Z}} \sum_{n<0} c^+(n,h) Z_{\Delta,r}(n,h).$$

Since the principal part of $\widehat{H}^{(m)}$ is $-2q^{-h^2/4m}(\mathfrak{e}_1 - \mathfrak{e}_{2m-1})$, this means that

$$Z_{\Delta,r}(\widehat{H}^{(m)}) = 2Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{-1}{2m}\right) - 2Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{1}{2m}\right).$$

Now, we just have to compute the divisors $Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{h}{2m}\right)$. They are defined as follows.

$$Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{h}{2m}\right) := \sum_{Q \in Q_{\Delta,hr}/\Gamma_0(m)} \frac{\chi_\Delta(Q)}{w(Q)} \alpha_Q,$$

where $w(Q) = 2$ for $\Delta < -4$, χ_Δ is the generalized genus character defined in Gross-Kohnen-Zagier, and α_Q is the unique root of $Q(x,1)$ in $\mathbb{H}$.

5.2. Proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1: Theorem 5.1 gives us that $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ is a weight 0 meromorphic modular function on $\Gamma_0(m)$ with specified divisor, which is a discriminant Δ Heegner divisor. For all of our m, $\Gamma_0(m)$ has genus zero. Therefore, $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ is a rational function in the Hauptmodul for $\Gamma_0(m)$. The normalized Hauptmodul, which we call $j_m(\tau)$, is defined by

$$j_m(\tau) := \frac{\eta(\tau)^{24/(m-1)}}{\eta(m\tau)^{24/(m-1)}} + \frac{24}{m-1}. \tag{5.1}$$

But using Table 1, we see that $j_m(\tau)$ is equal to $T_{g(X)}(\tau)$, the graded trace of $g(X) \in M$ on V. □

Proof of Corollary 1.2: From Theorem 1.1, we have that

$$\prod_{n=1}^{\infty} P_\Delta(q^n)^{c^+\left(\frac{|\Delta|n^2}{4m}, \frac{rn}{2m}\right)} = \prod_i (T_g(\tau) - T_g(\alpha_i))^{\gamma_i}.$$

We equate the q^1 Fourier coefficients of each side, using Table 2 to get the Fourier expansion

$$T_g(\tau) = \frac{1}{q} + O(q).$$

□

5.3. **Examples.** For each pure A-type case X with coxeter number m, we illustrate how to write $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ as a rational function in j_m. Note that here $\Delta < 0$ is a fundamental discriminant and $r \in \mathbb{Z}$ is such that $\Delta \equiv r^2 \pmod{4m}$.

First we work out an example for $m = 2$ in some detail, then list one example for each m. In Section 5.4, we explain how to find representatives of $Q_{\Delta,r}/\Gamma_0(m)$ using a method of Gross, Kohen, and Zagier.

Consider the case $m = 2, \Delta = -7, r = 1$. Using the method of Section 5.4, we compute that $Q_{-7,1}/\Gamma_0(2) = \{Q_1, Q_2\}$ and that $Q_{-7,-1}/\Gamma_0(2) = \{-Q_1, -Q_2\}$, where the quadratic forms Q, their Heenger points α_Q, and their generalized genus characters $\chi_\Delta(Q)$ are given in Table 5.3. We also include the value of j_2 at each Heegner point. Using the table, the divisor of $\Psi_{-7,1}(\tau)$

TABLE 6. Quadratic forms needed for $m = 2$, $\Delta = -7, r = 1$ case

quadratic form $= Q$	α_Q	$\chi_\Delta(Q)$	$j_2(\alpha_Q)$
$Q_1 = [2, 1, 1]$	$\alpha_1 = \frac{-1+\sqrt{-7}}{4}$	1	$\gamma_1 := \frac{1+45\sqrt{-7}}{2}$
$Q_2 = [-2, 1, -1]$	$\alpha_2 = \frac{1+\sqrt{-7}}{4}$	-1	$\gamma_2 := \frac{1-45\sqrt{-7}}{2}$
$-Q_2$	α_2	1	γ_2
$-Q_1$	α_1	-1	γ_1

is given by:

$$(-\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) = 2\alpha_2 - 2\alpha_1.$$

Therefore,

$$\Psi_{-7,1}(\tau, \widehat{H}^{(2)}) = \frac{(j_2(\tau) - \gamma_2)^2}{(j_2(\tau) - \gamma_1)^2}.$$

Similarly, for each value of m corresponding to a pure A-type case, we demonstrate in Table 7 how to write $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ as a rational function in j_m for some nice choice of Δ, r. In all the examples we consider,

$$\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)}) = \frac{(j_m(\tau) - \gamma_2)^2}{(j_m(\tau) - \gamma_1)^2}$$

for some $\gamma_1, \gamma_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{\Delta})}$. Note that $\Psi_{\Delta,r}$ will not always be a rational function of this particular form - we always picked Δ with class number 1.

TABLE 7. Examples

m	Δ	r	γ_1	γ_2
2	-7	1	$\frac{1+45\sqrt{-7}}{2}$	$\frac{1-45\sqrt{-7}}{2}$
3	-11	1	$17+8\sqrt{-11}$	$17-8\sqrt{-11}$
4	-7	3	$\frac{-15+3\sqrt{-7}}{2}$	$\frac{-15-3\sqrt{-7}}{2}$
5	-11	3	$-3+2\sqrt{-11}$	$-3-2\sqrt{-11}$
7	-19	3	$\frac{3+3\sqrt{-19}}{2}$	$\frac{3-3\sqrt{-19}}{2}$
9	-11	5	$-1+\sqrt{-11}$	$-1-\sqrt{-11}$
13	-43	3	$\frac{7+\sqrt{-43}}{2}$	$\frac{7-\sqrt{-43}}{2}$
25	-19	9	$\frac{\sqrt{-19}}{2}$	$\frac{-\sqrt{-19}}{2}$

5.4. **Computing the elements in $Q_{\Delta,r}/\Gamma_0(m)$.** In this section, we explain how to compute $Q_{\Delta,r}/\Gamma_0(m)$, following [20].

Let $Q^0_{\Delta,r}$ be the subset of primitive forms. Then we have a $\Gamma_0(m)$-invariant bijection of sets

$$Q_{\Delta,r} = \bigcup_{\ell^2|\Delta} \left(\bigcup_{h\in S(\ell)} \ell Q^0_{\Delta/\ell^2,h} \right),$$

where $S(\ell) := \{h \in \mathbb{Z}/2m\mathbb{Z} : h^2 \equiv \Delta/\ell^2 \pmod{4m}, \ell h \equiv r \pmod{2m}\}$. Since we pick Δ to be a fundamental discriminant, the only possible prime we need to worry about is $\ell = 2$. In our examples, we always choose Δ, r such that $S(2) = \emptyset$. In this case, we just need to work with $Q^0_{\Delta,r}$.

Now, let $n := \left(m, r, \frac{r^2-\Delta}{4m}\right)$. Then for $Q = [mc, b, a] \in Q^0_{\Delta,r}$, define $n_1 := (m, b, a), n_2 := (m, b, c)$, which are coprime and have product n. We have the following result:

Lemma 5.2. *(Section 1.1 of [20]) Define n as above and fix a decomposition $n = n_1n_2$ with n_1, n_2 positive and relatively prime. Then there is a 1:1 correspondence between the $\Gamma_0(m)$-equivalence classes of forms $[cm, b, a] \in Q^0_{\Delta,r}$ satisfying $(m, b, a) = n_1, (m, b, c) = n_2$ and the $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes of forms in Q^0_Δ given by $Q = [mc, b, a] \mapsto \tilde{Q} = [cm_1, b, am_2]$, where*

$m_1 \cdot m_2$ is any decomposition of m into coprime positive factors satisfying $(n_1, m_2) = (n_2, m_1) = 1$. In particular, $|Q^0_{\Delta,r}/\Gamma_0(m)| = 2^v |Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})|$, where v is the number of prime factors of n.

Note that $|Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})|$ equals $2h(\Delta)$ for $\Delta < 0$, where the factor of 2 arises because Q^0_Δ also contains negative semi-definite forms.

In our examples, we always choose Δ, r such that $n = 1$, so that $|Q^0_{\Delta,r}/\Gamma_0(m)| = |Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})| = 2h(\Delta)$, where $h(\Delta)$ is the class number of $\mathbb{Q}(\sqrt{\Delta})$. The theory of reduced forms allows us to easily compute $Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})$.

6. p-ADIC PROPERTIES OF THE LOGARITHMIC DERIVATIVE

6.1. **p-adic modular forms.** For each $i \in \mathbb{N}$, let $f_i = \sum a_i(n) q^n$ be a modular form of weight k_i with $a_i(n) \in \mathbb{Q}$. If for each n, the $a_i(n)$ converge p-adically to $a(n) \in \mathbb{Q}_p$, then $f := \sum a(n) q^n$ is called a p-adic modular form. For $p \neq 2$, we define the weight space

$$W := \varprojlim_t \mathbb{Z}/\phi(p^t)\mathbb{Z} = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

For $p = 2$, we define

$$W := \varprojlim_t \mathbb{Z}/2^{t-2}\mathbb{Z} = \mathbb{Z}_2.$$

Then the k_i converge to an element $k \in W$, which we call the weight of f. We identify integers by their image in $\mathbb{Z}_p \times \{0\}$.

6.2. **Proof of Theorem 1.3.**

Proof of Theorem 1.3: By Theorem 1.1, $\Psi_{\Delta,r}(\tau)$ is a meromorphic modular function, so that $\Theta(\Psi_{\Delta,r}(\tau))$ is a weight 2 meromorphic modular form on $\Gamma_0(m)$. Thus, the logarithmic derivative $\frac{\Theta(\Psi_{\Delta,r}(\tau))}{\Psi_{\Delta,r}(\tau)}$ is a weight 2 meromorphic modular form on $\Gamma_0(m)$ whose poles are simple and are supported on Heegner points of discriminant Δ. □

6.3. **Proof of Theorem 1.4 and its corollary.**

Proof of Theorem 1.4: We show that if (Δ, r) is an admissible pair and p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$, that

$$f_{\Delta,r} := \frac{\Theta(\Psi_{\Delta,r}(\tau))}{\Psi_{\Delta,r}(\tau)}$$

is a p-adic modular form of weight 2. Say f has poles at $\alpha_1, \ldots, \alpha_n$, all of which are CM points of discriminant Δ. For each α_i, there is some zero β_i of E_{p-1} such that $j(\tau) - j(\alpha_i) \equiv j(\tau) - j(\beta_i)$ (see Theorem 1 of [21]). Then let

$$\mathcal{E} := E_{p-1} \prod_i \frac{(j(\tau) - j(\alpha_i))}{(j(\tau) - j(\beta_i))}.$$

This has weight $p-1$, is congruent to 1 modulo p, has zeros at $\alpha_1, \ldots, \alpha_n$, and has no poles. Let $f_t := f\mathcal{E}^{(p^t)}$. Then $f_t \equiv f \pmod{p^t}$ and is a modular form of weight $k_t = 2 + (p-1)p^t \equiv 2 \pmod{\phi(p^{t+1})}$, so f is a p-adic modular form of weight 2. □

Proof of Corollary 1.5: This corollary follows directly for the coefficients of any p-adic modular form using the following beautiful result, proven by Serre [24] using the theory of Galois representations.

Lemma 6.1 (Serre [24] Theorem 4.7 (I)). *Let K be a number field and $\mathcal{O}_k$ the ring of integers of K. Suppose $f(\tau) = \sum_{n\geq 0} a_n q^n \in \mathcal{O}_K[[q]]$ is a modular form of integer weight $k \geq 1$ on a congruence subgroup. For any prime p, let $\mathfrak{p}$ be a prime above p in $\mathcal{O}_K$. Let $m \geq 1$. Then there exists a positive constant α_m such that*

$$\#\left\{n \leq X \colon a_n \not\equiv 0 \pmod{\mathfrak{p}}^m\right\} = O\left(\frac{X}{(\log X)^{\alpha_m}}\right).$$

□

7. Appendix: Definitions of jacobi forms, theta functions, etc.

We define the Jacobi theta functions $\theta_i(\tau, z)$ as follows for $q := e(\tau)$ and $y := e(z)$.

$$\begin{aligned}
\theta_2(\tau, z) &:= q^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1-q^n)(1+yq^n)(1+y^{-1}q^{n-1}) \\
\theta_3(\tau, z) &:= \prod_{n=1}^{\infty} (1-q^n)(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2}) \\
\theta_4(\tau, z) &:= \prod_{n=1}^{\infty} (1-q^n)(1-yq^{n-1/2})(1-y^{-1}q^{n-1/2})
\end{aligned}$$

We use them to define weight zero index $m-1$ weak Jacobi forms $\varphi_1^{(m)}$ as follows. Let

$$\begin{aligned}
\varphi_1^{(2)} &:= 4(f_2^2 + f_3^2 + f_4^2), \\
\varphi_1^{(3)} &:= 2(f_2^2 f_3^2 + f_3^2 f_4^2 + f_4^2 f_2^2), \\
\varphi_1^{(4)} &:= 4 f_2^2 f_3^2 f_4^2, \\
\varphi_1^{(5)} &:= \frac{1}{4}\left(\varphi_1^{(4)}\varphi_1^{(2)} - (\varphi_1^{(3)})^2\right) \\
\varphi_1^{(7)} &:= \varphi_1^{(3)}\varphi_1^{(5)} - (\varphi_1^{(4)})^2 \\
\varphi_1^{(9)} &:= \varphi_1^{(3)}\varphi_1^{(7)} - (\varphi_1^{(5)})^2 \\
\varphi_1^{(13)} &:= \varphi_1^{(5)}\varphi_1^{(9)} - 2(\varphi_1^{(7)})^2
\end{aligned}$$

where $f_i(\tau, z) := \theta_i(\tau, z)/\theta_i(\tau, 0)$ for $i = 2, 3, 4$.

For the remaining positive integers m with $m \leq 25$, we define $\varphi_1^{(m)}$ recursively. For $(12, m-1) = 1$ and $m > 5$ we set

$$\varphi_1^{(m)} = (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}.$$

For $(12, m-1) = 2$ and $m > 10$ we set

$$\varphi_1^{(m)} = \frac{1}{2}\left((12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}\right).$$

For $(12, m-1) = 3$ and $m > 9$, we set

$$\varphi_1^{(m)} = \frac{2}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{3}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

For $(12, m-1) = 4$ and $m > 16$ we set

$$\varphi_1^{(m)} = \frac{1}{4}\left((12, m-13)\varphi_1^{(m-12)}\varphi_1^{(13)} + (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m-9)\varphi_1^{(m-8)}\varphi_1^{(9)}\right).$$

For $(12, m-1) = 6$ and $m > 18$ we set

$$\varphi_1^{(m)} = \frac{1}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{6}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - \frac{1}{2}(12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

For $m = 25$, we set

$$\varphi_1^{(25)} = \frac{1}{2}\varphi_1^{(21)}\varphi_1^{(5)} - \varphi_1^{(19)}\varphi_1^{(7)} + \frac{1}{2}(\varphi_1^{(13)})^2.$$

See the appendix of [10] for more information on the space of weight zero Jacobi forms.

We use two versions of an Appell-Lerch sum. The first is the generalized Appell-Lerch sum $\mu_{m,0}$, defined as in [10]. It is given by

$$\mu_{m,0}(\tau, z) := -\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{1 + yq^k}{1 - yq^k},$$

and is the holomorphic part of a weight 1 index m "real-analytic Jacobi form".

Zwegers [30] uses a slightly different version of the Appell-Lerch sum. He first defines the theta function

$$\vartheta(z, \tau) := \sum_{\nu\in 1/2+\mathbb{Z}} q^{\nu^2/2} y^\nu e(\nu/2).$$

Then he defines

$$\mu(u, v; \tau) := \frac{e(u/2)}{\vartheta(v; \tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(n^2+n)/2} e(nv)}{1 - q^n e(u)}.$$

This is completed to a "real-analytic Jacobi form" $\tilde{\mu}(u, v; \tau)$ of weight $1/2$ by letting

$$\tilde{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau),$$

where

$$R(z, \tau) := \sum_{\nu\in 1/2+\mathbb{Z}} \left\{\operatorname{sgn}(\nu) - E(\nu + a)\sqrt{2t}\right\} (-1)^{\nu-1/2} q^{-\nu^2/2} y^{-\nu},$$

$t := \Im(\tau)$, $a := \frac{\Im(u)}{\Im(\tau)}$, and $E(z) := 2\int_0^z e^{-\pi u^2} du$.

References

[1] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986.

[2] Richard E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.*, 109(2):405–444, 1992.

[3] Richard E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998.

[4] Kathrin Bringmann and Ken Ono. Arithmetic properties of coefficients of half-integral weight Maass-Poincaré series. *Math. Ann.*, 337(3):591–612, 2007.

[5] Jan Bruinier and Ken Ono. Heegner divisors, L-functions and harmonic weak Maass forms. *Ann. of Math. (2)*, 172(3):2135–2181, 2010.

[6] Jan Hendrik Bruinier and Jens Funke. Traces of CM values of modular functions. *J. Reine Angew. Math.*, 594:1–33, 2006.

[7] Imin Chen and Noriko Yui. Singular values of Thompson series. In *Groups, difference sets, and the Monster (Columbus, OH, 1993)*, volume 4 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 255–326. de Gruyter, Berlin, 1996.

[8] Miranda C. N. Cheng. $K3$ surfaces, $\mathcal{N} = 4$ dyons and the Mathieu group M_{24}. *Commun. Number Theory Phys.*, 4(4):623–657, 2010.

[9] Miranda C N Cheng, John F R Duncan, and Jeffrey A Harvey. Umbral Moonshine. `arXiv:1204.2779 [math.RT]`.

[10] Miranda C N Cheng, John F R Duncan, and Jeffrey A Harvey. Umbral Moonshine and the Niemeier Lattices. *Research in the Mathematical Sciences*, **1**:3, 2014.

[11] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bull. London Math. Soc.*, 11(3):308–339, 1979.

[12] Thomas Creutzig, Gerald Hoehn, and Tsuyoshi Miezaki. The McKay-Thompson series of Mathieu Moonshine modulo two. `arXiv:1211.3703 [math.NT]`.

[13] W. Duke, Ö. Imamoḡlu, and Á. Tóth. Cycle integrals of the j-function and mock modular forms. *Ann. of Math. (2)*, 173(2):947–981, 2011.

[14] Tohru Eguchi and Kazuhiro Hikami. Note on twisted elliptic genus of $K3$ surface. *Phys. Lett. B*, 694(4-5):446–455, 2011.

[15] Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the $K3$ surface and the Mathieu group M_{24}. *Exp. Math.*, 20(1):91–96, 2011.

[16] I. B. Frenkel, J. Lepowsky, and A. Meurman. A natural representation of the Fischer-Griess Monster with the modular function J as character. *Proc. Nat. Acad. Sci. U.S.A.*, 81(10, Phys. Sci.):3256–3260, 1984.

[17] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu Moonshine in the elliptic genus of $K3$. *J. High Energy Phys.*, (10):062, 24, 2010.

[18] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu twining characters for $K3$. *J. High Energy Phys.*, (9):058, 20, 2010.

[19] Terry Gannon. Much ado about Mathieu. `arXiv:1211.5531 [math.RT]`.

[20] B. Gross, W. Kohnen, and D. Zagier. Heegner points and derivatives of L-series. II. *Math. Ann.*, 278(1-4):497–562, 1987.

[21] M. Kaneko and D. Zagier. Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials. In *Computational perspectives on number theory (Chicago, IL, 1995)*, volume 7 of *AMS/IP Stud. Adv. Math.*, pages 97–126. Amer. Math. Soc., Providence, RI, 1998.

[22] T Miezaki and M Waldherr. Congruences on the Fourier coefficients of the Mathieu mock theta function.

[23] Ken Ono. Parity of the partition function. *Adv. Math.*, 225(1):349–366, 2010.

[24] Jean-Pierre Serre. Divisibilité de certaines fonctions arithmétiques. *Séminaire Delange-Pisot-Poitou. Théorie des nombres*, 16(1):1–28, 1974.

[25] Goro Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.

[26] Stephen D. Smith. On the head characters of the Monster simple group. In *Finite groups—coming of age (Montreal, Que., 1982)*, volume 45 of *Contemp. Math.*, pages 303–313. Amer. Math. Soc., Providence, RI, 1985.

[27] J. G. Thompson. Finite groups and modular functions. *Bull. London Math. Soc.*, 11(3):347–351, 1979.

[28] J. G. Thompson. Some numerology between the Fischer-Griess Monster and the elliptic modular function. *Bull. London Math. Soc.*, 11(3):352–353, 1979.

[29] Don Zagier. Traces of Singular Moduli. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, pages 211–244. Int. Press, Somerville, MA, 2002.

[30] Sander Zwegers. *Mock Theta Functions*. PhD thesis, Utrecht University, July 2002.

Mathematics Institute, University of Cologne, Gyrhofstr. 8b, 50931 Cologne, Germany
E-mail address: lrolen@math.uni-koeln.de

Department of Mathematics, Emory University, Emory, Atlanta, GA 30322
E-mail address: ono@mathcs.emory.edu

Department of Mathematics, Emory University, Emory, Atlanta, GA 30322
E-mail address: strebat@emory.edu

arXiv:1204.2779v3 [math.RT] 17 Oct 2013

Umbral Moonshine*

Miranda C. N. Cheng†
John F. R. Duncan‡
Jeffrey A. Harvey§

2013 October 13

Abstract

We describe surprising relationships between automorphic forms of various kinds, imaginary quadratic number fields and a certain system of six finite groups that are parameterised naturally by the divisors of twelve. The Mathieu group correspondence recently discovered by Eguchi–Ooguri–Tachikawa is recovered as a special case. We introduce a notion of extremal Jacobi form and prove that it characterises the Jacobi forms arising by establishing a connection to critical values of Dirichlet series attached to modular forms of weight two. These extremal Jacobi forms are closely related to certain vector-valued mock modular forms studied recently by Dabholkar–Murthy–Zagier in connection with the physics of quantum black holes in string theory. In a manner similar to monstrous moonshine the automorphic forms we identify constitute evidence for the existence of infinite-dimensional graded modules for the six groups in our system. We formulate an umbral moonshine conjecture that is in direct analogy with the monstrous moonshine conjecture of Conway–Norton. Curiously, we find a number of Ramanujan's mock theta functions appearing as McKay–Thompson series. A new feature not apparent in the monstrous case is a property which allows us to predict the fields of definition of certain homogeneous submodules for the groups involved. For four of the groups in our system we find analogues of both the classical McKay correspondence and McKay's monstrous Dynkin diagram observation manifesting simultaneously and compatibly.

* *MSC2010:* 11F22, 11F37, 11F46, 11F50, 20C34, 20C35

† Université Paris 7, UMR CNRS 7586 and LPTHE, Université Paris 6, Paris, France.
E-mail: chengm@math.jussieu.fr

‡ Department of Mathematics, Case Western Reserve University, Cleveland, OH 44106, U.S.A.
E-mail: john.duncan@case.edu

§ Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, IL 60637, U.S.A.
E-mail: j-harvey@uchicago.edu

Contents

1 Introduction 4

2 Automorphic Forms 9

2.1 Mock Modular Forms . . . 10
2.2 Jacobi Forms . . . 13
2.3 Meromorphic Jacobi Forms . . . 15
2.4 Superconformal Algebra . . . 17
2.5 Extremal Jacobi Forms . . . 18
2.6 Siegel Modular Forms . . . 30

3 Finite Groups 33

3.1 Specification . . . 34
3.2 Signed Permutations . . . 36
3.3 Realisation Part I . . . 38
3.4 Realisation Part II . . . 41
3.5 Dynkin Diagrams Part I . . . 44
3.6 Dynkin Diagrams Part II . . . 47

4 McKay–Thompson Series 51

4.1 Forms of Higher Level . . . 52
4.2 Lambency Two . . . 56
4.3 Lambency Three . . . 57
4.4 Lambency Four . . . 59
4.5 Lambency Five . . . 61
4.6 Lambencies Seven and Thirteen . . . 62
4.7 Mock Theta Functions . . . 63
4.8 Automorphy . . . 64

5 Conjectures 67

5.1 Modules . . . 68
5.2 Moonshine . . . 69
5.3 Modularity . . . 71
5.4 Discriminants . . . 71
5.5 Geometry and Physics . . . 74

A Modular Forms **78**
A.1 Dedekind Eta Function . . . 78
A.2 Jacobi Theta Functions . . . 78
A.3 Higher Level Modular Forms . . . 79

B Characters **80**
B.1 Irreducible Characters . . . 81
B.2 Euler Characters . . . 85

C Coefficients **88**
C.1 Lambency Two . . . 89
C.2 Lambency Three . . . 90
C.3 Lambency Four . . . 92
C.4 Lambency Five . . . 95
C.5 Lambency Seven . . . 99
C.6 Lambency Thirteen . . . 102

D Decompositions **106**
D.1 Lambency Two . . . 107
D.2 Lambency Three . . . 108
D.3 Lambency Four . . . 109
D.4 Lambency Five . . . 110
D.5 Lambency Seven . . . 111
D.6 Lambency Thirteen . . . 112

References **114**

1 Introduction

The term *monstrous moonshine* was coined by Conway [1] in order to describe the unexpected and mysterious connections between the representation theory of the largest sporadic group—the *Fischer–Griess monster*, $\mathbb{M}$—and modular functions that stemmed from McKay's observation that $196883 + 1 = 196884$, where the summands on the left are degrees of irreducible representations of $\mathbb{M}$ and the number on the right is the coefficient of q in the Fourier expansion of the *elliptic modular invariant*

$$J(\tau) = \sum_{m\geq -1} a(m)q^m = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots . \tag{1.1}$$

Thompson expanded upon McKay's observation in [2] and conjectured the existence of an infinite-dimensional monster module

$$V = \bigoplus_{m\geq -1} V_m \tag{1.2}$$

with $\dim V_m = a(m)$ for all m. He also proposed [3] to consider the series, now known as *McKay–Thompson series*, given by

$$T_g(\tau) = \sum_{m\geq -1} \mathrm{tr}_{V_m}(g)\, q^m \tag{1.3}$$

for $g \in \mathbb{M}$, and detailed explorations [1] by Conway–Norton led to the astonishing *moonshine conjecture*:

For each $g \in \mathbb{M}$ the function T_g is a principal modulus for some genus zero group Γ_g.

(A discrete group $\Gamma < PSL_2(\mathbb{R})$ is said to have *genus zero* if the Riemann surface $\Gamma\backslash\mathbb{H}$ is isomorphic to the Riemann sphere minus finitely many points, and a holomorphic function f on $\mathbb{H}$ is called a *principal modulus* for a genus zero group Γ if it generates the field of Γ-invariant functions on $\mathbb{H}$.)

Thompson's conjecture was verified by Atkin, Fong and Smith (cf. [4, 5]). A more constructive verification was obtained by Frenkel–Lepowsky–Meurman [6, 7] with the explicit construction of a monster module $V = V^\natural$ with graded dimension given by the Fourier expansion (1.1) of the elliptic modular invariant. They used *vertex operators*—structures originating in the dual resonance theory of particle physics and finding contemporaneous application [8, 9] to affine Lie algebras—to recover the non-associative *Griess algebra* structure (developed in the first proof [10] of the existence of the monster) from a subspace of $V^\natural$. Borcherds found a way

to attach vertex operators to every element of $V^\natural$ and determined the precise sense in which these operators could be given a commutative associative composition law, and thus arrived at the notion of *vertex algebra* [11], an axiomatisation of the operator product expansion of chiral conformal field theory. The closely related notion of *vertex operator algebra (VOA)* was subsequently introduced by Frenkel–Lepowsky–Meurman [12] and they established that the monster is precisely the group of automorphisms of a VOA structure on $V^\natural$; the Frenkel–Lepowsky–Meurman construction of $V^\natural$ would ultimately prove to furnish the first example of an *orbifold conformal field theory*.

Borcherds introduced the notion of *generalised Kac–Moody algebra* in [13] and by using the VOA structure on $V^\natural$ was able to construct a particular example—the monster Lie algebra—and use the corresponding equivariant denominator identities to arrive at a proof [14] of the Conway–Norton moonshine conjectures. Thus by 1992 monstrous moonshine had already become a phenomenon encompassing elements of finite group theory, modular forms, vertex algebras and generalised Kac–Moody algebras, as well as aspects of conformal field theory and string theory.

Recently Eguchi–Ooguri–Tachikawa have presented evidence [15] for a new kind of moonshine involving the elliptic genus of $K3$ surfaces (the elliptic genus is a topological invariant and therefore independent of the choice of $K3$ surface) and the *largest Mathieu group* M_{24} (cf. §3.1). This connection between $K3$ surfaces and M_{24} becomes apparent only after decomposing the elliptic genus into characters of the $N = 4$ superconformal algebra (cf. [16, 17, 18]). This decomposition process (cf. §2.4) reveals the presence of a mock modular form of weight 1/2 (cf. §2.1) satisfying

$$H^{(2)}(\tau) = \sum_{n=0}^{\infty} c_1^{(2)}(n-1/8)q^{n-1/8} = 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \cdots) \quad (1.4)$$

and one recognises here the dimensions of several irreducible representations of M_{24} (cf. Table 8).

One is soon led to follow the path forged by Thompson in the case of the monster: to suspect the existence of a graded infinite-dimensional M_{24}-module

$$K^{(2)} = \bigoplus_{n=0}^{\infty} K^{(2)}_{n-1/8} \quad (1.5)$$

with $\dim K^{(2)}_{n-1/8} = c^{(2)}(n-1/8)$ for $n \geq 1$, and to study the analogues $H^{(2)}_g$ of the monstrous McKay–Thompson series obtained by replacing $c^{(2)}(n-1/8) = \dim K^{(2)}_{n-1/8}$ with $\mathrm{tr}_{K^{(2)}_{n-1/8}}(g)$ in (1.4). This idea has been implemented successfully in [19, 20, 21, 22] and provides strong

evidence for the existence of such an M_{24}-module $K^{(2)}$. A proof of the existence of $K^{(2)}$ has now been established in [23] although no explicit construction is yet known. In particular there is as yet no known analogue of the vertex operator algebra structure which conjecturally characterises [12] the monster module $V^\natural$.

The strong evidence in support of the M_{24} analogue of Thompson's conjecture invites us to consider the M_{24} analogue of the Conway–Norton moonshine conjectures—this will justify the use of the term moonshine in the M_{24} setting—except that it is not immediately obvious what the analogue should be. Whilst the McKay–Thompson series $H_g^{(2)}$ is a mock modular form of weight 1/2 on some $\Gamma_g < SL_2(\mathbb{Z})$ for every g in M_{24} [22], it is not the case that Γ_g is a genus zero group for every g, and even if it were, there is no obvious sense in which one mock modular form of weight 1/2 on some group can "generate" all the others, and thus no obvious analogue of the principal modulus property.

A solution to this problem—the formulation of the moonshine conjecture for M_{24}—was found in [24] (see also §5.2) via an extension of the program that was initiated in [25]; the antecedents of which include Rademacher's pioneering work [26] on the elliptic modular invariant $J(\tau)$, quantum gravity in three dimensions [27, 28, 29], the AdS/CFT correspondence in physics [30, 31, 32], and the application of *Rademacher sums* to these and other settings in string theory [33, 34, 35, 36, 37, 38, 39] (and in particular [40]). To explain the formulation of the moonshine conjecture for M_{24} we recall that in [25] a Rademacher sum $R_\Gamma(\tau)$ is defined for each discrete group $\Gamma < PSL_2(\mathbb{R})$ commensurable with the modular group in such a way as to naturally generalise Rademacher's Poincaré series-like expression for the elliptic modular invariant derived in [26]. It is then shown in [25] that a holomorphic function on the upper-half plane (with invariance group commensurable with $PSL_2(\mathbb{Z})$) is the principal modulus for its invariance group if and only if it coincides with the Rademacher sum attached to this group. Thus the genus property of monstrous moonshine may be reformulated:

For each g in $\mathbb{M}$ we have $T_g = R_{\Gamma_g}$ where Γ_g is the invariance group of T_g.

Write $R_\Gamma^{(2)}$ to indicate a weight 1/2 generalisation of the (weight 0) Rademacher sum construction R_Γ studied in [25]. (Note that a choice of multiplier system on Γ is also required.) Then a natural M_{24}-analogue of the Conway–Norton moonshine conjecture comes into view:

For each g in M_{24} we have $H_g^{(2)} = R_{\Gamma_g}^{(2)}$ where Γ_g is the invariance group of $H_g^{(2)}$.

This statement is confirmed in [24] for the functions $H_g^{(2)}$ that are, at this point, conjecturally attached to M_{24} via the conjectural M_{24}-module $K^{(2)}$.

Through these results we come to envisage the possibility that both monstrous moonshine and the M_{24} observation of Eguchi–Ooguri–Tachikawa will eventually be understood as aspects

of one underlying structure which will include finite groups, various kinds of automorphic forms, extended algebras and string theory, and will quite possibly be formulated in terms of the AdS/CFT correspondence in the context of some higher-dimensional string or gravitational theory.

In fact we can expect this *moonshine structure* to encompass more groups beyond the monster and M_{24}: In this paper we identify the M_{24} observation as one of a family of correspondences between finite groups and (vector-valued) mock modular forms, with each member in the family admitting a natural analogue of the Conway–Norton moonshine conjecture according to the philosophy of [25, 24] (see also [41]). For each ℓ in $\Lambda = \{2, 3, 4, 5, 7, 13\}$—the set of positive integers ℓ such that $\ell - 1$ divides 12—we identify a distinguished Jacobi form $Z^{(\ell)}$, a finite group $G^{(\ell)}$ and a family of vector-valued mock modular forms $H_g^{(\ell)}$ for $g \in G^{(\ell)}$. The Jacobi forms $Z^{(\ell)}$ satisfy an *extremal* condition formulated in terms of unitary irreducible characters of the $N = 4$ superconformal algebra (cf. §2.5), the mock modular form $H^{(\ell)} = H_e^{(\ell)}$ is related to $Z^{(\ell)}$ as $H^{(2)}$ is to the elliptic genus of a $K3$ surface (cf. §§2.4,2.5), the Fourier coefficients of the *McKay–Thompson series* $H_g^{(\ell)}$ support the existence of an infinite-dimensional graded module $K^{(\ell)}$ for $G^{(\ell)}$ playing a rôle analogous to that of $K^{(2)}$ for $G^{(2)} \simeq M_{24}$ (cf. §5.1), and the following *umbral moonshine conjecture* is predicted to hold where $R_\Gamma^{(\ell)}$ is an $(\ell-1)$-vector-valued generalisation (cf. §5.2) of the Rademacher sum construction $R_\Gamma^{(2)}$ studied in [24].

For each g in $G^{(\ell)}$ we have $H_g^{(\ell)} = R_{\Gamma_g}^{(\ell)}$ where Γ_g is the invariance group of $H_g^{(\ell)}$.

In addition to the above properties with monstrous analogues we find the following new *discriminant property*: The exponents of the powers of q having non-vanishing coefficient in the Fourier development of $H^{(\ell)}$ determine certain imaginary quadratic number fields and predict the existence of dual pairs of irreducible representations of $G^{(\ell)}$ that are defined over these fields and irreducible over $\mathbb{C}$. Moreover, these dual pairs consistently appear as irreducible constituents in homogeneous $G^{(\ell)}$-submodules of $K^{(\ell)}$ in such a way that the degree of the submodule determines the discriminant of the corresponding quadratic field (cf. §5.4).

A main result of this paper is Theorem 2.2, which states that the extremal condition formulated in §2.5 characterises the Jacobi forms $Z^{(\ell)}$ for $\ell \in \{2, 3, 4, 5, 7, 13\}$. To achieve this we establish a result of independent interest, which also serves to illustrate the depth of the characterisation problem: We show in Theorem 2.3 that the existence of an extremal Jacobi of index $m - 1$ implies the vanishing of $L(f, 1)$ for all new forms f of weight 2 and level m, where $L(f, s)$ is the Dirichlet series naturally attached to f. According to the Birch–Swinnerton-Dyer conjecture the vanishing of $L(f, 1)$ implies that the elliptic curve E_f, attached to f by Eichler–Shimura theory, has rational points of infinite order. Thus it is extremely unexpected that an

extremal Jacobi form can exist for all but finitely many values of m, and using estimates [42] due to Ellenberg together with some explicit computations we are able to verify that the only possible values are those for which no non-zero weight 2 forms exist; i.e. $\Gamma_0(m)$ has genus zero. The last step in our proof of Theorem 2.2 is to check the finitely many corresponding finite dimensional spaces of Jacobi forms, for which useful bases have been determined by Gritsenko in [43]. In this we way obtain that the $Z^{(\ell)}$ for $\ell \in \{2, 3, 4, 5, 7, 13\}$ are precisely the unique, up to scale, weak Jacobi forms of weight zero satisfying the extremal condition (2.37).

In contrast to the monstrous case the McKay–Thompson series $H_g^{(\ell)}$ arising here are mock modular forms, and these are typically not in fact modular but become so after *completion* with respect to a *shadow* function $S_g^{(\ell)}$. It turns out that all the McKay–Thompson series $H_g^{(\ell)}$ for fixed $\ell \in \Lambda$ have shadows that are (essentially) proportional to a single vector-valued *unary theta function* $S^{(\ell)}$ (cf. §2.2) and so it is in a sense the six *moonlight shadows* $S^{(\ell)}$ for $\ell \in \Lambda$ that provide the irreducible information required to uncover the structure that we reveal in this paper. We therefore refer to the phenomena investigated here as *umbral moonshine*.

According to the Oxford English Dictionary, a flame or light that is *lambent* is playing "lightly upon or gliding over a surface without burning it, like a 'tongue of fire'; shining with a soft clear light and without fierce heat." And since the light of umbral moonshine is apparently of this nature, we call the six values in $\Lambda = \{2, 3, 4, 5, 7, 13\}$ *lambent*, and we refer to the index $\ell \in \Lambda$ as the *lambency* of the connections relating the *umbral group* $G^{(\ell)}$ to the *umbral forms* $Z^{(\ell)}$ and $H_g^{(\ell)}$.

The rest of this paper is organised as follows. In §2 we discuss properties of Jacobi forms, Siegel forms, mock modular forms and mock theta functions. We explain two closely related ways in which Jacobi forms determine mock modular forms, one involving the decomposition into characters of the $N = 4$ superconformal algebra and the other involving a decomposition of meromorphic Jacobi forms into mock modular forms following [44] and [45]. In §2.5 we introduce the Jacobi forms $Z^{(\ell)}$ of weight 0 and index $\ell - 1$ for lambent ℓ and their associated vector-valued mock modular forms $H^{(\ell)}$. We prove (Theorem 2.2) that these functions are characterized by the extremal property (2.37) and we establish the connection (Theorem 2.6) to critical values of automorphic L-functions. We note that the coefficients in the q-expansions of the mock modular forms $H^{(\ell)}$ appear to be connected to the dimensions of irreducible representations of groups $G^{(\ell)}$ which we introduce and study in §3. In §§3.5,3.6 we discuss the remarkable fact that some of these groups manifest both the McKay correspondence relating ADE Dynkin diagrams to finite subgroups of $SU(2)$ as well as a generalisation of his monstrous E_8 observation. In §4 we discuss analytic properties of the McKay–Thompson series $H_g^{(\ell)}$ which are obtained by twisting the mock modular forms $H^{(\ell)}$ by elements $g \in G^{(\ell)}$. We determine the proposed McKay–Thompson series

$H_g^{(\ell)} = \big(H_{g,r}^{(\ell)}\big)$ precisely in terms of modular forms of weight 2 for all but a few g occurring for $\ell \in \{7, 13\}$, and we find that we can identify many of the component functions $H_{g,r}^{(\ell)}$ either with ratios of products of eta functions or with classical mock theta functions introduced by Ramanujan (and others). In §5 we collect our observations into a set of conjectures. These include the analogue of Thompson's conjecture (cf. §5.1), the umbral counterpart to the Conway–Norton moonshine conjecture (cf. §5.2), and a precise formulation of the discriminant property mentioned above (cf. §5.4). In §5.5 we mention some possible connections between our results, the geometry of complex surfaces, and string theory.

Our conventions for modular forms appear in §A, the character tables of the umbral groups $G^{(\ell)}$ appear in §B, tables of Fourier coefficients of low degree for all the proposed McKay–Thompson series $H_g^{(\ell)}$ appear in §C, and tables describing the $G^{(\ell)}$-module structures implied (for low degree) by the $H_g^{(\ell)}$ are collected in §D.

It is important to mention that much of the data presented in the tables of §C was first derived using certain vector-valued generalisations of the Rademacher sum construction that was applied to the functions of monstrous moonshine in [25], and in [24] to the functions attached to M_{24} via the observation of Eguchi–Ooguri–Tachikawa. In particular, these vector-valued Rademacher sums played an indispensable rôle in helping us arrive at the groups $G^{(\ell)}$ specified in §3, especially for $\ell > 3$, and also allowed us to formulate and test hypotheses regarding the modularity of the (vector-valued) functions $H_g^{(\ell)}$, including eta product expressions and the occurrences of classical mock theta functions; considerations which ultimately developed into the discussion of §4. A detailed discussion of the Rademacher construction is beyond the scope of this article but a full treatment is to be the focus of forthcoming work. The fact that the Rademacher sum approach proved so powerful may be taken as strong evidence in support of the *umbral moonshine conjecture*, Conjecture 5.4.

2 Automorphic Forms

In this section we discuss the modular objects that play a rôle in the connection between mock modular forms and finite groups that we will develop later in the paper. We also establish our notation and describe various relationships between Jacobi forms, theta functions, and vector-valued mock modular forms, including mock theta functions.

In what follows we take τ in the upper half-plane $\mathbb{H}$ and $z \in \mathbb{C}$, and adopt the shorthand

notation $e(x) = e^{2\pi i x}$. We also define $q = e(\tau)$ and $y = e(z)$ and write

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{2.1}$$

for the natural action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ and

$$\gamma(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \tag{2.2}$$

for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H} \times \mathbb{C}$.

2.1 Mock Modular Forms

Mock theta functions were first introduced in 1920 by S. Ramanujan in his last letter to Hardy. This letter contained 17 examples divided into four of *order* 3, ten of *order* 5 and three of *order* 7. Ramanujan did not define what he meant by the term order and to this day there seems to be no universally agreed upon definition. In this paper we use the term order only as a historical label. Ramanujan wrote his mock theta functions as what he termed "Eulerian series" that today would be recognised as specialisations of q-hypergeometric series. A well studied example is the order 3 mock theta function

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \,, \tag{2.3}$$

where we have introduced the *q-Pochhammer* symbol

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \,. \tag{2.4}$$

Interest in and applications of mock theta functions has burgeoned during the last decade following the work of Zwegers [44] who found an intrinsic definition of mock theta functions and their near modular behavior, and many applications of his work can be found in combinatorics [46, 47], characters of infinite-dimensional Lie superalgebras [48, 49], topological field theory [50, 51, 52, 53], the computation of quantum invariants of three-dimensional manifolds [54], and the counting of black hole states in string theory [45]. Descriptions of this breakthrough and some of the history of mock theta functions can be found in [55], [56] and [57].

Mock theta function are now understood as a special case of more general objects known as mock modular forms. A holomorphic function $h(\tau)$ on $\mathbb{H}$ is called a *(weakly holomorphic) mock*

modular form of weight k for a discrete group Γ (e.g. a congruence subgroup of $SL_2(\mathbb{Z})$) if it has at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$, and if there exists a holomorphic modular form $f(\tau)$ of weight $2-k$ on Γ such that the *completion* of h given by

$$\hat{h}(\tau) = h(\tau) + (4i)^{k-1} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k} \overline{f(-\bar{z})} \mathrm{d}z, \tag{2.5}$$

is a (non-holomorphic) modular form of weight k for Γ for some multiplier system ν say. In this case the function f is called the *shadow* of the mock modular form h. Even though h is not a modular form, it is common practice to call ν the multiplier system of h. One can show that ν is the conjugate of the multiplier system of f. In most of the examples in this paper we will deal with vector-valued mock modular forms so that the completion in fact transforms as $\nu(\gamma)\hat{h}(\gamma\tau)(c\tau+d)^{-k} = \hat{h}(\tau)$ in case the weight is k for all $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma$ where ν is a matrix-valued function on Γ.

The completion $\hat{h}(\tau)$ satisfies interesting differential equations. For instance, completions of mock modular forms were identified as weak Maass forms (non-holomorphic modular forms which are eigenfunctions of the Laplace operator) in [46] as a part of their solution to the longstanding Andrews–Dragonette conjecture. Note that we have the identity

$$2^{1-k}\pi\Im(\tau)^k \frac{\partial \hat{h}(\tau)}{\partial \bar{\tau}} = -2\pi i \overline{f(\tau)} \tag{2.6}$$

when f is the shadow of h.

Thanks to Zweger's work we may define a *mock theta function* to be a q-series $h = \sum_n a_n q^n$ such that for some $\lambda \in \mathbb{Q}$ the assignment $\tau \mapsto q^\lambda h|_{q=e(\tau)}$ defines a mock modular form of weight $1/2$ whose shadow is a unary (i.e. attached to a quadratic form in one variable) theta series of weight $3/2$.

In this paper we add one more rôle for mock theta functions to the list mentioned earlier; namely we conjecture that specific sets of mock theta functions appear as McKay–Thompson series associated to (also conjectural) infinite-dimensional modules for a sequence of groups $G^{(\ell)}$ which we refer to as the *umbral groups* and label by the *lambent* integers $\ell \in \Lambda = \{2,3,4,5,7,13\}$, which are just those positive integers that are one greater than a divisor of 12.

Many of the mock theta functions that appear later in this paper appear either in Ramanujan's last letter to Hardy or in his lost notebook [58]. These include an order 2 mock theta function

$$\mu(q) = \sum_{n\geq 0} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q^2;q^2)_n^2} \tag{2.7}$$

and an order 8 mock theta function

$$U_0(q) = \sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}, \tag{2.8}$$

both of which appear at lambency 2 in connection with $G^{(2)} \simeq M_{24}$. The function $f(q)$ of (2.3) together with

$$\begin{aligned}
\phi(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})} \\
\chi(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})} \\
\omega(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2} \\
\rho(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})}
\end{aligned} \tag{2.9}$$

constitute five order 3 mock theta functions appearing at lambency 3, and the four order 10 mock theta functions

$$\begin{aligned}
\phi_{10}(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}} \\
\psi_{10}(q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}} \\
X(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \\
\chi_{10}(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}
\end{aligned} \tag{2.10}$$

appear at lambency 5.

More mock theta functions were found later by others. At lambency 4 we will encounter

order 8 mock theta functions discussed in [59] with q-expansions

$$\begin{aligned} S_0(\tau) &= \sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \\ S_1(\tau) &= \sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n} \\ T_0(\tau) &= \sum_{n\geq 0} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_n} \\ T_1(\tau) &= \sum_{n\geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_n}. \end{aligned} \tag{2.11}$$

It is curious to note that the order is divisible by the lambency in each example.

2.2 Jacobi Forms

We now discuss Jacobi forms following [60]. We say a holomorphic function $\phi : \mathbb{H}\times\mathbb{C} \to \mathbb{C}$ is an *unrestricted Jacobi form* of weight k and index m for $SL_2(\mathbb{Z})$ if it transforms under the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ as

$$\phi(\tau,z) = (c\tau+d)^{-k} e(-m\tfrac{cz^2}{c\tau+d})\, \phi(\gamma(\tau,z)) \tag{2.12}$$

$$\phi(\tau,z) = e(m(\lambda^2\tau+2\lambda z))\, \phi(\tau, z+\lambda\tau+\mu) \tag{2.13}$$

where $\gamma \in SL_2(\mathbb{Z})$ and $\lambda,\mu \in \mathbb{Z}$. In what follows we refer to the transformations (2.12) and (2.13) as the *modular* and *elliptic* transformations, respectively. The invariance of $\phi(\tau,z)$ under $\tau \to \tau+1$ and $z \to z+1$ implies a Fourier expansion

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}} c(n,r) q^n y^r \tag{2.14}$$

and the elliptic transformation can be used to show that $c(n,r)$ depends only on the *discriminant* r^2-4mn and r mod $2m$, and so we have $c(n,r) = C(r^2-4mn,\tilde{r})$ for some function $D \mapsto C(D,\tilde{r})$ where $\tilde{r} \in \{-m,\ldots,m-1\}$, for example. An unrestricted Jacobi form is called a *weak Jacobi form*, a *(strong) Jacobi form*, or a *Jacobi cusp form* according as the Fourier coefficients satisfy $c(n,r)=0$ whenever $n<0$, $C(D,\tilde{r})=0$ whenever $D>0$, or $C(D,\tilde{r})=0$ whenever $D\geq 0$, respectively. In a slight departure from [60] we denote the space of weak Jacobi forms of weight k and index m by $J_{k,m}$.

In what follows we will need two further generalisations of the above definitions. The first

is straightforward and replaces $SL_2(\mathbb{Z})$ by a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$ in the modular transformation law. The second is more subtle and leads to *meromorphic Jacobi forms* which obey the modular and elliptic transformation laws but are such that the functions $z \mapsto \phi(\tau, z)$ are allowed to have poles lying at values of $z \in \mathbb{C}$ that map to torsion points of the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. Our treatment of meromorphic Jacobi forms follows [44] and [45].

A property of (weak) Jacobi forms that will be important for us later is that they admit an expansion in terms of the *index m theta functions*,

$$\theta_r^{(m)}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{(2mn+r)^2/4m} y^{2mn+r}, \tag{2.15}$$

given by

$$\phi(\tau, z) = \sum_{r (\mathrm{mod}\ 2m)} \tilde{h}_r(\tau) \theta_r^{(m)}(\tau, z) \tag{2.16}$$

in case the index of ϕ is m, where the *theta-coefficients* $\tilde{h}_r(\tau)$ constitute the components of a vector-valued modular form of weight $k - 1/2$ when k is the weight of ϕ. Recall that a vector valued function $\tilde{h} = (\tilde{h}_r)$ is called a *vector-valued modular form* of weight k for $\Gamma \subset SL_2(\mathbb{Z})$ if

$$\tilde{h}_r(\tau) = \frac{1}{(c\tau + d)^k} \sum_s \nu_{rs}(\gamma) \tilde{h}_s(\tau) \tag{2.17}$$

for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$ for some matrix-valued function $\nu = (\nu_{rs})$ on Γ called the *multiplier system* for $\tilde{h}$.

The modular transformation law (2.12) with $\gamma = -I_2$ implies that $\phi(\tau, -z) = (-1)^k \phi(\tau, z)$. Combining this with the identity $\theta_{-r}^{(m)}(\tau, z) = \theta_r^{(m)}(\tau, -z)$ we see that $\tilde{h}_r(\tau) = (-1)^k \tilde{h}_{-r}(\tau)$ and in particular we can recover a weak Jacobi form of weight k and index m from the $m - 1$ theta-coefficients $\{\tilde{h}_1, \cdots, \tilde{h}_{m-1}\}$ in case k is odd.

In what follows we will encounter only weight 0 and weight 1 Jacobi forms; a typical such form will be denoted by $\phi(\tau, z)$ or $\psi(\tau, z)$ according as the weight is 0 or 1 and we will write the theta-expansion of a weight 1 Jacobi form ψ as

$$\psi(\tau, z) = \sum_{r=1}^{m-1} \tilde{h}_r(\tau) \hat{\theta}_r^{(m)}(\tau, z) \tag{2.18}$$

where $\hat{\theta}_r^{(m)}(\tau, z) = \theta_{-r}^{(m)}(\tau, z) - \theta_r^{(m)}(\tau, z)$ (cf. (2.15)) for $r \in \{1, 2, \cdots, m-1\}$.

2.3 Meromorphic Jacobi Forms

We now explain a connection between the vector-valued mock modular forms we shall consider in this paper and meromorphic Jacobi forms. We specialize our discussion to weight 1 meromorphic Jacobi forms of a particular form that arise in our pairing of Jacobi forms with groups $G^{(\ell)}$, and later in our computation of McKay–Thompson series; namely, we consider meromorphic weight 1 and index m Jacobi forms which can be written as

$$\psi(\tau, z) = \Psi_{1,1}(\tau, z)\phi(\tau, z) \tag{2.19}$$

for some weight 0 index $m-1$ (holomorphic) weak Jacobi form ϕ where $\Psi_{1,1}$ is the specific meromorphic Jacobi form of weight 1 and index 1 given by

$$\Psi_{1,1}(\tau, z) = -i\,\frac{\theta_1(\tau, 2z)\,\eta(\tau)^3}{(\theta_1(\tau, z))^2} = \frac{y+1}{y-1} - (y^2 - y^{-2})q + \cdots . \tag{2.20}$$

We note that $\psi(\tau, z)$ has a simple pole at $z = 0$ with residue $\phi(\tau, 0)/\pi i$. Since $\phi(\tau, z)$ is a weak Jacobi form of weight 0 the function $\tau \mapsto \phi(\tau, 0)$ is a modular form of weight 0 (with no poles at any cusps) and is hence equal to a constant; we denote this constant by

$$\chi = \phi(\tau, 0). \tag{2.21}$$

It was shown by Zwegers [44] that meromorphic Jacobi forms have a modified theta-expansion (cf. (2.16)) in terms of vector-valued mock modular forms; in [45] this expansion was recast as follows. Define the averaging operator

$$\mathrm{Av}^{(m)}\Big[F(y)\Big] = \sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} F(q^k y) \tag{2.22}$$

which takes a function of $y = e(z)$ with polynomial growth and returns a function of z which transforms like an index m Jacobi form under the elliptic transformations (2.13). Now define the *polar part* of $\psi = \Psi_{1,1}\phi$ by

$$\psi^P(\tau, z) = \chi \mathrm{Av}^{(m)}\Big[\frac{y+1}{y-1}\Big] \tag{2.23}$$

where $\chi = \phi(\tau, 0)$ and define the *finite part* of ψ by

$$\psi^F(\tau, z) = \psi(\tau, z) - \psi^P(\tau, z). \tag{2.24}$$

The term finite is appropriate because with the polar part subtracted the finite part no longer has a pole at $z = 0$.

It follows from the analysis in [45] that $\psi^F(\tau, z)$ is a weight 1 index m *mock Jacobi form*, meaning that it has a theta-expansion

$$\psi^F(\tau, z) = \sum_{r=1}^{m-1} h_r(\tau)\hat{\theta}_r^{(m)}(\tau, z) \tag{2.25}$$

(cf. (2.18)) where the theta-coefficients h_r comprise the components of a vector-valued mock modular form of weight 1/2.

In the above we have again used the fact that ψ, ψ^P and hence also ψ^F pick up a minus sign under the transformation $z \to -z$. Moreover, the vector-valued mock modular forms obtained in this way always have shadow function given by the *unary theta series*

$$S_r^{(m)}(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_r^{(m)}(\tau, z)\bigg|_{z=0} = \sum_{n\in\mathbb{Z}}(2mn + r)q^{(2mn+r)^2/4m}. \tag{2.26}$$

To see why this is so, and for later use, we introduce the functions

$$\mu_j^{(m)}(\tau, z) = (-1)^{1+2j}\sum_{k\in\mathbb{Z}} q^{mk^2}y^{2mk}\frac{(yq^k)^{-2j} + (yq^k)^{-2j+1} + \cdots + (yq^k)^{1+2j}}{1 - yq^k} \tag{2.27}$$

for m a positive integer and $2j \in \{0, 1, \ldots, m-1\}$. Note that $\mu_0^{(m)}(\tau, z)$ is proportional to the polar part of ψ, as identified in (2.23),

$$\mu_0^{(m)}(\tau, z) = \mathrm{Av}^{(m)}\Big[\frac{y+1}{y-1}\Big]. \tag{2.28}$$

Remark 2.1. The function $\mu_0^{(2)}$ is closely related to the Appell-Lerch sum $\mu(t, z)$ which features prominently in [44].

The $\mu_0^{(m)}$ enjoy the following relation to the modular group $SL_2(\mathbb{Z})$. Define the *completion* of $\mu_0^{(m)}(\tau, z)$ by setting

$$\hat{\mu}^{(m)}(\tau, \bar{\tau}, z) = \mu_0^{(m)}(\tau, z) + \frac{1}{\sqrt{2m}}\frac{1}{(4i)^{1/2}}\sum_{r=-m}^{m-1}\theta_r^{(m)}(\tau, z)\int_{-\bar{\tau}}^{i\infty}(z+\tau)^{-1/2}\overline{S_r^{(m)}(-\bar{z})}\,\mathrm{d}z. \tag{2.29}$$

Then $\hat{\mu}^{(m)}$ transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z})$ but is not holo-

morphic. Therefore, from the transformation of the polar part

$$\psi^P(\tau,z) = \chi\mu_0^{(m)}(\tau,z) \tag{2.30}$$

we see that the shadow of $h = (h_r)$ is given by $\chi S^{(m)} = \left(\chi S_r^{(m)}\right)$, as we claimed. This means that the vector-valued mock modular forms arising in this way are closely related to mock theta functions: By the definition given in §2.1 we have $h_r(\tau) = q^\lambda M_r$ for some $\lambda \in \mathbb{Q}$ with M_r a mock theta function.

2.4 Superconformal Algebra

In §2.3 we saw how to associate a vector-valued mock modular form (h_r) to a weight 1 meromorphic Jacobi form ψ satisfying $\psi = \Psi_{1,1}\phi$ for some weak Jacobi form ϕ via the theta-expansion of the finite part of ψ. It will develop that the weight 0 forms ϕ of relevance to us have a close relation to the representation theory of the 2-dimensional $N = 4$ superconformal algebra. To see this recall (cf. [16, 17, 18]) that this algebra contains subalgebras isomorphic to the affine Lie algebra $\hat{\mathfrak{sl}}_2$ and the Virasoro algebra, and in a unitary representation the former of these acts with level $m-1$, for some integer $m > 1$, and the latter with central charge $c = 6(m-1)$, and the unitary irreducible highest weight representations $V_{h,j}^{(m)}$ are labelled by the two *quantum numbers* h and j which are the eigenvalues of L_0 and $\frac{1}{2}J_0^3$, respectively, when acting on the highest weight state. (We adopt a normalisation of the $SU(2)$ current J^3 such that the zero mode J_0^3 has integer eigenvalues.) In the Ramond sector of the superconformal algebra there are two types of highest weight representations: the *massless* (or *BPS*, *supersymmetric*) ones with $h = \frac{m-1}{4}$ and $j \in \{0, \frac{1}{2}, \cdots, \frac{m-1}{2}\}$, and the *massive* (or *non-BPS*, *non-supersymmetric*) ones with $h > \frac{m-1}{4}$ and $j \in \{\frac{1}{2}, 1, \cdots, \frac{m-1}{2}\}$. Their *(Ramond) characters*, defined as

$$\mathrm{ch}_{h,j}^{(m)}(\tau,z) = \mathrm{tr}_{V_{h,j}^{(m)}}\left((-1)^{J_0^3} y^{J_0^3} q^{L_0 - c/24}\right), \tag{2.31}$$

are given by

$$\mathrm{ch}_{\frac{m-1}{4},j}^{(m)}(\tau,z) = (\Psi_{1,1}(\tau,z))^{-1}\mu_j^{(m)}(\tau,z) \tag{2.32}$$

and

$$\mathrm{ch}_{h,j}^{(m)}(\tau,z) = (-1)^{2j+1}(\Psi_{1,1}(\tau,z))^{-1}\, q^{h-\frac{m-1}{4}-\frac{j^2}{m}}\, \hat{\theta}_{2j}^{(m)}(\tau,z) \tag{2.33}$$

in the massless and massive cases, respectively, [17] where the function $\mu_j^{(m)}(\tau,z)$ is defined as in (2.27) and $\hat{\theta}_r^{(m)}(\tau,z)$ is as in the sentence following (2.18).

We can use the above results to derive a decomposition of an arbitrary weight 0 index $m-1$

Jacobi form $\phi(\tau,z)$ into $N=4$ characters as follows. Set $\psi(\tau,z)=\Psi_{1,1}(\tau,z)\phi(\tau,z)$ as in §2.3 and write

$$h_r(\tau)=\sum_n c_r(n-r^2/4m)q^{n-r^2/4m} \tag{2.34}$$

for the Fourier expansion of the theta-coefficient h_r of the finite part ψ^F of ψ (cf. (2.25)). Then use (2.24), (2.25) and (2.30) along with (2.32) and (2.33) to obtain

$$\begin{aligned}\phi =&\chi\,\mathrm{ch}^{(m)}_{\frac{m-1}{4},0}+\sum_{r=1}^{m-1}(-1)^{r+1}c_r(-\tfrac{r^2}{4m})\left(\mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{r}{2}}+2\mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{r-1}{2}}+\mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{r-2}{2}}\right)\\ &+\sum_{r=1}^{m-1}(-1)^{r+1}\sum_{n=1}^{\infty}c_r(n-\tfrac{r^2}{4m})\,\mathrm{ch}^{(m)}_{\frac{m-1}{4}+n,\frac{r}{2}}\end{aligned} \tag{2.35}$$

where $\chi=\phi(\tau,0)$ is the constant such that $\chi S^{(m)}$ is the shadow of $h=(h_r)$ (cf. §2.3). In deriving (2.35) we have used the relation

$$\mu^{(m)}_{\frac{r-2}{2}}+2\mu^{(m)}_{\frac{r-1}{2}}+\mu^{(m)}_{\frac{r}{2}}=(-1)^{r+1}\,q^{-\frac{r^2}{4m}}\hat{\theta}^{(m)}_r \tag{2.36}$$

subject to the understanding that $\mu^{(m)}_{-\frac{1}{2}}=\mathrm{ch}^{(m)}_{\frac{m-1}{4},-\frac{1}{2}}=0$.

One way a weak Jacobi form of weight 0 and index $m-1$ having integer coefficients can arise in nature is as the elliptic genus of a 2-dimensional $N=4$ superconformal field theory with central charge $c=6(m-1)$. There is a unique weak Jacobi form of weight 0 and index 1 up to scale and this (suitably scaled) turns out to be the elliptic genus of a superconformal field theory attached to a $K3$ surface. Then the above analysis at $m=2$ recovers the mock modular form $H^{(2)}$ (the vectors have $m-1=1$ components in this case) exhibiting the connection to the Mathieu group M_{24} observed by Eguchi–Ooguri–Tachikawa in [15].

In the next section we will construct a distinguished family of extremal weight 0 weak Jacobi forms $\phi^{(\ell)}(\tau,z)$ with corresponding vector-valued weight 1/2 mock modular forms $H^{(\ell)}(\tau)$ according to the procedure (2.19–2.25) of §2.3. In §3 we will specify finite groups $G^{(\ell)}$ for which the forms $H^{(\ell)}$ will serve as generating functions for the graded dimensions of conjectural bi-graded infinite-dimensional $G^{(\ell)}$-modules.

2.5 Extremal Jacobi Forms

In this section we identify the significance of the divisors of 12 from the point of view of the $N=4$ superconformal algebra. We introduce the notion of an extremal (weak) Jacobi form

of weight 0 and integral index and we prove that the space $J^{\text{ext}}_{0,m-1}$ of extremal forms with index $m-1$ has dimension 1 if $m-1$ divides 12, and has dimension 0 otherwise. Several of the extremal Jacobi forms we identify have appeared earlier in the literature in the context of studying decompositions of elliptic genera of Calabi–Yau manifolds [43, 61, 48] and in a recent study of the connection between black hole counting in superstring theory and mock modular forms [45].

For m a positive integer and ϕ a weak Jacobi form with weight 0 and index $m-1$ say ϕ is *extremal* if it admits an expression

$$\phi = a_{\frac{m-1}{4},0}\text{ch}^{(m)}_{\frac{m-1}{4},0} + a_{\frac{m-1}{4},\frac{1}{2}}\text{ch}^{(m)}_{\frac{m-1}{4},\frac{1}{2}} + \sum_{0<r<m}\sum_{\substack{n\in\mathbb{Z}\\ r^2-4mn<0}} a_{\frac{m-1}{4}+n,\frac{r}{2}}\text{ch}^{(m)}_{\frac{m-1}{4}+n,\frac{r}{2}} \tag{2.37}$$

for some $a_{h,j}\in\mathbb{C}$ (cf. (2.35)) where the $N=4$ characters are as defined in (2.32) and (2.33). Write $J^{\text{ext}}_{0,m-1}$ for the subspace of $J_{0,m-1}$ consisting of extremal weak Jacobi forms. Note that the extremal condition restricts both the massless and massive $N=4$ representations that can appear, for generally there are non-zero massive characters $\text{ch}^{(m)}_{\frac{m-1}{4}+n,\frac{r}{2}}$ with $r^2-4mn\geq 0$.

We observe here that the extremal condition has a very natural interpretation in terms of the mock modular forms of weight 1/2 attached to weak Jacobi forms of weight 0 via the procedure detailed in §2.3. By comparing with (2.35) we find that the condition (2.37) on a weak Jacobi form ϕ of index $m-1$ is equivalent to requiring that the corresponding vector-valued mock modular form (h_r) obtained from the theta-expansion (2.25) of the finite part of the weight 1 Jacobi form $\psi=\Psi_{1,1}\phi$ has a single polar term $q^{-\frac{1}{4m}}$ in the first component h_1 and has all other components vanishing as $\tau\to i\infty$.

Our main result in this section is the following characterisation of extremal Jacobi forms.

Theorem 2.2. *If m is a positive integer then* $\dim J^{\text{ext}}_{0,m-1}=1$ *in case $m-1$ divides* 12 *and* $\dim J^{\text{ext}}_{0,m-1}=0$ *otherwise.*

In preparation for the proof of Theorem 2.2 we now summarise (aspects of) a useful construction given in [43]. The graded ring

$$J_{0,*} = \bigoplus_{m\geq 1} J_{0,m-1} \tag{2.38}$$

of (weak) Jacobi forms of weight 0 and integral index has an ideal

$$J_{0,*}(q) = \bigoplus_{m>1} J_{0,m-1}(q) = \left\{ \phi \in J_{0,*} \mid \phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ n>0}} c(n,r) q^n y^r \right\} \tag{2.39}$$

consisting of Jacobi forms that vanish in the limit as $\tau \to i\infty$ (i.e. have vanishing coefficient of $q^0 y^r$, for all r, in their Fourier development). This ideal is principal and generated by a weak Jacobi form of weight 0 and index 6 given by

$$\zeta(\tau,z) = \frac{\theta_1(\tau,z)^{12}}{\eta(\tau)^{12}} \tag{2.40}$$

(cf. §A for θ_1 and η). Gritsenko shows [43] that for any positive integer m the quotient $J_{0,m-1}/J_{0,m-1}(q)$ is a vector space of dimension $m-1$ admitting a basis consisting of weight 0 index $m-1$ weak Jacobi forms $\varphi_n^{(m)}$ (denoted $\psi_{0,m-1}^{(n)}$ in [43]) for $1 \le n \le m-1$ such that the coefficient of $q^0 y^k$ in $\varphi_n^{(m)}$ vanishes for $|k| > n$ but does not vanish for $|k| = n$. In fact Gritsenko works in the subring $J_{0,*}^{\mathbb{Z}}$ of Jacobi forms having integer Fourier coefficients and his $\varphi_n^{(m)}$ furnish a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $J_{0,*}^{\mathbb{Z}}/J_{0,*}^{\mathbb{Z}}(q)$.

We show now that there are no non-zero extremal Jacobi forms in the ideal $J_{0,*}(q)$.

Lemma 2.3. *If ϕ is an extremal weak Jacobi form belonging to $J_{0,*}(q)$ then $\phi = 0$.*

Proof. If ϕ belongs to $J_{0,*}(q)$ then the coefficients of $q^0 y^k$ in the Fourier development of ϕ vanish for all k. This implies the vanishing of $a_{\frac{m-1}{4},0}$ and $a_{\frac{m-1}{4},\frac{1}{2}}$ in (2.37) where $m-1$ is the index of ϕ, and this in turn implies that the meromorphic Jacobi form $\psi = \Psi_{1,1}\phi$ of weight 1 and index m coincides with its finite part $\psi = \psi^F = \sum_r h_r \hat{\theta}_r^{(m)}$ (cf. (2.21), (2.23), (2.25)) and has theta-coefficients h_r that remain bounded as $\tau \to i\infty$. In particular, ψ is a (strong) Jacobi form of weight 1 and integral index but the space of such forms vanishes according to [62], so ϕ must vanish also. □

As a consequence of Lemma 2.3 we obtain that there are no extremal Jacobi forms with vanishing massless contribution at spin $j = 1/2$.

Lemma 2.4. *If ϕ is an extremal weak Jacobi form of index $m-1$ such that $a_{\frac{m-1}{4},\frac{1}{2}} = 0$ in (2.37) then $\phi = 0$.*

Proof. If ϕ is of weight 0 and index $m-1$ satisfying (2.37) then $\phi = a + b(y + y^{-1}) + O(q)$ as

$\tau \to i\infty$ where $a = a_{\frac{m-1}{4},0}$ and $b = -a_{\frac{m-1}{4},\frac{1}{2}}$. Following [43] we set

$$\tilde{\phi}(\tau, z) = \exp(-8\pi^2(m-1)G_2(\tau)z^2)\phi(\tau, z) \tag{2.41}$$

where $G_2(\tau) = -\frac{1}{24} + \sum_{n>0} \sigma_1(n)q^n$ is the unique up to scale mock modular form of weight 2 for $SL_2(\mathbb{Z})$ (with shadow a constant function) and consider the *Taylor expansion* $\tilde{\phi}(\tau, z) = \sum_{n\geq 0} f_n(\tau)z^n$. Then $f_0(\tau) = \chi = a + 2b$ and more generally $f_n(\tau)$ is a modular form of weight n for $SL_2(\mathbb{Z})$. In particular $f_2(\tau) = 0$. On the other hand the constant term of $f_2(\tau)$ is $\frac{1}{3}\pi^2(m-1)\chi - 4\pi^2 b$ so $(m-1)\chi = 12b$. By hypothesis $b = 0$ so $\chi = 0$ and ϕ belongs to $J_{0,m-1}(q)$ and thus vanishes according to Lemma 2.3. □

Applying Lemma 2.4 we obtain that the dimension of $J^{\text{ext}}_{0,m-1}$ is at most 1 for any m.

Proposition 2.5. *We have* $\dim J^{\text{ext}}_{0,m-1} \leq 1$ *for any positive integer* m.

Proof. If ϕ' and ϕ'' are two elements of $J^{\text{ext}}_{0,m-1}$ then there exists $c \in \mathbb{C}$ such that $\phi = \phi' - c\phi''$ is extremal and has vanishing $a_{\frac{m-1}{4},\frac{1}{2}}$ in (2.37). Then $\phi = 0$ according to Lemma 2.4 and thus ϕ' belongs to the linear span of ϕ''. □

We now present a result which ultimately shows that there are only finitely many m for which $J^{\text{ext}}_{0,m-1} \neq \{0\}$, and also illustrates the depth of the characterisation problem. In preparation for it let us write $S_2(m)$ for the space of cusp forms of weight 2 for $\Gamma_0(m)$ and recall that $f \in S_2(m)$ is called a *newform* if it is a Hecke eigenform, satisfying $T(n)f = \lambda_f(n)f$ whenever $(n, m) = 1$ for some $\lambda_f(n) \in \mathbb{C}$, that is uniquely specified (up to scale) in $S_2(m)$ by its Hecke eigenvalues $\{\lambda_f(n) \mid (n, m) = 1\}$ (cf. [63, §IX.7]). Write $S_2^{\text{new}}(m)$ for the subspace of $S_2(m)$ spanned by newforms. Given $f \in S_2(m)$ define $L(f, s) = \sum_{n>0} a_f(n)n^{-s}$ for $\Re(s) > 1$ when $f(\tau) = \sum_{n>0} a_f(n)q^n$. Then $L(f, s)$ is called the *Dirichlet L-function* attached to f and admits an analytic continuation to $s \in \mathbb{C}$ (cf. [63, §IX.4] or [64, §3.6].) To a newform f in $S_2(m)$ (i.e. a Hecke eigenform in $S_2^{\text{new}}(m)$) Eichler–Shimura theory attaches an Elliptic curve E_f defined over $\mathbb{Q}$ (cf. [63, §XI.11]), and the Birch–Swinnerton-Dyer conjecture predicts that E_f has rational points of infinite order whenever $L(f, s)$ vanishes at $s = 1$ (cf. [65, 66]).

Theorem 2.6. *If* m *is a positive integer and* $J^{\text{ext}}_{0,m-1} \neq \{0\}$ *then* $L(f, 1) = 0$ *for every* $f \in S_2^{\text{new}}(m)$.

Proof. Let m be a positive integer. For $L = \sqrt{2m}\mathbb{Z}$ the spaces $H^+_{k,L}$ and S_{2-k,L^-} of [67, §3] are naturally isomorphic to $\mathbb{J}_{k+1/2,m}$ and $S^{\text{skew}}_{5/2-k,m}$, respectively, where $\mathbb{J}_{k,m}$ denotes the space of weak (pure) mock Jacobi forms of weight k and index m (cf. [45, §7.2]) and $S^{\text{skew}}_{k,m}$ denotes the

space of cuspidal skew-holomorphic Jacobi forms of weight k and index m (cf. [68]). Translating the construction (3.9) of [67] into this language, and taking $k = 1/2$, we obtain a pairing

$$\{\cdot\,,\cdot\} : S^{\text{skew}}_{2,m} \times \mathbb{J}_{1,m} \to \mathbb{C}. \tag{2.42}$$

If $J^{\text{ext}}_{0,m-1} \neq \{0\}$ then let $\phi \in J^{\text{ext}}_{0,m-1}$ be a non-zero extremal Jacobi form and set $\psi = \Psi_{1,1}\phi$ as in §2.3. Then the finite part ψ^F of ψ belongs to $\mathbb{J}_{1,m}$ and the shadow of ψ^F (cf. [45, §7.2]) is the cuspidal skew-holomorphic Jacobi form $\chi\sigma^{(m)} \in S^{\text{skew}}_{2,m}$ where $\chi = \phi(\tau, 0)$ and

$$\sigma^{(m)}(\tau, z) = \sum_{r \bmod 2m} \overline{S^{(m)}_r(\tau)}\theta^{(m)}_r(\tau, z). \tag{2.43}$$

Consider the linear functional λ_ϕ on $S^{\text{skew}}_{2,m}$ defined by setting $\lambda_\phi(\varphi) = \{\varphi, \psi^F\}$. According to the definition of (2.42) (i.e. (3.9) of [67]) we have

$$\lambda_\phi(\varphi) = C_m\langle\varphi, \chi\sigma^{(m)}\rangle \tag{2.44}$$

for some (non-zero) constant C_m (depending only on m) where $\langle\cdot\,,\cdot\rangle$ denotes the Petersson inner product on $S^{\text{skew}}_{2,m}$ (cf. [68]). On the other hand, inspection reveals that $h_{\pm 1}(\tau) = \mp a_{\frac{m-1}{4},\frac{1}{2}} q^{-1/4m}(1 + O(q))$ when $\psi^F = \sum_{r \bmod 2m} h_r\theta^{(m)}_r$ so

$$\lambda_\phi(\varphi) = -2a_{\frac{m-1}{4},\frac{1}{2}} C_\varphi(1, 1) \tag{2.45}$$

according to Proposition 3.5 of [67] where $\varphi(\tau, z) = \sum C_\varphi(\Delta, r)\bar{q}^{\Delta/4m} q^{r^2/4m} y^r$ is the Fourier expansion of φ (cf. [68]). Applying Lemma 2.4 we deduce from (2.45) that $\lambda_\phi(\varphi) = 0$ for $\varphi \in S^{\text{skew}}_{2,m}$ if and only if $C_\varphi(1, 1) = 0$.

Now let f be a newform in $S_2(m)$. If $\Lambda(f, s) = (2\pi)^{-s} m^{s/2}\Gamma(s)L(f, s)$ then either $\Lambda(f, s) = \Lambda(f, 2-s)$ or $\Lambda(f, s) = -\Lambda(f, 2-s)$. In the latter case $L(f, 1)$ necessarily vanishes. In the former case Theorem 1 of [69] attaches a skew-holomorphic Jacobi form $\varphi_f \in P^{\text{skew}}_{2,m}$ to f having the same eigenvalues as f under the Hecke operators $T(n)$ for $(n, m) = 1$. (See [69] for the action of Hecke operators on skew-holmorphic Jacobi forms.) Here $P^{\text{skew}}_{2,m}$ denotes the subspace of $S^{\text{skew}}_{2,m}$ spanned by cuspidal Hecke eigenforms which are not of the trivial type: say $\varphi \in S^{\text{skew}}_{2,m}$ is of the *trivial type* if the Fourier coefficients $C_\varphi(\Delta, r)$ are non-vanishing only when Δ is a perfect square. Cusp forms of the trivial type are orthogonal to $P^{\text{skew}}_{2,m}$ with respect to the Petersson inner product (cf. the proof of Theorem 3 in [69]) and $\sigma^{(m)}$ is a cusp form of the trivial type, so we have $\lambda_\phi(\varphi_f) = 0$ by virtue of (2.44) and hence $C_{\varphi_f}(1, 1) = 0$ according to the previous

paragraph. The proposition now follows by taking $(\Delta, r) = (1, 1)$ in the formula at the bottom of p. 503 of [68]. $\square$

Applying Theorem 2.6 we see, for example, that there are no non-zero extremal Jacobi forms of index 10, for it is known that $L(f, 1) \neq 0$ for $f = \eta(\tau)^2\eta(11\tau)^2 \in S_2(11)$. (Cf. [68, p. 504] where the corresponding Jacobi form φ_f is computed explicitly.)

Duke showed [70] that there is a constant C such that at least $Cm/\log^2 m$ newforms $f \in S_2(m)$ satisfy $L(f, 1) \neq 0$ when m is a sufficiently large prime, and Ellenberg proved an effective version of this result [42] that is valid also for composite m. We will apply the formulas of Ellenberg momentarily in order to deduce an upper bound on the m for which $J^{\text{ext}}_{0,m-1} \neq \{0\}$, but since these formulas give better results for values of m with fewer divisors we first show that $J^{\text{ext}}_{0,m-1}$ must vanish whenever m is divisible by more than one prime, and whenever $m = p^\nu$ is a prime power with $\nu > 2$.

Proposition 2.7. *If m is divisible by more than one prime then $J^{\text{ext}}_{0,m-1} = \{0\}$.*

Proof. Suppose that $m = p^\nu d$ with p prime, $\nu > 0$, $d > 1$ and $(p, d) = 1$. With notation as in the proof of Theorem 2.6 we suppose $\phi \in J^{\text{ext}}_{0,m-1}$ is non-zero and consider the linear functional $\lambda_\phi : S^{\text{skew}}_{2,m} \to \mathbb{C}$. The Atkin–Lehner involution W_{p^∞} (cf. [45, §4.4]) acts on $S^{\text{skew}}_{2,m}$ in such a way that

$$(W_{p^\infty}\varphi)(\tau, z) = \sum_{r \bmod 2m} \overline{g_{r^*}(\tau)}\theta^{(m)}_r(\tau, z) \tag{2.46}$$

when $\varphi = \sum_{r \bmod 2m} \overline{g_r}\theta^{(m)}_r$ is the theta-decomposition of $\varphi \in S^{\text{skew}}_{2,m}$ and where the map $r \mapsto r^*$ is defined so that r^* is the unique solution $(\text{mod } 2m)$ to $r^* \equiv -r \ (\text{mod } 2p^\nu)$ and $r^* \equiv r \ (\text{mod } 2d)$.

By the definition of λ_ϕ we have $\lambda_\phi(\varphi) = \{\varphi, \psi^F\} = \overline{\chi}\sum_r \langle g_r, S^{(m)}_r\rangle$ with φ as above since the shadow of ψ^F is $\chi\sigma^{(m)} = \chi\sum_r \overline{S^{(m)}_r}\theta^{(m)}_r$. Taking $\varphi = W_{p^\infty}\sigma^{(m)}$ we see that $\lambda_\phi(W_{p^\infty}\sigma^{(m)}) = \overline{\chi}\sum_r \langle S^{(m)}_{r^*}, S^{(m)}_r\rangle$ is not zero according to the computation

$$\langle S^{(m)}_{r^*}, S^{(m)}_r\rangle = \frac{1}{2^6\pi m^{3/2}}\left(\delta_{r^*,r \ (\text{mod } 2m)} - \delta_{r^*,-r \ (\text{mod } 2m)}\right) \tag{2.47}$$

which can be obtained using the Rankin–Selberg formula. (Cf., e.g., (3.15) of [45]. A very similar computation is carried out in Proposition 10.2 of [45].) On the other hand for $\varphi = W_{p^\infty}\sigma^{(m)}$ we have $C_\varphi(1, 1) = 0$ since $r^* \not\equiv \pm 1 \ (\text{mod } 2m)$ for $r \equiv \pm 1 \ (\text{mod } 2m)$, and $S^{(m)}_r = O(q^{2/4m})$ for $r \not\equiv \pm 1 \ (\text{mod } 2m)$. So $\lambda_\phi(W_{p^\infty}\sigma^{(m)}) = 0$ in light of (2.45). This contradiction proves the claim. $\square$

Proposition 2.8. *If m is a prime power $m = p^\nu$ and $\nu > 2$ then $J^{\mathrm{ext}}_{0,m-1} = \{0\}$.*

Proof. The proof is very similar to that of Proposition 2.7, computing $\lambda_\phi(\varphi)$ two ways using (2.44) and (2.45), but taking now $\varphi(\tau, z) = \sigma^{(p)}(\tau, p^\mu z)$ or $\varphi(\tau, z) = \sigma^{(p^2)}(\tau, p^\mu z)$ according as $\nu = 2\mu + 1$ or $\nu = 2\mu + 2$ with $\mu > 0$. The transformation $\varphi(\tau, z) \mapsto \varphi(\tau, tz)$ maps Jacobi forms of index m to Jacobi forms of index $t^2 m$, and is one of the Hecke-like operators of [45, §4.4]. We leave the remaining details—the expression of φ in terms of $S^{(m)}_r$ and $\theta^{(m)}_r$, and an application of (2.47)—to the reader. □

We require one more result in advance of the proof of Theorem 2.2.

Lemma 2.9. *If $m = p^2$ for some prime p and there exists a newform $f \in S_2(p)$ with $L(f, 1) \neq 0$ then $J^{\mathrm{ext}}_{0,m-1} = \{0\}$.*

Proof. Let m, p and $f \in S_2(p)$ be as in the statement of the lemma. Then, as in the proof of Theorem 2.6, there exists a uniquely determined $\varphi_f \in P^{\mathrm{skew}}_{2,p}$ with the same Hecke eigenvalues as f according to the results of [68, 69], and since $L(f, 1) \neq 0$ we have $C_{\varphi_f}(1, 1) \neq 0$. Now set $\varphi'_f = \varphi_f | V_p$ where V_p is the Hecke-like operator (cf. [45, (4.37)]) that maps Jacobi forms of index m to forms of index pm, and whose action is given explicitly by

$$C_{\varphi|V_p}(\Delta, r) = \sum_{d \mid \left(\frac{\Delta - r^2}{4p^2}, r, p\right)} d C_\varphi\left(\frac{\Delta}{d^2}, \frac{r}{d}\right) \tag{2.48}$$

in our special case that $\varphi = \varphi_f \in J^{\mathrm{skew}}_{2,p}$ when $\varphi = \sum C_\varphi(\Delta, r) \bar{q}^{\Delta/4p} q^{r^2/4p} y^r$ is the Fourier expansion of φ.

To prove that $J^{\mathrm{ext}}_{0,m-1} = \{0\}$ suppose that ϕ is a non-zero extremal form of index $m-1 = p^2-1$ and consider the functional $\lambda_\phi : S^{\mathrm{skew}}_{2,p^2} \to \mathbb{C}$ constructed in the proof of Theorem 2.6. Since $C_{\varphi_f}(1, 1) \neq 0$ we have $C_{\varphi'_f}(1, 1) \neq 0$ according to (2.48). So $\lambda_\phi(\varphi'_f) \neq 0$ thanks to (2.45) and Lemma 2.4. On the other hand the Hecke-like operator V_p commutes with all Hecke operators $T(n)$ (with, in our case, $(n, p) = 1$) and so restricts to a map $P^{\mathrm{skew}}_{2,p} \to P^{\mathrm{skew}}_{2,p^2}$. Thus φ'_f is a Hecke-eigenform in $P^{\mathrm{skew}}_{2,p^2}$ and we have $\lambda_\phi(\varphi'_f) = 0$ by virtue of (2.44) and the fact that $P^{\mathrm{skew}}_{2,p^2}$ is orthogonal to the cusp form $\sigma^{(p^2)}$ of the trivial type. This contradiction completes the proof. □

Proof of Theorem 2.2. We have that $\dim J^{\mathrm{ext}}_{0,m-1} \leq 1$ for all m according to Proposition 2.5, and $\dim J^{\mathrm{ext}}_{0,m-1} = 0$ unless $m = p$ or $m = p^2$ for some prime p by virtue of Propositions 2.7 and 2.8. Applying Theorem 1 of [42] with $\sigma = 13/25\pi$, for example, shows that some $f \in S^{\mathrm{new}}_2(p)$ satisfies $L(f, 1) \neq 0$ whenever p is a prime greater than or equal to 3001. The tables of arithmetic data

on newforms posted at [71] give the non-vanishing or otherwise for every newform of level up to 5134, and inspecting the entries for the 430 prime levels less than 3000 we see that $S_2(p)$ has a newform f with $L(f,1) \neq 0$ for every prime p so long as $S_2(p)$ has a newform. We have $S_2^{\mathrm{new}}(p) = S_2(p) = \{0\}$ just when $p \in \{2,3,5,7,13\}$, so we conclude from Theorem 2.6 that $J^{\mathrm{ext}}_{0,m-1} = \{0\}$ for every prime $m = p$ such that $p-1$ does not divide 12. Also, we conclude from Lemma 2.9 that $J^{\mathrm{ext}}_{0,m-1} = \{0\}$ whenever $m = p^2$ is the square of a prime and $p-1$ does not divide 12. We inspect the tables of [71] again to find newforms with non-vanishing critical central value at levels $7^2 = 49$ and $13^2 = 169$, and apply Theorem 2.6 to conclude $J^{\mathrm{ext}}_{0,48} = J^{\mathrm{ext}}_{0,168} = \{0\}$.

So we now require to determine the dimension of $J^{\mathrm{ext}}_{0,m-1}$ for $m \in \{2,3,4,5,7,9,13,25\}$, which are the primes and squares of primes m for which $S_2(m) = \{0\}$. For this we utilise the basis $\{\varphi_n^{(m)}\}$ for $J_{0,*}$ determined by Gritsenko [43], and discussed, briefly, after the statement of Theorem 2.2 above. In more detail, the ring $J_{0,*}$ is finitely generated, by $\varphi_1^{(2)}$, $\varphi_1^{(3)}$ and $\varphi_1^{(4)}$, where

$$\begin{aligned}
\varphi_1^{(2)} &= 4\left(f_2^2 + f_3^2 + f_4^2\right),\\
\varphi_1^{(3)} &= 2\left(f_2^2 f_3^2 + f_3^2 f_4^2 + f_4^2 f_2^2\right),\\
\varphi_1^{(4)} &= 4 f_2^2 f_3^2 f_4^2,
\end{aligned} \tag{2.49}$$

and $f_i(\tau,z) = \theta_i(\tau,z)/\theta_i(\tau,0)$ for $i \in \{2,3,4\}$ (cf. §A.2). If we work over $\mathbb{Z}$ then we must include $\varphi_1^{(5)}$

$$\varphi_1^{(5)} = \frac{1}{4}\left(\varphi_1^{(4)}\varphi_1^{(2)} - (\varphi_1^{(3)})^2\right) \tag{2.50}$$

as a generator also, so that $J^{\mathbb{Z}}_{0,*} = \mathbb{Z}[\varphi_1^{(2)}, \varphi_1^{(3)}, \varphi_1^{(4)}, \varphi_1^{(5)}]$. Following [43] we define

$$\begin{aligned}
\varphi_1^{(7)} &= \varphi_1^{(3)}\varphi_1^{(5)} - (\varphi_1^{(4)})^2,\\
\varphi_1^{(9)} &= \varphi_1^{(3)}\varphi_1^{(7)} - (\varphi_1^{(5)})^2,\\
\varphi_1^{(13)} &= \varphi_1^{(5)}\varphi_1^{(9)} - 2(\varphi_1^{(7)})^2,
\end{aligned} \tag{2.51}$$

and define $\varphi_1^{(m)}$ for the remaining positive integers m according to the following recursive procedure. For $(12, m-1) = 1$ and $m > 5$ we set

$$\varphi_1^{(m)} = (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}. \tag{2.52}$$

For $(12, m-1) = 2$ and $m > 10$ we set

$$\varphi_1^{(m)} = \frac{1}{2}\big((12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}\big). \tag{2.53}$$

For $(12, m-1) = 3$ and $m > 9$ we set

$$\varphi_1^{(m)} = \frac{2}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{3}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}. \quad (2.54)$$

For $(12, m-1) = 4$ and $m > 16$ we set

$$\varphi_1^{(m)} = \frac{1}{4}\big((12, m-13)\varphi_1^{(m-12)}\varphi_1^{(13)} + (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m-9)\varphi_1^{(m-8)}\varphi_1^{(9)}\big). \quad (2.55)$$

For $(12, m-1) = 6$ and $m > 18$ we set

$$\varphi_1^{(m)} = \frac{1}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{6}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - \frac{1}{2}(12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}. \quad (2.56)$$

Finally, for $(12, m-1) = 12$ and $m > 24$ we set[1]

$$\varphi_1^{(m)} = \frac{1}{6}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} - \frac{1}{4}(12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + \frac{1}{12}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)}. \quad (2.57)$$

The $\varphi_2^{(m)}$ are defined by setting

$$\begin{aligned} \varphi_2^{(3)} &= (\varphi_1^{(2)})^2 - 24\,\varphi_1^{(3)}, \\ \varphi_2^{(4)} &= \varphi_1^{(2)}\varphi_1^{(3)} - 18\,\varphi_1^{(4)}, \\ \varphi_2^{(5)} &= \varphi_1^{(2)}\varphi_1^{(4)} - 16\,\varphi_1^{(5)}, \end{aligned} \quad (2.58)$$

and

$$\varphi_2^{(m)} = (12, m-4)\,\varphi_1^{(m-3)}\varphi_1^{(4)} - (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m-1)\varphi_1^{(m)} \quad (2.59)$$

for $m > 5$, and the remaining $\varphi_n^{(m)}$ for $2 \leq m \leq 25$ are given by

$$\begin{aligned} \varphi_n^{(m)} &= \varphi_{n-1}^{(m-3)}\varphi_1^{(4)}, \\ \varphi_{m-2}^{(m)} &= (\varphi_1^{(2)})^{m-3}\varphi_1^{(3)}, \\ \varphi_{m-1}^{(m)} &= (\varphi_1^{(2)})^{m-1}, \end{aligned} \quad (2.60)$$

where the first equation of (2.60) holds for $3 \leq n \leq m-3$.

Next we calculate that $\varphi_1^{(m)}$ defines a non-zero element of $J^{\mathrm{ext}}_{0,m-1}$ when $m-1$ divides 12, so $\dim J^{\mathrm{ext}}_{0,m-1} = 1$ for $m \in \{2, 3, 4, 5, 7, 13\}$, and it remains to show that $J^{\mathrm{ext}}_{0,m-1} = \{0\}$ for $m \in \{9, 25\}$. To do this we first observe that a weight zero form with vanishing Fourier coefficients

[1] Our expression gives one possible correction for a small error in the last line on p. 10 of [43].

of q^0y^k for $|k|>1$ necessarily has the form

$$c_{0,1}\,\varphi_1^{(m)}+\sum_{i=1}^{\lfloor\frac{m-1}{6}\rfloor}\sum_{j=1}^{m-6i-1}c_{i,j}\,\zeta^i\varphi_j^{(m-6i)} \tag{2.61}$$

for some $c_{i,j}\in\mathbb{C}$. One has to show that the only such linear combination satisfying the extremal condition (2.37) necessarily has $c_{i,j}=0$ for all i,j (including $(i,j)=(0,1)$). An explicit inspection of the coefficients of terms q^ny^r with $r^2-4mn\geq 0$ for $n=1,2,3,4$ is sufficient to establish that $\dim J^{\text{ext}}_{0,m-1}=0$ when $m=9$. For $m=25$ we notice that there are at least as many possible polar terms in the above sum as the number of parameters $c_{i,j}$. Indeed, by explicitly solving the system of linear equations we find that there is no accidental cancellation and the only solution to (2.61) that also satisfies the extremal condition (2.37) is zero. In this way we conclude that there is no solution to (2.37) in $J_{0,m-1}$ for $m\in\{9,25\}$. This completes the proof of the Theorem. □

The quantity $Z^{(2)}(\tau,z)=2\varphi_1^{(2)}(\tau,z)$ is equal to the elliptic genus of a(ny) $K3$ surface and the factor of two relating $Z^{(2)}$ to $\varphi_1^{(2)}$ is required in the $K3/M_{24}$ connection in order for the mock modular form $H^{(2)}=(H_1^{(2)})$ derived from $Z^{(2)}$ to have coefficients compatible with an interpretation as dimensions of representations of M_{24} for which the corresponding McKay–Thompson series $H_g^{(2)}$ have integer Fourier coefficients. Inspired by this we define the *umbral Jacobi forms*

$$Z^{(\ell)}(\tau,z)=2\varphi_1^{(\ell)}(\tau,z) \tag{2.62}$$

for $\ell\in\Lambda=\{2,3,4,5,7,13\}$. We also set $\chi^{(\ell)}$ to be the constant $Z^{(\ell)}(\tau,0)$ and find that $\chi^{(\ell)}=24/(\ell-1)$ for all $\ell\in\Lambda$.

Remark 2.10. The identity $(m-1)\chi=12b$ for $\phi=a+b(y+y^{-1})+O(q)$ established in Lemma 2.4 shows that there is no such Jacobi form with $b=1$ and a an integer unless $m-1$ divides 12. In particular, there is no extremal Jacobi form $\phi\in J^{\text{ext}}_{0,m-1}$ such that $a_{\frac{m-1}{4},\frac{1}{2}}=1$ and $a_{\frac{m-1}{4},0}$ is an integer unless $m-1$ divides 12. Inspecting the $Z^{(\ell)}$ we conclude that the divisors of 12 are exactly the values of $m-1$ for which such a Jacobi form exists.

Following the discussion in §§2.3,2.4, each of the umbral Jacobi forms $Z^{(\ell)}$ leads to an $(\ell-1)$-vector-valued mock modular form $H^{(\ell)}=\big(H_r^{(\ell)}\big)$ through the decomposition of $Z^{(\ell)}$ into $N=4$ characters, or equivalently through the decomposition of $\psi^{(\ell)}=\Psi_{1,1}Z^{(\ell)}$ into its polar and finite

parts and the theta-expansion of the finite part,

$$\psi^{(\ell),F}(\tau,z)=\sum_{r=1}^{\ell-1}H_r^{(\ell)}(\tau)\hat{\theta}_r^{(\ell)}(\tau,z). \tag{2.63}$$

As we have explained above, the extremal condition translates into a natural condition on the vector-valued mock modular forms $H^{(\ell)}$: The requirement that $Z^{(\ell)}$ takes the form (2.37) is equivalent to requiring that the only polar term in $H^{(\ell)}=\left(H_r^{(\ell)}\right)$ is $-2q^{-\frac{1}{4\ell}}$ in the component $H_1^{(\ell)}$ and all the other components $H_r^{(\ell)}$ for $r\neq 1$ vanish as $\tau\to i\infty$, so that

$$H_r^{(\ell)}(\tau)=-2\delta_{r,1}q^{-\frac{1}{4\ell}}+O(q^{\frac{1}{4\ell}}) \tag{2.64}$$

for $0<r<\ell$.

Some low order terms in the Fourier expansions of the component functions $H_r^{(\ell)}$ obtained from the extremal forms $Z^{(\ell)}$ for $\ell\in\{2,3,4\}$ are given as follows.

$$H_1^{(2)}(\tau)=2q^{-1/8}\left(-1+45q+231q^2+770q^3+2277q^4+5796q^5+\cdots\right) \tag{2.65}$$

$$\begin{aligned}H_1^{(3)}(\tau)&=2q^{-1/12}\left(-1+16q+55q^2+144q^3+330q^4+704q^5+\cdots\right)\\ H_2^{(3)}(\tau)&=2q^{2/3}\left(10+44q+110q^2+280q^3+572q^4+1200q^5+\cdots\right)\end{aligned} \tag{2.66}$$

$$\begin{aligned}H_1^{(4)}(\tau)&=2q^{-1/16}\left(-1+7q+21q^2+43q^3+94q^4+168q^5+\cdots\right)\\ H_2^{(4)}(\tau)&=2q^{3/4}\left(8+24q+56q^2+112q^3+216q^4+392q^5+\cdots\right)\\ H_3^{(4)}(\tau)&=2q^{7/16}\left(3+14q+28q^2+69q^3+119q^4+239q^5+\cdots\right)\end{aligned} \tag{2.67}$$

As a prelude to the next section we note that the coefficients 16, 55 and 144 appearing in $H_1^{(3)}$ are dimensions of irreducible representations of the Mathieu group M_{12} and that $\chi^{(3)}=12$ is the dimension of the defining permutation representation of M_{12} just as $\chi^{(2)}=24$ is the dimension of the defining permutation representation of M_{24}. We further note that the low-lying coefficients appearing in $H_2^{(3)}$ are dimensions of faithful irreducible representations of a group $2.M_{12}$. Here the notation $2.G$ denotes a group with a $\mathbb{Z}/2\mathbb{Z}$ normal subgroup such that $2.G/(\mathbb{Z}/2\mathbb{Z})=G$. Note also that $2.M_{12}$ has a faithful (and irreducible) 12-dimensional signed permutation representation (cf. §3.2). The pattern that the coefficients of $H_r^{(\ell)}$ for r odd are dimensions of representations of a group $\bar{G}^{(\ell)}$ with a permutation representation of dimension $\chi^{(\ell)}$ and the coefficients of $H_r^{(\ell)}$ for r even are dimensions of faithful representations of a group $G^{(\ell)}=2.\bar{G}^{(\ell)}$ with a signed permutation representation of degree $\chi^{(\ell)}$ persists for all $\ell\in\Lambda$; detailed descriptions of the (unsigned) permutation and signed permutation representations of

$\bar{G}^{(\ell)}$ and $G^{(\ell)}$ are given in §3.3 and §3.4.

The leading terms in the q-expansions of the mock modular forms $H_1^{(\ell)}$ that correspond via the procedure described earlier to the Jacobi forms $Z^{(\ell)}$ for $\ell \in \{5, 7, 13\}$ are

$$\begin{aligned} H_1^{(5)}(\tau) &= 2q^{-1/20}\left(-1 + 4q + 9q^2 + 20q^3 + 35q^4 + 60q^5 + \cdots\right), \\ H_1^{(7)}(\tau) &= 2q^{-1/28}\left(-1 + 2q + 3q^2 + 5q^3 + 10q^4 + 15q^5 + 21q^6 + \cdots\right), \\ H_1^{(13)}(\tau) &= 2q^{-1/52}\left(-1 + q + q^2 + q^4 + q^5 + 2q^6 + 3q^7 + \cdots\right). \end{aligned} \tag{2.68}$$

To avoid clutter we refrain from describing low order terms in the q-expansions of $H_r^{(\ell)}$ for $\ell \in \{5, 7, 13\}$ and $r > 1$ here, but these can be read off from the 1A entries in Tables 27-29, 31-35, and 37-47.

We conclude this section with a comparison of our condition (2.37) with other notions of extremal in the literature. A notion of *extremal holomorphic conformal field theory* was given in [27] following earlier related work on vertex operator superalgebras in [72]. Extremal CFT's have central charge $c = 24k$ with k a positive integer and are defined in such a way that their partition function satisfies

$$Z = q^{-k}\prod_{n>1}^{\infty}\frac{1}{1-q^n} + O(q)\prod_{n>0}^{\infty}\frac{1}{1-q^n} \tag{2.69}$$

as $\tau \to i\infty$ where the term $O(q)$ is presumed to be a series in q with integer coefficients, so that the second term $O(q)\prod_{n>0}(1-q^n)^{-1}$ in (2.69) represents an integer combination of irreducible characters of the Virasoro algebra. The first term in (2.69) is the vacuum character of the Virasoro algebra, generated by the vacuum state, and thus any Virasoro primaries above the vacuum are only allowed to contribute positive powers of q to the partition function. At this time the only known example of an extremal CFT is that determined by the monster vertex operator algebra $V^\natural$ whose partition function has $k = 1$ in (2.69) so this notion is, at the very least, good at singling out extraordinary structure.

If ϕ is a weak Jacobi form of weight 0 and index $m-1$ that is extremal in our sense then we have

$$\phi = a_{\frac{m-1}{4},0}\mathrm{ch}^{(m)}_{\frac{m-1}{4},0} + a_{\frac{m-1}{4},\frac{1}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{1}{2}} + \sum_{0<r<m} O(q)\frac{\hat{\theta}_r^{(m)}}{\Psi_{1,1}} \tag{2.70}$$

as $\tau \to i\infty$ for some $a_{\frac{m-1}{4},0}$ and $a_{\frac{m-1}{4},\frac{1}{2}}$, and the third term in (2.70) is a natural counterpart to the second term in (2.69) since the massive $N = 4$ characters (in the Ramond sector with $(-1)^F$ insertion) are all of the form $q^\alpha\hat{\theta}_r^{(m)}/\Psi_{1,1}$ for some α and some r. It is harder to argue that the massless $N = 4$ contributions in (2.70) are in direct analogy with the first term in

(2.69) since the vacuum state in a superconformal field theory with $N = 4$ supersymmetry will generally (i.e. when the index is greater than 1) give rise to non-zero massless $N = 4$ character contributions with spin greater than 1/2. We remark that the stronger condition

$$\phi = a_{\frac{m-1}{4},0}\mathrm{ch}^{(m)}_{\frac{m-1}{4},0} + \sum_{0<r<m} O(q)\frac{\hat{\theta}^{(m)}_r}{\Psi_{1,1}} \tag{2.71}$$

has no solutions according to Lemma 2.4. According to Theorem 2.2 the six Jacobi forms $Z^{(\ell)}$ of umbral moonshine are the unique solutions to (2.70) having $a_{\frac{m-1}{4},\frac{1}{2}} = -2$.

A notion of *extremal conformal field theory with* $N = (2, 2)$ *superconformal symmetry* was introduced in [73] and the Ramond sector partition function of such an object defines a weak Jacobi form of weight 0 and some index but will generally not coincide with a Jacobi form that is extremal in our sense since, as has been mentioned, our functions are free from contributions arising from states in the Ramond sector that are related by spectral flow to the vacuum state in the Neveu–Schwarz sector, except in the case of index 1.

Despite the absence of a notion of extremal conformal field theory underlying our extremal Jacobi forms it is interesting to reflect on the fact that the one known example of an extremal CFT is that (at $k = 1$) determined by the moonshine vertex operator algebra of [12] with partition function $Z(\tau) = J(\tau)$. This function encodes dimensions of irreducible representations of the monster and, as we have seen to some extent above and will see in more detail below, the quantities which play the same role with respect to the groups $G^{(\ell)}$ of umbral moonshine are precisely the mock modular forms $H^{(\ell)}(\tau) = \big(H^{(\ell)}_r(\tau)\big)$ which arise naturally from the extremal Jacobi forms $Z^{(\ell)}(\tau, z)$ according to the procedure of §2.3. One of the main motivations for the notion of extremal CFT introduced in [27] was a possible connection to pure (chiral) gravity theory in 3-dimensional Anti-de Sitter space via the AdS/CFT correspondence and it is interesting to compare this with the important rôle that Rademacher sum constructions play in monstrous moonshine [25], umbral moonshine at $\ell = 2$ [24], and in umbral moonshine more generally (cf. §1 and §5.2). We refer to [28, 29, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85] for further discussions on the AdS/CFT correspondence and extremal CFT's.

2.6 Siegel Modular Forms

Siegel modular forms are automorphic forms which generalise modular forms by replacing the modular group $SL_2(\mathbb{Z})$ by the *genus n Siegel modular group* $Sp_{2n}(\mathbb{Z})$ and the upper half-plane $\mathbb{H}$

by the *genus n Siegel upper half-plane* $\mathbb{H}_n$. Here we restrict ourselves to the case $n = 2$. Define

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \tag{2.72}$$

with I_2 the unit 2×2 matrix and let $Sp_4(\mathbb{Z})$ be the group of 4×4 matrices $\gamma \in M_4(\mathbb{Z})$ satisfying $\gamma J \gamma^t = J$. If we write γ in terms of 2×2 matrices with integer entries A, B, C, D,

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.73}$$

then the condition $\gamma J \gamma^t = J$ becomes

$$AB^t = BA^t, \qquad CD^t = DC^t, \qquad AD^t - BC^t = 1. \tag{2.74}$$

Just as $SL_2(\mathbb{Z})$ has a natural action on $\mathbb{H}$, the (genus 2) Siegel modular group $Sp_4(\mathbb{Z})$ has a natural action on the (genus 2) Siegel upper half-plane, $\mathbb{H}_2$, defined as the set of 2×2 complex, symmetric matrices

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \tag{2.75}$$

obeying

$$\mathrm{Im}(\tau) > 0, \qquad \mathrm{Im}(\sigma) > 0, \qquad \det(\mathrm{Im}(\Omega)) > 0. \tag{2.76}$$

The action of $\gamma \in Sp_4(\mathbb{Z})$ is given by

$$\gamma\Omega = (A\Omega + B)(C\Omega + D)^{-1} \tag{2.77}$$

when γ is given by (2.73). A *Siegel modular form* of weight k for $\Gamma \subset Sp_4(\mathbb{Z})$ is a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ obeying the transformation law

$$F((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^k F(\Omega) \tag{2.78}$$

for $\gamma \in \Gamma$. We can write a *Fourier-Jacobi decomposition* of $F(\Omega)$ in terms of $p = e(\sigma)$ as

$$F(\Omega) = \sum_{m=0}^{\infty} \varphi_m(\tau, z) p^m \tag{2.79}$$

and the transformation law for $F(\Omega)$ can be used to show that the Fourier-Jacobi coefficient $\varphi_m(\tau, z)$ is a Jacobi form of weight k and index m. (This is one of the main motivations for the notion of Jacobi form; cf. [86] for an early analysis with applications to affine Lie algebras.) We can thus write

$$\varphi_m(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn-r^2\geq 0}} c(m,n,r)q^n y^r \tag{2.80}$$

and then

$$F(\Omega) = \sum_{m,n,r} c(m,n,r)p^m q^n y^r. \tag{2.81}$$

A special class of Siegel modular forms, called *Spezialschar* by Maass, arise by taking the Jacobi-Fourier coefficients φ_m to be given by $\varphi_m = \varphi_1|V_m$ where $\varphi_1 \in J_{k,1}$ and V_m is the *Hecke-like operator* defined so that if $\varphi_1 = \sum_{n,r} c(n,r)q^n y^r$ then

$$\varphi_1|V_m = \sum_{n,r}\left(\sum_{j|(n,r,m)} j^{k-1} c\left(\frac{nm}{j^2}, \frac{r}{j}\right)\right) q^n y^r. \tag{2.82}$$

It develops that the function

$$F(\Omega) = \sum_{m=0}^{\infty} (\varphi_1|V_m)(\tau, z)p^m \tag{2.83}$$

is a weight k Siegel modular form known as the *Saito–Kurokawa* or *additive lift* of the weight k and index 1 Jacobi form φ_1. An important example arises by taking the additive lift of the Jacobi form

$$\varphi_{10,1}(\tau, z) = -\eta(\tau)^{18}\theta_1(\tau, z)^2 \tag{2.84}$$

(cf. §A) which produces the *Igusa cusp form*

$$\Phi_{10}(\Omega) = \sum_{m=1}^{\infty} (\varphi_{10,1}|V_m)(\tau, z)p^m \tag{2.85}$$

The Igusa cusp form also admits a product representation obtained from the *Borcherds* or *exponential lift* of the umbral Jacobi form $Z^{(2)} = 2\varphi_{2,1}$ (cf. (2.49)) which is

$$\Phi_{10}(\Omega) = pqy \prod_{\substack{m,n,r\in\mathbb{Z} \\ (m,n,r)>0}} (1 - p^m q^n y^r)^{c^{(2)}(4mn-r^2)} \tag{2.86}$$

where the coefficients $c^{(2)}$ are defined by the Fourier expansion of $Z^{(2)}$

$$Z^{(2)}(\tau,z)=\sum_{n,r}c^{(2)}(4n-r^2)q^ny^r \tag{2.87}$$

and where the condition $(m,n,r)>0$ is that either $m>0$ or $m=0$ and $n>0$ or $m=n=0$ and $r<0$.

According to Theorem 2.1 in [87] the umbral Jacobi form $Z^{(\ell)}$ defines a Siegel modular form $\Phi^{(\ell)}$ on the *paramodular group* $\Gamma^+_{\ell-1}<Sp_4(\mathbb{Q})$ via the Borcherds lift for each $\ell\in\Lambda$. (We refer the reader to [87] for details.) For small values of ℓ these Siegel forms $\Phi^{(\ell)}$ have appeared in the literature (in particular in the work [88, 87] of Gritsenko–Nikulin). Taking $\ell\in\Lambda$ and defining $c^{(\ell)}(n,r)$ so that

$$Z^{(\ell)}(\tau,z)=\sum_{n,r}c^{(\ell)}(n,r)q^ny^r \tag{2.88}$$

we have the exponential lift $\Phi^{(\ell)}$ of weight $k=c^{(\ell)}(0,0)/2$ for $\Gamma^+_{\ell-1}$ given by

$$\Phi^{(\ell)}(\Omega)=p^{A(\ell)}q^{B(\ell)}y^{C(\ell)}\prod_{(m,n,r)>0}(1-p^mq^ny^r)^{c^{(\ell)}(mn,r)} \tag{2.89}$$

where

$$A(\ell)=\frac{1}{24}\sum_r c^{(\ell)}(0,r),\quad B(\ell)=\frac{1}{2}\sum_{r>0}rc^{(\ell)}(0,r),\quad C(\ell)=\frac{1}{4}\sum_r r^2c^{(\ell)}(0,r). \tag{2.90}$$

Note that for $\ell\in\{2,3,4,5\}$ we have $\Phi^{(\ell)}=(\Delta_k)^2$ in the notation of [87] where k is given by $k=(7-\ell)/(\ell-1)$. These four functions Δ_k appear as denominator functions for Lorentzian Kac–Moody Lie (super)algebras in [87, §5.1] (see also [89]) and in connection with mirror symmetry for $K3$ surfaces in [87, §5.2]. We refer the reader to §5.5 for more discussion on this.

3 Finite Groups

In this section we introduce the *umbral groups* $G^{(\ell)}$ for $\ell\in\Lambda=\{2,3,4,5,7,13\}$. It will develop that the representation theory of $G^{(\ell)}$ is intimately related to the vector-valued mock modular form $H^{(\ell)}$ of §2.5.

We specify the abstract isomorphism types of the groups explicitly in §3.1. Each group admits a quotient $\bar{G}^{(\ell)}$ which is naturally realised as a permutation group on $24/(\ell-1)$ points. We construct these permutations explicitly in §3.3. In order to construct the $G^{(\ell)}$ we use signed permutations—a notion we discuss in §3.2—and explicit generators for the $G^{(\ell)}$ are finally

obtained in §3.4. The signed permutation constructions are then used (again in §3.4) to define characters for $G^{(\ell)}$—the *twisted Euler characters*—which turn out to encode the automorphy of the mock modular forms $H_g^{(\ell)}$ that are be described in detail in §4. In §§3.5-3.6 we describe curious connections between the $G^{(\ell)}$ and certain Dynkin diagrams.

3.1 Specification

For $\ell \in \Lambda$ let abstract groups $G^{(\ell)}$ and $\bar{G}^{(\ell)}$ be as specified in Table 1. We will now explain the notation used therein (moving from right to left).

We write n as a shorthand for $\mathbb{Z}/n\mathbb{Z}$ (in the second and third rows of Table 1) and Sym_n denotes the symmetric group on n points. We write Alt_n for the alternating group on n points, which is the subgroup of Sym_n consisting of all even permutations.

We say that G is a *double cover* of a group H and write $G \simeq 2.H$ (cf. [90, §5.2]) in case G has a subgroup Z of order 2 that is normal (and therefore central, being of order two) with the property that G/Z is isomorphic to H. We say that G is a *non-trivial* double cover of H if it is a double cover that is not isomorphic to the direct product $2 \times H$. (We don't usually write $G \simeq 2.H$ unless G is a non-trivial cover of H.) The cyclic group of order 4 is a non-trivial double cover of the group of order 2.

For n a positive integer and q a prime power we write $GL_n(q)$ for the general linear group of invertible $n \times n$ matrices with coefficients in the finite field $\mathbb{F}_q$ with q elements, and we write $SL_n(q)$ for the subgroup consisting of matrices with determinant 1. We write $PGL_n(q)$ for the quotient of $GL_n(q)$ by its centre but we adopt the ATLAS convention (cf. [90, §2.1]) of writing $L_n(q)$ as a shorthand for $PSL_n(q)$—being the quotient of $SL_n(q)$ by its centre—since this group is typically simple. In the case that $q = 3$ and n is even the centre of $SL_n(q)$ has order 2 and $SL_n(q) \simeq 2.L_n(q)$ is a non-trivial double cover of $L_n(q)$.

In case $q = 5$ the centre of $GL_n(q)$ is a cyclic group of order 4 and so $GL_n(5)$ has a unique central subgroup of order 2. We write $GL_n(5)/2$ for the quotient of $GL_n(5)$ by this subgroup. In case $n = 2$ the exceptional isomorphism $PGL_2(5) \simeq Sym_5$ tells us then that $GL_2(5)/2$ is a double cover of the symmetric group Sym_5. For $n \geq 4$ there are two isomorphism classes of non-trivial double covers of Sym_n with the property that the central subgroup of order 2 is contained in the commutator subgroup of the double cover—these are the so-called *Schur double covers* (cf. [90, §4.1, §6.7]). The group $GL_2(5)/2$ is curious in that its only subgroup of index 2 is a direct product $2 \times Alt_5$ and its commutator subgroup is a copy of Alt_5, so it is a non-trivial double cover of Sym_5 that is not isomorphic to either of the Schur double covers.

The symbols $AGL_n(q)$ denote the *affine linear group* generated by the natural action of

Table 1: The Umbral Groups

ℓ	2	3	4	5	7	13
$G^{(\ell)}$	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$SL_2(3)$	4
$\bar{G}^{(\ell)}$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$L_2(3)$	2

$GL_n(q)$ on $\mathbb{F}_q^n$ and the translations $x \mapsto x + v$ for $v \in \mathbb{F}_q^n$. The group $AGL_n(q)$ may be realised as a subgroup of $GL_{n+1}(q)$ so in particular $AGL_3(2)$ embeds in $GL_4(2)$. The group $GL_4(2)$ is unusual amongst the $GL_n(2)$ in that it admits a non-trivial double cover $2.GL_4(2)$, which is a manifestation of the exceptional isomorphism $GL_4(2) \simeq Alt_8$. There are two conjugacy classes of subgroups of $2.Alt_8$ of order 2688, which is twice the order of $AGL_3(2)$. The groups in both classes are isomorphic to a particular non-trivial double cover of $AGL_3(2)$ and this is the group we denote $2.AGL_3(2)$ in Table 1.

The symbols M_{24} and M_{12} denote the sporadic simple Mathieu groups attached to the binary and ternary Golay codes, respectively. The group M_{24} is the automorphism group of the binary Golay code, which is the unique self-dual linear binary code of length 24 with minimum weight 8 (cf. [91]). The ternary Golay code is the unique self-dual linear ternary code of length 12 with minimum weight 6 (cf. [91]) and its automorphism group is a non-trivial double cover of M_{12}. There is a unique such group up to isomorphism (cf. [90]) which we denote by $2.M_{12}$ in Table 1.

Observe that for $\ell > 2$ the group $G^{(\ell)}$ has a unique central subgroup of order 2. Then $\bar{G}^{(\ell)}$ may be described for $\ell > 2$ by $\bar{G}^{(\ell)} = G^{(\ell)}/2$ and $G^{(\ell)}$ is a non-trivial double cover of $\bar{G}^{(\ell)}$ for each $\ell > 2$. It will develop in §3.2 that $\bar{G}^{(\ell)}$ is a permutation group of degree $n = 24/(\ell - 1)$ for each $\ell \in \Lambda$. Since $G^{(2)} \simeq M_{24}$ has trivial centre and is already a permutation group of degree $24 = 24/(2-1)$ we set $\bar{G}^{(2)} = G^{(2)}$.

Generalising the notation used above for double covers we write $G \simeq N.H$ to indicate that G fits into a short exact sequence

$$1 \to N \to G \to H \to 1 \tag{3.1}$$

for some groups N and H. We write $G \simeq N:H$ for a group of the form $N.H$ for which the sequence (3.1) splits (i.e. in case G is a semi-direct product $N \rtimes H$). Then we have $AGL_3(2) \simeq 2^3{:}L_2(7)$ according to the exceptional isomorphism $L_3(2) \simeq L_2(7)$ where 2^3 denotes an elementary abelian group of order 8.

3.2 Signed Permutations

The group of *signed permutations of degree* n or *hyperoctahedral group of degree* n, denoted here by Oct_n, is a semi-direct product $2^n{:}Sym_n$ where 2^n denotes an elementary abelian group of order 2^n. We may realise it explicitly as the subgroup of invertible linear transformations of an n-dimensional vector space generated by sign changes and permutations in a chosen basis. (This recovers $2^n{:}Sym_n$ so long as the vector space is defined over a field of characteristic other than 2. In case the characteristic is 2 the sign changes are trivial and we recover Sym_n.) When we write Oct_n we usually have in mind the data of fixed subgroups $N \simeq 2^n$ and $H \simeq Sym_n$ such that N is normal and $Oct_n/N \simeq H$, such as are given by sign changes and coordinate permutations, respectively, when we realise Oct_n explicitly as described above. In what follows we assume such data to be chosen for each n and write $Oct_n \to Sym_n$ for the composition $Oct_n \to Oct_n/N \simeq Sym_n$.

We will show in §3.4 that each $G^{(\ell)}$ for $\ell \in \Lambda$ admits a realisation as a subgroup of Oct_n for $n = 24/(\ell - 1)$ such that the image of $G^{(\ell)}$ under the map $Oct_n \to Sym_n$ is $\bar{G}^{(\ell)}$.

In §3.4 we use the following modification of the usual cycle notation for permutations to denote elements of Oct_n. Suppose Ω is a set with n elements and V is the vector space generated over a field $\mathbf{k}$ by the symbols $\{e_x\}_{x\in\Omega}$. Whereas the juxtaposition xy occurring in a cycle $(...xy...)$ indicates a coordinate permutation mapping e_x to e_y, we write $x\bar{y}$ to indicate that a sign change is applied so that e_x is mapped to $-e_y$. Then, for example, if $\Omega = \{\infty, 0, 1, 2, 3, 4\}$ we write $\sigma = (\bar{\infty}2\bar{3}4)(\bar{0})$ for the element of $GL(V)$ determined by

$$\sigma : e_\infty \mapsto e_2, \quad e_2 \mapsto -e_3, \quad e_3 \mapsto e_4, \quad e_4 \mapsto -e_\infty, \quad e_0 \mapsto -e_0. \tag{3.2}$$

(We think of the bar over the 3 in $(...2\bar{3}4...)$ as "occurring between" the 2 and 3.) We call the symbol 1 a *fixed point* of σ and we call 0 an *anti-fixed point*.

We define the *signed permutation character* of Oct_n by setting

$$\begin{aligned} \chi : Oct_n &\to \mathbb{Z} \\ g &\mapsto \chi_g = h_{g,+} - h_{g,-} \end{aligned} \tag{3.3}$$

where $h_{g,+}$ is the number of fixed points of g and $h_{g,-}$ is the number of anti-fixed points. Then χ is just the character of the ordinary representation of Oct_n furnished by V that obtains when the field $\mathbf{k}$ is taken to be $\mathbb{C}$. Writing $\bar{\chi}$ for the composition of $Oct_n \to Sym_n$ with the usual

permutation character of Sym_n we have

$$\begin{aligned} \bar{\chi} : Oct_n &\to \mathbb{Z} \\ g &\mapsto \bar{\chi}_g = h_{g,+} + h_{g,-} \end{aligned} \tag{3.4}$$

and we call $\bar{\chi}$ the *unsigned permutation character* of Oct_n. Then the signed and unsigned permutation characters χ and $\bar{\chi}$ together encode the number of fixed and anti-fixed points for each $g \in Oct_n$.

Let us take $\mathbf{k} = \mathbb{C}$ in the realisation of Oct_n as a subgroup of $GL(V)$. Then to each element $g \in Oct_n$ we assign a *signed permutation Frame shape* Π_g which encodes the eigenvalues of the corresponding (necessarily diagonalisable) linear transformation of V in the following way. An expression

$$\Pi_g = \prod_{k \geq 1} k^{m_g(k)} \tag{3.5}$$

with $m_g(k) \in \mathbb{Z}$ and $m_g(k) = 0$ for all but finitely many k indicates that g defines a linear transformation with $m_g(k)$ eigenvalues equal to $\mathrm{e}(j/k)$ for each $0 \leq j < k$ in the case that all the $m_g(k)$ are non-negative. If some $m_g(k)$ are negative then we find the number of eigenvalues equal to ξ say by looking at how many copies of ξ are present in $\Pi_g^+ = \prod_{k \geq 1,\, m_g(k) > 0} k^{m_g(k)}$ and subtracting the number of copies indicated by $\Pi_g^- = \prod_{k \geq 1,\, m_g(k) < 0} k^{-m_g(k)}$. Observe that the signed permutation character of Oct_n is recovered from the signed permutation Frame shapes via $\chi_g = m_g(1)$. Also, the Frame shape is invariant under conjugacy.

The map $Oct_n \to Sym_n$ (which may be realised by "ignoring" sign changes) furnishes a (non-faithful) permutation representation of Oct_n on n points; we call it the *unsigned permutation representation* of Oct_n. We write $g \mapsto \bar{\Pi}_g$ for the corresponding cycle shapes and call them the *unsigned permutation Frame shapes* attached to Oct_n. Observe that the unsigned permutation character of Oct_n is recovered from the unsigned permutation Frame shapes via $\bar{\chi}_g = \bar{m}_g(1)$ when $\bar{\Pi}_g = \prod_{k \geq 1} k^{\bar{m}_g(k)}$.

Observe that we obtain a faithful permutation representation $Oct_n \to Sym_{2n}$ by regarding $g \in Oct_n$ as a permutation of the $2n$ points $\{\pm e_x\}_{x \in \Omega}$. We denote the corresponding cycle shapes $\tilde{\Pi}_g$ and call them the *total permutation Frame shapes* attached to Oct_n. Then the total permutation Frame shapes can be recovered from the signed and unsigned permutation Frame

shapes for if we define a formal product on Frame shapes by setting

$$\Pi\,\Pi' = \prod_{k\geq 1} k^{m(k)+m'(k)}. \tag{3.6}$$

in case $\Pi = \prod_{k\geq 1} k^{m(k)}$ and $\Pi' = \prod_{k\geq 1} k^{m'(k)}$ then we have $\tilde{\Pi}_g = \Pi_g \bar{\Pi}_g$ for all $g \in Oct_n$.

Suppose that G is a subgroup of Oct_n with the property that the intersection $G \cap N$ has order 2. (This will be the case for $G = G^{(\ell)}$ when $\ell \in \Lambda$ and $\ell \neq 2$.) Let z be the unique and necessarily central involution in $G \cap N$ and write $\bar{G}$ for the image of G under the map $Oct_n \to Sym_n$. Then the conjugacy class of zg depends only on the conjugacy class of g and we obtain an involutory map $[g] \mapsto [zg]$ on the conjugacy classes of G. We call $[g]$ and $[zg]$ *paired* conjugacy classes and we say that $[g]$ is *self-paired* if $[g] = [zg]$. Observe that z must act as -1 times the identity in any faithful irreducible representation of G so if $g \mapsto \chi(g)$ is the character of such a representation then $\chi(zg) = -\chi(g)$. In particular, $\chi(g) = 0$ if g is self-paired.

3.3 Realisation Part I

In this section we construct the groups $\bar{G}^{(\ell)}$ as subgroups of Sym_n where $n = 24/(\ell - 1)$. This construction will be nested in the sense that each group $\bar{G}^{(\ell)}$ will be realised as a subgroup of some $\bar{G}^{(\ell')}$ for $\ell' - 1$ a divisor of $\ell - 1$.

For $\ell = 2$ let $\Omega^{(2)}$ be a set with 24 elements and let $\mathcal{G} \subset \mathcal{P}(\Omega^{(2)})$ be a copy of the Golay code on $\Omega^{(2)}$. Then the subgroup of the symmetric group $Sym_{\Omega^{(2)}}$ of permutations of the set $\Omega^{(2)}$ that preserves $\mathcal{G}$ is isomorphic to M_{24}. So we may set

$$\bar{G}^{(2)} = \{\sigma \in Sym_{\Omega^{(2)}} \mid \sigma(C) \in \mathcal{G},\, \forall C \in \mathcal{G}\}\,.$$

For $\ell = 3$ the largest proper divisor of $\ell - 1 = 2$ is $1 = 2 - 1$. Choose a partition of $\Omega^{(2)}$ into $2/1 = 2$ subsets of $24/2 = 12$ elements such that each 12-element set belongs to $\mathcal{G}$ and denote it $P^{(3)} = \{\Omega^{(3)}, \Omega^{(3)'}\}$. Equivalently, take $\Omega^{(3)}$ to be the symmetric difference of the sets of fixed-points of elements $\sigma, \sigma' \in \bar{G}^{(2)}$ of cycle shape $1^8 2^8$ such that $\sigma\sigma'$ has order 6. (That is, $\Omega^{(3)}$ consists of the points that are fixed by σ or σ' but not both. In turns out that if the product $\sigma\sigma'$ has order 6 then its cycle shape is $1^2 2^2 3^2 6^2$ and so the fixed-point sets of σ and σ' have intersection of size 2 and $\Omega^{(3)}$ and $\Omega^{(3)'} = \Omega^{(2)} \setminus \Omega^{(3)}$ both have 12 elements.) Then the group of permutations of $\Omega^{(3)}$ induced by the subgroup of $\bar{G}^{(2)}$ that fixes this partition is a copy of $\bar{G}^{(3)} \simeq M_{12}$. Explicitly, and to explain what we mean here by fix, if $\bar{G}^{(2)}_{P^{(3)}}$ denotes the

subgroup of $\bar{G}^{(2)}$ that fixes the partition $P^{(3)}$ in the sense that

$$\bar{G}^{(2)}_{P^{(3)}} = \left\{\sigma \in \bar{G}^{(2)} \mid \sigma(C) = C,\, \forall C \in P^{(3)}\right\}, \tag{3.7}$$

and if $\varrho : \bar{G}^{(2)}_{P^{(3)}} \to Sym_{\Omega^{(3)}}$ denotes the natural map obtained by restricting elements of $\bar{G}^{(2)}_{P^{(3)}}$ to $\Omega^{(3)}$,

$$\begin{aligned} \varrho : \bar{G}^{(2)}_{P^{(3)}} &\to Sym_{\Omega^{(3)}} \\ \sigma &\mapsto \sigma|_{\Omega^{(3)}}, \end{aligned} \tag{3.8}$$

then $\bar{G}^{(3)}$ is the image of ϱ. The kernel of this map is the subgroup

$$\bar{G}^{(2)}_{P^{(3)},\Omega^{(3)}} = \left\{\sigma \in \bar{G}^{(2)}_{P^{(3)}} \mid \sigma(x) = x,\, \forall x \in \Omega^{(3)}\right\} \tag{3.9}$$

comprised of permutations in $\bar{G}^{(2)}_{P^{(3)}}$ that fix every element of $\Omega^{(3)}$ so we have

$$\bar{G}^{(3)} = \varrho\left(\bar{G}^{(2)}_{P^{(3)}}\right) \simeq \bar{G}^{(2)}_{P^{(3)}}/\bar{G}^{(2)}_{P^{(3)},\Omega^{(3)}} \simeq M_{12}.$$

For $\ell = 4$ the largest proper divisor of $\ell - 1 = 3$ is $1 = 2 - 1$. Choose a partition of $\Omega^{(2)}$ into $3/1 = 3$ subsets of $24/3 = 8$ elements that belong to $\mathcal{G}$ and denote it $P^{(4)} = \{\Omega^{(4)}, \Omega^{(4)'}, \Omega^{(4)''}\}$. Equivalently, choose $\Omega^{(4)}$ and $\Omega^{(4)'}$ to be the fixed-point sets of respective elements σ and σ' of order 2 in $\bar{G}^{(2)}$ having cycle shapes $1^8 2^8$ and disjoint fixed-point sets and the property that $\sigma\sigma'$ has order 3. Then the group of permutations of $\Omega^{(4)}$ induced by the subgroup of $\bar{G}^{(2)}$ that fixes this partition is a copy of the group $\bar{G}^{(4)} \simeq AGL_3(2)$ and we have

$$\bar{G}^{(4)} = \varrho\left(\bar{G}^{(2)}_{P^{(4)}}\right) \simeq \bar{G}^{(2)}_{P^{(4)}}/\bar{G}^{(2)}_{P^{(4)},\Omega^{(4)}} \simeq AGL_3(2)$$

where $\bar{G}^{(2)}_{P^{(4)}}$, the map ϱ and the subgroup $\bar{G}^{(2)}_{P^{(4)},\Omega^{(4)}}$ are defined according to the natural analogues of (3.7), (3.8) and (3.9), respectively.

For $\ell = 5$ the largest proper divisor of $\ell - 1 = 4$ is $2 = 3 - 1$. Choose a partition $P^{(5)} = \{\Omega^{(5)}, \Omega^{(5)'}\}$ of $\Omega^{(3)}$ into $4/2 = 2$ subsets of $12/2 = 6$ elements such that neither of the sets in $P^{(5)}$ is contained in a set of size 8 in $\mathcal{G}$. Equivalently, take $\Omega^{(5)}$ to be the symmetric difference of the sets of fixed-points of elements $\sigma, \sigma' \in \bar{G}^{(3)}$ of cycle shape $1^4 2^4$ such that $\sigma\sigma'$ has order 6. (This condition forces the fixed-point sets of σ and σ' to have a single point of intersection so that $\Omega^{(5)}$ and $\Omega^{(5)'} = \Omega^{(3)} \setminus \Omega^{(5)}$ both have 6 elements.) Then the group of permutations of

$\Omega^{(5)}$ induced by the subgroup of $\bar{G}^{(3)}$ that fixes this partition is a copy of $\bar{G}^{(5)} \simeq PGL_2(5)$.

$$\bar{G}^{(5)} = \varrho\left(\bar{G}^{(3)}_{P^{(5)}}\right) \simeq \bar{G}^{(3)}_{P^{(5)}} / \bar{G}^{(3)}_{P^{(5)},\Omega^{(5)}} \simeq PGL_2(5)$$

For $\ell = 7$ the largest proper divisor of $\ell - 1 = 6$ is $3 = 4 - 1$. Choose a partition $P^{(7)} = \{\Omega^{(7)}, \Omega^{(7)'}\}$ of $\Omega^{(4)}$ into $6/3 = 2$ subsets of $8/2 = 4$ elements such that $\Omega^{(7)}$ (and therefore also $\Omega^{(7)'} = \Omega^{(4)} \setminus \Omega^{(7)}$) is the set of fixed points of an element of order 2 in $\bar{G}^{(4)}$ having cycle shape $1^4 2^2$. (There are three conjugacy classes of elements of order 2 in $\bar{G}^{(4)}$ but for just one of these classes do the elements have the cycle shape $1^4 2^2$.) Then the group of even permutations of $\Omega^{(7)}$ induced by the subgroup of $\bar{G}^{(4)}$ that fixes this partition is a copy of the group $\bar{G}^{(7)} \simeq L_2(3)$, which is of course isomorphic to the alternating group on 4 points.

$$\bar{G}^{(7)} = \varrho\left(\bar{G}^{(4)}_{P^{(7)}}\right) \cap Alt_{\Omega^{(7)}} \simeq L_2(3)$$

(The quotient $\bar{G}^{(4)}_{P^{(7)}} / \bar{G}^{(4)}_{P^{(7)},\Omega^{(7)}}$ is isomorphic to the full symmetric group on 4 points so the restriction to even permutations is not redundant.)

The next largest divisor of $7 - 1 = 6$ is $2 = 3 - 1$. An alternative construction of $\bar{G}^{(7)}$ is obtained by choosing a partition $P^{(7)} = \{\Omega^{(7)}, \Omega^{(7)'}, \Omega^{(7)''}\}$ of $\Omega^{(3)}$ into $6/2 = 3$ subsets of $12/3 = 4$ elements such that $\Omega^{(7)}$ and $\Omega^{(7)'}$ are the fixed-point sets of respective elements σ and σ' of order 2 in $\bar{G}^{(3)}$ having cycle shapes $1^4 2^4$ and disjoint fixed-point sets and the property that $\sigma\sigma'$ has order 3. Then the group of permutations of $\Omega^{(7)}$ induced by the subgroup of $\bar{G}^{(3)}$ that fixes this partition is again a copy of the group $\bar{G}^{(7)} \simeq L_2(3)$.

$$\bar{G}^{(7)} = \varrho\left(\bar{G}^{(3)}_{P^{(7)}}\right) \simeq \bar{G}^{(3)}_{P^{(7)}} / \bar{G}^{(3)}_{P^{(7)},\Omega^{(7)}} \simeq L_2(3)$$

For $\ell = 13$ the largest proper divisor of $\ell - 1 = 12$ is $6 = 7 - 1$. Choose a partition $P^{(13)} = \{\Omega^{(13)}, \Omega^{(13)'}\}$ of $\Omega^{(7)}$ into $12/6 = 2$ subsets of $4/2 = 2$ elements such that $\Omega^{(13)}$ is the orbit of an element of order 2 in $\bar{G}^{(7)}$. (That is, choose any partition of $\Omega^{(7)}$ into subsets of size 2. The elements of order 2 in $\bar{G}^{(7)}$ all have cycle shape 2^2.) Then the group of permutations of $\Omega^{(13)}$ induced by the subgroup of $\bar{G}^{(7)}$ that fixes this partition is a copy of the group $\bar{G}^{(13)} \simeq Sym_2$.

$$\bar{G}^{(13)} = \varrho\left(\bar{G}^{(7)}_{P^{(13)}}\right) \simeq \bar{G}^{(7)}_{P^{(13)}} / \bar{G}^{(7)}_{P^{(13)},\Omega^{(13)}} \simeq Sym_2$$

The next largest divisor of $13 - 1 = 12$ is $4 = 5 - 1$. An alternative construction of $\bar{G}^{(13)}$ is obtained by choosing a partition $P^{(13)} = \{\Omega^{(13)}, \Omega^{(13)'}, \Omega^{(13)''}\}$ of $\Omega^{(5)}$ into $12/4 = 3$ subsets of $6/3 = 2$ elements such that $\Omega^{(13)}$ and $\Omega^{(13)'}$ are the fixed-point sets of respective elements σ

and σ' of order 2 in $\bar{G}^{(3)}$ having cycle shapes $1^2 2^2$ and disjoint fixed-point sets and the property that $\sigma\sigma'$ has order 3. Then the group of permutations of $\Omega^{(13)}$ induced by the subgroup of $\bar{G}^{(5)}$ that fixes this partition is a copy of the group $\bar{G}^{(13)} \simeq Sym_2$.

$$\bar{G}^{(13)} = \varrho\left(\bar{G}^{(5)}_{P^{(13)}}\right) \simeq \bar{G}^{(5)}_{P^{(13)}} / \bar{G}^{(5)}_{P^{(13)},\Omega^{(13)}} \simeq Sym_2$$

Remark 3.1. The group $\bar{G}^{(\ell')}_{P^{(\ell)},\Omega^{(\ell)}}$ is trivial in every instance except for when (ℓ', ℓ) is $(2, 4)$ or $(4, 7)$ so apart from these cases we have $\bar{G}^{(\ell)} \simeq \bar{G}^{(\ell')}_{P^{(\ell)}}$. The group $\bar{G}^{(2)}_{P^{(4)},\Omega^{(4)}}$ has order 8 and $\bar{G}^{(4)}_{P^{(7)},\Omega^{(7)}}$ has order 4.

Remark 3.2. A *Steiner system* with parameters (t, k, n) is the data of an n-element set Ω together with a collection of subsets of Ω of size k, called the *blocks* of the system, with the property that any t-element subset of Ω is contained in a unique block. It is well-known that $M_{24} \simeq \bar{G}^{(2)}$ may be realised as the automorphism group of a Steiner system with parameters $(5, 8, 24)$ and $M_{12} \simeq \bar{G}^{(3)}$ may be described as the automorphism group of a Steiner system with parameters $(5, 6, 12)$. Indeed, for $\bar{G}^{(2)} \simeq M_{24}$ we may take the blocks to be the $\binom{24}{5}/\binom{8}{5} = 759$ sets of size 8 in a copy $\mathcal{G}$ of the Golay code on the 24 element set $\Omega = \Omega^{(2)}$ and for $\bar{G}^{(3)} \simeq M_{12}$ regarded as the subgroup of $\bar{G}^{(2)}$ preserving a set $\Omega^{(3)}$ of size 12 in $\mathcal{G}$ we may take the blocks to be the $\binom{12}{5}/\binom{6}{5} = 132$ subsets of size 6 that arise as an intersection $C \cap \Omega^{(3)}$ for $C \in \mathcal{G}$ having size 8. The group $\bar{G}^{(4)} \simeq AGL_3(2)$ also admits such a description: it is the automorphism group of the unique Steiner system with parameters $(3, 4, 8)$. The blocks of such a system constitute the $\binom{8}{3}/\binom{4}{3} = 14$ codewords of weight 4 in the unique doubly even linear binary code of length 8, the length 8 *Hamming code*.

Remark 3.3. Following [92, Chp. 11] let n be a divisor of 12, take n cards labelled by 0 through $n-1$ and consider the group $M_n < Sym_n$ generated by the *reverse shuffle* $r_n : t \mapsto n-1-t$ and the *Mongean shuffle* $s_n : t \mapsto \min\{2t, 2n-1-2t\}$. Then the notation is consistent with that used above, for the group M_{12} just defined is indeed isomorphic to the Mathieu group of permutations on 12 points. So we have $M_{12} \simeq \bar{G}^{(3)}$ according to Table 1 and it turns out that in fact $M_n \simeq \bar{G}^{(\ell)}$ whenever $n = 24/(\ell - 1)$, and so this shuffle construction recovers all of the $\bar{G}^{(\ell)}$ except for $G^{(2)} \simeq M_{24}$ and $G^{(4)} \simeq AGL_3(2)$ (i.e. all $\bar{G}^{(\ell)}$ for ℓ odd).

3.4 Realisation Part II

Having constructed the $\bar{G}^{(\ell)}$ as permutation groups we now describe the umbral groups $G^{(\ell)}$ as groups of signed permutations. We retain the notation of §3.3.

To construct $G^{(3)}$ choose an element τ of order 11 in $\bar{G}^{(3)}$ and (re)label the elements of $\Omega^{(3)}$

so that

$$\Omega^{(3)} = \{\infty, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X\}$$

and the action of τ on $\Omega^{(3)}$ is given by $x \mapsto x+1$ modulo 11 (where $X = 10$). Then for any set $C \subset \Omega^{(3)}$ of size 4 there is a unique element $\sigma \in \bar{G}^{(3)}$ with cycle shape $1^4 2^4$ such that the fixed-point set of σ is precisely C. In case $C = \{0, 1, 4, 9\}$ for example $\sigma = (\infty 6)(2X)(35)(78)$ and $\bar{G}^{(3)}$ is generated by σ and τ. Let $\hat{\Omega}^{(3)} = \{e_x \mid x \in \Omega^{(3)}\}$ be a basis for a 12-dimensional vector space over $\mathbb{C}$ say, let $\hat{\tau}$ be the element of the hyperoctahedral group $Oct_{\hat{\Omega}^{(3)}} \simeq Oct_{12}$ given by $\hat{\tau} : e_x \mapsto e_{x+1}$, so that $\hat{\tau} = (0123456789X)$ in (signed) cycle notation (cf. §3.2), and set $\hat{\sigma} = (\infty 6)(\bar{2}\bar{X})(35)(\bar{7}\bar{8})$. Then for $G^{(3)}$ the group generated by $\hat{\sigma}$ and $\hat{\tau}$ we have that $G^{(3)} \simeq 2.M_{12}$ and the image of $G^{(3)}$ under the map $Oct_{\hat{\Omega}^{(3)}} \to S_{\Omega^{(3)}}$ is $\bar{G}^{(3)}$.

To construct $G^{(4)}$ choose an element τ of order 7 in $\bar{G}^{(4)}$ and (re)label the elements of $\Omega^{(4)}$ so that

$$\Omega^{(4)} = \{\infty, 0, 1, 2, 3, 4, 5, 6\}$$

and the action of τ on $\Omega^{(4)}$ is given by $x \mapsto x+1$ modulo 7. There is a unique conjugacy class of elements of $\bar{G}^{(4)}$ having cycle shape $1^4 2^2$ and the fixed-point sets of these elements are the 14 subsets of $\Omega^{(4)}$ of size 4 comprising the blocks of a $(3, 4, 8)$ Steiner system preserved by $\bar{G}^{(4)}$ (cf. Remark 3.2). The particular block $C = \{0, 1, 3, 6\}$ is the unique one with the property that σ and τ generate $\bar{G}^{(4)}$ in case σ is any (of the 3) involution(s) with C as its fixed-point set. Take $\sigma = (\infty 5)(24)$ and let $\hat{\Omega}^{(4)} = \{e_x \mid x \in \Omega^{(4)}\}$ be a basis for an 8-dimensional vector space over $\mathbb{C}$ say, let $\hat{\tau}$ be the element of the hyperoctahedral group $Oct_{\hat{\Omega}^{(4)}} \simeq Oct_8$ given by $\hat{\tau} : e_x \mapsto e_{x+1}$, so that $\hat{\tau} = (0123456)$, and set $\hat{\sigma} = (\infty 5)(\bar{0})(\bar{1})(24)$. Then for $G^{(4)}$ the group generated by $\hat{\sigma}$ and $\hat{\tau}$ we have that $G^{(4)} \simeq 2.AGL_3(2)$ and the image of $G^{(4)}$ under the map $Oct_{\hat{\Omega}^{(4)}} \to S_{\Omega^{(4)}}$ is $\bar{G}^{(4)}$.

For the remaining $\ell \geq 5$ we can construct $G^{(\ell)}$ from $G^{(3)}$ by proceeding in analogy with the constructions of $\bar{G}^{(\ell)}$ from $\bar{G}^{(3)}$ (and its subgroups) given earlier in this section. Concretely, for $\ell = 5$ we may set

$$G^{(5)} = \hat{\varrho}\left(G^{(3)}_{\hat{P}^{(5)}}\right) \simeq G^{(3)}_{\hat{P}^{(5)}} / G^{(3)}_{\hat{P}^{(5)}, \hat{\Omega}^{(5)}} \simeq GL_2(5)/2$$

where $\hat{\Omega}^{(5)} = \{e_x \mid x \in \Omega^{(5)}\}$ is a basis for $24/(5-1) = 6$-dimensional vector space over $\mathbb{C}$, we define $\hat{C} = \{e_x \mid x \in C\}$ for $C \subset \Omega^{(5)}$ and set $\hat{P}^{(5)} = \{\hat{\Omega}^{(5)}, \hat{\Omega}^{(5)\prime}\}$ for $P^{(5)} = \{\Omega^{(5)}, \Omega^{(5)\prime}\}$ as in

the construction of $\bar{G}^{(5)}$ given above, the groups $G^{(3)}_{\hat{P}^{(5)}}$ and $G^{(3)}_{\hat{P}^{(5)},\hat{\Omega}^{(5)}}$ are defined by

$$G^{(3)}_{\hat{P}^{(5)}} = \left\{\sigma \in G^{(3)} \mid \sigma(\operatorname{Span}\hat{C}) \subset \operatorname{Span}\hat{C},\, \forall \hat{C} \in \hat{P}^{(5)}\right\}, \tag{3.10}$$

$$G^{(3)}_{\hat{P}^{(5)},\hat{\Omega}^{(5)}} = \left\{\sigma \in G^{(3)}_{\hat{P}^{(5)}} \mid \sigma(e_x) = e_x,\, \forall x \in \Omega^{(5)}\right\}, \tag{3.11}$$

and $\hat{\varrho}$ is the map $G^{(3)}_{\hat{P}^{(5)}} \to Oct_{\hat{\Omega}^{(5)}}$ obtained by restriction.

$$\begin{aligned} \hat{\varrho} : G^{(3)}_{\hat{P}^{(5)}} &\to Oct_{\hat{\Omega}^{(5)}} \\ \sigma &\mapsto \sigma|_{\operatorname{Span}\hat{\Omega}^{(5)}} \end{aligned} \tag{3.12}$$

In direct analogy with this we have that

$$\begin{aligned} G^{(7)} &= \hat{\varrho}\left(G^{(3)}_{\hat{P}^{(7)}}\right) \simeq G^{(3)}_{\hat{P}^{(7)}}/G^{(3)}_{\hat{P}^{(7)},\hat{\Omega}^{(7)}} \simeq SL_2(3), \\ G^{(13)} &= \hat{\varrho}\left(G^{(5)}_{\hat{P}^{(13)}}\right) \simeq G^{(5)}_{\hat{P}^{(13)}}/G^{(5)}_{\hat{P}^{(13)},\hat{\Omega}^{(13)}} \simeq 4, \end{aligned}$$

when $G^{(\ell')}_{\hat{P}^{(\ell)}}$ and $G^{(\ell')}_{\hat{P}^{(\ell)},\hat{\Omega}^{(\ell)}}$ are defined as in (3.10) and (3.11), respectively, for $P^{(\ell)} = \{\Omega^{(\ell)}, \cdots\}$ as specified in the construction of $\bar{G}^{(\ell')}$ given above, and $\hat{\varrho}$ as in (3.12).

We conclude this section with explicit signed permutation presentations of the $G^{(\ell)}$ as subgroups of the $Oct_{\hat{\Omega}^{(\ell)}}$ for $\ell \geq 3$. The presentations for $\ell = 3$ and $\ell = 4$ were obtained above, and the remaining ones can be found in a similar manner. In each case we label the index set $\Omega^{(\ell)}$ so that $\Omega^{(\ell)} = \{\infty, 0, \cdots, n-1\}$ where $n = 24/(\ell-1) - 1 = (25-\ell)/(\ell-1)$ and seek a presentation for which the coordinate permutation $e_x \mapsto e_{x+1}$ (with indices read modulo n) is an element of order n in $G^{(\ell)}$ (although we must take this element to be trivial in case $\ell = 13$).

$$G^{(3)} = \left\langle (\infty 6)(\bar{2}\bar{X})(35)(\bar{7}\bar{8}), (0123456789X)\right\rangle \tag{3.13}$$

$$G^{(4)} = \langle (\infty 5)(\bar{0})(\bar{1})(24), (0123456)\rangle \tag{3.14}$$

$$G^{(5)} = \langle (\infty\bar{0})(3\bar{1})(2\bar{4}), (01234)\rangle \tag{3.15}$$

$$G^{(7)} = \langle (\infty\bar{0})(\bar{1}2), (012)\rangle \tag{3.16}$$

$$G^{(13)} = \langle (\infty\bar{0})\rangle \tag{3.17}$$

Equipped now with explicit realisations $G^{(\ell)} < Oct_n$ of the umbral groups $G^{(\ell)}$ as signed permutation groups we define symbols $\Pi^{(\ell)}_g$, $\chi^{(\ell)}_g$,$\bar{\Pi}^{(\ell)}_g$ and $\bar{\chi}^{(\ell)}_g$ as follows for $\ell \in \Lambda$ and $g \in G^{(\ell)}$. We write $\Pi^{(\ell)}_g$ for the signed permutation Frame shape attached to $g \in G^{(\ell)}$ (when regarded as an element of Oct_n) as defined in §3.2 and we write $\bar{\Pi}^{(\ell)}_g$ for the cycle shape attached to the image

of $g \in G^{(\ell)}$ under the composition $G^{(\ell)} \to Oct_n \to Sym_n$. We define $g \mapsto \chi_g^{(\ell)}$ by restricting the signed permutation character (cf. (3.3)) to $G^{(\ell)}$ and we define $g \mapsto \bar{\chi}_g^{(\ell)}$ by restricting the unsigned permutation character (cf. (3.4)) to $G^{(\ell)}$. We call $\chi_g^{(\ell)}$ the *signed twisted Euler character* attached to $g \in G^{(\ell)}$ and we call $\bar{\chi}_g^{(\ell)}$ the *unsigned twisted Euler character* attached to $g \in G^{(\ell)}$.

As is explained in §3.2 the data $g \mapsto \Pi_g^{(\ell)}$ is sufficient to determine the cycle shapes $\bar{\Pi}_g^{(\ell)}$ and the twisted Euler characters $\chi_g^{(\ell)}$ and $\bar{\chi}_g^{(\ell)}$. We will attach vector-valued mock modular forms $H_g^{(\ell)}$ to each $g \in G^{(\ell)}$ in §4 and it will develop that the shadows and multiplier systems of these functions are encoded in the Frame shapes $\Pi_g^{(\ell)}$. We list the Frame shapes $\Pi_g^{(\ell)}$ and $\bar{\Pi}_g^{(\ell)}$ and the twisted Euler characters $\chi_g^{(\ell)}$ and $\bar{\chi}_g^{(\ell)}$ for all $g \in G^{(\ell)}$ and $\ell \in \Lambda$ in the tables of §B.2.

We conclude this section with an extraordinary property relating the Frame shapes $\Pi_g^{(\ell)}$ and $\bar{\Pi}_g^{(\ell)}$ attached to the umbral groups $G^{(\ell)}$ at $\ell = 2$ and $\ell = 4$ which can be verified explicitly using the tables of §B.2.

Proposition 3.4. *Let $g \in G^{(4)}$ and suppose that the Frame shape $\Pi_g^{(4)} = \prod_k k^{m_g(k)}$ is a cycle shape, so that $m_g(k) \geq 0$ for all k. Then there exists $g' \in G^{(2)}$ with $o(g') = 2o(g)$ such that*

$$\bar{\Pi}_{g'}^{(2)} = \prod_k k^{\bar{m}_g(k)} (2k)^{m_g(k)} \tag{3.18}$$

when $\bar{\Pi}_g^{(4)} = \prod_k k^{\bar{m}_g(k)}$ except in case g is of class 4B.

As will be explained in §4 this result implies direct relationships between the functions $H_{g'}^{(2)}$ and $H_g^{(4)}$ when g and g' are as in the statement of the proposition.

3.5 Dynkin Diagrams Part I

The *McKay correspondence* [93] relates finite subgroups of $SU(2)$ to the affine Dynkin diagrams of ADE type by associating irreducible representations of the finite groups to nodes of the corresponding diagrams and by now is well understood in terms of resolutions of *simple singularities* $\mathbb{C}^2/G$ for $G < SU(2)$ [94, 95]. A more mysterious Dynkin diagram correspondence also due to McKay is his *monstrous E_8 observation* [93] (see also [96, §14]) which associates nine of the conjugacy classes of the monster group to the nodes of the affine E_8 Dynkin diagram and extends to similar correspondences relating certain subgroups of the Monster to other affine Dynkin diagrams [97]. We find analogues of both McKay's Dynkin diagram observations manifesting in the groups $G^{(\ell)}$, as we shall now explain.

For $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$ the number $(25 - \ell)/(\ell - 1)$ is an odd integer p such that $p + 1$ divides 24 and is a prime in case ℓ is not 13. Recall the construction of $\bar{G}^{(\ell)}$ as permutations of

$\Omega^{(\ell)}$ from §3.3. By inspection, with the assistance of [98], we obtain the following lemma.

Lemma 3.5. *Let $\ell \in \{2,3,4,5,7\}$ and set $p = (25-\ell)/(\ell-1)$. Then the group $\bar{G}^{(\ell)}$ has a unique conjugacy class of subgroups isomorphic to $L_2(p)$ that act transitively on the degree $p+1 = 24/(\ell-1)$ permutation representation of $\bar{G}^{(\ell)}$ on $\Omega^{(\ell)}$.*

Armed with Lemma 3.5 we pick a transitive subgroup of $\bar{G}^{(\ell)}$ isomorphic to $L_2(p)$ for each ℓ in $\{2,3,4,5,7\}$—these being the cases that $(25-\ell)/(\ell-1)$ is prime—and denote it $\bar{L}^{(\ell)}$. For future reference we note that there are two conjugacy classes of subgroups of $\bar{G}^{(\ell)}$ isomorphic to $L_2(p)$ in case $\ell \in \{3,4\}$ but in each case only one of these classes contains subgroups acting transitively.

Lemma 3.6. *For $\ell \in \{3,4\}$ and $p = (25-\ell)/(\ell-1)$ there is a unique conjugacy class of subgroups of $\bar{G}^{(\ell)}$ isomorphic to $L_2(p)$ that does not act transitively in the degree $p+1 = 24/(\ell-1)$ permutation representation of $\bar{G}^{(\ell)}$ on the set $\Omega^{(\ell)}$.*

It is a result due to Galois (proven in a letter to Chevalier written on the eve of his deadly duel [92, Chp. 10]) that the group $L_2(p)$ has no transitive permutation representation of degree less than $p+1$ if $p > 11$. However for the remaining primes p not exceeding 11 there are transitive permutation representations on exactly p points, and in fact we have the following statement.

Lemma 3.7. *Let $p \in \{2,3,5,7,11\}$ and set $\bar{L} = L_2(p)$. Then there is a subgroup $\bar{D} < \bar{L}$ of index p with the property that $\bar{L} = \langle\sigma\rangle\bar{D}$ for σ an element of order p in $\bar{L}$, so that every element $g \in \bar{L}$ admits a unique expression $g = sd$ where $s \in \langle\sigma\rangle$ and $d \in \bar{D}$.*

Remark 3.8. See [92, Chp. 10] for applications of the result of Lemma 3.7 to exceptional isomorphisms among finite simple groups of small order, and see [99, 100] for an application of the case that $p = 11$ to the truncated icosahedron (which describes the structure of buckminsterfullerenes, a.k.a. buckyballs).

According to Lemma 3.7 we obtain a permutation representation of degree p for $\bar{L}$ by taking the natural action of $\bar{L}$ on cosets of $\bar{D}$ and this action is transitive since σ induces a p-cycle. Using this result we choose a copy of $\bar{D}$ in $\bar{L} = \bar{L}^{(\ell)}$ for each $\ell \in \{3,4,5,7\}$ and denote it $\bar{D}^{(\ell)}$. (Such subgroups are uniquely determined up to isomorphism, but there are as many conjugacy classes of subgroups of $\bar{L}$ isomorphic to $\bar{D}$ as there are conjugacy classes of elements of order p in $\bar{L}$, which is to say there are 2 classes in case $p \in \{11,7\}$ and a unique class in case $p \in \{5,3\}$.) We describe the groups $\bar{D}^{(\ell)}$ in Table 2 and observe that each one is isomorphic to a finite group $\bar{D}_0 < SO_3(\mathbb{R})$. As such there is a corresponding finite group $D_0 < SU(2)$ that maps onto $\bar{D}_0$

Table 2: The Umbral Groups and Dynkin Data

ℓ	2	3	4	5	7	13
p	23	11	7	5	3	1
$G^{(\ell)}$	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$SL_2(3)$	4
$L^{(\ell)}$		$SL_2(11)$	$SL_2(7)$		$SL_2(3)$	
$D^{(\ell)}$		$2.Alt_5$	$2.Sym_4$		Q_8	
$\bar{G}^{(\ell)}$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$L_2(3)$	2
$\bar{L}^{(\ell)}$	$L_2(23)$	$L_2(11)$	$L_2(7)$	$L_2(5)$	$L_2(3)$	
$\bar{D}^{(\ell)}$		Alt_5	Sym_4	Alt_4	2^2	
$\Delta^{(\ell)}$		$\hat{E}_8$	$\hat{E}_7$	$\hat{E}_6$	$\hat{D}_4$	

via the 2-fold covering $SU(2) \to SO_3(\mathbb{R})$, and a corresponding Dynkin diagram $\Delta^{(\ell)}$ according to the McKay correspondence. We list the Dynkin diagrams $\Delta^{(\ell)}$ also in Table 2.

Traditionally finite subgroups of $SO_3(\mathbb{R})$ are called *ternary*, owing to the fact that their elements are described using 3×3 matrices, and finite subgroups of $SU(2)$ are called *binary*. The map $SU(2) \to SO_3(\mathbb{R})$ determines a correspondence between binary and ternary groups whereby the ternary polyhedral groups (of orientation preserving symmetries of regular polyhedra) correspond to binary groups of the form $2.Alt_5$, $2.Sym_4$ and $2.Alt_4$ (depending upon the polyhedron), a dihedral subgroup $Dih_n < SO_3(\mathbb{R})$ corresponds to a generalised quaternion group of order $4n$ in $SU(2)$, and a binary cyclic group corresponds to a ternary cyclic group of the same order. Thus, given a finite subgroup $\bar{D}_0 < SO_3(\mathbb{R})$, we may speak of the *associated binary group* $D_0 < SU(2)$.

Next we observe that each group $G^{(\ell)}$ contains a subgroup $D^{(\ell)}$ isomorphic to the binary group associated to $\bar{D}^{(\ell)}$ (when $\bar{D}^{(\ell)}$ is regarded as a subgroup of $SO_3(\mathbb{R})$) for all of the above cases except when $\ell = p = 5$. More particularly, for $\ell \in \{3, 7\}$ there is a unique conjugacy class of subgroups isomorphic to the binary group attached to $\bar{D}^{(\ell)}$ while for $\ell = 4$ there are two such conjugacy classes and for $\ell = 5$ there are none. For each $\ell \in \{3, 4, 7\}$ we pick a subgroup isomorphic to the binary group attached to $\bar{D}^{(\ell)}$ and denote it $D^{(\ell)}$ and we display the (isomorphism types of the) groups $D^{(\ell)}$ in Table 2. We write Q_8 there for the quaternion group of order 8.

To see how the $D^{(\ell)}$ arise in $G^{(\ell)}$ (and fail to do so in the case that $\ell = 5$) recall from Lemma 3.5 that $\bar{G}^{(\ell)}$ contains a unique transitive subgroup isomorphic to $L_2(p)$ up to conjugacy for $\ell \in \{3, 4, 5, 7\}$. The preimage of such a subgroup under the natural map $G^{(\ell)} \to \bar{G}^{(\ell)}$ (cf. §3.4) is a double cover of $L_2(p)$ that is in fact isomorphic to $SL_2(p)$ (cf. §3.1) except when $\ell = 5$. In case $\ell = 5$ we have $G^{(5)} \simeq GL_2(5)/2$ which has the same order as $SL_2(5)$ but is not isomorphic

to it. We see then that for $\ell \in \{3,4,7\}$ we may find a copy of $SL_2(p)$ in the preimage of $\bar{L}^{(\ell)}$ under the map $G^{(\ell)} \to \bar{G}^{(\ell)}$; we do so and denote it $L^{(\ell)}$. Then we may take $D^{(\ell)}$ to be the preimage of $\bar{D}^{(\ell)}$ under the map $L^{(\ell)} \to \bar{L}^{(\ell)}$. The fact that there is no $SL_2(5)$ in $G^{(5)}$ explains why there is no group $D^{(5)}$.

Observe that the rank of $\Delta^{(\ell)}$ is given by $11-\ell$ for each $\ell \in \{3,4,5,7\}$. This may be taken as a hint to the following uniform construction of the $\Delta^{(\ell)}$. Starting with the (finite type) E_8 Dynkin diagram, being star shaped with three *branches*, construct a sequence of diagrams iteratively by removing the end node from a branch of maximal length at each iteration. In this way we obtain the sequence E_8, E_7, E_6, D_5, D_4, A_3, A_2, A_1, and it is striking to observe that our list $\Delta^{(\ell)}$, obtained by applying the McKay correspondence to distinguished subgroups of the $G^{(\ell)}$, is a subsequence of the one obtained from this by affinisation.

We summarise the main observations of this section as follows.

For $\ell \in \{3,4,5,7\}$ the group $\bar{G}^{(\ell)}$ admits a distinguished isomorphism class of subgroups $\bar{D}^{(\ell)}$. This connects $\bar{G}^{(\ell)}$ to a Dynkin diagram of rank $11-\ell$, for the group $\bar{D}^{(\ell)}$ is the image in $SO_3(\mathbb{R})$ of a finite subgroup $D^{(\ell)}$ in $SU(2)$ which corresponds to the affinisation $\Delta^{(\ell)}$ of a Dynkin diagram of rank $11-\ell$ according to McKay's correspondence. The Dynkin diagrams arising belong to a naturally defined sequence.

3.6 Dynkin Diagrams Part II

Recall from Lemma 3.6 that the cases $\ell \in \{3,4\}$ are distinguished in that for such ℓ the group $\bar{G}^{(\ell)}$ has a unique conjugacy class of subgroups isomorphic to $L_2(p)$ and not acting transitively in the degree $p+1$ permutation representation (cf. §3.3). Since such an $L_2(p)$ subgroup acts non-trivially on $\Omega^{(\ell)}$ it follows from Lemma 3.7 that it must have one fixed point and act transitively on the remaining p points of $\Omega^{(\ell)}$, and thus we have witnesses within $G^{(\ell)}$ to the exceptional degree p permutation representations of $L_2(p)$ in case $\ell \in \{3,4\}$ and $p \in \{11,7\}$. For these two special cases of lambency 3 and 4 we find direct analogues of McKay's monstrous E_8 observation [93] attaching certain conjugacy classes of $G^{(\ell)}$ to the nodes of $\Delta^{(\ell)}$. At lambency 5 and 7 (where p is 5 and 3, respectively) we find similar analogues where the diagram $\Delta^{(\ell)}$ is replaced by one obtained via folding with respect to a diagram automorphism of order 3.

In case $\ell = 3$ let T denote the conjugacy class of elements g of order 2 in $G^{(3)} \simeq 2.M_{12}$ such that g has 4 fixed points in both the signed and unsigned permutation representations of $G^{(3)}$. This is the conjugacy class labelled $2B$ in Table 9 and we have $\chi_g^{(3)} = \bar{\chi}_g^{(3)} = 4$ in the notation of Table 15. Then $T^2 = \{gh \mid g,h \in T\}$ is a union of conjugacy classes of $G^{(3)}$ and in fact there are exactly nine of the twenty-six conjugacy classes of $G^{(3)}$ that appear. In (3.19) we use these

classes (and the notation of Table 9) to label the affine E_8 Dynkin diagram.

$$\begin{array}{ccccccccccccc} & & & & & & 3B & & & & & & \\ & & & & & & | & & & & & & \\ 1A & — & 2B & — & 3A & — & 4C & — & 5A & — & 6C & — & 4B & — & 2C \end{array} \tag{3.19}$$

Observe that the labelling of (3.19) recovers the highest root labelling when the classes are replaced with their orders. In (3.20) we replace the labels of (3.19) with the cycle shapes attached to these classes via the total permutation action (cf. §3.2) of $G^{(3)}$ on the $24 = 2.24/(\ell - 1)$ elements $\{\pm e_i \mid i \in \Omega^{(3)}\}$ appearing in the signed permutation representation of $G^{(3)}$ (cf. §3.4).

$$\begin{array}{ccccccccccccc} & & & & & & 3^8 & & & & & & \\ & & & & & & | & & & & & & \\ 1^{24} & — & 1^8 2^8 & — & 1^6 3^6 & — & 1^4 2^2 4^4 & — & 1^4 5^4 & — & 1^2 2^2 3^2 6^2 & — & 2^4 4^4 & — & 2^{12} \end{array} \tag{3.20}$$

Explicitly, the cycle shape attached to $g \in G^{(3)}$ is the total permutation Frame shape $\tilde{\Pi}_g^{(3)} = \Pi_g^{(3)} \bar{\Pi}_g^{(3)}$ realised as the product (cf. §3.2) of the signed and unsigned permutation Frame shapes attached to $G^{(3)}$ (cf. Table 15).

Remark 3.9. The conjugacy class labelled $2C$ in Table 9 consists of elements of the form gz where g belongs to the class $2B$ (denoted T above) and z is the central involution of $G^{(3)}$, so we obtain exactly the nine conjugacy classes of (3.19) by considering products gh where g and h are involutions in the class $2C$.

Remark 3.10. In [101] an analogue of McKay's monstrous E_8 observation is found for M_{24}. Namely, it is observed that there are exactly nine conjugacy classes of elements of M_{24} having representatives of the form gh where g and h belong to the class labelled 2A in Table 8—this is the class with cycle shape $1^8 2^8$ in the defining degree 24 permutation representation—and the nodes of the affine E_8 Dynkin diagram can be labelled by these nine classes in such a way that their orders recover the highest root labelling. This condition leaves some ambiguity as to the correct placement of the two classes of order 2, and similarly for the pairs of orders 3 and 4, but an analogue of the procedure described in [102] is shown to determine a particular choice that recovers the original correspondence for the monster group via the modular functions of monstrous moonshine. Observe that if we express the $G^{(2)} \simeq M_{24}$ labelling of the affine E_8 diagram given in [101] using cycle shapes to name the conjugacy classes then we recover exactly the labelling (3.20) obtained here from $G^{(3)} \simeq 2.M_{12}$.

In case $\ell = 4$ let T denote the conjugacy class of elements g of order 2 in $G^{(4)}$ such that

g has 4 fixed points in the unsigned permutation representation of $G^{(4)}$ but has 2 fixed points and 2 anti-fixed points in the signed permutation representation. This is the conjugacy class labelled $2C$ in Table 10 and we have $\chi_g^{(4)} = 0$ and $\bar{\chi}_g^{(4)} = 4$ in the notation of Table 16. Then T^2 is the union of eight of the sixteen conjugacy classes of $G^{(4)}$ and we use these classes (and the notation of Table 10) to label the nodes of the affine E_7 Dynkin diagram in (3.21).

$$\begin{array}{ccccccccccccc} & & & & & & 2B & & & & & & \\ & & & & & & | & & & & & & \\ 1A & — & 2C & — & 3A & — & 4C & — & 6A & — & 4A & — & 2A \end{array} \tag{3.21}$$

In (3.22) we replace the labels of (3.21) with the cycle shapes $\bar{\Pi}_g^{(4)}$ (cf. Table 16) attached to these classes via the degree 8 permutation action of $G^{(4)}$ on $\Omega^{(4)}$. The orders of these permutations are the orders of the images of the corresponding elements of $G^{(4)}$ under the map $G^{(4)} \to \bar{G}^{(4)}$ so the labelling (3.22) demonstrates that we recover the highest root labelling of the affine E_7 Dynkin diagram when we replace the conjugacy classes of (3.21) with the orders of their images in $\bar{G}^{(4)}$.

$$\begin{array}{ccccccccccccc} & & & & & & 2^4 & & & & & & \\ & & & & & & | & & & & & & \\ 1^8 & — & 1^4 2^2 & — & 1^2 3^2 & — & 1^2 2^1 4^1 & — & 1^2 3^2 & — & 2^4 & — & 1^8 \end{array} \tag{3.22}$$

In (3.23) we replace the labels of (3.21) with the cycle shapes of degree 24 given by the products $\Pi_g^{(4)}\bar{\Pi}_g^{(4)}\bar{\Pi}_g^{(4)}$ (cf. Table 16) for g an element of $G^{(4)}$ arising in (3.21). Observe that all the cycle shapes in (3.23) are *balanced*, meaning that they are invariant (and well-defined) under the operation $\prod_{k\geq 1} k^{m(k)} \mapsto \prod_{k\geq 1}(N/k)^{m(k)}$ for some integer $N > 1$, and constitute a subset of the cycle shapes attached to $G^{(3)}$ in (3.20). Observe also that the order two symmetry of the affine E_7 diagram that identifies the two long branches is realised by squaring: the cycle shapes obtained by squaring permutations represented by the cycle shapes on the right-hand branch of (3.23) are just those that appear on the left-hand branch.

$$\begin{array}{ccccccccccccc} & & & & & & 2^{12} & & & & & & \\ & & & & & & | & & & & & & \\ 1^{24} & — & 1^8 2^8 & — & 1^6 3^6 & — & 1^4 2^2 4^4 & — & 1^2 2^2 3^2 6^2 & — & 2^4 4^4 & — & 1^8 2^8 \end{array} \tag{3.23}$$

Remark 3.11. It is interesting to note that while the conjugacy class $2C$ is stable under multiplication by the central involution there are just eight conjugacy classes of $G^{(4)}$ that are contained in T^2 when T is the class labelled $4A$ in Table 10. In fact the eight conjugacy classes appearing

are $8A$ together with all those of (3.21) except for $4C$. Thus we obtain analogues of the labelings (3.21), (3.22) and (3.23) where $4C$, $1^2 2^1 4^1$ and $1^4 2^2 4^4$, respectively, are replaced by $8A$, 4^2 and $4^2 8^2$, respectively. Under this correspondence the highest root labelling is again obtained by taking orders in $\bar{G}^{(4)}$, but while the eta product attached to $4^2 8^2$ is multiplicative it only has weight 2 (i.e. less than 4) and so does not appear in [101].

Remark 3.12. The correspondence of [101] may be viewed as attaching the nine multiplicative eta products of weight at least 4 to the nodes of the affine E_8 Dynkin diagram when we regard a cycle shape $\prod_{k\geq 1} k^{m(k)}$ as a shorthand for the *eta product* function $\prod_{k\geq 1} \eta(k\tau)^{m(k)}$ (cf. §A.1). Observe that the cycle shapes appearing in (3.23) are just those whose corresponding eta products are multiplicative, have weight at least 4, and have level dividing 24. (The eta products defined by $1^4 5^4$ and 3^8 have level 5 and 9, respectively.)

For $\ell = 5$ the Dynkin diagram $\Delta^{(5)} = \hat{E}_6$ admits a Sym_3 group of automorphisms. Folding by either of the three-fold symmetries we obtain the affine Dynkin diagram of type G_2. Let T be either of the two conjugacy class of elements g of order 4 in $G^{(5)}$ such that g^2 is central. Then we have $\chi_g^{(5)} = \bar{\chi}_g^{(5)} = 0$ in the notation of Table 17 and T is either the class labelled $4A$ in Table 11 or the class labelled $4B$, and there are exactly three of the fourteen conjugacy classes of $G^{(5)}$ that are subsets of T^2. In (3.24) we use these classes (and the notation of Table 11) to label the affine G_2 Dynkin diagram.

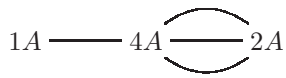
$$2A \text{ ——— } 2C \text{ ——— } 6A \tag{3.24}$$

In (3.25) we replace the labels of (3.24) with the cycle shapes $\bar{\Pi}_g^{(5)}$ (cf. Table 17) attached to these classes via the degree 6 permutation action of $G^{(5)}$ on $\Omega^{(5)}$. The orders of these permutations are the orders of the images of the corresponding elements of $G^{(5)}$ under the map $G^{(5)} \to \bar{G}^{(5)}$ so the labelling (3.25) demonstrates that we recover the highest root labelling of the affine G_2 Dynkin diagram—which is the labelling induced form the highest root labelling of $\hat{E}_6$—when we replace the conjugacy classes of (3.24) with the orders of their images in $\bar{G}^{(5)}$.

$$1^6 \text{ ——— } 1^2 2^2 \text{ ——— } 3^2 \tag{3.25}$$

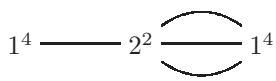
For $\ell = 7$ the Dynkin diagram $\Delta^{(7)} = \hat{D}_4$ admits a Sym_4 group of automorphisms. Folding by any three-fold symmetry we again obtain the affine Dynkin diagram of type G_2. Let T be the unique conjugacy class of elements of order 4 in $G^{(7)}$. Then we have $\chi_g^{(7)} = \bar{\chi}_g^{(7)} = 0$ in the notation of Table 18 and T is the class labelled $4A$ in Table 12, and there are exactly three of

the seven conjugacy classes of $G^{(7)}$ that have a representative of the form gh for some $g, h \in T$ (and T^2 is the union of these three conjugacy classes). In (3.26) we use these classes (and the notation of Table 12) to label the affine G_2 Dynkin diagram.



(3.26)

In (3.27) we replace the labels of (3.26) with the cycle shapes $\bar{\Pi}_g^{(7)}$ (cf. Table 18) attached to these classes via the degree 4 permutation action of $G^{(7)}$ on $\Omega^{(7)}$. The orders of these permutations are the orders of the images of the corresponding elements of $G^{(7)}$ under the map $G^{(7)} \to \bar{G}^{(7)}$ so the labelling (3.27) demonstrates that we recover the labelling of the affine G_2 Dynkin diagram that is induced by the highest root labelling of $\hat{D}_4$ when we replace the conjugacy classes of (3.26) with the orders of their images in $\bar{G}^{(7)}$.



(3.27)

We summarise the results of this section with the following statement.

For $\ell \in \{3, 4, 5, 7\}$ there is an analogue of McKay's monstrous E_8 observation that relates $G^{(\ell)}$ to (a folding of) the affinisation of a Dynkin diagram of rank $11 - \ell$, and the diagrams arising are precisely those that appear in §3.5.

We conclude by mentioning that McKay's monstrous observation has partially been explained, using vertex operator algebra theory, by the work of Sakuma [103] and Lam–Yamada–Yamauchi [104, 105]. Important related work appears in [106, 107].

4 McKay–Thompson Series

In §2.5 we made the observation that the first few positive degree Fourier coefficients of the vector-valued mock modular forms $H^{(\ell)} = \left(H_r^{(\ell)}\right)$ coincide (up to a factor of 2) with dimensions of irreducible representations of the group $G^{(\ell)}$ described in the previous section. This observation suggests the possibility that

$$H_r^{(\ell)}(\tau) = \sum_{k\in\mathbb{Z}} c_r^{(\ell)}(k - r^2/4\ell) q^{k-r^2/4\ell} = -2\delta_{r,1} q^{-1/4\ell} + \sum_{\substack{k\in\mathbb{Z} \\ r^2-4k\ell<0}} \dim K_{r,k-r^2/4\ell}^{(\ell)} q^{k-r^2/4\ell} \quad (4.1)$$

for some $\mathbb{Z} \times \mathbb{Q}$-graded infinite-dimensional $G^{(\ell)}$-module $K^{(\ell)} = \bigoplus_{r,d} K_{r,d}^{(\ell)}$. To further test this possibility we would like to see if there are similar vector-valued mock modular forms

$H_g^{(\ell)} = \left(H_{g,r}^{(\ell)}\right)$ whose positive degree Fourier coefficients recover characters of representations of $G^{(\ell)}$. In other words, for an element g of the group $G^{(\ell)}$ we would like to see if we can find a mock modular form $H_g^{(\ell)}$ compatible with the hypothesis that

$$H_{g,r}^{(\ell)}(\tau) = \sum_k c_{g,r}^{(\ell)}(k - r^2/4\ell) q^{k-r^2/4\ell} = -2\delta_{r,1} q^{-1/4\ell} + \sum_{\substack{k\in\mathbb{Z} \\ r^2-4k\ell<0}} \mathrm{tr}_{K_{r,k-r^2/4\ell}^{(\ell)}}(g) q^{k-r^2/4\ell} \quad (4.2)$$

for a hypothetical bi-graded $G^{(\ell)}$-module $K^{(\ell)}$. We refer to such a generating function as a *McKay–Thompson series*. Notice that we recover the generating functions of the $\dim K_{r,d}^{(\ell)}$ when g is the identity element. Moreover, the McKay–Thompson series attached to $g \in G^{(\ell)}$ is invariant under conjugation, since the trace is such, so $H_g^{(\ell)}$ depends only on the conjugacy class $[g]$ of g.

Having such McKay–Thompson series for each conjugacy class of $G^{(\ell)}$ not only provides strong evidence for the existence of the $G^{(\ell)}$-module $K^{(\ell)}$ but in fact it uniquely specifies it up to $G^{(\ell)}$-module isomorphism. This is because, since there are as many irreducible representations as conjugacy classes of a finite group, given the characters $\mathrm{tr}_{K_{r,d}^{(\ell)}}(g)$ for all $[g] \subset G^{(\ell)}$ we can simply invert the character table to have a unique decomposition of the conjectural module $K_{r,d}^{(\ell)}$ into irreducible representations. What is not clear, a priori, is that we will end up with a decomposition into non-negative integer multiplies of irreducible representations of $G^{(\ell)}$, but remarkably it appears that this property holds for all $\ell \in \Lambda$ and all bi-degrees (r, d). For evidence in support of this see the explicit decompositions for small degrees tabulated in §D.

In this section we construct a set of vector-valued mock modular forms $H_g^{(\ell)} = \left(H_{g,r}^{(\ell)}\right)$ for each lambency ℓ and we formulate a precise conjecture implying (4.2) in §5.1.

4.1 Forms of Higher Level

In §2 we have discussed the relation between certain vector-valued mock modular forms, meromorphic Jacobi forms of weight 1 and Jacobi forms of weight 0 under the group $SL_2(\mathbb{Z})$. In order to investigate the McKay–Thompson series of the groups $G^{(\ell)}$ we need to generalise the discussion in §2.5 to modular forms of higher level and consider forms transforming under $\Gamma_0(N)$ (cf. (A.2)) with $N > 1$.

As in §2 we would like to consider $(\ell - 1)$-vector-valued mock modular forms $\left(H_{g,r}^{(\ell)}\right)$ for a group $\Gamma_0(N_g) < SL_2(\mathbb{Z})$ with shadow given by the unary theta series $S^{(\ell)}$ (cf. (2.26)). The levels N_g will be specified explicitly for all g in §4.8. A first difference between the case of $N_g = 1$ and $N_g > 1$ is the following. Since the components of $S^{(\ell)} = \left(S_r^{(\ell)}\right)$ have more than one orbit under $\Gamma_0(N_g)$ when N_g is even it is natural to consider mock modular forms with shadows given by

$\big(\chi^{(\ell)}_{g,r}S^{(\ell)}_r\big)$ where the $\chi^{(\ell)}_{g,r}$ are not necessarily equal for different values of r. It will develop that in the cases of interest to us $\chi^{(\ell)}_{g,r}$ depends only on r modulo 2. In fact, we will find that

$$\chi^{(\ell)}_{g,r} = \begin{cases} \chi^{(\ell)}_g, & \text{for } r \equiv 0 \pmod 2; \\ \bar{\chi}^{(\ell)}_g, & \text{for } r \equiv 1 \pmod 2; \end{cases} \tag{4.3}$$

where $\chi^{(\ell)}_g$ and $\bar{\chi}^{(\ell)}_g$ are as defined in §3.4 and as tabulated in §B.2.

Group theoretically the multiplicities $\bar{\chi}^{(\ell)}_g$ and $\chi^{(\ell)}_g$ appearing in the shadow of $H^{(\ell)}_g$ are determined by the number of fixed points and anti-fixed points in the signed permutation representation of the group $G^{(\ell)}$, as explained in §3.2. For instance, we have $\bar{\chi}^{(\ell)}_g = \chi^{(\ell)}_g = \chi^{(\ell)}$ for the identity element $g = e$. From this interpretation we can deduce that the vanishing of $\bar{\chi}^{(\ell)}_g$ implies the vanishing of $\chi^{(\ell)}_g$, while it is possible to have $\chi^{(\ell)}_g = 0$ and $\bar{\chi}^{(\ell)}_g \neq 0$. It will also turn out that

$$\bar{\chi}^{(\ell)}_g = \chi^{(\ell)}_g \quad \text{unless} \quad 2|N_g. \tag{4.4}$$

For later use we define the combinations

$$\chi^{(\ell)}_{g,+} = \tfrac{1}{2}\big(\bar{\chi}^{(\ell)}_g + \chi^{(\ell)}_g\big), \quad \chi^{(\ell)}_{g,-} = \tfrac{1}{2}\big(\bar{\chi}^{(\ell)}_g - \chi^{(\ell)}_g\big), \tag{4.5}$$

which enumerate the number of fixed and anti-fixed points, respecitlvely, in the signed permutation representation of $G^{(\ell)}$ (cf. (3.3-3.4)). Of course in the cases where $\bar{\chi}^{(\ell)}_g = \chi^{(\ell)}_g = 0$ the function $H^{(\ell)}_g = \big(H^{(\ell)}_{g,r}\big)$ has vanishing shadow and is a vector-valued modular form in the usual sense.

Interestingly, just as in the $SL_2(\mathbb{Z})$ case that was considered in §2.3, the higher level vector-valued mock modular forms with shadows as described above are again closely related to the finite parts of meromorphic Jacobi forms of weight 1. More explicitly, from the transformation (2.29) and the simple fact $\theta^{(\ell)}_r(\tau, z + 1/2) = (-1)^r\theta^{(\ell)}_r(\tau, z)$ it is not difficult to check that the function

$$\psi^{(\ell)}_g(\tau, z) = \chi^{(\ell)}_{g,+}\mu^{(\ell)}_0(\tau, z) - \chi^{(\ell)}_{g,-}\mu^{(\ell)}_0(\tau, z + 1/2) + \sum_{r=1}^{\ell-1} H^{(\ell)}_{g,r}(\tau)\hat{\theta}^{(\ell)}_r(\tau, z) \tag{4.6}$$

transforms as a Jacobi form of weight 1 and index ℓ under the congruence subgroup $\Gamma_0(N_g)$ when $H^{(\ell)}_g$ is a mock modular form of weight 1/2 for $\Gamma_0(N_g)$ with shadow $S^{(\ell)}_g = \big(\chi^{(\ell)}_{g,r}S^{(\ell)}_r\big)$. Observe that, for those g with both $\chi^{(\ell)}_{g,+}$ and $\chi^{(\ell)}_{g,-}$ being non-zero, the weak Jacobi form $\psi^{(\ell)}_g(\tau, z)$ has

a pole not only at $z=0$ but also at $z=1/2$. From

$$\mu_0^{(\ell)}(\tau,z)=\mathrm{Av}^{(\ell)}\left[\frac{y+1}{y-1}\right],\quad -\mu_0^{(\ell)}(\tau,z+1/2)=\mathrm{Av}^{(\ell)}\left[\frac{1-y}{1+y}\right],\tag{4.7}$$

we see that the last two terms of the decomposition given in (4.6) have the interpretation as the polar parts at the poles $z=0$ and $z=1/2$, respectively. In other words, we again have a decomposition $\psi_g^{(\ell)}=\psi_g^{(\ell),P}+\psi_g^{(\ell),F}$ into polar and finite parts given by

$$\begin{aligned}\psi_g^{(\ell),P}(\tau,z)&=\chi_{g,+}^{(\ell)}\mu_0^{(\ell)}(\tau,z)-\chi_{g,-}^{(\ell)}\mu_0^{(\ell)}(\tau,z+1/2),\\ \psi_g^{(\ell),F}(\tau,z)&=\sum_{r=1}^{\ell-1}H_{g,r}^{(\ell)}(\tau)\hat{\theta}_r^{(\ell)}(\tau,z),\end{aligned}\tag{4.8}$$

and the components $H_{g,r}^{(\ell)}$ of the mock modular form $H_g^{(\ell)}$ may again be interpreted as the theta-coefficients of a meromorphic Jacobi form; namely, $\psi_g^{(\ell)}(\tau,z)$.

Moreover, analogous to the $SL_2(\mathbb{Z})$ case, the mock modular form $H_g^{(\ell)}$ also enjoys a close relationship with a weight 0 index $\ell-1$ weak Jacobi form $Z_g^{(\ell)}$ which admits a decomposition into characters of the $N=4$ superconformal algebra at level $\ell-1$. To see this, observe that

$$\psi_g^{(\ell)}(\tau,z)=\frac{\chi_{g,+}^{(\ell)}}{\bar{\chi}_g^{(\ell)}}\,\Psi_{1,1}(\tau,z)\,Z_g^{(\ell)}(\tau,z)-\frac{\chi_{g,-}^{(\ell)}}{\bar{\chi}_g^{(\ell)}}\,\Psi_{1,1}(\tau,z+1/2)\,Z_g^{(\ell)}(\tau,z+1/2)\tag{4.9}$$

when $\bar{\chi}_g^{(\ell)}\neq 0$ and

$$\psi_g^{(\ell)}(\tau,z)=\Psi_{1,1}(\tau,z)\,Z_g^{(\ell)}(\tau,z)\tag{4.10}$$

when $\bar{\chi}_g^{(\ell)}=0$, where $\Psi_{1,1}(\tau,z)$ is as in (2.20) and $Z_g^{(\ell)}$ is the weak Jacobi form of weight 0 and index $\ell-1$ given by

$$Z_g^{(\ell)}(\tau,z)=\frac{1}{\Psi_{1,1}(\tau,z)}\left(\bar{\chi}_g^{(\ell)}\mu_0^{(\ell)}(\tau,z)+\sum_{0<r<\ell}\left(1+\frac{\bar{\chi}_g^{(\ell)}-\chi_{g,r}^{(\ell)}}{\chi_g^{(\ell)}}\right)H_{g,r}^{(\ell)}(\tau)\hat{\theta}_r^{(\ell)}(\tau,z)\right)\tag{4.11}$$

for $\chi_g^{(\ell)}\neq 0$ and

$$Z_g^{(\ell)}(\tau,z)=\frac{1}{\Psi_{1,1}(\tau,z)}\left(\bar{\chi}_g^{(\ell)}\mu_0^{(\ell)}(\tau,z)+\sum_{0<r<\ell}H_{g,r}^{(\ell)}(\tau)\hat{\theta}_r^{(\ell)}(\tau,z)\right)\tag{4.12}$$

otherwise.

Anticipating their relation to the conjugacy classes $[g]$ of $G^{(\ell)}$, another comment on the weight 0 forms $Z_g^{(\ell)}$ is in order here. As discussed in §3.2, if $\ell\in\Lambda$ and $\ell>2$ then $G^{(\ell)}$ has a

unique central element z of order 2 and for any $g \in G^{(\ell)}$ the conjugacy casses $[g]$ and $[zg]$ are said to be paired, and a class $[g]$ is said to be self-paired if $[g] = [zg]$. We have

$$\bar{\chi}_g^{(\ell)} = \bar{\chi}_{zg}^{(\ell)}, \quad \chi_g^{(\ell)} = -\chi_{zg}^{(\ell)}, \tag{4.13}$$

and as a consequence $Z_g^{(\ell)}(\tau, z) = Z_{zg}^{(\ell)}(\tau, z)$. Therefore, while the vector-valued mock modular forms $H_g^{(\ell)}$ and the weight 1 meromorphic Jacobi forms $\psi_g^{(\ell)}(\tau, z)$ are generally distinct for different conjugacy classes $[g]$ of the group $G^{(\ell)}$, the weight 0 index $\ell - 1$ weak Jacobi forms $Z_g^{(\ell)}$ cannot distinguish between two classes that are paired in the above sense and are more naturally associated to the group $\bar{G}^{(\ell)}$ which is the quotient of $G^{(\ell)}$ by the subgroup $\langle z \rangle$.

Equipped with the $H^{(\ell)}$ defined in terms of decompositions of Jacobi forms as discussed in §2.5 it will develop that a convenient way to specify the vector-valued mock modular forms $H_g^{(\ell)} = \left(H_{g,r}^{(\ell)}\right)$ corresponding to non-identity conjugacy classes of $G^{(\ell)}$ will be to specify certain sets of weight 2 modular forms. To see how a weight 2 modular form is naturally associated with a vector-valued modular form with the properties described above, observe that we can eliminate the presence of the polar part in the weight 1 Jacobi form $\psi_g^{(\ell)}(\tau, z)$ (cf. (4.6)) by taking a linear combination with $\psi^{(\ell)}(\tau, z)$. To be more precise, note that

$$\hat{\psi}_g^{(\ell)}(\tau, z) = \sum_{r=1}^{\ell-1} \hat{H}_{g,r}^{(\ell)}(\tau)\hat{\theta}_r^{(\ell)}(\tau, z) = \psi_g^{(\ell)}(\tau, z) - \frac{\chi_{g,+}^{(\ell)}}{\chi^{(\ell)}}\psi^{(\ell)}(\tau, z) + \frac{\chi_{g,-}^{(\ell)}}{\chi^{(\ell)}}\psi^{(\ell)}(\tau, z + 1/2) \tag{4.14}$$

is a weight 1 index ℓ Jacobi form for $\Gamma_0(N_g)$ with no poles in z, and the functions

$$\hat{H}_{g,r}^{(\ell)}(\tau) = H_{g,r}^{(\ell)}(\tau) - \frac{\chi_{g,r}^{(\ell)}}{\chi^{(\ell)}} H_r^{(\ell)}(\tau) \tag{4.15}$$

are the components of a vector-valued modular form $\hat{H}_g^{(\ell)}$ for $\Gamma_0(N_g)$ in the usual sense (i.e. a mock modular form with vanishing shadow). From this we readily conclude that the function

$$F_g^{(\ell)}(\tau) = \sum_{r=1}^{\ell-1} \hat{H}_{g,r}^{(\ell)}(\tau) S_r^{(\ell)}(\tau) = -\frac{1}{4\pi i}\frac{\partial}{\partial z}\hat{\psi}_g^{(\ell)}(\tau, z)\big|_{z=0} \tag{4.16}$$

is a weight 2 modular form for the group $\Gamma_0(N_g)$.

For general values of ℓ, specifying the weight 2 form $F_g^{(\ell)}(\tau)$ is not sufficient to specify the whole vector-valued modular form $\hat{H}_g^{(\ell)}$ since we have evidently collapsed information in taking the particular combination (4.16). However, for $\ell \in \{2, 3\}$ we are in the privileged situation that specifying the weight 2 form $F_g^{(\ell)}(\tau)$ completely specifies all the components of $H_g^{(\ell)}$, as will be

explained in more detail in the following section. For $\ell > 3$ we need more weight 2 forms, and we will also consider

$$F_g^{(\ell),2}(\tau) = \sum_{r=1}^{\ell-1} (-1)^{r+1} \hat{H}_{g,r}^{(\ell)}(\tau) S_{\ell-r}^{(\ell)}(\tau). \tag{4.17}$$

In the next section we give our concrete proposals for the McKay–Thompson series $H_g^{(\ell)}$ for all but a few of the conjugacy classes $[g]$ arising. We give closed expressions for all the $H_g^{(\ell)}$ in case $\ell \in \{2,3,4,5\}$ and we partially determine the $H_g^{(\ell)}$ for $\ell = \{7,13\}$. Although we do not offer analytic expressions for all the $H_g^{(\ell)}$ with $\ell = 7$ or $\ell = 13$ we have predictions for the low degree terms in the Fourier developments of all the McKay–Thompson series at all lambencies $\ell \in \Lambda$ and these are detailed in the tables of §C.

4.2 Lambency Two

$[g]$	N_g	$F_g^{(2)}(\tau)$
$1A$	1	0
$2A$	2	$-16\Lambda_2$
$2B$	4	$24\Lambda_2 - 8\Lambda_4 = -2\eta(\tau)^8/\eta(2\tau)^4$
$3A$	3	$-6\Lambda_3$
$3B$	9	$-2\eta(\tau)^6/\eta(3\tau)^2$
$4A$	8	$-4\Lambda_2 + 6\Lambda_4 - 2\Lambda_8 = -2\eta(2\tau)^8/\eta(4\tau)^4$
$4B$	4	$4(\Lambda_2 - \Lambda_4)$
$4C$	16	$-2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
$5A$	5	$-2\Lambda_5$
$6A$	6	$2(\Lambda_2 + \Lambda_3 - \Lambda_6)$
$6B$	36	$-2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2/\eta(6\tau)^2$
$7AB$	7	$-\Lambda_7$
$8A$	8	$\Lambda_4 - \Lambda_8$
$10A$	20	$-2\eta(\tau)^3\eta(2\tau)\eta(5\tau)/\eta(10\tau)$
$11A$	11	$\frac{2}{5}(-\Lambda_{11} + 11f_{11})$
$12A$	24	$-2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(3\tau)\eta(12\tau)^2$
$12B$	144	$-2\eta(\tau)^4\eta(4\tau)\eta(6\tau)/\eta(2\tau)\eta(12\tau)$
$14AB$	14	$\frac{1}{3}(\Lambda_2 + \Lambda_7 - \Lambda_{14} + 14f_{14})$
$15AB$	15	$\frac{1}{4}(\Lambda_3 + \Lambda_5 - \Lambda_{15} + 15f_{15})$
$21AB$	63	$\frac{1}{3}(-7\eta(\tau)^3\eta(7\tau)^3/\eta(3\tau)\eta(21\tau) + \eta(\tau)^6/\eta(3\tau)^2)$
$23AB$	23	$\frac{1}{11}(-\Lambda_{23} + 23f_{23,a} + 69f_{23,b})$

Table 3: The list of weight 2 modular forms $F_g^{(2)}(\tau)$ for $\Gamma_0(N_g)$.

When $\ell = 2$ the vector-valued mock modular form $H_g^{(2)}$ has only one component $H_{g,1}^{(2)}$ and our

conjecture relating the $H_g^{(2)}$ and the group $G^{(2)}$ is nothing but the conjecture relating (scalar-valued) mock modular forms and the largest Mathieu group M_{24} that has been investigated recently [15, 19, 24, 20, 21, 22, 108, 109]. See also [41, 110] for reviews and [111, 112] for related discussions. Explicit expressions for the McKay–Thompson series arising from the M_{24}-module that is conjectured to underlie this connection have been proposed in [19, 20, 21, 22]. As mentioned before, one convenient way to express them is via a set of weight 2 modular forms $F_g^{(2)}(\tau)$. In this case (4.15-4.16) simply reduces to

$$H_{g,1}^{(2)}(\tau) = \frac{\chi_g^{(2)}}{24} H_1^{(2)}(\tau) + \frac{F_g^{(2)}(\tau)}{\eta(\tau)^3} \,, \tag{4.18}$$

where we have used $\chi^{(2)} = 24$ and $S_1^{(2)}(\tau) = \eta(\tau)^3$. For later use and for the sake of completeness we collect the explicit expressions for $F_g^{(2)}(\tau)$ for all conjugacy classes $[g] \subset M_{24}$ in Table 3. They are given in terms of eta quotients and standard generators of the weight 2 modular forms of level N. Among the latter are those denoted here by $\Lambda_N(\tau)$ and $f_N(\tau)$ and given explicitly in Appendix A.

4.3 Lambency Three

We would like to specify the two components $H_{g,1}^{(3)}(\tau)$ and $H_{g,2}^{(3)}(\tau)$ of the vector-valued mock modular form $\big(H_{g,r}^{(3)}(\tau)\big)$ which we propose to be the McKay–Thompson series arising from the $G^{(3)}$-module $K^{(3)}$. As mentioned earlier, to specify $H_{g,1}^{(3)}(\tau)$ and $H_{g,2}^{(3)}(\tau)$ it is sufficient to specify the weight 2 forms defined in (4.16). To see this, recall that to any given conjugacy class $[g]$ we can associate a conjugacy class $[zg]$ such that (4.13) holds. From this we obtain

$$\begin{aligned} H_{g,1}^{(3)}(\tau) &= \frac{\bar{\chi}_g^{(3)}}{\chi^{(3)}} H_1^{(3)}(\tau) + \frac{1}{2}\frac{\eta(4\tau)^2}{\eta(2\tau)^5}\big(F_g^{(3)}(\tau) + F_{zg}^{(3)}(\tau)\big), \\ H_{g,2}^{(3)}(\tau) &= \frac{\chi_g^{(3)}}{\chi^{(3)}} H_2^{(3)}(\tau) + \frac{1}{4}\frac{\eta(2\tau)}{\eta(\tau)^2\eta(4\tau)^2}\big(F_g^{(3)}(\tau) - F_{zg}^{(3)}(\tau)\big), \end{aligned} \tag{4.19}$$

where we have used the eta quotient expressions

$$S_1^{(3)}(\tau) = \frac{\eta(2\tau)^5}{\eta(4\tau)^2}, \quad S_2^{(3)}(\tau) = 2\frac{\eta(\tau)^2\eta(4\tau)^2}{\eta(2\tau)}, \tag{4.20}$$

for the components of the unary theta series at $\ell = 3$.

The explicit expressions for $F_g^{(3)}(\tau)$ are listed in Table 4.

We also note that for the classes $3B$ and $6B$ we also have the following alternative expressions

$[g]$	N_g	$F_g^{(3)}(\tau)$
1A	1	0
2A	4	0
4A	16	$-2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
2B	2	$-16\Lambda_2$
2C	4	$16(\Lambda_2-\Lambda_4/3)$
3A	3	$-6\Lambda_3$
6A	12	$-9\Lambda_2-2\Lambda_3+3\Lambda_4+3\Lambda_6-\Lambda_{12}$
3B	9	$8\Lambda_3-2\Lambda_9+2\,\eta^6(\tau)/\eta^2(3\tau)$
6B	36	$-2\eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)$
4B	8	$-2\eta(2\tau)^8/\eta(4\tau)^4$
4C	4	$-8\Lambda_4/3$
5A	5	$-2\Lambda_5$
10A	20	$\sum_{d\mid 20} c_{10A}(d)\Lambda_d+\frac{20}{3}f_{20}$
12A	144	$-2\eta(\tau)\eta(2\tau)^5\eta(3\tau)/\eta(4\tau)^2\eta(6\tau)$
6C	6	$2(\Lambda_2+\Lambda_3-\Lambda_6)$
6D	12	$-5\Lambda_2-2\Lambda_3+\frac{5}{3}\Lambda_4+3\Lambda_6-\Lambda_{12}$
8AB	32	$-2\eta(2\tau)^4\eta(4\tau)^2/\eta(8\tau)^2$
8CD	8	$-2\Lambda_2+\frac{5}{3}\Lambda_4-\Lambda_8$
20AB	80	$-2\eta(2\tau)^7\eta(5\tau)/\eta(\tau)\eta(4\tau)^2\eta(10\tau)$
11AB	11	$-\frac{2}{5}\Lambda_{11}-\frac{33}{5}f_{11}$
22AB	44	$\sum_{d\mid 44} c_{22AB}(d)\Lambda_d(\tau)-\frac{11}{5}\sum_{d\mid 4} c'_{22AB}(d)f_{11}(d\tau)+\frac{22}{3}f_{44}(\tau)$

$c_{10A}(d)=-5,\frac{5}{3},-\frac{2}{3},1,-\frac{1}{3}$ for $d=2,4,5,10,20$
$c_{22AB}(d)=-\frac{11}{5},\frac{11}{15},-\frac{2}{15},\frac{1}{5},-\frac{1}{15}$ for $d=2,4,11,22,44$
$c'_{22AB}(d)=1,4,8$ for $d=1,2,4$

Table 4: The list of weight 2 modular forms $F_g^{(3)}(\tau)$ for $\Gamma_0(N_g)$.

for the McKay–Thompson series in terms of eta quotients:

$$H^{(3)}_{3B,1}(\tau) = H^{(3)}_{6B,1}(\tau) = -2\frac{\eta(\tau)\eta(6\tau)^5}{\eta(3\tau)^3\eta(12\tau)^2}, \quad H^{(3)}_{3B,2}(\tau) = -H^{(3)}_{6B,2}(\tau) = -4\frac{\eta(\tau)\eta(12\tau)^2}{\eta(3\tau)\eta(6\tau)}. \tag{4.21}$$

Coincidences with Ramanujan's mock theta functions will be discussed (4.7).

4.4 Lambency Four

The McKay–Thompson series for lambency 4 display an interesting relation with those for lambency 2. To see this, notice the following relation among the theta functions

$$S^{(4)}_1(2\tau) - S^{(4)}_3(2\tau) = S^{(2)}_1(\tau). \tag{4.22}$$

Given this relation it is natural to consider the function

$$H^{(4)}_{g,*}(\tau) := H^{(4)}_{g,1}(\tau) - H^{(4)}_{g,3}(\tau) \tag{4.23}$$

for each conjugacy classe $[g]$ of $G^{(4)}$. Note that $H^{(4)}_{g,1}(\tau)$ and $H^{(4)}_{g,3}(\tau)$ can be reconstructed from $H^{(4)}_{g,*}(\tau)$ since they are q-series of the form $q^{-1/16}$ times a series of even or odd powers of $q^{1/2}$, respectively. Explicitly, we have

$$\begin{aligned} H^{(4)}_{g,1}(\tau) &= \tfrac{1}{2}\big(H^{(4)}_{g,*}(\tau) + e(\tfrac{1}{16})H^{(4)}_{g,*}(\tau+1)\big), \\ H^{(4)}_{g,3}(\tau) &= \tfrac{1}{2}\big(-H^{(4)}_{g,*}(\tau) + e(\tfrac{1}{16})H^{(4)}_{g,*}(\tau+1)\big). \end{aligned} \tag{4.24}$$

In order to obtain an expression for $H^{(4)}_{g,*}(\tau)$ we rely on the following two observations. First, recall in §3.4 we have observed a relation between the Frame shapes of $g \in G^{(4)}$ and $g' \in G^{(2)}$. It turns out that for a pair of group elements $g \in G^{(4)}$ and $g' \in G^{(2)}$ related in the way described in Proposition 3.4 their McKay–Thompson series $H^{(4)}_g(2\tau)$ and $H^{(2)}_{g'}(\tau)$ are also related in a

simple way. As examples of this we have

$$\begin{aligned}
H^{(4)}_{1A,*}(\tau) &= H^{(4)}_{2A,*}(\tau) = H^{(2)}_{2A}(\tau/2)\\
H^{(4)}_{2B,*}(\tau) &= H^{(2)}_{4A}(\tau/2)\\
H^{(4)}_{2C,*}(\tau) &= H^{(2)}_{4B}(\tau/2)\\
H^{(4)}_{3A,*}(\tau) &= H^{(4)}_{6A,*}(\tau) = H^{(2)}_{6A}(\tau/2)\\
H^{(4)}_{4C,*}(\tau) &= H^{(2)}_{8A}(\tau/2)\\
H^{(4)}_{6BC,*}(\tau) &= H^{(2)}_{12A}(\tau/2)\\
H^{(4)}_{7AB,*}(\tau) &= H^{(4)}_{14AB,*}(\tau) = H^{(2)}_{14AB}(\tau/2).
\end{aligned} \tag{4.25}$$

This leaves us just the classes $4A$, $4B$, and $8A$ that are not related to any element of $G^{(2)} \simeq M_{24}$ in the way described in Proposition 3.4. All of these classes have $\chi^{(4)}_g = \bar{\chi}^{(4)}_g = 0$. A second observation is that all the $H^{(3)}_{g,r}(\tau)$ for those classes $[g] \subset G^{(3)}$ with $\chi^{(3)}_g = \bar{\chi}^{(3)}_g = 0$ have an expression in terms of eta quotients according to Table 4, and so do the $H^{(2)}_g(\tau)$ for those classes $[g] \subset G^{(2)}$ with $\chi^{(2)}_g = 0$. This is consistent with our expectation that the shadow of the vector-valued mock modular form $\big(H^{(\ell)}_{g,r}(\tau)\big)$ is given by $\big(\chi^{(\ell)}_{g,r} S^{(\ell)}_r(\tau)\big)$, and hence $\big(H^{(\ell)}_{g,r}(\tau)\big)$ is nothing but a vector-valued modular form in the usual sense when $\chi^{(\ell)}_g = \bar{\chi}^{(\ell)}_g = 0$. Given this it is natural to ask whether we can find similar eta quotient expressions for $H^{(4)}_{g,*}(\tau)$ with $\chi^{(\ell)}_g = \bar{\chi}^{(\ell)}_g = 0$ when $\ell > 3$. We find

$$\begin{aligned}
H^{(4)}_{4A,*}(\tau) &= -2\frac{\eta(\tau/2)\eta(\tau)^2}{\eta(2\tau)^2},\\
H^{(4)}_{4B,*}(\tau) &= -2\frac{\eta(\tau/2)\eta(\tau)^4}{\eta(\tau)^2\eta(4\tau)^2},\\
H^{(4)}_{8A,*}(\tau) &= -2\frac{\eta(\tau)^3}{\eta(\tau/2)\eta(4\tau)}.
\end{aligned} \tag{4.26}$$

Together with (4.24) and Table 3, the above equations completely specify $H^{(4)}_{g,1}(\tau)$ and $H^{(4)}_{g,3}(\tau)$ for all conjugacy classes $[g]$ of $G^{(4)}$. We are left to determine the second components $H^{(4)}_{g,2}(\tau)$. To start with, we have $H^{(4)}_2(\tau) = H^{(4)}_{1A,2}(\tau) = -H^{(4)}_{2A,2}(\tau)$ given in terms of the decomposition of the weight 0 Jacobi form as explained in §2.5. To determine the rest, notice that (4.13) and (4.16) implies that

$$S^{(4)}_2(\tau)\left(H^{(4)}_{g,2}(\tau) - \frac{\chi^{(4)}_{g,2}}{\chi^{(4)}} H^{(4)}_2(\tau)\right) \tag{4.27}$$

should be a weight 2 modular form. Employing this consideration we arrive at

$$\begin{aligned} H^{(4)}_{3A,2}(\tau) &= -H^{(4)}_{6A,2}(\tau) \\ &= \frac{1}{4}H^{(4)}_2(\tau) + \frac{1}{2\eta(2\tau)^3}\Big(-3\Lambda_2(\tau) - 4\Lambda_3(\tau) + \Lambda_6(\tau)\Big), \\ H^{(4)}_{7AB,2}(\tau) &= -H^{(4)}_{14AB,2}(\tau) \\ &= \frac{1}{8}H^{(4)}_2(\tau) + \frac{1}{12\,\eta(2\tau)^3}\Big(-7\Lambda_2(\tau) - 4\Lambda_7(\tau) + \Lambda_{14}(\tau) + 28 f_{14}(\tau)\Big), \end{aligned} \tag{4.28}$$

where we have used $S^{(4)}_2(\tau) = 2\eta(2\tau)^3$ and the hypothesis that

$$H^{(4)}_{g,2}(\tau) = 0 \tag{4.29}$$

for all classes not in the set $\{1A, 2A, 3A, 6A, 7AB, 14AB\}$. This finishes our proposal for the McKay–Thompson series $\big(H^{(4)}_{g,r}\big)$ at $\ell = 4$. Coincidences between some of the $H^{(4)}_{g,r}$ and Ramanujan's mock theta functions will be discussed in §4.7.

4.5 Lambency Five

As mentioned before, for $\ell = 5$ it is no longer sufficient to specify the weight 2 forms $F^{(\ell)}_g(\tau)$ as defined in (4.16). In this case we also need to specify the weight 2 forms $F^{(\ell),2}_g(\tau)$ defined in (4.17).

With the data of $F^{(5)}_g(\tau)$ and $F^{(5),2}_g(\tau)$ we can determine the $H^{(5)}_{g,r}$ using the relations

$$H^{(5)}_{g,r}(\tau) = \hat{H}^{(5)}_{g,r}(\tau) + \frac{\chi^{(5)}_{g,r}}{6} H^{(5)}_r(\tau) \tag{4.30}$$

and

$$\begin{aligned} \hat{H}^{(5)}_{g,1}(\tau)S^{(5)}_1(\tau) + \hat{H}^{(5)}_{g,3}(\tau)S^{(5)}_3(\tau) &= \tfrac{1}{2}\Big(F^{(5)}_g(\tau) + F^{(5)}_{zg}(\tau)\Big) \\ \hat{H}^{(5)}_{g,1}(\tau)S^{(5)}_4(\tau) + \hat{H}^{(5)}_{g,3}(\tau)S^{(5)}_2(\tau) &= \tfrac{1}{2}\Big(F^{(5),2}_g(\tau) + F^{(5),2}_{zg}(\tau)\Big) \\ \hat{H}^{(5)}_{g,2}(\tau)S^{(5)}_2(\tau) + \hat{H}^{(5)}_{g,4}(\tau)S^{(5)}_4(\tau) &= \tfrac{1}{2}\Big(F^{(5)}_g(\tau) - F^{(5)}_{zg}(\tau)\Big) \\ \hat{H}^{(5)}_{g,2}(\tau)S^{(5)}_3(\tau) + \hat{H}^{(5)}_{g,4}(\tau)S^{(5)}_1(\tau) &= \tfrac{1}{2}\Big(-F^{(5),2}_g(\tau) + F^{(5),2}_{zg}(\tau)\Big) \end{aligned} \tag{4.31}$$

where g and zg again form a pair satisfying (4.13).

Moreover, notice that in $F^{(5),2}_g(\tau)$ the part of the sum involving even r has an expansion in $q^{1/4+k}$ for non-negative integers k, while the sum over odd r has an expansion in $q^{3/4+k}$. As

$[g]$	N_g	$F_g^{(5)}(\tau)$	$F_g^{(5),2}(\tau)$
1A	1	0	0
2A	4	0	0
2B	4	$16(\Lambda_2 - \Lambda_4/3)$	$-\frac{8}{3}\eta(\tau)^8/\eta(\tau/2)^4$
2C	2	$-16\Lambda_2$	$e(\frac{1}{4})\, F_{2B}^{(5),2}(\tau+1)$
3A	9	$-2\eta^6(\tau)/\eta^2(3\tau)$	$-2\frac{\eta(\tau)^8\eta(3\tau/2)^2\eta(6\tau)^2}{\eta(\tau/2)^2\eta(2\tau)^2\eta(3\tau)^4}$
6A	36	$-2\,\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2/\eta(6\tau)^2$	$2\,\eta(\tau/2)^2\eta(\tau)^2\eta(3\tau)^2/\eta(3\tau/2)^2$
4AB	16	$-2\eta(2\tau)^{14}/\eta(\tau)^4\eta(4\tau)^6$	$-16\eta(\tau)^2\eta(4\tau)^6/\eta(2\tau)^4$
4CD	8	$-8/3\Lambda_4(\tau)$	$-\frac{32}{3}\,\eta(2\tau)^2\eta(4\tau)^2/\eta(\tau)^2$
5A	5	$-2\Lambda_5$	$e(\frac{1}{4})\, F_{10A}^{(5),2}(\tau+1)$
10A	20	$\sum_{d\|20} c_{10A}(d)\Lambda_d - \frac{40}{3} f_{20}$	$\sum_{d\|20} c_{10A,2}(d)\Lambda_d(\tau/4) + \frac{10}{3} f_{20}(\tau/4)$
12AB	144	$-2\eta(\tau)^2\eta(2\tau)^2\eta(6\tau)^4/\eta(3\tau)^2\eta(12\tau)^2$	$-4\eta(\tau)^2\eta(2\tau)^2\eta(12\tau)^2/\eta(6\tau)^2$
$c_{10A}(d) = -5, \frac{5}{3}, -\frac{2}{3}, 1, -\frac{1}{3}$ for $d = 2, 4, 5, 10, 20$			
$c_{10A,2}(d) = -\frac{5}{4}, \frac{5}{24}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{24}$ for $d = 2, 4, 5, 10, 20$			

Table 5: The list of weight 2 modular forms $F_g^{(5)}(\tau)$ on $\Gamma_0(N_g)$ and the other set of weight 2 modular forms $F_g^{(5),2}(\tau)$.

a result, it is sufficient to specify $F_g^{(\ell),2}(\tau)$ for one of the paired classes g and zg as they are related to each other by

$$F_{zg}^{(5),2}(\tau) = e(\tfrac{1}{4})\, F_g^{(\ell),2}(\tau+1)\ . \tag{4.32}$$

Using (4.30-4.32), the data recorded in Table 5 are sufficient to explicitly specify all the mock modular forms $H_g^{(5)}(\tau)$ for all $[g] \subset G^{(5)}$ at lambency 5.

4.6 Lambencies Seven and Thirteen

For $\ell = 7$, apart from the $1A$ and $2A$ classes whose McKay-Thomson series have been given in terms of the weight 0 Jacobi form $Z^{(7)}$ in §2.5, there are five more classes $3AB$, $4A$, and $6AB$ whose McKay-Thomson series $H_g^{(7)}$ we would like to identify. We specify the weight 2 forms associated to these classes. Notice that these expressions are not sufficient to completely determine all the components $H_{g,r}^{(7)}$. Nevertheless, they provide strong evidence for the mock modular properties of the proposed McKay–Thompson series.

$$
\begin{aligned}
F_{4A}^{(7)}(\tau) &= -2\frac{\eta(\tau)^4\eta(2\tau)^2}{\eta(4\tau)^2},\\
F_{4A}^{(7),2}(\tau) &= 4\frac{\eta(\tau)^6\eta(4\tau)^2}{\eta(2\tau)^4},\\
F_{3AB}^{(7)}(\tau) &= -6\Lambda_3(\tau),\\
F_{6AB}^{(7)}(\tau) &= -9\Lambda_2(\tau) - 2\Lambda_3(\tau) + 3\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau),\\
F_{6AB}^{(7),2}(4\tau) &= -\frac{9}{4}\Lambda_2(\tau) - \Lambda_3(\tau) + \frac{3}{8}\Lambda_4(\tau) + \frac{3}{4}\Lambda_6(\tau) - \frac{1}{8}\Lambda_{12}(\tau).
\end{aligned}
\tag{4.33}
$$

Similarly, for lambency 13 we have

$$
\begin{aligned}
F_{4AB}^{(13)}(\tau) &= -2\frac{\eta(2\tau)^{14}}{\eta(\tau)^4\eta(4\tau)^6},\\
F_{4AB}^{(13),2}(\tau) &= -16\frac{\eta(\tau)^2\eta(4\tau)^6}{\eta(2\tau)^4}.
\end{aligned}
\tag{4.34}
$$

4.7 Mock Theta Functions

As mentioned in §2.1 we observe that in many cases across different lambencies the components $H_{g,r}^{(\ell)}$ of the McKay–Thompson series coincide with known mock theta functions. In particular, we often encounter Ramanujan's mock theta functions, and always in such a way that the order is divisible by the lambency. In this section we will list these conjectural identities between $H_{g,r}^{(\ell)}$ and mock theta functions identified previously in the literature.

For $\ell = 2$ two of the functions $H_g^{(2)}(\tau)$ are related to Ramanujan's mock theta functions of orders 2 and 8 (cf. (2.7)) through

$$
\begin{aligned}
H_{4B}^{(2)}(\tau) &= -2q^{-1/8}\mu(q),\\
H_{8A}^{(2)}(\tau) &= -2q^{-1/8}U_0(q).
\end{aligned}
\tag{4.35}
$$

For $\ell = 3$ we encounter the following order 3 mock theta functions of Ramanujan:

$$
\begin{aligned}
&H_{2B,1}^{(3)}(\tau) = H_{2C,1}^{(3)}(\tau) = H_{4C,1}^{(3)}(\tau) = -2q^{-1/12}f(q^2),\\
&H_{6C,1}^{(3)}(\tau) = H_{6D,1}^{(3)}(\tau) = -2q^{-1/12}\chi(q^2),\\
&H_{8C,1}^{(3)}(\tau) = H_{8D,1}^{(3)}(\tau) = -2q^{-1/12}\phi(-q^2),\\
&H_{2B,2}^{(3)}(\tau) = -H_{2C,2}^{(3)}(\tau) = -4q^{2/3}\omega(-q),\\
&H_{6C,2}^{(3)}(\tau) = -H_{6D,2}^{(3)}(\tau) = 2q^{2/3}\rho(-q).
\end{aligned}
\tag{4.36}
$$

(Cf. (2.9).) The description of the shadow of $H_g^{(3)}$ is consistent with the fact that, among the seven order 3 mock theta functions of Ramanujan, $f(\tau)$, $\phi(\tau)$, $\psi(\tau)$ and $\chi(\tau)$ form a group with the same shadow $(S_1^{(3)}(\tau))$ while the other three $\omega(\tau)$, $\nu(\tau)$ and $\rho(\tau)$ form another group with another shadow $(S_2^{(3)}(\tau))$. Moreover, various relations among these order 3 mock theta functions can be obtained as consequences of the above identification.

For $\ell = 4$ we encounter the following order 8 mock theta functions

$$\begin{aligned} H_{2C,1}^{(4)}(\tau) &= q^{-\frac{1}{16}}\big(-2S_0(q) + 4T_0(q)\big), \\ H_{2C,3}^{(4)}(\tau) &= q^{\frac{7}{16}}\big(2S_1(q) - 4T_1(q)\big), \\ H_{4C,1}^{(4)}(\tau) &= -2\,q^{-\frac{1}{16}}S_0(q), \\ H_{4C,3}^{(4)}(\tau) &= 2\,q^{\frac{7}{16}}\,S_1(q). \end{aligned} \tag{4.37}$$

(Cf. (2.11).) Comparing with (2.7) we see that our proposal implies the identities

$$\begin{aligned} \mu(q) &= S_0(q^2) - 2T_0(q^2) + q\big(S_1(q^2) - 2T_1(q^2)\big) \\ &= U_0(q) - 2U_1(q), \\ U_0(q) &= S_0(q^2) + q\,S_1(q^2), \\ U_1(\tau) &= T_0(q^2) + q\,T_1(q^2), \end{aligned} \tag{4.38}$$

between different mock theta functions of Ramanujan. See, for instance, [113] for a collection of such identities.

For $\ell = 5$ we encounter four of Ramanujan's order 10 mock theta functions:

$$\begin{aligned} H_{2BC,1}^{(5)}(\tau) &= H_{4CD,1}^{(5)}(\tau) = -2q^{-\frac{1}{20}}\,X(q^2), \\ H_{2BC,3}^{(5)}(\tau) &= H_{4CD,3}^{(5)}(\tau) = -2q^{-\frac{9}{20}}\,\chi_{10}(q^2), \\ H_{2C,2}^{(5)}(\tau) &= -H_{2B,2}^{(5)}(\tau) = 2q^{-\frac{1}{5}}\,\psi_{10}(-q), \\ H_{2C,4}^{(5)}(\tau) &= -H_{2B,4}^{(5)}(\tau) = -2q^{\frac{1}{5}}\,\phi_{10}(-q). \end{aligned} \tag{4.39}$$

(Cf. (2.10).)

4.8 Automorphy

In this section we discuss the automorphy of the proposed McKay–Thompson series $H_g^{(\ell)} = \big(H_{g,r}^{(\ell)}\big)$. As mentioned in §4.1, the function $H_g^{(\ell)}$ is a vector-valued mock modular form with shadow $\big(\chi_{g,r}S_r^{(\ell)}\big)$ for some $\Gamma_g \subset SL_2(\mathbb{Z})$ with a certain (matrix-valued) multiplier ν_g. In this subsection we will specify the group Γ_g and the multiplier ν_g. The multipliers we specify here

can be verified explicitly using the data given in §§4.2-4.6 (except for the few conjugacy classes at $\ell = 7$ and $\ell = 13$ for which the McKay–Thompson series $H_g^{(\ell)}$ have not been completely determined).

We find that the automorphy of the vector-valued function $H_g^{(\ell)}$ is governed, in way that we shall describe presently, by the signed permutation representation of $G^{(\ell)}$ arising from the construction (as a subgroup of Oct_m for $m = 24/(\ell-1)$) given in §3.4. We use this representation to define the symbols $n_g|h_g$ which appear in the second row of each table 20-47, and also in the twisted Euler character tables 14-19, and we will explain below how to use these symbols to determine the multiplier system for each $H_g^{(\ell)}$. It will develop also that the twisted Euler character $g \mapsto \chi_g^{(\ell)}$ (cf. Tables 14-19) attached to the signed permutation representation of $G^{(\ell)}$ determines the shadow of $H_g^{(\ell)}$.

Recall from §3.4 that the signed permutation representation of $G^{(\ell)}$ naturally induces a permutation representation of the same degree, and this permutation representation factors through the quotient $\bar{G}^{(\ell)} = G^{(\ell)}/2$ of $G^{(\ell)}$ by its unique central subgroup of order 2 (except in case $\ell = 2$ when the signed and unsigned permutation representations coincide, and we have $G^{(2)} = \bar{G}^{(2)}$). Let $g \mapsto \bar{g}$ denote the natural map $G^{(\ell)} \to \bar{G}^{(\ell)}$ and observe that (for $\ell > 2$) if the unique central involution of $G^{(\ell)}$ belongs to the cyclic subgroup $\langle g \rangle$ generated by g then $o(g) = 2o(\bar{g})$ and otherwise $o(g) = o(\bar{g})$. We say that $g \in G^{(\ell)}$ is *split* over $\bar{G}^{(\ell)}$ in case $o(g) = o(\bar{g})$ and we call g *non-split* otherwise. (By this definition every element of $G^{(2)} = \bar{G}^{(2)}$ is split.)

Recall from §3.4 that $g \mapsto \Pi_g^{(\ell)}$ denotes the map attaching signed permutation Frame shapes to elements of $G^{(\ell)}$ and $g \mapsto \bar{\Pi}_g^{(\ell)}$ denotes the Frame shapes (actually cycle shapes) arising from the (unsigned) permutation representation (on $m = 24/(\ell-1)$ points). Taking a formal product of Frame shapes $\tilde{\Pi}_g^{(\ell)} = \Pi_g^{(\ell)}\bar{\Pi}_g^{(\ell)}$ (defined so that $j^{m_1}j^{m_2} = j^{m_1+m_2}$, &c.) we obtain the Frame shape of $g \in G^{(\ell)}$ regarded as a permutation of the $2m$ points $\{\pm e_i\}$ for $i \in \Omega^{(\ell)}$ (cf. §3.4). In particular, $\tilde{\Pi}_g$ is a cycle shape and none of the exponents appearing in $\tilde{\Pi}_g$ are negative. Given $g \in G^{(\ell)}$ and $\tilde{\Pi}_g = j_1^{m_1} \cdots j_k^{m_l}$ with $j_1 < \cdots < j_k$ and $m_i > 0$ define $N_g = j_1 j_k$. That is, set N_g to be the product of the shortest and longest cycle lengths appearing in a cycle decomposition for g regarded as a permutation on the $2m$ points $\{\pm e_i\}$. Now define the symbols $n_g|h_g$ by setting $n_g = o(\bar{g})$ for all $g \in G^{(\ell)}$ and all $\ell \in \Lambda$, and by setting $h_g = N_g/n_g$ for all $g \in G^{(\ell)}$ and $\ell \in \Lambda$ except when $\ell = 4$ and g is non-split in which case set $h_g = N_g/2n_g$. The symbols $n_g|h_g$ are specified in Tables 14-19, and also in Tables 20-47, in the rows labelled Γ_g. We omit the $|h_g$ when $h_g = 1$, so n_g is a shorthand for $n_g|1$ in these tables.

The significance of the value n_g is that it is the minimal positive integer n for which $H_g^{(\ell)}$ is a mock modular form (of weight 1/2) on $\Gamma_0(n)$, and the significance of h_g is that, as we shall see momentarily, it is the minimal positive integer h for which the multiplier for $H_g^{(\ell)}$ coincides

with the conjugate multiplier for the (vector-valued) cusp form $S^{(\ell)}$ when restricted to $\Gamma_0(nh)$ (for $n = n_g$). Since $n_g h_g \leq N_g$ for all g the multiplier for $H_g^{(\ell)}$ coincides with the conjugate multiplier for $S^{(\ell)}$ when regarded as a mock modular form on $\Gamma_0(N_g)$ for all g. It is very curious that this coincidence of multipliers extends to the larger group $\Gamma_0(N_g/2)$ in the case that $\ell = 4$ and g is non-split (i.e. $o(g) = 2o(\bar{g})$).

Table 6: Admissible $v^{(\ell)}$

ℓ	2	3	4	5	7	13
$v^{(\ell)}$	1	5	3	7	1	7

Given a pair of positive integers (n, h) we define a matrix-valued function $\rho_{n|h}^{(\ell)}$ on $\Gamma_0(n)$ as follows. For each $\ell \in \Lambda$ let the integer $v^{(\ell)}$ be as specified in Table 6. When h divides n we set

$$\rho_{n|h}^{(\ell)}(\gamma) = \mathrm{e}\left(-v^{(\ell)}\frac{cd}{nh}\right)\mathrm{I}_{\ell-1} \tag{4.40}$$

where $\mathrm{I}_{\ell-1}$ denotes the $(\ell-1)\times(\ell-1)$ identity matrix. When h does not divide n and n is even we set

$$\rho_{n|h}^{(\ell)}(\gamma) = \mathrm{e}\left(-v^{(\ell)}\frac{cd}{nh}\frac{(n,h)}{n}\right)\mathrm{J}_{\ell-1}^{c(d+1)/n}\mathrm{K}_{\ell-1}^{c/n} \tag{4.41}$$

where (n, h) denotes the greatest common divisor of n and h, and $\mathrm{J}_{\ell-1}$ and $\mathrm{K}_{\ell-1}$ are the $(\ell-1)\times(\ell-1)$ matrices given by

$$\mathrm{J}_{\ell-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^{\ell} \end{pmatrix}, \quad \mathrm{K}_{\ell-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix}, \tag{4.42}$$

and when h does not divide n and n is odd we set

$$\rho_{n|h}^{(\ell)}(\gamma) = \mathrm{e}\left(-v^{(\ell)}\frac{cd}{nh}\frac{n}{(n,h)}\right)\mathrm{J}_{\ell-1}^{c(d+1)/n}\mathrm{K}_{\ell-1}^{c/n}. \tag{4.43}$$

Now the multiplier system $\nu_g^{(\ell)}$ for the umbral mock modular form $H_g^{(\ell)}$ is the matrix-valued function on $\Gamma_0(n) = \Gamma_0(n_g)$ given simply by

$$\nu_g^{(\ell)} = \nu_{n|h}^{(\ell)} = \rho_{n|h}^{(\ell)}\overline{\sigma^{(\ell)}} \tag{4.44}$$

where $n = n_g$ and $h = h_g$ and $\sigma^{(\ell)} = (\sigma^{(\ell)}_{ij})$ denotes the (matrix-valued) multiplier system for the (vector-valued) theta series $S^{(\ell)}$ (cf. (2.26)) and satisfies

$$\sigma^{(\ell)}(\gamma)S^{(\ell)}(\gamma\tau)\mathrm{jac}(\gamma,\tau)^{3/4} = S^{(\ell)}(\tau) \tag{4.45}$$

for $\gamma \in \Gamma_0(1) = SL_2(\mathbb{Z})$ where $\mathrm{jac}(\gamma,\tau) = (c\tau + d)^{-2}$ in case γ has lower row (c, d).

Recall from §3.4 that the character of $G^{(\ell)}$ attached to its signed permutation representation is denoted $g \mapsto \chi^{(\ell)}_g$ in Tables 14-19 and that of the (unsigned) permutation representation is denoted $g \mapsto \bar{\chi}^{(\ell)}_g$. Define $\chi^{(\ell)}_{g,r}$ for $0 < r < \ell$ by setting $\chi^{(\ell)}_{g,r} = \chi^{(\ell)}_g$ in case r is even and $\chi^{(\ell)}_{g,r} = \bar{\chi}^{(\ell)}_g$ otherwise. Then the shadow of $H^{(\ell)}_g$ is the function $S^{(\ell)}_g = \big(S^{(\ell)}_{g,r}\big)$ with components related to those of $S^{(\ell)}$ by $S^{(\ell)}_{g,r} = \chi^{(\ell)}_{g,r} S^{(\ell)}_r$. In particular, $H^{(\ell)}_g$ is a vector-valued modular form of weight 1/2 for $\Gamma_0(n_g)$ when $\chi^{(\ell)}_g = \bar{\chi}^{(\ell)}_g = 0$.

To summarise, we claim that our proposed McKay–Thompson series $H^{(\ell)}_g$ are such that if we define $\hat{H}^{(\ell)}_g = \big(\hat{H}^{(\ell)}_{g,r}\big)$ by setting

$$\hat{H}^{(\ell)}_{g,r}(\tau) = H_{g,r}(\tau) - \frac{\chi^{(\ell)}_{g,r}}{\sqrt{2\ell}} \frac{1}{(4i)^{1/2}} \int_{-\bar{\tau}}^{i\infty} (z+\tau)^{-1/2} S^{(\ell)}_r(z)\mathrm{d}z \tag{4.46}$$

for $0 < r < \ell$ then $\hat{H}^{(\ell)}_g$ is invariant for the weight 1/2 action of $\Gamma_0(n)$ on $(\ell-1)$-vector-valued functions on $\mathbb{H}$ given by

$$\left(\hat{H}^{(\ell)}_g\Big|_{1/2,n|h}\gamma\right)(\tau) = \nu_{n|h}(\gamma)\hat{H}^{(\ell)}_g(\gamma\tau)\,\mathrm{jac}(\gamma,\tau)^{1/4} \tag{4.47}$$

where $n = n_g$ and $h = h_g$ and $\nu_{n|h}$ is defined by (4.44). This statement completely describes the (conjectured) automorphy of the mock modular forms $H^{(\ell)}_g$ (cf. §5.3).

5 Conjectures

We have described the umbral forms $Z^{(\ell)}$, $H^{(\ell)}$ and $\Phi^{(\ell)}$ in §2 and the umbral groups $G^{(\ell)}$ in §3 and we have introduced families $\{H^{(\ell)}_g \mid g \in G^{(\ell)}\}$ of vector-valued mock modular forms in §4. The discussions of those sections clearly demonstrate the distinguished nature of these objects, and we have mentioned some coincidences relating the groups $G^{(\ell)}$ and the forms $H^{(\ell)}_g$ directly. In this section we present, in a more systematic fashion, evidence that the relationship between the $G^{(\ell)}$ and the $H^{(\ell)}$ is more than coincidental. Our observations lead naturally to conjectures that we hope will serve as first steps in revealing the structural nature of the mechanism underlying umbral moonshine.

5.1 Modules

As was mentioned in §4, after comparison of the character tables (cf. §B.1) of the umbral groups $G^{(\ell)}$ with the Fourier coefficient tables (cf. §C) for the forms $H_g^{(\ell)}$ it becomes apparent that the low degree Fourier coefficients of $H^{(\ell)} = H_e^{(\ell)}$ may be interpreted as degrees of representations of $G^{(\ell)}$ in such a way that the corresponding coefficients of $H_g^{(\ell)}$ are recovered by substituting character values at g; we have tabulated evidence for this in the form of explicit combinations of irreducible representations in §D. This observation suggests the existence of bi-graded $G^{(\ell)}$-modules $K^{(\ell)} = \bigoplus K_{r,d}^{(\ell)}$ whose bi-graded dimensions are recovered via Fourier coefficients from the vector-valued mock modular forms $H^{(\ell)} = \left(H_r^{(\ell)}\right)$.

Conjecture 5.1. *We conjecture that for $\ell \in \Lambda = \{2,3,4,5,7,13\}$ there exist naturally defined $\mathbb{Z}\times\mathbb{Q}$-graded $G^{(\ell)}$-modules*

$$K^{(\ell)} = \bigoplus_{\substack{r\in\mathbb{Z}\\ 0<r<\ell}} K_r^{(\ell)} = \bigoplus_{\substack{r,k\in\mathbb{Z}\\ 0<r<\ell}} K_{r,k-r^2/4\ell}^{(\ell)} \tag{5.1}$$

such that the graded dimension of $K^{(\ell)}$ is related to the vector-valued mock modular form $H^{(\ell)} = \left(H_r^{(\ell)}\right)$ by

$$H_r^{(\ell)}(\tau) = -2\delta_{r,1}q^{-1/4\ell} + \sum_{\substack{k\in\mathbb{Z}\\ r^2-4k\ell<0}} \dim\left(K_{r,k-r^2/4\ell}^{(\ell)}\right) q^{k-r^2/4\ell} \tag{5.2}$$

for $q = \mathrm{e}(\tau)$ and such that the vector-valued mock modular forms $H_g^{(\ell)} = \left(H_{g,r}^{(\ell)}\right)$ described in §4 (and partially in the tables of §C) are recovered from $K^{(\ell)}$ via graded trace functions according to

$$H_{g,r}^{(\ell)}(\tau) = -2\delta_{r,1}q^{-1/4\ell} + \sum_{\substack{k\in\mathbb{Z}\\ r^2-4k\ell<0}} \mathrm{tr}_{K_{r,k-r^2/4\ell}^{(\ell)}}(g) q^{k-r^2/4\ell}. \tag{5.3}$$

Remark 5.2. Recall that a *superspace* is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space and if $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is such an object and $T : V \to V$ is a linear operator preserving the grading then the *supertrace* of T is given by $\mathrm{str}_V T = \mathrm{tr}_{V_{\bar{0}}} T - \mathrm{tr}_{V_{\bar{1}}} T$ where $\mathrm{tr}_W T$ denotes the usual trace of T on W. Since the coefficient of $q^{-1/4\ell}$ in $H_{g,1}^{(\ell)}$ is $c_{g,1}^{(\ell)}(-1/4\ell) = -2$ for all $g \in G^{(\ell)}$ for all ℓ it is natural to expect that this term may be interpreted as the supertrace of $g \in G^{(\ell)}$ on a trivial $G^{(\ell)}$-supermodule $K_{1,-1/4\ell}^{(\ell)}$ with vanishing even part and 2-dimensional odd part. Thus Conjecture 5.1 implies the existence of a $G^{(\ell)}$-supermodule $K^{(\ell)} = \bigoplus K_{r,d}^{(\ell)}$ such that $K_{r,d}^{(\ell)}$ is purely even or purely odd

according as d is positive or negative, and such that $H^{(\ell)}_{g,r} = \sum_k \mathrm{str}_{K^{(\ell)}_{r,k-r^2/4\ell}}(g) q^{k-r^2/r\ell}$ for each $0 < r < \ell$ and $g \in G^{(\ell)}$.

Remark 5.3. For $\ell \in \Lambda$ and $0 < r, s < \ell$ we have that $r^2 \equiv s^2 \pmod{4\ell}$ implies $r = s$ so the first index r in $K^{(\ell)}_{r,d}$ can be deduced from d since it is the unique $0 < r < \ell$ such that $d + r^2/4\ell$ is an integer. Thus we may dispense with the first of the two gradings on the conjectural $G^{(\ell)}$-modules $K^{(\ell)}$ and regard them as $\mathbb{Q}$-graded $K^{(\ell)} = \bigoplus_d K^{(\ell)}_d$ by the rationals of the form $d = n - r^2/4\ell$ for $n \in \mathbb{Z}$ and $0 < r < \ell$ without introducing any ambiguity.

5.2 Moonshine

In the case of monstrous moonshine the McKay–Thompson series T_g for g in the monster group have the astonishing property that they all serve as generators for the function fields of their invariance groups (cf. [1]). In other words, if Γ_g is the subgroup of $PSL_2(\mathbb{R})$ consisting of the isometries $\gamma : \mathbb{H} \to \mathbb{H}$ such that $T_g(\gamma\tau) = T_g(\tau)$ for all $\tau \in \mathbb{H}$ then T_g induces an isomorphism from the compactification of $\Gamma_g \backslash \mathbb{H}$ to the Riemann sphere (being the one point compactification of $\mathbb{C}$). In particular, Γ_g is a genus zero subgroup of $PSL_2(\mathbb{R})$, and so this property is commonly referred to as the *genus zero property* of monstrous moonshine.

By now there are many methods extant for constructing graded vector spaces with algebraic structure whose graded dimensions are modular functions—suitable classes of vertex algebras, for example, serve this purpose (cf. [114, 115])—and so one can expect to obtain analogues of the McKay–Thompson series T_g by equipping such an algebraic structure with the action of a group. But there is no guarantee that such a procedure will result in functions that have the genus zero property of monstrous moonshine, and so it is this genus zero property that distinguishes the observations of McKay, Thompson, Conway and Norton regarding the Monster group from any number of more generic connections between finite groups and modular functions. In what follows we will propose a conjecture that may be regarded as the natural analogue of the Conway–Norton conjecture for umbral moonshine.

Suppose that $T : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with invariance group $\Gamma < PSL_2(\mathbb{R})$ and a simple pole in $q = e(\tau)$ as $\tau \to i\infty$. Suppose also that Γ is commensurable with $PSL_2(\mathbb{Z})$ and that the *translation subgroup* Γ_∞, consisting of the elements of Γ with upper-triangular preimages in $SL_2(\mathbb{R})$, is generated by $\tau \mapsto \tau + 1$. It is shown in [25] that such a function T is a generator for the field of Γ-invariant functions on $\mathbb{H}$ if and only if $T(\tau)$ coincides with the weight

0 *Rademacher sum*

$$R_\Gamma(\tau) = \mathrm{Reg}\left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} q^{-1}\big|_0\,\gamma\right) \tag{5.4}$$

attached to Γ. The sum here is over representatives for the cosets of Γ_∞ in Γ and $f \mapsto f|_0\gamma$ denotes the weight 0 action of γ on holomorphic functions (viz., $(f|_0\gamma)(\tau) = f(\gamma\tau)$). We write $\mathrm{Reg}(\cdot)$ to indicate a regularisation procedure first realised by Rademacher (for the case that $\Gamma = PSL_2(\mathbb{Z})$) in [26]. According then to the result of [25] the genus zero property of monstrous moonshine may be reformulated in the following way:

> For each element g in the monster group the McKay–Thompson series T_g satisfies $T_g = R_{\Gamma_g}$ when Γ_g is the invariance group of T_g.

This may be compared with the article [24] which considers the mock modular forms $H_g^{(2)}$ attached to the largest Mathieu group $G^{(2)} \simeq M_{24}$ via the observation of Eguchi–Ooguri–Tachikawa and applies some of the philosophy of [25] to the problem of finding a uniform construction for these functions. The solution developed in [24] is that the function $H_g^{(2)}$ coincides with the weight 1/2 Rademacher sum

$$R^{(2)}_{n|h}(\tau) = \mathrm{Reg}\left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(n)} -2q^{-1/8}\Big|_{1/2,n|h}\,\gamma\right) \tag{5.5}$$

where $n = n_g$ and $h = h_g$ are integers determined by the defining permutation representation of M_{24} (cf. Table 14) and $f \mapsto f|_{1/2,n|h}\gamma$ denotes a certain weight 1/2 action of $\Gamma_0(n)$ determined by n and h. (We refer to [24] for full details.) By comparison with the previous paragraph we thus arrive at a direct analogue of the genus zero property of monstrous moonshine that holds for all the umbral forms with $\ell = 2$:

> For each element g in the largest Mathieu group the McKay–Thompson series $H_g^{(2)}$ satisfies $H_g^{(2)} = R^{(2)}_{n|h}$ when $n = n_g$ and $h = h_g$.

We conjecture that this genus zero property extends to all the functions of umbral moonshine.

Conjecture 5.4 (umbral moonshine)**.** *We conjecture that for each $\ell \in \Lambda$ and each $g \in G^{(\ell)}$ we*

have $H_g^{(\ell)} = R_{n|h}^{(\ell)}$ *where* $n = n_g$ *and* $h = h_g$ *are as specified in* §B.2 *and*

$$R_{n|h}^{(\ell)} = \mathrm{Reg}\left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(n)} \begin{pmatrix} -2q^{-1/4\ell} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Bigg|_{1/2,n|h} \gamma \right) \tag{5.6}$$

is a vector-valued generalisation of the Rademacher sum $R_{n|h}^{(2)}$ *adapted to the weight* $1/2$ *action of* $\Gamma_0(n)$ *on* $(\ell-1)$*-vector-valued functions on* $\mathbb{H}$ *that is defined in* (4.47)*.*

5.3 Modularity

A beautiful feature of the umbral moonshine conjecture is that it implies the precise nature of the modularity of the Thompson series $H_g^{(\ell)}$. We record these implications explicitly as conjectures in this short section.

Conjecture 5.5. *We conjecture that the graded supertrace functions* (5.3) *for fixed* $\ell \in \Lambda$ *and* $g \in G^{(\ell)}$ *and varying* $0 < r < \ell$ *define the components of a vector-valued mock modular form* $H_g^{(\ell)}$ *of weight* $1/2$ *on* $\Gamma_0(n_g)$ *with shadow function* $S_g^{(\ell)} = (S_{g,r}^{(\ell)}) = (\chi_{g,r}^{(\ell)} S_r^{(\ell)})$ *where* $S^{(\ell)} = (S_r^{(\ell)})$ *is the vector-valued theta series described in* §2.3*, the* $\chi_{g,r}^{(\ell)}$ *are determined from the twisted Euler characters of* $G^{(\ell)}$ *(cf.* §B.2*) by* $\chi_{g,r}^{(\ell)} = \bar{\chi}_g^{(\ell)}$ *for* r *odd and* $\chi_{g,r}^{(\ell)} = \chi_g^{(\ell)}$ *for* r *even, and* n_g *denotes the order of the image of* $g \in G^{(\ell)}$ *in the factor group* $\bar{G}^{(\ell)}$ *(cf.* §3.1, 3.4*).*

Recall from §4.8 that for $\ell \in \Lambda$ and a pair (n, h) of positive integers we have the matrix-valued function $\nu_{n|h}^{(\ell)}$ on $\Gamma_0(n)$ defined in (4.44). Recall also that we have attached a pair (n_g, h_g) to each $g \in G^{(\ell)}$ for each $\ell \in \Lambda$ in §4.8.

Conjecture 5.6. *We conjecture that the multiplier system of* $H_g^{(\ell)}$ *is given by* $\nu_{n|h}^{(\ell)}$ *when* $n = n_g$ *and* $h = h_g$ *for all* $g \in G^{(\ell)}$ *for all* $\ell \in \Lambda$*.*

5.4 Discriminants

One of the most striking features of umbral moonshine is the apparently intimate relation between the number fields on which the irreducible representations of $G^{(\ell)}$ are defined and the discriminants of the vector-valued mock modular form $H^{(\ell)}$. We discuss the evidence for this relation and formulate conjectures about it in this section.

First we observe that the discriminants of the components $H_r^{(\ell)}$ of the mock modular form $H^{(\ell)} = H_e^{(\ell)}$ determine some important properties of the representations of $G^{(\ell)}$, where we

say that an integer D is a *discriminant of* $H^{(\ell)}$ if there exists a term $q^d = q^{-\frac{D}{4\ell}}$ with non-vanishing Fourier coefficient in at least one of the components. The following result can be verified explicitly using the tables in §§B.C.

Proposition 5.7. *Let $\ell \in \Lambda$. If $n > 1$ is an integer satisfying*

1. *there exists an element of $G^{(\ell)}$ of order n, and*
2. *there exists an integer λ that is co-prime to n such that $D = -n\lambda^2$ is a discriminant of $H^{(\ell)}$,*

then there exists at least one pair of irreducible representations ϱ and ϱ^* of $G^{(\ell)}$ and at least one element $g \in G^{(\ell)}$ such that $\mathrm{tr}_\varrho(g)$ is not rational but

$$\mathrm{tr}_\varrho(g), \mathrm{tr}_{\varrho^*}(g) \in \mathbb{Q}(\sqrt{-n}) \tag{5.7}$$

and n divides $o(g)$.

The finite list of integers n satisfying the two conditions of Proposition 5.7 is given in Table 7.

From now on we say that an irreducible representation ϱ of the umbral group $G^{(\ell)}$ is of *type* n if n is an integer satisfying the two conditions of Proposition 5.7 and the character values of ϱ generate the field $\mathbb{Q}(\sqrt{-n})$. Evidently, irreducible representations of type n come in pairs (ϱ, ϱ^*) with $\mathrm{tr}_{\varrho^*}(g)$ the complex conjugate of $\mathrm{tr}_\varrho(g)$ for all $g \in G^{(\ell)}$. The list of all irreducible representations of type n is given in Table 7. (See §B.1 for the character tables of the $G^{(\ell)}$ and our notation for irreducible representations.)

Recall that the *Frobenius–Schur indicator* of an irreducible ordinary representation of a finite group is 1, -1 or 0 according as the representation admits an invariant symmetric bilinear form, an invariant skew-symmetric bilinear form, or no invariant bilinear form, respectively. The representations admitting no invariant bilinear form are precisely those whose character values are not all real. We can now state the next observation.

Proposition 5.8. *For each $\ell \in \Lambda$ an irreducible representation ϱ of $G^{(\ell)}$ has Frobenius–Schur indicator 0 if and only if it is of type n for some n.*

The *Schur index* of an irreducible representation ϱ of a finite group G is the smallest positive integer s such that there exists a degree s extension k of the field generated by the character values $\mathrm{tr}_\varrho(g)$ for $g \in G$ such that ϱ can be realised over k. Inspired by Proposition 5.8 we make the following conjecture.

Conjecture 5.9. *If ϱ is an irreducible representation of $G^{(\ell)}$ of type n then the Schur index of ϱ is equal to* 1.

In other words, we conjecture that the irreducible $G^{(\ell)}$-representations of type n can be realised over $\mathbb{Q}(\sqrt{-n})$. For $\ell = 2$ this speculation is in fact a theorem, since it is known [116] that the Schur indices for M_{24} are always 1. For $\ell = 3$ it is also known [116] that the Schur indices for $\bar{G}^{(3)} = M_{12}$ are also always 1. Moreover, the representations of $G^{(3)} \simeq 2.\bar{G}^{(3)}$ with characters χ_{16} and χ_{17} in the notation of Table 9 have been constructed explicitly over $\mathbb{Q}(\sqrt{-2})$ in [117]. Finally, Proposition 5.8 constitutes a non-trivial consistency check for Conjecture 5.9 since the Schur index is at least 2 for a representation with Frobenius–Schur indicator equal to -1.

ℓ	n	(ϱ, ϱ^*)
2	$7, 15, 23$	$(\chi_3, \chi_4), (\chi_5, \chi_6), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13}), (\chi_{15}, \chi_{16})$
3	$5, 8, 11, 20$	$(\chi_4, \chi_5), (\chi_{16}, \chi_{17}), (\chi_{20}, \chi_{21}), (\chi_{22}, \chi_{23}), (\chi_{25}, \chi_{26})$
4	$3, 7$	$(\chi_2, \chi_3), (\chi_{13}, \chi_{14}), (\chi_{15}, \chi_{16})$
5	4	$(\chi_8, \chi_9), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13})$
7	3	$(\chi_2, \chi_3), (\chi_6, \chi_7)$
13	4	(χ_3, χ_4)

Table 7: The irreducible representations of type n.

Armed with the preceding discussion we are now ready to state our main observation for the discriminant property of umbral moonshine. For the purpose of stating this we temporarily write $K^{(\ell)}_{r,d}$ for the ordinary representation of $G^{(\ell)}$ with character $g \mapsto c^{(\ell)}_{g,r}(d)$ where the coefficients $c^{(\ell)}_{g,r}(d)$ are assumed to be those given in §C.

Proposition 5.10. *Let n be one of the integers in Table 7 and let λ_n be the smallest positive integer such that $D = -n\lambda_n^2$ is a discriminant of $H^{(\ell)}$. Then $K^{(\ell)}_{r,-D/4\ell} = \varrho_n \oplus \varrho_n^*$ where ϱ_n and ϱ_n^* are dual irreducible representations of type n. Conversely, if ϱ is an irreducible representation of type n and $-D$ is the smallest positive integer such that $K^{(\ell)}_{r,-D/4\ell}$ has ϱ as an irreducible constituent then there exists an integer λ such that $D = -n\lambda^2$.*

Extending this we make the following conjecture.

Conjecture 5.11. *If D is a discriminant of $H^{(\ell)}$ which satisfies $D = -n\lambda^2$ for some integer λ then the representation $K^{(\ell)}_{r,-D/4\ell}$ has at least one dual pair of irreducible representations of type n arising as irreducible constituents.*

We conclude this section with conjectures arising from the observation (cf. §D) that the conjectural $G^{(\ell)}$-module $K^{(\ell)}_{r,d}$ is typically isomorphic to several copies of a single representation. Say a G-module V is a *doublet* if it is isomorphic to the direct sum of two copies of a single representation of G.

Conjecture 5.12. *For $\ell \in \Lambda = \{2,3,4,5,7,13\}$ the representation $K^{(\ell)}_{r,-D/4\ell}$ is a doublet if and only if $D \neq -n\lambda^2$ for any integer λ for any n satisfying the conditions of Proposition 5.7.*

To see some evidence for Conjecture 5.12 one can inspect the proposed decompositions of the representations $K^{(\ell)}_{r,d}$ in the tables in §D for the following discriminants:

- $-D = 7, 15, 23, 63, 135, 175, 207$ for $\ell = 2$,
- $-D = 8, 11, 20, 32, 44, 80$ for $\ell = 3$,
- $-D = 7, 12, 28, 63, 108$ for $\ell = 4$,
- $-D = 4, 16, 64, 144, 196$ for $\ell = 5$,
- $-D = 3\lambda^2$, $\lambda = 1, \ldots, 9$, $\lambda \neq 7$ for $\ell = 7$,
- $-D = 4\lambda^2$, $\lambda = 1, \ldots, 11$ for $\ell = 13$.

5.5 Geometry and Physics

Beyond the conjectures already mentioned above several interesting and important questions remain regarding the structural nature of umbral moonshine. In the case that $\ell = 2$ there are strong indications that a deep relationship to $K3$ surfaces—extending in some way the relation [118, 119] between finite groups of $K3$ surface symplectomorphisms and subgroups of M_{23}—is responsible for the relationship between $G^{(2)} \simeq M_{24}$ and $H^{(2)}$. One such indication is the fact that the Jacobi form $Z^{(2)}$, from which $H^{(2)}$ may be obtained by decomposing with respect to superconformal characters, coincides with the elliptic genus of a(ny) complex $K3$ surface. It is natural then to ask if there are analogous geometric interpretations for the remaining extremal Jacobi forms $Z^{(\ell)}$ for $\ell \in \Lambda$, and a positive answer to this question will be a first step in determining the geometric significance of the umbral groups $G^{(\ell)}$ and the attached mock modular forms $H^{(\ell)}_g$.

In a series of papers [120, 121, 88, 87, 89] Gritsenko–Nikulin develop applications of Siegel modular forms to mirror symmetry for $K3$ surfaces (cf. [122]). Many of the Siegel forms arising are realised as additive or exponential lifts of (weak) Jacobi forms, and amongst many examples the exponential lifts $\Phi^{(\ell)}$ of the particular forms $Z^{(\ell)}$ for $\ell \in \{2,3,4,5\} \subset \Lambda$ appear in connection with explicitly defined families of lattice polarised $K3$ surfaces in [87]. It is natural to ask if

there is an analogous relationship between the umbral $Z^{(\ell)}$ and suitably defined mirror families of $K3$ surfaces for the remaining ℓ in Λ.

As mentioned in the introduction, monstrous moonshine involves aspects of conformal field theory and string theory and there are several hints that umbral moonshine will also play a role in string theory. The most obvious hint at $\ell = 2$ is through the fact mentioned above that the weak Jacobi form $Z^{(2)}$ coincides with the elliptic genus of a $K3$ surface and the fact that $K3$ surfaces play a prominent role in the study of superstring compactification. The Siegel form $\Phi^{(2)} = (\Delta_5)^2$ which is the multiplicative lift of the weak Jacobi form $Z^{(2)}$ also plays a distinguished role in type II string theory on $K3 \times E$ where E is an elliptic curve and $1/\Phi^{(2)}$ occurs as the generating function which counts the number of 1/4 BPS dyon states [123]. BPS states in a supersymmetric theory are states that are annihilated by some of the supercharges of the theory; the 1/4 BPS states of interest here are annihilated by 4 of the 16 supercharges of a theory with $N = 4$ supersymmetry in four spacetime dimensions and are termed *dyons* because they necessarily carry both electric and magnetic charges. The relation between the zeros of $\Phi^{(2)}$ and the wall-crossing phenomenon in this theory has been studied in [124, 125, 126, 127] and this relation leads to a connection between the counting of BPS black hole states in string theory and mock modular forms [45]. For a pedagogical review of this material see [128, 38].

There are several hints that umbral moonshine for $\ell > 2$ will also play a role in string theory. The first of these occurs in the study of $N = 2$ dual pairs of string theories [129, 130]. An $N = 2$ dual pair consists of a compactification of the heterotic string on a product of a $K3$ surface with an elliptic curve E with a specific choice of $E_8 \times E_8$ gauge bundle on $K3 \times E$ and a conjectured dual description of this model in terms of IIA string theory on a Calabi–Yau threefold X which admits a $K3$ fibration. The low-energy description of such theories involves a vector-multiplet moduli space which is a special Kähler manifold of the form

$$\mathcal{M}_{vm}^{s+2,2} = \frac{SU(1,1)}{U(1)} \times \mathcal{N}^{s+2,2} \tag{5.8}$$

where $\mathcal{N}^{s+2,2}$ is the quotient $\Gamma\backslash\mathcal{H}^{s+1,1}$ of the generalised upper half-plane

$$\mathcal{H}^{s+1,1} = O(s+2,2;\mathbb{R})/(O(s+2) \times O(2)) \tag{5.9}$$

by an arithmetic subgroup Γ in $SO(s+2,2;\mathbb{R})$, with Γ depending on the specific model in question. One-loop string computations in the heterotic string lead to automorphic forms on $\mathcal{N}^{s+2,2}$ via a generalised theta lift constructed in [131, 132], see [133] for a review. In models with $s = 1$ this leads to automorphic forms on $\Gamma\backslash O(3,2;\mathbb{R})/(O(3) \times O(2))$, that is to Siegel

modular forms on the genus 2 Siegel upper half-space $\mathbb{H}_2 \cong \mathcal{H}^{2,1}$.

A particular example of such an $N = 2$ dual pair was studied in [134, 135] with $s = 1$ and involving a dual description in terms of Type II string theory on a Calabi–Yau 3-fold with Hodge numbers $(h^{1,1}, h^{2,1}) = (4, 148)$. In the heterotic description the arithmetic subgroup which appears is the paramodular group Γ_2, the Siegel modular form which occurs is the exponential lift of $Z^{(3)}$ described in §2.6 and the Calabi–Yau 3-fold is also elliptically fibered and the elliptic fiber is of E_7 type. We thus see many of the ingredients of $\ell = 3$ umbral moonshine appearing in the context of a specific dual pair of string theories with $N = 2$ spacetime supersymmetry. It will be very interesting to investigate whether other aspects of $\ell = 3$ umbral moonshine can be realized in these models and whether $N = 2$ dual pairs exist which exhibit elements of umbral moonshine for $\ell > 3$.

A second way that elements of umbral moonshine for $\ell > 2$ are likely to appear in string theory involves a generalisation of the generating function discussed above that counts 1/4 BPS states. Type II string theory on $K3 \times E$ preserves $N = 4$ spacetime supersymmetry. There exist orbifold versions of this model that also preserve $N = 4$ spacetime supersymmetry that are known in the physics literature as CHL models. To construct such a model one chooses a $K3$ surface with a $\mathbb{Z}/m\mathbb{Z}$ hyper-Kähler automorphism and constructs the orbifold theory $(K3 \times E)/(\mathbb{Z}/m\mathbb{Z})$ where $\mathbb{Z}/m\mathbb{Z}$ is a freely acting symmetry realised as the product of a hyper-Kähler automorphism of the $K3$ surface and an order m translation along E. For $m = 2, 3, 4$ it is possible to find $K3$ surfaces with $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$ symmetry, and in this case one can construct a CHL model that utilises the first $\mathbb{Z}/m\mathbb{Z}$ factor in the orbifold construction and has the second $\mathbb{Z}/m\mathbb{Z}$ factor acting as a symmetry which preserves the holomorphic 3-form of the Calabi–Yau space $(K3 \times E)/(\mathbb{Z}/m\mathbb{Z})$. It was proposed in [136] that the generating function which counts 1/4 BPS states weighted by an element of $\mathbb{Z}/m\mathbb{Z}$ is the reciprocal of the Siegel form $\Phi^{(m+1)}$ for $m \in \{2, 3, 4\}$. In both this construction and in the study of $N = 2$ dual pairs the appearance of some of the umbral Siegel forms $\Phi^{(\ell)}$ can be anticipated, although many details remain to be worked out. The action of the umbral groups $G^{(\ell)}$ remains more elusive, and deeper insight into possible connections between string theory and umbral moonshine will undoubtedly require progress in understanding the actions of these groups in terms of their action on BPS states.

We plan to elaborate further on the topics mentioned above in future work.

Acknowledgements

We thank Chris Cummins, Atish Dabholkar, Noam Elkies, Igor Frenkel, George Glauberman, Ian Grojnowski, Kobi Kremnizer, Anthony Licata, Greg Moore, Sameer Murthy, Ken Ono, Cumrun Vafa, Erik Verlinde, Edward Witten, Shing-Tung Yau and Don Zagier for helpful comments and discussions. We thank Noam Elkies and Ken Ono in particular for communication regarding the non-vanishing of critical central values for modular L-functions.

We are especially grateful to Don Zagier for showing us his computation of the first few Fourier coefficients of the functions (that we have here denoted) $H^{(\ell)}$ for $\ell \in \{2, 3, 4, 5, 7\}$ at the "Workshop on Mathieu Moonshine" held in July 2011 at the Swiss Federal Institute of Technology (ETH), Zürich. This event was an important catalyst for precipitating the idea that there could be a family of groups attached to a distinguished family of vector-valued modular forms, in a way that generalises the connection between M_{24} and the (scalar-valued) mock modular form $H^{(2)}$. We are also grateful to Kazuhiro Hikami for his talk at the ETH meeting on character decompositions of elliptic genera, and for kindly sharing his slides from that presentation. The analysis described therein proved useful for us in developing a computational approach to approximate Fourier coefficients of the $H^{(\ell)}$ and this furnished an important source of experimental evidence to guide our umbral explorations. We warmly thank the organisers of the ETH meeting for making these exchanges possible and for providing an excellent workshop experience more generally.

The research of MC is partially supported by the NSF grant FRG 0854971. The work of JH was supported by the NSF grant 0855039, the hospitality of the Aspen Center for Physics, the LPTHE of the Université Pierre et Marie Curie, and the Isaac Newton Institute for Mathematical Sciences. We also thank George Armhold for instruction on the efficient implementation of Pari in OS X, and Jia-Chen Fu for the provision of computational resources at a crucial stage in our investigations.

We are grateful to George Glauberman, Sander Mack-Crane, and the referee, for pointing out errors in an earlier draft. We are grateful to the referee also for suggesting many valuable improvements to the text.

A Modular Forms

A.1 Dedekind Eta Function

The *Dedekind eta function*, denoted $\eta(\tau)$, is a holomorphic function on the upper half-plane defined by the infinite product

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

where $q = \mathrm{e}(\tau) = e^{2\pi \mathbf{i} \tau}$. It is a modular form of weight 1/2 for the modular group $SL_2(\mathbb{Z})$ with multiplier $\epsilon : SL_2(\mathbb{Z}) \to \mathbb{C}^*$, which means that

$$\epsilon(\gamma)\eta(\gamma\tau)\mathrm{jac}(\gamma, \tau)^{1/4} = \eta(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, where $\mathrm{jac}(\gamma, \tau) = (c\tau + d)^{-2}$. The *multiplier system* ϵ may be described explicitly as

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \mathrm{e}(-b/24), & c = 0,\ d = 1 \\ \mathrm{e}(-(a+d)/24c + s(d,c)/2 + 1/8), & c > 0 \end{cases} \tag{A.1}$$

where $s(d, c) = \sum_{m=1}^{c-1} (d/c)((md/c))$ and $((x))$ is 0 for $x \in \mathbb{Z}$ and $x - \lfloor x \rfloor - 1/2$ otherwise. We can deduce the values $\epsilon(a, b, c, d)$ for $c < 0$, or for $c = 0$ and $d = -1$, by observing that $\epsilon(-\gamma) = \epsilon(\gamma)\,\mathrm{e}(1/4)$ for $\gamma \in SL_2(\mathbb{Z})$.

Let T denote the element of $SL_2(\mathbb{Z})$ such that $\mathrm{tr}(T) = 2$ and $T\tau = \tau + 1$ for $\tau \in \mathbb{H}$. Observe that

$$\epsilon(T^m\gamma) = \epsilon(\gamma T^m) = \mathrm{e}(-m/24)\epsilon(\gamma)$$

for $m \in \mathbb{Z}$.

A.2 Jacobi Theta Functions

We define the *Jacboi theta functions* $\theta_i(\tau, z)$ as follows for $q = e(\tau)$ and $y = e(z)$.

$$\theta_1(\tau, z) = -iq^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^{n-1})$$

$$\theta_2(\tau, z) = q^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1}q^{n-1})$$

$$\theta_3(\tau,z) = \prod_{n=1}^{\infty}(1-q^n)(1+y\,q^{n-1/2})(1+y^{-1}q^{n-1/2})$$

$$\theta_4(\tau,z) = \prod_{n=1}^{\infty}(1-q^n)(1-y\,q^{n-1/2})(1-y^{-1}q^{n-1/2})$$

Note that there are competing conventions for $\theta_1(\tau,z)$ in the literature and our normalisation may differ from another by a factor of -1 (or possibly $\pm i$).

A.3 Higher Level Modular Forms

The congruence subgroups of the modular group $SL_2(\mathbb{Z})$ that are most relevant for this paper are

$$\Gamma_0(N) \quad = \quad \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), c = 0 \bmod N \right\}. \tag{A.2}$$

For $N > 1$ a (non-zero) modular form of weight 2 for $\Gamma_0(N)$ is given by

$$\begin{aligned} \Lambda_N(\tau) &= N\,q\partial_q \log\left(\frac{\eta(N\tau)}{\eta(\tau)}\right) \qquad (A.3) \\ &= \frac{N(N-1)}{24}\left(1+\frac{24}{N-1}\sum_{k>0}\sigma(k)(q^k - Nq^{Nk})\right) \end{aligned}$$

where $\sigma(k)$ is the divisor function $\sigma(k) = \sum_{d|k} d$.

Observe that a modular form on $\Gamma_0(N)$ is a modular form on $\Gamma_0(M)$ whenever $N|M$, and for some small N the space of forms of weight 2 is spanned by the $\Lambda_d(\tau)$ for d a divisor of N. By contrast, in the case that $N = 11$ we have the *newform*

$$f_{11}(\tau) = \eta^2(\tau)\eta^2(11\tau) \tag{A.4}$$

which is a cusp form of weight 2 for $\Gamma_0(11)$ that is not a multiple of $\Lambda_{11}(\tau)$. We meet the newforms

$$f_{14}(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau,) \tag{A.5}$$

$$f_{15}(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau), \tag{A.6}$$

$$f_{20}(\tau) = \eta(2\tau)^2\eta(10\tau)^2, \tag{A.7}$$

at $N = 14$, $N = 15$ and $N = 20$, respectively, and together with f_N the functions $\Lambda_d(\tau)$ for $d|N$

span the space of weight 2 forms on $\Gamma_0(N)$ for $N = 11, 14, 15$.

For $N = 23$ there is a two dimensional space of newforms. We may use the basis

$$\begin{aligned} f_{23,a}(\tau) &= \frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 3\eta(\tau)^2\eta(23\tau)^2 + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta(2\tau)^2\eta(46\tau)^2 \\ f_{23,b}(\tau) &= \eta(\tau)^2\eta(23\tau)^2 \end{aligned} \tag{A.8}$$

satisfying $f_{23,a} = q + O(q^3)$ and $f_{23,b} = q^2 + O(q^3)$. (The normalised Hecke-eigenforms of weight 2 for $\Gamma_0(23)$ are $f_{23,a} - \frac{1}{2}(1 \pm \sqrt{5})f_{23,b}$.)

For $N = 44$ there is a unique newform up to a multiplicative constant. It satisfies

$$f_{44}(\tau) = q + q^3 - 3q^5 + 2q^7 - 2q^9 - q^{11} - 4q^{13} - 3q^{15} + 6q^{17} + 8q^{19} + 2q^{21} - 3q^{23} + 4q^{25} - 5q^{27} + O(q^{28}).$$

See [137] and Chapter 4.D of [138] for more details. A discussion of the ring of weak Jacobi forms of higher level can be found in [139].

B Characters

In §B.1 we give character tables (with power maps and Frobenius–Schur indicators) for each group $G^{(\ell)}$. These were computed with the aid of the computer algebra package GAP4 [98] using the explicit presentations for the $G^{(\ell)}$ that appear in §3.4. We use the abbreviations $a_n = \sqrt{-n}$ and $b_n = (-1 + \sqrt{-n})/2$ in these tables.

The tables in §B.2 furnish the Frame shapes $\Pi_g^{(\ell)}$ and character values $\chi_g^{(\ell)}$ attached to the signed permutation representations (cf. §3.2) of the groups $G^{(\ell)}$ given in §3.4. These Frame shapes and character values can easily be computed by hand; we detail them here explicitly since they can be used to define symbols $n_g|h_g$ which encode the automorphy of the vector-valued mock modular forms $H_g^{(\ell)}$ according to the prescription of §4.8. The symbols $n_g|h_g$ are given in the rows labelled Γ_g in §B.2.

B.1 Irreducible Characters

Table 8: Character table of $G^{(2)} \simeq M_{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
χ_3	∘	45	-3	5	0	3	-3	1	1	0	0	-1	b_7	$\overline{b_7}$	-1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	-1	-1
χ_4	∘	45	-3	5	0	3	-3	1	1	0	0	-1	$\overline{b_7}$	b_7	-1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	-1	-1
χ_5	∘	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	∘	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	+	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
χ_9	+	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	∘	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	∘	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	∘	990	-18	-10	0	3	6	2	-2	0	0	-1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	∘	990	-18	-10	0	3	6	2	-2	0	0	-1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	+	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
χ_{15}	∘	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2b_7$	$2\overline{b_7}$	-1	0	1	0	-1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	∘	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\overline{b_7}$	$2b_7$	-1	0	1	0	-1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	+	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0
χ_{18}	+	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0
χ_{19}	+	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	+	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
χ_{21}	+	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0
χ_{22}	+	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1
χ_{23}	+	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	+	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
χ_{25}	+	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0
χ_{26}	+	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

Table 9: Character table of $G^{(3)} \simeq 2.M_{12}$

$[g]$	FS	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^2]$		1A	1A	2A	1A	1A	3A	3A	3B	3B	2B	2B	5A	5A	6B	3A	3A	4B	4B	4C	4C	10A	10A	11B	11B	11A	11A
$[g^3]$		1A	2A	4A	2B	2C	1A	2A	1A	2A	4B	4C	5A	10A	4A	2B	2C	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^5]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	1A	2A	12A	6C	6D	8B	8A	8D	8C	4A	4A	11A	22A	11B	22B
$[g^{11}]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20B	20A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	11	11	-1	3	3	2	2	-1	-1	-1	3	1	1	-1	0	0	-1	-1	1	1	-1	-1	0	0	0	0
χ_3	$+$	11	11	-1	3	3	2	2	-1	-1	3	-1	1	1	-1	0	0	1	1	-1	-1	-1	-1	0	0	0	0
χ_4	$\circ$	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	b_{11}	b_{11}	$\overline{b_{11}}$	$\overline{b_{11}}$
χ_5	$\circ$	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	$\overline{b_{11}}$	$\overline{b_{11}}$	b_{11}	b_{11}
χ_6	$+$	45	45	5	-3	-3	0	0	3	3	1	1	0	0	-1	0	0	-1	-1	-1	-1	0	0	1	1	1	1
χ_7	$+$	54	54	6	6	6	0	0	0	0	2	2	-1	-1	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
χ_8	$+$	55	55	-5	7	7	1	1	1	1	-1	-1	0	0	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
χ_9	$+$	55	55	-5	-1	-1	1	1	1	1	3	-1	0	0	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	$+$	55	55	-5	-1	-1	1	1	1	1	-1	3	0	0	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
χ_{11}	$+$	66	66	6	2	2	3	3	0	0	-2	-2	1	1	0	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{12}	$+$	99	99	-1	3	3	0	0	3	3	-1	-1	-1	-1	-1	0	0	1	1	1	1	-1	-1	0	0	0	0
χ_{13}	$+$	120	120	0	-8	-8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	-1	-1	-1
χ_{14}	$+$	144	144	4	0	0	0	0	-3	-3	0	0	-1	-1	1	0	0	0	0	0	0	-1	-1	1	1	1	1
χ_{15}	$+$	176	176	-4	0	0	-4	-4	-1	-1	0	0	1	1	-1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	$\circ$	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	a_2	$\overline{a_2}$	a_2	$\overline{a_2}$	0	0	-1	1	-1	1
χ_{17}	$\circ$	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	$\overline{a_2}$	a_2	$\overline{a_2}$	a_2	0	0	-1	1	-1	1
χ_{18}	$+$	12	-12	0	4	-4	3	-3	0	0	0	0	2	-2	0	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{19}	$-$	32	-32	0	0	0	-4	4	2	-2	0	0	2	-2	0	0	0	0	0	0	0	0	0	-1	1	-1	1
χ_{20}	$\circ$	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	a_5	$\overline{a_5}$	0	0	0	0
χ_{21}	$\circ$	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	$\overline{a_5}$	a_5	0	0	0	0
χ_{22}	$\circ$	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	a_2	$\overline{a_2}$	$\overline{a_2}$	a_2	0	0	0	0	0	0
χ_{23}	$\circ$	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	$\overline{a_2}$	a_2	a_2	$\overline{a_2}$	0	0	0	0	0	0
χ_{24}	$+$	120	-120	0	8	-8	3	-3	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	-1	1	-1	1
χ_{25}	$\circ$	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-b_{11}$	b_{11}	$-\overline{b_{11}}$	$\overline{b_{11}}$
χ_{26}	$\circ$	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\overline{b_{11}}$	$\overline{b_{11}}$	$-b_{11}$	b_{11}

Table 10: Character table of $G^{(4)} \simeq 2.AGL_3(2)$

$[g]$	FS	1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	7A	14A	7B	14B
$[g^2]$		1A	1A	1A	2A	2B	1A	3A	3A	3A	3A	4A	2C	7A	7A	7B	7B
$[g^3]$		1A	2A	2B	4A	4B	2C	1A	2A	2B	2B	8A	4C	7B	14B	7A	14A
$[g^7]$		1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	b_7	b_7	$\overline{b_7}$	$\overline{b_7}$
χ_3	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	$\overline{b_7}$	$\overline{b_7}$	b_7	b_7
χ_4	+	6	6	6	2	2	2	0	0	0	0	0	0	−1	−1	−1	−1
χ_5	+	7	7	7	−1	−1	−1	1	1	1	1	−1	−1	0	0	0	0
χ_6	+	8	8	8	0	0	0	−1	−1	−1	−1	0	0	1	1	1	1
χ_7	+	7	7	−1	3	−1	−1	1	1	−1	−1	1	−1	0	0	0	0
χ_8	+	7	7	−1	−1	−1	3	1	1	−1	−1	−1	1	0	0	0	0
χ_9	+	14	14	−2	2	−2	2	−1	−1	1	1	0	0	0	0	0	0
χ_{10}	+	21	21	−3	1	1	−3	0	0	0	0	−1	1	0	0	0	0
χ_{11}	+	21	21	−3	−3	1	1	0	0	0	0	1	−1	0	0	0	0
χ_{12}	+	8	−8	0	0	0	0	2	−2	0	0	0	0	1	−1	1	−1
χ_{13}	∘	8	−8	0	0	0	0	−1	1	a_3	$\overline{a_3}$	0	0	1	−1	1	−1
χ_{14}	∘	8	−8	0	0	0	0	−1	1	$\overline{a_3}$	a_3	0	0	1	−1	1	−1
χ_{15}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	$\overline{b_7}$	$-\overline{b_7}$	b_7	$-b_7$
χ_{16}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	b_7	$-b_7$	$\overline{b_7}$	$-\overline{b_7}$

Table 11: Character table of $G^{(5)} \simeq GL_2(5)/2$

$[g]$	FS	1A	2A	2B	2C	3A	6A	5A	10A	4A	4B	4C	4D	12A	12B
$[g^2]$		1A	1A	1A	1A	3A	3A	5A	5A	2A	2A	2C	2C	6A	6A
$[g^3]$		1A	2A	2B	2C	1A	2A	5A	10A	4B	4A	4D	4C	4B	4A
$[g^5]$		1A	2A	2B	2C	3A	6A	1A	2A	4A	4B	4C	4D	12A	12B
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_3	+	4	4	0	0	1	1	−1	−1	2	2	0	0	−1	−1
χ_4	+	4	4	0	0	1	1	−1	−1	−2	−2	0	0	1	1
χ_5	+	5	5	1	1	−1	−1	0	0	1	1	−1	−1	1	1
χ_6	+	5	5	1	1	−1	−1	0	0	−1	−1	1	1	−1	−1
χ_7	+	6	6	−2	−2	0	0	1	1	0	0	0	0	0	0
χ_8	∘	1	−1	1	−1	1	−1	1	−1	a_1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$
χ_9	∘	1	−1	1	−1	1	−1	1	−1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$	a_1
χ_{10}	∘	4	−4	0	0	1	−1	−1	1	$2a_1$	$-2a_1$	0	0	$-a_1$	a_1
χ_{11}	∘	4	−4	0	0	1	−1	−1	1	$-2a_1$	$2a_1$	0	0	a_1	$-a_1$
χ_{12}	∘	5	−5	1	−1	−1	1	0	0	a_1	$-a_1$	$-a_1$	a_1	a_1	$-a_1$
χ_{13}	∘	5	−5	1	−1	−1	1	0	0	$-a_1$	a_1	a_1	$-a_1$	$-a_1$	a_1
χ_{14}	+	6	−6	−2	2	0	0	1	−1	0	0	0	0	0	0

Table 12: Character table of $G^{(7)} \simeq SL_2(3)$

$[g]$	FS	1A	2A	4A	3A	6A	3B	6B
$[g^2]$		1A	1A	2A	3B	3A	3A	3B
$[g^3]$		1A	2A	4A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1
χ_2	$\circ$	1	1	1	b_3	$\overline{b_3}$	$\overline{b_3}$	b_3
χ_3	$\circ$	1	1	1	$\overline{b_3}$	b_3	b_3	$\overline{b_3}$
χ_4	$+$	3	3	-1	0	0	0	0
χ_5	$-$	2	-2	0	-1	1	-1	1
χ_6	$\circ$	2	-2	0	$-\overline{b_3}$	b_3	$-b_3$	$\overline{b_3}$
χ_7	$\circ$	2	-2	0	$-b_3$	$\overline{b_3}$	$-\overline{b_3}$	b_3

Table 13: Character table of $G^{(13)} \simeq 4$

$[g]$	FS	1A	2A	4A	4B
$[g^2]$		1A	1A	2A	2A
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	$\circ$	1	-1	a_1	$\overline{a_1}$
χ_4	$\circ$	1	-1	$\overline{a_1}$	a_1

B.2 Euler Characters

The tables in this section describe the Frame shapes $\Pi_g^{(\ell)}$ and twisted Euler characters $\chi_g^{(\ell)}$ attached to each group $G^{(\ell)}$ via the signed permutation representations given in §3.4. The rows labelled $\bar{\Pi}_g^{(\ell)}$ and $\bar{\chi}_g^{(\ell)}$ describe the corresponding data for the (unsigned) permutation representations. According to the discussion of §4.8 the Frame shapes $\Pi_g^{(\ell)}$ and $\bar{\Pi}_g^{(\ell)}$ (or even just the $\Pi_g^{(\ell)}$) can be used to define symbols $n_g|h_g$ which encode the automorphy of the vector-valued mock modular form $H_g^{(\ell)}$; these symbols are given in the rows labelled Γ_g. We write n_g here as a shorthand for $n_g|1$.

Table 14: Twisted Euler characters and Frame shapes at $\ell = 2$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
Γ_g	1	2	2|2	3	3|3	4|2	4	4|4	5	6	6|6
$\chi_g^{(2)}$	24	8	0	6	0	0	4	0	4	2	0
$\Pi_g^{(2)}$	1^{24}	$1^8 2^8$	2^{12}	$1^6 3^6$	3^8	$2^4 4^4$	$1^4 2^2 4^4$	4^6	$1^4 5^4$	$1^2 2^2 3^2 6^2$	6^4

$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
Γ_g	7	8	10|2	11	12|2	12|12	14	15	21|3	23
$\chi_g^{(2)}$	3	2	0	2	0	0	1	1	0	1
$\Pi_g^{(2)}$	$1^3 7^3$	$1^2 2^1 4^1 8^2$	$2^2 10^2$	$1^2 11^2$	$2^1 4^1 6^1 12^1$	12^2	$1^1 2^1 7^1 14^1$	$1^1 3^1 5^1 15^1$	$3^1 21^1$	$1^1 23^1$

We have $\chi_g^{(2)} = \chi_1(g) + \chi_2(g)$ in the notation of Table 8.

Table 15: Twisted Euler characters and Frame shapes at $\ell = 3$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
Γ_g	1	1|4	2|8	2	2|2	3	3|4	3|3	3|12	4|2	4	5	5|4	6|24	6	6|2	8|4	8	10|8	11	11|4
$\bar{\chi}_g^{(3)}$	12	12	0	4	4	3	3	0	0	0	4	2	2	0	1	1	0	2	0	1	1
$\chi_g^{(3)}$	12	-12	0	4	-4	3	-3	0	0	0	0	2	-2	0	1	-1	0	0	0	1	-1
$\bar{\Pi}_g^{(3)}$	1^{12}	1^{12}	2^6	$1^4 2^4$	$1^4 2^4$	$1^3 3^3$	$1^3 3^3$	3^4	3^4	$2^2 4^2$	$1^4 4^2$	$1^2 5^2$	$1^2 5^2$	6^2	$1^1 2^1 3^1 6^1$	$1^1 2^1 3^1 6^1$	$4^1 8^1$	$1^2 2^1 8^1$	$2^1 10^1$	$1^1 11^1$	$1^1 11^1$
$\Pi_g^{(3)}$	1^{12}	$\frac{2^{12}}{1^{12}}$	$\frac{4^6}{2^6}$	$1^4 2^4$	$\frac{2^8}{1^4}$	$1^3 3^3$	$\frac{2^3 6^3}{1^3 3^3}$	3^4	$\frac{6^4}{3^4}$	$2^2 4^2$	$2^2 4^2$	$1^2 5^2$	$\frac{2^2 10^2}{1^2 5^2}$	$\frac{12^2}{6^2}$	$1^1 2^1 3^1 6^1$	$\frac{2^2 6^2}{1^1 3^1}$	$4^1 8^1$	$4^1 8^1$	$\frac{4^1 20^1}{2^1 10^1}$	$1^1 11^1$	$\frac{2^1 22^1}{1^1 11^1}$

We have $\chi_g^{(3)} = \chi_{18}(g)$ and $\bar{\chi}_g^{(3)} = \chi_1(g) + \chi_2(g)$ in the notation of Table 9.

Table 16: Twisted Euler characters and Frame shapes at $\ell = 4$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
Γ_g	1	1|2	2|2	2|4	4|4	2	3	3|2	6|2	4|8	4	7	7|2
$\bar{\chi}_g^{(4)}$	8	8	0	0	0	4	2	2	0	0	2	1	1
$\chi_g^{(4)}$	8	-8	0	0	0	0	2	-2	0	0	0	1	-1
$\bar{\Pi}_g^{(4)}$	1^8	1^8	2^4	2^4	4^2	$1^4 2^2$	$1^2 3^2$	$1^2 3^2$	$2^1 6^1$	4^2	$1^2 2^1 4^1$	$1^1 7^1$	$1^1 7^1$
$\Pi_g^{(4)}$	1^8	$\frac{2^8}{1^8}$	2^4	$\frac{4^4}{2^4}$	4^2	2^4	$1^2 3^2$	$\frac{2^2 6^2}{1^2 3^2}$	$2^1 6^1$	$\frac{8^2}{4^2}$	4^2	$1^1 7^1$	$\frac{2^1 14^1}{1^1 7^1}$

We have $\chi_g^{(4)} = \chi_{12}$ and $\bar{\chi}_g^{(4)} = \chi_1(g) + \chi_8(g)$ in the notation of Table 10.

Table 17: Twisted Euler characters and Frame shapes at $\ell = 5$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
$\bar{\chi}_g^{(5)}$	6	6	2	2	0	0	1	1	0	2	0
$\chi_g^{(5)}$	6	-6	-2	2	0	0	1	-1	0	0	0
$\bar{\Pi}_g^{(5)}$	1^6	1^6	$1^2 2^2$	$1^2 2^2$	3^2	3^2	$1^1 5^1$	$1^1 5^1$	2^3	$1^2 4^1$	6^1
$\Pi_g^{(5)}$	1^6	$\frac{2^6}{1^6}$	$\frac{2^4}{1^2}$	$1^2 2^2$	3^2	$\frac{6^2}{3^2}$	$1^1 5^1$	$\frac{2^1 10^1}{1^1 5^1}$	$\frac{4^3}{2^3}$	$2^1 4^1$	$\frac{12^1}{6^1}$

We have $\chi_g^{(5)} = \chi_{14}(g)$ and $\bar{\chi}_g^{(5)} = \chi_1(g) + \chi_6(g)$ in the notation of Table 11.

Table 18: Twisted Euler characters and Frame shapes at $\ell = 7$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
$\bar{\chi}_g^{(7)}$	4	4	0	1	1
$\chi_g^{(7)}$	4	-4	0	1	-1
$\bar{\Pi}_g^{(7)}$	1^4	1^4	2^2	$1^1 3^1$	$1^1 3^1$
$\Pi_g^{(7)}$	1^4	$\frac{2^4}{1^4}$	$\frac{4^2}{2^2}$	$1^1 3^1$	$\frac{2^1 6^1}{1^1 3^1}$

We have $\chi_g^{(7)} = \chi_6(g) + \chi_7(g)$ in the notation of Table 12.

Table 19: Twisted Euler characters and Frame shapes at $\ell = 13$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
$\bar{\chi}_g^{(13)}$	2	2	0
$\chi_g^{(13)}$	2	-2	0
$\bar{\Pi}_g^{(13)}$	1^2	1^2	2^1
$\Pi_g^{(13)}$	1^2	$\frac{2^2}{1^2}$	$\frac{4^1}{2^1}$

We have $\chi_g^{(13)} = \chi_3(g) + \chi_4(g)$ in the notation of Table 13.

C Coefficients

In this section we furnish tables of Fourier coefficients of small degree for the vector-valued mock modular forms $H_g^{(\ell)}$ that we attach to the conjugacy classes of the groups $G^{(\ell)}$ for $\ell \in \Lambda$. For each ℓ and $0 < r < \ell$ we give a table that displays the coefficients of $H_{g,r}^{(\ell)}$ for (g ranging over a set of representatives for) each conjugacy class $[g]$ in $G^{(\ell)}$. The first row of each table labels the conjugacy classes, and the first column labels exponents of q (or rather $q^{1/4\ell}$), so that for the table captioned $H_{g,r}^{(\ell)}$ (for some $\ell \in \Lambda$ and $0 < r < \ell$) the entry in the row labelled d and the column labelled nZ is the coefficient of $q^{d/4\ell}$ in the Fourier expansion of $H_{g,r}^{(\ell)}$ for $[g] = nZ$. Occasionally the functions $H_g^{(\ell)}$ and $H_{g'}^{(\ell)}$ coincide for non-conjugate g and g' and when this happens we condense information into a single column, writing $7AB$ in Table 20, for example, to indicate that the entries in that column are Fourier coefficients for both $H_{7A}^{(2)}$ and $H_{7B}^{(2)}$.

C.1 Lambency Two

Table 20: McKay–Thompson series $H^{(2)}_{g,1}$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
Γ_g	1	2	2|2	3	3|3	4|2	4	4|4	5	6	6|6	7	8	10|2	11	12|2	12|12	14	15	21|3	23
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
7	90	-6	10	0	6	-6	2	2	0	0	-2	-1	-2	0	2	0	2	1	0	-1	-2
15	462	14	-18	-6	0	-2	-2	6	2	2	0	0	-2	2	0	-2	0	0	-1	0	2
23	1540	-28	20	10	-14	4	-4	-4	0	2	2	0	0	0	0	-2	2	0	0	0	-1
31	4554	42	-38	0	12	-6	2	-6	-6	0	4	4	-2	2	0	0	0	0	0	-2	0
39	11592	-56	72	-18	0	-8	8	0	2	-2	0	0	0	2	-2	-2	0	0	2	0	0
47	27830	86	-90	20	-16	6	-2	6	0	-4	0	-2	2	0	0	0	0	2	0	-2	0
55	61686	-138	118	0	30	6	-10	-2	6	0	-2	2	-2	-2	-2	0	-2	2	0	2	0
63	131100	188	-180	-30	0	-4	4	-12	0	2	0	-3	0	0	2	2	0	-1	0	0	0
71	265650	-238	258	42	-42	-14	10	10	-10	2	6	0	-2	-2	0	-2	-2	0	2	0	0
79	521136	336	-352	0	42	0	-8	16	6	0	2	0	-4	-2	0	0	-2	0	0	0	2
87	988770	-478	450	-60	0	18	-14	-6	0	-4	0	6	2	0	2	0	0	-2	0	0	0
95	1830248	616	-600	62	-70	-8	8	-16	8	-2	-6	0	0	0	2	-2	2	0	2	0	0
103	3303630	-786	830	0	84	-18	22	6	0	0	-4	-6	2	0	0	0	0	-2	0	0	2
111	5844762	1050	-1062	-90	0	10	-6	18	-18	6	0	0	2	-2	0	-2	0	0	0	0	2
119	10139734	-1386	1334	118	-110	22	-26	-10	4	6	2	-4	-2	4	0	-2	2	0	-2	2	0
127	17301060	1764	-1740	0	126	-12	12	-28	0	0	6	0	0	0	-4	0	2	0	0	0	0
135	29051484	-2212	2268	-156	0	-36	28	12	14	-4	0	0	-4	-2	0	0	0	0	-1	0	0
143	48106430	2814	-2850	170	-166	14	-18	38	0	-6	-6	8	-2	0	-2	2	2	0	0	2	-2
151	78599556	-3612	3540	0	210	36	-36	-20	-24	0	-6	0	0	0	2	0	-2	0	0	0	0
159	126894174	4510	-4482	-228	0	-18	14	-42	14	4	0	-6	-2	-2	0	0	0	2	2	0	0
167	202537080	-5544	5640	270	-282	-40	48	16	0	6	6	4	4	0	-2	2	-2	0	0	-2	0
175	319927608	6936	-6968	0	300	24	-16	48	18	0	4	-7	4	2	0	0	0	-1	0	-1	0
183	500376870	-8666	8550	-360	0	54	-58	-18	0	-8	0	0	-2	0	4	0	0	0	0	0	2
191	775492564	10612	-10556	400	-392	-28	28	-60	-36	-8	-8	0	0	4	0	-4	0	0	0	0	0
199	1191453912	-12936	13064	0	462	-72	64	32	12	0	-10	12	-4	4	0	0	2	0	0	0	0
207	1815754710	15862	-15930	-510	0	22	-34	78	0	10	0	0	-6	0	0	-2	0	0	0	0	-1
215	2745870180	-19420	19268	600	-600	84	-76	-36	30	8	8	-10	4	-2	-2	0	0	-2	0	2	0
223	4122417420	23532	-23460	0	660	-36	36	-84	0	0	12	2	0	0	0	0	0	-2	0	2	0
231	6146311620	-28348	28548	-762	0	-92	100	36	-50	-10	0	-6	4	-2	-2	-2	0	2	-2	0	0
239	9104078592	34272	-34352	828	-840	48	-40	96	22	-12	-8	0	4	-2	4	0	0	0	-2	0	0
247	13401053820	-41412	41180	0	966	108	-116	-44	0	0	-10	0	-4	0	0	0	-2	0	0	0	-2

C.2 Lambency Three

Table 21: McKay–Thompson series $H^{(3)}_{g,1}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
Γ_g	1	1|4	2|8	2	2|2	3	3|4	3|3	3|12	4|2	4	5	5|4	6|24	6	6|2	8|4	8	10|8	11	11|4
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
11	32	32	8	0	0	-4	-4	2	2	0	0	2	2	2	0	0	0	0	-2	-1	-1
23	110	110	-10	-2	-2	2	2	2	2	6	-2	0	0	2	-2	-2	-2	2	0	0	0
35	288	288	8	0	0	0	0	-6	-6	0	0	-2	-2	2	0	0	0	0	-2	2	2
47	660	660	-20	4	4	-6	-6	6	6	-4	4	0	0	-2	-2	-2	4	0	0	0	0
59	1408	1408	32	0	0	4	4	4	4	0	0	-2	-2	-4	0	0	0	0	2	0	0
71	2794	2794	-30	-6	-6	4	4	-8	-8	2	-6	4	4	0	0	0	2	-2	0	0	0
83	5280	5280	40	0	0	-12	-12	6	6	0	0	0	0	-2	0	0	0	0	0	0	0
95	9638	9638	-58	6	6	8	8	2	2	-10	6	-2	-2	2	0	0	-2	-2	2	2	2
107	16960	16960	80	0	0	4	4	-14	-14	0	0	0	0	2	0	0	0	0	0	-2	-2
119	29018	29018	-102	-6	-6	-16	-16	8	8	10	-6	-2	-2	0	0	0	2	2	-2	0	0
131	48576	48576	112	0	0	12	12	6	6	0	0	6	6	-2	0	0	0	0	2	0	0
143	79530	79530	-150	10	10	6	6	-24	-24	-6	10	0	0	0	-2	-2	2	2	0	0	0
155	127776	127776	200	0	0	-24	-24	18	18	0	0	-4	-4	2	0	0	0	0	0	0	0
167	202050	202050	-230	-14	-14	18	18	12	12	10	-14	0	0	4	-2	-2	2	-2	0	2	2
179	314688	314688	272	0	0	12	12	-30	-30	0	0	-2	-2	2	0	0	0	0	2	0	0
191	483516	483516	-348	12	12	-36	-36	24	24	-12	12	6	6	0	0	0	-4	0	2	0	0
203	733920	733920	440	0	0	24	24	12	12	0	0	0	0	-4	0	0	0	0	0	0	0
215	1101364	1101364	-508	-12	-12	16	16	-44	-44	20	-12	-6	-6	-4	0	0	-4	4	2	0	0
227	1635680	1635680	600	0	0	-52	-52	32	32	0	0	0	0	0	0	0	0	0	0	2	2
239	2406116	2406116	-740	20	20	38	38	20	20	-20	20	-4	-4	4	2	2	4	0	0	-2	-2
251	3507680	3507680	888	0	0	20	20	-64	-64	0	0	10	10	0	0	0	0	0	-2	0	0
263	5071000	5071000	-1040	-24	-24	-68	-68	46	46	16	-24	0	0	-2	0	0	0	-4	0	0	0
275	7274464	7274464	1208	0	0	52	52	28	28	0	0	-6	-6	-4	0	0	0	0	-2	-1	-1
287	10359030	10359030	-1450	22	22	30	30	-84	-84	-26	22	0	0	-4	-2	-2	-2	-2	0	0	0
299	14650176	14650176	1744	0	0	-96	-96	60	60	0	0	-4	-4	4	0	0	0	0	4	2	2
311	20585334	20585334	-2018	-26	-26	66	66	36	36	30	-26	14	14	4	-2	-2	-2	2	2	0	0
323	28747840	28747840	2320	0	0	40	40	-116	-116	0	0	0	0	4	0	0	0	0	0	0	0
335	39914402	39914402	-2750	34	34	-130	-130	86	86	-30	34	-8	-8	-2	-2	-2	2	2	0	0	0
347	55114400	55114400	3240	0	0	92	92	50	50	0	0	0	0	-6	0	0	0	0	0	0	0
359	75704904	75704904	-3712	-40	-40	54	54	-156	-156	32	-40	-6	-6	-4	2	2	0	-4	-2	0	0

Table 22: McKay–Thompson series $H^{(3)}_{g,2}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
Γ_g	1	1|4	2|8	2	2|2	3	3|4	3|3	3|12	4|2	4	5	5|4	6|24	6	6|2	8|4	8	10|8	11	11|4
8	20	-20	0	-4	4	2	-2	-4	4	0	0	0	0	0	2	-2	0	0	0	-2	2
20	88	-88	0	8	-8	-2	2	4	-4	0	0	-2	2	0	2	-2	0	0	0	0	0
32	220	-220	0	-12	12	4	-4	4	-4	0	0	0	0	0	0	0	0	0	0	0	0
44	560	-560	0	16	-16	2	-2	-4	4	0	0	0	0	0	-2	2	0	0	0	-1	1
56	1144	-1144	0	-24	24	-8	8	4	-4	0	0	4	-4	0	0	0	0	0	0	0	0
68	2400	-2400	0	32	-32	6	-6	0	0	0	0	0	0	0	2	-2	0	0	0	2	-2
80	4488	-4488	0	-40	40	6	-6	-12	12	0	0	-2	2	0	2	-2	0	0	0	0	0
92	8360	-8360	0	56	-56	-10	10	8	-8	0	0	0	0	0	2	-2	0	0	0	0	0
104	14696	-14696	0	-72	72	8	-8	8	-8	0	0	-4	4	0	0	0	0	0	0	0	0
116	25544	-25544	0	88	-88	2	-2	-16	16	0	0	4	-4	0	-2	2	0	0	0	2	-2
128	42660	-42660	0	-116	116	-18	18	12	-12	0	0	0	0	0	-2	2	0	0	0	2	-2
140	70576	-70576	0	144	-144	16	-16	4	-4	0	0	-4	4	0	0	0	0	0	0	0	0
152	113520	-113520	0	-176	176	12	-12	-24	24	0	0	0	0	0	4	-4	0	0	0	0	0
164	180640	-180640	0	224	-224	-26	26	16	-16	0	0	0	0	0	2	-2	0	0	0	-2	2
176	281808	-281808	0	-272	272	18	-18	12	-12	0	0	8	-8	0	-2	2	0	0	0	-1	1
188	435160	-435160	0	328	-328	10	-10	-32	32	0	0	0	0	0	-2	2	0	0	0	0	0
200	661476	-661476	0	-404	404	-42	42	24	-24	0	0	-4	4	0	-2	2	0	0	0	2	-2
212	996600	-996600	0	488	-488	30	-30	12	-12	0	0	0	0	0	2	-2	0	0	0	0	0
224	1482536	-1482536	0	-584	584	20	-20	-52	52	0	0	-4	4	0	4	-4	0	0	0	0	0
236	2187328	-2187328	0	704	-704	-50	50	40	-40	0	0	8	-8	0	2	-2	0	0	0	0	0
248	3193960	-3193960	0	-840	840	40	-40	28	-28	0	0	0	0	0	0	0	0	0	0	0	0
260	4629152	-4629152	0	992	-992	20	-20	-64	64	0	0	-8	8	0	-4	4	0	0	0	0	0
272	6650400	-6650400	0	-1184	1184	-78	78	48	-48	0	0	0	0	0	-2	2	0	0	0	-2	2
284	9490536	-9490536	0	1400	-1400	54	-54	24	-24	0	0	-4	4	0	2	-2	0	0	0	0	0
296	13441032	-13441032	0	-1640	1640	36	-36	-96	96	0	0	12	-12	0	4	-4	0	0	0	0	0
308	18920240	-18920240	0	1936	-1936	-100	100	68	-68	0	0	0	0	0	4	-4	0	0	0	-2	2
320	26457464	-26457464	0	-2264	2264	74	-74	44	-44	0	0	-6	6	0	-2	2	0	0	0	0	0
332	36792560	-36792560	0	2640	-2640	38	-38	-124	124	0	0	0	0	0	-6	6	0	0	0	2	-2
344	50865232	-50865232	0	-3088	3088	-140	140	88	-88	0	0	-8	8	0	-4	4	0	0	0	0	0
356	69966336	-69966336	0	3584	-3584	102	-102	48	-48	0	0	16	-16	0	2	-2	0	0	0	0	0

C.3 Lambency Four

Table 23: McKay–Thompson series $H^{(4)}_{g,1}$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
Γ_g	1	1|2	2|2	2|4	4|4	2	3	3|2	6|2	4|8	4	7	7|2
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
15	14	14	-2	6	-2	-2	2	2	-2	2	-2	0	0
31	42	42	-6	-6	2	2	0	0	0	2	-2	0	0
47	86	86	6	6	-2	-2	-4	-4	0	-2	2	2	2
63	188	188	-4	-12	-4	4	2	2	2	0	0	-1	-1
79	336	336	0	16	0	-8	0	0	0	0	-4	0	0
95	616	616	-8	-16	0	8	-2	-2	-2	4	0	0	0
111	1050	1050	10	18	2	-6	6	6	-2	-2	2	0	0
127	1764	1764	-12	-28	-4	12	0	0	0	-4	0	0	0
143	2814	2814	14	38	-2	-18	-6	-6	2	2	-2	0	0
159	4510	4510	-18	-42	6	14	4	4	0	2	-2	2	2
175	6936	6936	24	48	0	-16	0	0	0	-4	4	-1	-1
191	10612	10612	-28	-60	-4	28	-8	-8	-4	-4	0	0	0
207	15862	15862	22	78	-2	-34	10	10	-2	2	-6	0	0
223	23532	23532	-36	-84	4	36	0	0	0	4	0	-2	-2
239	34272	34272	48	96	0	-40	-12	-12	0	0	4	0	0
255	49618	49618	-46	-126	-6	50	10	10	2	-6	2	2	2
271	70758	70758	54	150	-2	-66	0	0	0	6	-6	2	2
287	100310	100310	-74	-170	6	70	-10	-10	-2	6	-2	0	0
303	140616	140616	88	192	0	-72	18	18	-2	-4	8	0	0
319	195888	195888	-96	-232	-8	96	0	0	0	-4	0	0	0
335	270296	270296	104	272	0	-120	-22	-22	2	4	-8	-2	-2
351	371070	371070	-130	-306	6	126	18	18	2	6	-2	0	0
367	505260	505260	156	348	4	-140	0	0	0	-4	8	0	0
383	684518	684518	-170	-410	-10	174	-22	-22	-2	-10	2	2	2
399	921142	921142	182	486	-2	-202	28	28	-4	6	-10	-2	-2
415	1233708	1233708	-228	-540	12	220	0	0	0	8	-4	0	0
431	1642592	1642592	272	608	0	-248	-34	-34	2	-8	12	0	0
447	2177684	2177684	-284	-708	-12	292	32	32	4	-8	4	-2	-2
463	2871918	2871918	318	814	-2	-346	0	0	0	6	-14	0	0
479	3772468	3772468	-380	-908	12	380	-38	-38	-2	12	0	0	0
495	4932580	4932580	436	1020	4	-412	46	46	-2	-8	12	2	2
511	6425466	6425466	-486	-1174	-14	490	0	0	0	-14	2	-2	-2
527	8335418	8335418	538	1338	-6	-566	-52	-52	4	10	-14	0	0
543	10776290	10776290	-622	-1494	18	610	50	50	2	14	-6	0	0
559	13879290	13879290	714	1666	2	-678	0	0	0	-10	18	-2	-2
575	17818766	17818766	-786	-1898	-18	790	-58	-58	-6	-14	2	0	0
591	22798188	22798188	860	2148	-4	-900	72	72	-4	8	-20	0	0

Table 24: McKay–Thompson series $H^{(4)}_{g,2}$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
Γ_g	1	1|2	2|2	2|4	4|4	2	3	3|2	6|2	4|8	4	7	7|2
12	16	-16	0	0	0	0	-2	2	0	0	0	2	-2
28	48	-48	0	0	0	0	0	0	0	0	0	-1	1
44	112	-112	0	0	0	0	4	-4	0	0	0	0	0
60	224	-224	0	0	0	0	-4	4	0	0	0	0	0
76	432	-432	0	0	0	0	0	0	0	0	0	-2	2
92	784	-784	0	0	0	0	4	-4	0	0	0	0	0
108	1344	-1344	0	0	0	0	-6	6	0	0	0	0	0
124	2256	-2256	0	0	0	0	0	0	0	0	0	2	-2
140	3680	-3680	0	0	0	0	8	-8	0	0	0	-2	2
156	5824	-5824	0	0	0	0	-8	8	0	0	0	0	0
172	9072	-9072	0	0	0	0	0	0	0	0	0	0	0
188	13872	-13872	0	0	0	0	12	-12	0	0	0	-2	2
204	20832	-20832	0	0	0	0	-12	12	0	0	0	0	0
220	30912	-30912	0	0	0	0	0	0	0	0	0	0	0
236	45264	-45264	0	0	0	0	12	-12	0	0	0	2	-2
252	65456	-65456	0	0	0	0	-16	16	0	0	0	-1	1
268	93744	-93744	0	0	0	0	0	0	0	0	0	0	0
284	132944	-132944	0	0	0	0	20	-20	0	0	0	0	0
300	186800	-186800	0	0	0	0	-22	22	0	0	0	-2	2
316	260400	-260400	0	0	0	0	0	0	0	0	0	0	0
332	360208	-360208	0	0	0	0	28	-28	0	0	0	2	-2
348	494624	-494624	0	0	0	0	-28	28	0	0	0	4	-4
364	674784	-674784	0	0	0	0	0	0	0	0	0	-2	2
380	914816	-914816	0	0	0	0	32	-32	0	0	0	0	0
396	1232784	-1232784	0	0	0	0	-36	36	0	0	0	0	0
412	1652208	-1652208	0	0	0	0	0	0	0	0	0	-2	2
428	2202704	-2202704	0	0	0	0	44	-44	0	0	0	0	0
444	2921856	-2921856	0	0	0	0	-48	48	0	0	0	0	0
460	3857760	-3857760	0	0	0	0	0	0	0	0	0	4	-4
476	5070560	-5070560	0	0	0	0	56	-56	0	0	0	-2	2
492	6636000	-6636000	0	0	0	0	-60	60	0	0	0	0	0
508	8649648	-8649648	0	0	0	0	0	0	0	0	0	0	0
524	11230448	-11230448	0	0	0	0	68	-68	0	0	0	-2	2
540	14526848	-14526848	0	0	0	0	-76	76	0	0	0	0	0
556	18724176	-18724176	0	0	0	0	0	0	0	0	0	2	-2
572	24051808	-24051808	0	0	0	0	88	-88	0	0	0	4	-4
588	30793712	-30793712	0	0	0	0	-94	94	0	0	0	-2	2
604	39301584	-39301584	0	0	0	0	0	0	0	0	0	0	0

Table 25: McKay–Thompson series $H^{(4)}_{g,3}$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
Γ_g	1	1|2	2|2	2|4	4|4	2	3	3|2	6|2	4|8	4	7	7|2
7	6	6	6	-2	-2	-2	0	0	0	2	2	-1	-1
23	28	28	-4	4	-4	4	-2	-2	2	0	0	0	0
39	56	56	8	0	0	-8	2	2	2	-4	0	0	0
55	138	138	-6	2	2	10	0	0	0	-2	2	-2	-2
71	238	238	14	-10	-2	-10	-2	-2	2	2	2	0	0
87	478	478	-18	6	-2	14	4	4	0	2	-2	2	2
103	786	786	18	-6	2	-22	0	0	0	-2	-2	2	2
119	1386	1386	-22	10	2	26	-6	-6	2	2	2	0	0
135	2212	2212	36	-12	-4	-28	4	4	0	4	4	0	0
151	3612	3612	-36	20	-4	36	0	0	0	0	0	0	0
167	5544	5544	40	-16	0	-48	-6	-6	-2	-4	-4	0	0
183	8666	8666	-54	18	2	58	8	8	0	-2	2	0	0
199	12936	12936	72	-32	0	-64	0	0	0	4	4	0	0
215	19420	19420	-84	36	-4	76	-8	-8	0	0	-4	2	2
231	28348	28348	92	-36	4	-100	10	10	2	-4	-4	-2	-2
247	41412	41412	-108	44	4	116	0	0	0	0	4	0	0
263	59178	59178	138	-62	-6	-126	-12	-12	0	6	6	0	0
279	84530	84530	-158	66	-6	154	14	14	-2	2	-2	-2	-2
295	118692	118692	180	-68	4	-188	0	0	0	-8	-4	0	0
311	166320	166320	-208	88	8	216	-18	-18	2	-4	4	0	0
327	230092	230092	252	-108	-4	-244	16	16	0	8	4	2	2
343	317274	317274	-294	122	-6	282	0	0	0	2	-6	-1	-1
359	432964	432964	324	-132	4	-340	-20	-20	0	-8	-8	0	0
375	588966	588966	-378	150	6	390	24	24	0	-2	6	0	0
391	794178	794178	450	-190	-6	-430	0	0	0	10	10	0	0
407	1067220	1067220	-508	220	-12	500	-30	-30	2	0	-4	0	0
423	1423884	1423884	572	-228	4	-588	30	30	2	-12	-8	0	0
439	1893138	1893138	-654	266	10	666	0	0	0	-2	6	2	2
455	2501434	2501434	762	-326	-6	-742	-32	-32	0	10	10	-2	-2
471	3294256	3294256	-864	360	-8	848	40	40	0	4	-8	0	0
487	4314912	4314912	960	-392	8	-984	0	0	0	-12	-12	0	0
503	5633596	5633596	-1092	452	12	1108	-44	-44	0	0	8	-4	-4
519	7320670	7320670	1262	-522	-10	-1234	46	46	2	18	14	0	0
535	9483336	9483336	-1416	592	-16	1400	0	0	0	4	-8	2	2
551	12233330	12233330	1570	-646	10	-1598	-58	-58	-2	-18	-14	4	4
567	15734606	15734606	-1778	726	14	1798	62	62	-2	-6	10	-1	-1
583	20161302	20161302	2022	-850	-10	-1994	0	0	0	18	14	0	0
599	25761288	25761288	-2264	944	-16	2240	-72	-72	3	4	-12	0	0

C.4 Lambency Five

Table 26: McKay–Thompson series $H^{(5)}_{g,1}$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
19	8	8	0	0	2	2	-2	-2	4	0	-2
39	18	18	2	2	0	0	-2	-2	-6	2	0
59	40	40	0	0	-2	-2	0	0	4	0	-2
79	70	70	-2	-2	4	4	0	0	-6	-2	0
99	120	120	0	0	0	0	0	0	12	0	0
119	208	208	0	0	-2	-2	-2	-2	-8	0	-2
139	328	328	0	0	4	4	-2	-2	12	0	0
159	510	510	-2	-2	0	0	0	0	-18	-2	0
179	792	792	0	0	-6	-6	2	2	20	0	2
199	1180	1180	4	4	4	4	0	0	-24	4	0
219	1728	1728	0	0	0	0	-2	-2	24	0	0
239	2518	2518	-2	-2	-8	-8	-2	-2	-30	-2	0
259	3600	3600	0	0	6	6	0	0	40	0	-2
279	5082	5082	2	2	0	0	2	2	-42	2	0
299	7120	7120	0	0	-8	-8	0	0	48	0	0
319	9838	9838	-2	-2	10	10	-2	-2	-58	-2	2
339	13488	13488	0	0	0	0	-2	-2	72	0	0
359	18380	18380	4	4	-10	-10	0	0	-80	4	-2
379	24792	24792	0	0	12	12	2	2	84	0	0
399	33210	33210	-6	-6	0	0	0	0	-102	-6	0
419	44248	44248	0	0	-14	-14	-2	-2	116	0	2
439	58538	58538	2	2	14	14	-2	-2	-130	2	2
459	76992	76992	0	0	0	0	2	2	144	0	0
479	100772	100772	-4	-4	-16	-16	2	2	-168	-4	0
499	131160	131160	0	0	18	18	0	0	196	0	-2
519	169896	169896	8	8	0	0	-4	-4	-216	8	0
539	219128	219128	0	0	-22	-22	-2	-2	236	0	2
559	281322	281322	-6	-6	24	24	2	2	-270	-6	0
579	359712	359712	0	0	0	0	2	2	312	0	0
599	458220	458220	4	4	-24	-24	0	0	-336	4	0
619	581416	581416	0	0	28	28	-4	-4	372	0	0
639	735138	735138	-6	-6	0	0	-2	-2	-426	-6	0
659	926472	926472	0	0	-30	-30	2	2	476	0	2
679	1163674	1163674	10	10	34	34	4	4	-526	10	2
699	1457040	1457040	0	0	0	0	0	0	576	0	0
719	1819056	1819056	-8	-8	-42	-42	-4	-4	-644	-8	-2
739	2264376	2264376	0	0	42	42	-4	-4	724	0	-2

Table 27: McKay–Thompson series $H^{(5)}_{g,2}$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
16	10	-10	2	-2	-2	2	0	0	0	0	0
36	30	-30	-2	2	0	0	0	0	0	0	0
56	52	-52	4	-4	4	-4	2	-2	0	0	0
76	108	-108	-4	4	0	0	-2	2	0	0	0
96	180	-180	4	-4	0	0	0	0	0	0	0
116	312	-312	-8	8	0	0	2	-2	0	0	0
136	488	-488	8	-8	-4	4	-2	2	0	0	0
156	792	-792	-8	8	0	0	2	-2	0	0	0
176	1180	-1180	12	-12	4	-4	0	0	0	0	0
196	1810	-1810	-14	14	-2	2	0	0	0	0	0
216	2640	-2640	16	-16	0	0	0	0	0	0	0
236	3868	-3868	-20	20	4	-4	-2	2	0	0	0
256	5502	-5502	22	-22	-6	6	2	-2	0	0	0
276	7848	-7848	-24	24	0	0	-2	2	0	0	0
296	10912	-10912	32	-32	4	-4	2	-2	0	0	0
316	15212	-15212	-36	36	-4	4	2	-2	0	0	0
336	20808	-20808	40	-40	0	0	-2	2	0	0	0
356	28432	-28432	-48	48	4	-4	2	-2	0	0	0
376	38308	-38308	52	-52	-8	8	-2	2	0	0	0
396	51540	-51540	-60	60	0	0	0	0	0	0	0
416	68520	-68520	72	-72	12	-12	0	0	0	0	0
436	90928	-90928	-80	80	-8	8	-2	2	0	0	0
456	119544	-119544	88	-88	0	0	4	-4	0	0	0
476	156728	-156728	-104	104	8	-8	-2	2	0	0	0
496	203940	-203940	116	-116	-12	12	0	0	0	0	0
516	264672	-264672	-128	128	0	0	2	-2	0	0	0
536	341188	-341188	148	-148	16	-16	-2	2	0	0	0
556	438732	-438732	-164	164	-12	12	2	-2	0	0	0
576	560958	-560958	182	-182	0	0	-2	2	0	0	0
596	715312	-715312	-208	208	16	-16	2	-2	0	0	0
616	907720	-907720	232	-232	-20	20	0	0	0	0	0
636	1148928	-1148928	-256	256	0	0	-2	2	0	0	0
656	1447904	-1447904	288	-288	20	-20	4	-4	0	0	0
676	1820226	-1820226	-318	318	-18	18	-4	4	0	0	0
696	2279520	-2279520	352	-352	0	0	0	0	0	0	0
716	2847812	-2847812	-396	396	20	-20	2	-2	0	0	0
736	3545636	-3545636	436	-436	-28	28	-4	4	0	0	0
756	4404384	-4404384	-480	480	0	0	4	-4	0	0	0

Table 28: McKay–Thompson series $H^{(5)}_{g,3}$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
11	8	8	0	0	2	2	-2	-2	-4	0	2
31	22	22	-2	-2	-2	-2	2	2	2	-2	2
51	48	48	0	0	0	0	-2	-2	0	0	0
71	90	90	2	2	0	0	0	0	6	2	0
91	160	160	0	0	-2	-2	0	0	-8	0	-2
111	270	270	-2	-2	0	0	0	0	6	-2	0
131	440	440	0	0	2	2	0	0	-4	0	2
151	700	700	4	4	-2	-2	0	0	8	4	2
171	1080	1080	0	0	0	0	0	0	-12	0	0
191	1620	1620	-4	-4	6	6	0	0	16	-4	-2
211	2408	2408	0	0	-4	-4	-2	-2	-12	0	0
231	3522	3522	2	2	0	0	2	2	18	2	0
251	5048	5048	0	0	2	2	-2	-2	-28	0	2
271	7172	7172	-4	-4	-4	-4	2	2	24	-4	0
291	10080	10080	0	0	0	0	0	0	-24	0	0
311	13998	13998	6	6	6	6	-2	-2	34	6	-2
331	19272	19272	0	0	-6	-6	2	2	-44	0	-2
351	26298	26298	-6	-6	0	0	-2	-2	42	-6	0
371	35600	35600	0	0	8	8	0	0	-48	0	0
391	47862	47862	6	6	-6	-6	2	2	62	6	2
411	63888	63888	0	0	0	0	-2	-2	-72	0	0
431	84722	84722	-6	-6	8	8	2	2	78	-6	0
451	111728	111728	0	0	-10	-10	-2	-2	-80	0	-2
471	146520	146520	8	8	0	0	0	0	96	8	0
491	191080	191080	0	0	10	10	0	0	-124	0	2
511	248008	248008	-8	-8	-14	-14	-2	-2	128	-8	2
531	320424	320424	0	0	0	0	4	4	-132	0	0
551	412088	412088	8	8	14	14	-2	-2	160	8	-2
571	527800	527800	0	0	-14	-14	0	0	-188	0	-2
591	673302	673302	-10	-10	0	0	2	2	198	-10	0
611	855616	855616	0	0	16	16	-4	-4	-216	0	0
631	1083444	1083444	12	12	-18	-18	4	4	248	12	2
651	1367136	1367136	0	0	0	0	-4	-4	-288	0	0
671	1719362	1719362	-14	-14	20	20	2	2	314	-14	-4
691	2155592	2155592	0	0	-22	-22	2	2	-332	0	-2
711	2694276	2694276	12	12	0	0	-4	-4	384	12	0
731	3357664	3357664	0	0	28	28	4	4	-440	0	4
751	4172746	4172746	-14	-14	-26	-26	-4	-4	470	-14	2

Table 29: McKay–Thompson series $H^{(5)}_{g,4}$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
4	2	-2	2	-2	2	-2	2	-2	0	0	0
24	12	-12	-4	4	0	0	2	-2	0	0	0
44	20	-20	4	-4	-4	4	0	0	0	0	0
64	50	-50	-6	6	2	-2	0	0	0	0	0
84	72	-72	8	-8	0	0	2	-2	0	0	0
104	152	-152	-8	8	-4	4	2	-2	0	0	0
124	220	-220	12	-12	4	-4	0	0	0	0	0
144	378	-378	-14	14	0	0	-2	2	0	0	0
164	560	-560	16	-16	-4	4	0	0	0	0	0
184	892	-892	-20	20	4	-4	2	-2	0	0	0
204	1272	-1272	24	-24	0	0	2	-2	0	0	0
224	1940	-1940	-28	28	-4	4	0	0	0	0	0
244	2720	-2720	32	-32	8	-8	0	0	0	0	0
264	3960	-3960	-40	40	0	0	0	0	0	0	0
284	5500	-5500	44	-44	-8	8	0	0	0	0	0
304	7772	-7772	-52	52	8	-8	2	-2	0	0	0
324	10590	-10590	62	-62	0	0	0	0	0	0	0
344	14668	-14668	-68	68	-8	8	-2	2	0	0	0
364	19728	-19728	80	-80	12	-12	-2	2	0	0	0
384	26772	-26772	-92	92	0	0	2	-2	0	0	0
404	35624	-35624	104	-104	-16	16	4	-4	0	0	0
424	47592	-47592	-120	120	12	-12	2	-2	0	0	0
444	62568	-62568	136	-136	0	0	-2	2	0	0	0
464	82568	-82568	-152	152	-16	16	-2	2	0	0	0
484	107502	-107502	174	-174	18	-18	2	-2	0	0	0
504	140172	-140172	-196	196	0	0	2	-2	0	0	0
524	180940	-180940	220	-220	-20	20	0	0	0	0	0
544	233576	-233576	-248	248	20	-20	-4	4	0	0	0
564	298968	-298968	280	-280	0	0	-2	2	0	0	0
584	382632	-382632	-312	312	-24	24	2	-2	0	0	0
604	486124	-486124	348	-348	28	-28	4	-4	0	0	0
624	617112	-617112	-392	392	0	0	2	-2	0	0	0
644	778768	-778768	432	-432	-32	32	-2	2	0	0	0
664	981548	-981548	-484	484	32	-32	-2	2	0	0	0
684	1230732	-1230732	540	-540	0	0	2	-2	0	0	0
704	1541244	-1541244	-596	596	-36	36	4	-4	0	0	0
724	1921240	-1921240	664	-664	40	-40	0	0	0	0	0
744	2391456	-2391456	-736	736	0	0	-4	4	0	0	0

C.5 Lambency Seven

Table 30: $H^{(7)}_{g,1}$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
-1	-2	-2	-2	-2	-2
27	4	4	4	1	1
55	6	6	-2	0	0
83	10	10	2	-2	-2
111	20	20	-4	2	2
139	30	30	6	0	0
167	42	42	-6	0	0
195	68	68	4	2	2
223	96	96	-8	0	0
251	130	130	10	-2	-2
279	188	188	-12	2	2
307	258	258	10	0	0
335	350	350	-10	-4	-4
363	474	474	18	3	3
391	624	624	-16	0	0
419	826	826	18	-2	-2
447	1090	1090	-22	4	4
475	1410	1410	26	0	0
503	1814	1814	-26	-4	-4
531	2338	2338	26	4	4
559	2982	2982	-34	0	0
587	3774	3774	38	-6	-6
615	4774	4774	-42	4	4
643	5994	5994	42	0	0
671	7494	7494	-50	-6	-6
699	9348	9348	60	6	6
727	11586	11586	-62	0	0
755	14320	14320	64	-8	-8
783	17654	17654	-74	8	8
811	21654	21654	86	0	0
839	26488	26488	-88	-8	-8
867	32334	32334	94	9	9
895	39324	39324	-108	0	0
923	47680	47680	120	-8	-8
951	57688	57688	-128	10	10
979	69600	69600	136	0	0
1007	83760	83760	-152	-12	-12
1035	100596	100596	172	12	12

Table 31: $H^{(7)}_{g,2}$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
24	4	-4	0	-2	2
52	12	-12	0	0	0
80	20	-20	0	2	-2
108	32	-32	0	-1	1
136	48	-48	0	0	0
164	80	-80	0	2	-2
192	108	-108	0	-3	3
220	168	-168	0	0	0
248	232	-232	0	4	-4
276	328	-328	0	-2	2
304	444	-444	0	0	0
332	620	-620	0	2	-2
360	812	-812	0	-4	4
388	1104	-1104	0	0	0
416	1444	-1444	0	4	-4
444	1904	-1904	0	-4	4
472	2460	-2460	0	0	0
500	3208	-3208	0	4	-4
528	4080	-4080	0	-6	6
556	5244	-5244	0	0	0
584	6632	-6632	0	8	-8
612	8400	-8400	0	-6	6
640	10524	-10524	0	0	0
668	13224	-13224	0	6	-6
696	16408	-16408	0	-8	8
724	20436	-20436	0	0	0
752	25216	-25216	0	10	-10
780	31120	-31120	0	-8	8
808	38148	-38148	0	0	0
836	46784	-46784	0	8	-8
864	56976	-56976	0	-12	12
892	69432	-69432	0	0	0
920	84144	-84144	0	12	-12
948	101904	-101904	0	-12	12
976	122868	-122868	0	0	0
1004	148076	-148076	0	14	-14
1032	177656	-177656	0	-16	16
1060	213072	-213072	0	0	0

Table 32: $H^{(7)}_{g,3}$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
19	6	6	-2	0	0
47	12	12	4	0	0
75	22	22	-2	1	1
103	36	36	4	0	0
131	58	58	-6	-2	-2
159	90	90	2	0	0
187	132	132	-4	0	0
215	190	190	6	-2	-2
243	274	274	-6	1	1
271	384	384	8	0	0
299	528	528	-8	0	0
327	722	722	10	2	2
355	972	972	-12	0	0
383	1300	1300	12	-2	-2
411	1724	1724	-12	2	2
439	2256	2256	16	0	0
467	2938	2938	-22	-2	-2
495	3806	3806	22	2	2
523	4890	4890	-22	0	0
551	6244	6244	28	-2	-2
579	7940	7940	-28	2	2
607	10038	10038	30	0	0
635	12620	12620	-36	-4	-4
663	15814	15814	38	4	4
691	19722	19722	-46	0	0
719	24490	24490	50	-2	-2
747	30310	30310	-50	4	4
775	37362	37362	58	0	0
803	45908	45908	-68	-4	-4
831	56236	56236	68	4	4
859	68646	68646	-74	0	0
887	83556	83556	84	-6	-6
915	101436	101436	-92	6	6
943	122790	122790	102	0	0
971	148254	148254	-106	-6	-6
999	178566	178566	118	6	6
1027	214548	214548	-132	0	0
1055	257190	257190	142	-6	-6

Table 33: $H^{(7)}_{g,4}$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
12	4	-4	0	1	-1
40	12	-12	0	0	0
68	16	-16	0	-2	2
96	36	-36	0	0	0
124	48	-48	0	0	0
152	84	-84	0	0	0
180	116	-116	0	2	-2
208	180	-180	0	0	0
236	244	-244	0	-2	2
264	360	-360	0	0	0
292	480	-480	0	0	0
320	676	-676	0	-2	2
348	896	-896	0	2	-2
376	1224	-1224	0	0	0
404	1588	-1588	0	-2	2
432	2128	-2128	0	1	-1
460	2736	-2736	0	0	0
488	3588	-3588	0	0	0
516	4576	-4576	0	4	-4
544	5904	-5904	0	0	0
572	7448	-7448	0	-4	4
600	9500	-9500	0	2	-2
628	11892	-11892	0	0	0
656	14992	-14992	0	-2	2
684	18628	-18628	0	4	-4
712	23256	-23256	0	0	0
740	28688	-28688	0	-4	4
768	35532	-35532	0	3	-3
796	43560	-43560	0	0	0
824	53528	-53528	0	-4	4
852	65256	-65256	0	6	-6
880	79656	-79656	0	0	0
908	96564	-96564	0	-6	6
936	117196	-117196	0	4	-4
964	141360	-141360	0	0	0
992	170600	-170600	0	-4	4
1020	204848	-204848	0	8	-8
1048	245988	-245988	0	0	0

Table 34: $H^{(7)}_{g,5}$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
3	2	2	2	-1	-1
31	6	6	-2	0	0
59	14	14	-2	2	2
87	22	22	-2	-2	-2
115	36	36	4	0	0
143	56	56	0	2	2
171	82	82	2	-2	-2
199	126	126	-2	0	0
227	182	182	6	2	2
255	250	250	-6	-2	-2
283	354	354	2	0	0
311	490	490	-6	4	4
339	656	656	8	-4	-4
367	882	882	-6	0	0
395	1180	1180	4	4	4
423	1550	1550	-10	-4	-4
451	2028	2028	12	0	0
479	2638	2638	-10	4	4
507	3394	3394	10	-5	-5
535	4362	4362	-14	0	0
563	5562	5562	18	6	6
591	7032	7032	-16	-6	-6
619	8886	8886	14	0	0
647	11166	11166	-18	6	6
675	13940	13940	28	-7	-7
703	17358	17358	-26	0	0
731	21536	21536	24	8	8
759	26594	26594	-30	-10	-10
787	32742	32742	38	0	0
815	40180	40180	-36	10	10
843	49124	49124	36	-10	-10
871	59916	59916	-44	0	0
899	72852	72852	52	12	12
927	88296	88296	-56	-12	-12
955	106788	106788	52	0	0
983	128816	128816	-64	14	14
1011	154948	154948	76	-14	-14

Table 35: $H^{(7)}_{g,6}$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
20	4	-4	0	-2	2
48	4	-4	0	1	-1
76	12	-12	0	0	0
104	12	-12	0	0	0
132	32	-32	0	2	-2
160	36	-36	0	0	0
188	64	-64	0	-2	2
216	80	-80	0	2	-2
244	132	-132	0	0	0
272	160	-160	0	-2	2
300	252	-252	0	3	-3
328	312	-312	0	0	0
356	448	-448	0	-2	2
384	572	-572	0	2	-2
412	792	-792	0	0	0
440	992	-992	0	-4	4
468	1348	-1348	0	4	-4
496	1680	-1680	0	0	0
524	2220	-2220	0	-6	6
552	2776	-2776	0	4	-4
580	3600	-3600	0	0	0
608	4460	-4460	0	-4	4
636	5712	-5712	0	6	-6
664	7044	-7044	0	0	0
692	8892	-8892	0	-6	6
720	10932	-10932	0	6	-6
748	13656	-13656	0	0	0
776	16672	-16672	0	-8	8
804	20672	-20672	0	8	-8
832	25116	-25116	0	0	0
860	30856	-30856	0	-8	8
888	37352	-37352	0	8	-8
916	45564	-45564	0	0	0
944	54884	-54884	0	-10	10
972	66572	-66572	0	11	-11
1000	79848	-79848	0	0	0
1028	96256	-96256	0	-14	14

C.6 Lambency Thirteen

Table 36: $H^{(13)}_{g,1}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
-1	-2	-2	-2
51	2	2	2
103	2	2	-2
155	0	0	0
207	2	2	-2
259	2	2	2
311	4	4	0
363	6	6	2
415	6	6	-2
467	8	8	4
519	12	12	-4
571	14	14	2
623	14	14	-2
675	20	20	4
727	24	24	-4
779	28	28	4
831	36	36	-4
883	42	42	6
935	50	50	-6
987	62	62	6
1039	70	70	-6
1091	84	84	8
1143	102	102	-6
1195	118	118	6
1247	136	136	-8
1299	162	162	10
1351	190	190	-10
1403	216	216	8
1455	254	254	-10
1507	292	292	12
1559	336	336	-12
1611	392	392	12
1663	446	446	-14
1715	510	510	14
1767	592	592	-16
1819	672	672	16
1871	764	764	-16
1923	876	876	20

Table 37: $H^{(13)}_{g,2}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
48	0	0	0
100	2	-2	0
152	4	-4	0
204	4	-4	0
256	6	-6	0
308	8	-8	0
360	8	-8	0
412	12	-12	0
464	16	-16	0
516	20	-20	0
568	24	-24	0
620	32	-32	0
672	36	-36	0
724	48	-48	0
776	56	-56	0
828	68	-68	0
880	80	-80	0
932	100	-100	0
984	112	-112	0
1036	140	-140	0
1088	164	-164	0
1140	192	-192	0
1192	224	-224	0
1244	268	-268	0
1296	306	-306	0
1348	364	-364	0
1400	420	-420	0
1452	488	-488	0
1504	560	-560	0
1556	656	-656	0
1608	744	-744	0
1660	864	-864	0
1712	988	-988	0
1764	1134	-1134	0
1816	1292	-1292	0
1868	1484	-1484	0
1920	1676	-1676	0
1972	1920	-1920	0

Table 38: $H^{(13)}_{g,3}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
43	2	2	-2
95	2	2	2
147	4	4	0
199	6	6	2
251	8	8	-4
303	10	10	2
355	14	14	-2
407	18	18	2
459	22	22	-2
511	26	26	2
563	34	34	-2
615	44	44	4
667	52	52	-4
719	64	64	4
771	78	78	-2
823	96	96	4
875	114	114	-6
927	136	136	4
979	164	164	-4
1031	194	194	6
1083	230	230	-6
1135	270	270	6
1187	318	318	-6
1239	374	374	6
1291	434	434	-10
1343	506	506	10
1395	592	592	-8
1447	686	686	10
1499	792	792	-12
1551	914	914	10
1603	1054	1054	-10
1655	1214	1214	14
1707	1394	1394	-14
1759	1594	1594	14
1811	1822	1822	-14
1863	2084	2084	16
1915	2374	2374	-18
1967	2698	2698	18

Table 39: $H^{(13)}_{g,4}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
36	2	-2	0
88	4	-4	0
140	4	-4	0
192	8	-8	0
244	8	-8	0
296	12	-12	0
348	16	-16	0
400	22	-22	0
452	24	-24	0
504	36	-36	0
556	40	-40	0
608	52	-52	0
660	64	-64	0
712	80	-80	0
764	92	-92	0
816	116	-116	0
868	136	-136	0
920	168	-168	0
972	196	-196	0
1024	238	-238	0
1076	272	-272	0
1128	332	-332	0
1180	384	-384	0
1232	456	-456	0
1284	528	-528	0
1336	620	-620	0
1388	712	-712	0
1440	840	-840	0
1492	960	-960	0
1544	1120	-1120	0
1596	1280	-1280	0
1648	1484	-1484	0
1700	1688	-1688	0
1752	1952	-1952	0
1804	2216	-2216	0
1856	2544	-2544	0
1908	2888	-2888	0
1960	3304	-3304	0

Table 40: $H^{(13)}_{g,5}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
27	2	2	2
79	4	4	0
131	6	6	2
183	6	6	-2
235	10	10	2
287	14	14	-2
339	16	16	0
391	22	22	-2
443	30	30	2
495	36	36	-4
547	46	46	2
599	58	58	-2
651	68	68	4
703	86	86	-2
755	106	106	2
807	124	124	-4
859	152	152	4
911	184	184	-4
963	216	216	4
1015	258	258	-6
1067	308	308	4
1119	362	362	-6
1171	426	426	6
1223	502	502	-6
1275	584	584	8
1327	684	684	-8
1379	798	798	6
1431	920	920	-8
1483	1070	1070	10
1535	1238	1238	-10
1587	1422	1422	10
1639	1638	1638	-10
1691	1884	1884	12
1743	2156	2156	-12
1795	2468	2468	12
1847	2822	2822	-14
1899	3212	3212	16
1951	3660	3660	-16

Table 41: $H^{(13)}_{g,6}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
16	2	-2	0
68	4	-4	0
120	4	-4	0
172	8	-8	0
224	8	-8	0
276	16	-16	0
328	16	-16	0
380	24	-24	0
432	28	-28	0
484	38	-38	0
536	44	-44	0
588	60	-60	0
640	68	-68	0
692	88	-88	0
744	104	-104	0
796	132	-132	0
848	152	-152	0
900	190	-190	0
952	220	-220	0
1004	268	-268	0
1056	312	-312	0
1108	376	-376	0
1160	432	-432	0
1212	520	-520	0
1264	596	-596	0
1316	708	-708	0
1368	812	-812	0
1420	956	-956	0
1472	1092	-1092	0
1524	1280	-1280	0
1576	1460	-1460	0
1628	1696	-1696	0
1680	1932	-1932	0
1732	2236	-2236	0
1784	2536	-2536	0
1836	2924	-2924	0
1888	3308	-3308	0
1940	3792	-3792	0

Table 42: $H^{(13)}_{g,7}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
3	2	2	-2
55	2	2	2
107	4	4	0
159	8	8	0
211	10	10	-2
263	12	12	0
315	16	16	0
367	22	22	2
419	26	26	-2
471	34	34	2
523	44	44	0
575	54	54	2
627	68	68	-4
679	82	82	2
731	102	102	-2
783	124	124	4
835	148	148	-4
887	176	176	4
939	214	214	-2
991	256	256	4
1043	300	300	-4
1095	356	356	4
1147	420	420	-4
1199	494	494	6
1251	580	580	-8
1303	674	674	6
1355	786	786	-6
1407	918	918	6
1459	1060	1060	-8
1511	1226	1226	6
1563	1418	1418	-6
1615	1632	1632	8
1667	1874	1874	-10
1719	2150	2150	10
1771	2464	2464	-8
1823	2816	2816	12
1875	3214	3214	-14

Table 43: $H^{(13)}_{g,8}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
40	4	-4	0
92	4	-4	0
144	6	-6	0
196	6	-6	0
248	12	-12	0
300	12	-12	0
352	20	-20	0
404	24	-24	0
456	32	-32	0
508	36	-36	0
560	52	-52	0
612	56	-56	0
664	76	-76	0
716	88	-88	0
768	112	-112	0
820	128	-128	0
872	164	-164	0
924	184	-184	0
976	232	-232	0
1028	268	-268	0
1080	324	-324	0
1132	372	-372	0
1184	452	-452	0
1236	512	-512	0
1288	616	-616	0
1340	704	-704	0
1392	832	-832	0
1444	950	-950	0
1496	1120	-1120	0
1548	1268	-1268	0
1600	1486	-1486	0
1652	1688	-1688	0
1704	1956	-1956	0
1756	2220	-2220	0
1808	2568	-2568	0
1860	2896	-2896	0
1912	3336	-3336	0

Table 44: $H^{(13)}_{g,9}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
23	2	2	-2
75	4	4	0
127	4	4	0
179	6	6	2
231	8	8	0
283	12	12	0
335	14	14	-2
387	20	20	0
439	26	26	-2
491	30	30	2
543	40	40	0
595	50	50	2
647	60	60	0
699	74	74	2
751	90	90	-2
803	108	108	4
855	134	134	-2
907	158	158	2
959	188	188	-4
1011	226	226	2
1063	266	266	-2
1115	314	314	2
1167	372	372	-4
1219	436	436	4
1271	508	508	-4
1323	596	596	4
1375	692	692	-4
1427	802	802	6
1479	932	932	-4
1531	1074	1074	6
1583	1238	1238	-6
1635	1430	1430	6
1687	1640	1640	-8
1739	1878	1878	6
1791	2150	2150	-6
1843	2456	2456	8
1895	2800	2800	-8

Table 45: $H^{(13)}_{g,10}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
4	2	-2	0
56	0	0	0
108	4	-4	0
160	4	-4	0
212	8	-8	0
264	8	-8	0
316	12	-12	0
368	12	-12	0
420	20	-20	0
472	20	-20	0
524	32	-32	0
576	34	-34	0
628	48	-48	0
680	52	-52	0
732	72	-72	0
784	78	-78	0
836	104	-104	0
888	116	-116	0
940	148	-148	0
992	164	-164	0
1044	208	-208	0
1096	232	-232	0
1148	288	-288	0
1200	324	-324	0
1252	396	-396	0
1304	444	-444	0
1356	536	-536	0
1408	604	-604	0
1460	720	-720	0
1512	812	-812	0
1564	960	-960	0
1616	1080	-1080	0
1668	1268	-1268	0
1720	1428	-1428	0
1772	1664	-1664	0
1824	1872	-1872	0

Table 46: $H^{(13)}_{g,11}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
35	2	2	2
87	2	2	2
139	2	2	-2
191	4	4	0
243	4	4	0
295	8	8	0
347	10	10	-2
399	10	10	2
451	16	16	0
503	20	20	0
555	22	22	-2
607	28	28	0
659	36	36	0
711	44	44	0
763	54	54	-2
815	64	64	0
867	76	76	0
919	94	94	2
971	114	114	-2
1023	130	130	2
1075	156	156	0
1127	188	188	0
1179	216	216	-4
1231	254	254	2
1283	300	300	0
1335	346	346	2
1387	404	404	-4
1439	470	470	2
1491	542	542	-2
1543	630	630	2
1595	724	724	-4
1647	828	828	4
1699	954	954	-2
1751	1100	1100	4
1803	1250	1250	-6
1855	1428	1428	4

Table 47: $H^{(13)}_{g,12}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
12	0	0	0
64	2	-2	0
116	0	0	0
168	4	-4	0
220	0	0	0
272	4	-4	0
324	2	-2	0
376	8	-8	0
428	4	-4	0
480	12	-12	0
532	8	-8	0
584	16	-16	0
636	16	-16	0
688	24	-24	0
740	20	-20	0
792	36	-36	0
844	32	-32	0
896	48	-48	0
948	48	-48	0
1000	68	-68	0
1052	68	-68	0
1104	96	-96	0
1156	98	-98	0
1208	128	-128	0
1260	136	-136	0
1312	176	-176	0
1364	184	-184	0
1416	240	-240	0
1468	252	-252	0
1520	312	-312	0
1572	340	-340	0
1624	416	-416	0
1676	448	-448	0
1728	548	-548	0
1780	592	-592	0
1832	708	-708	0

D Decompositions

As explained in §4 (see also §5.1) our conjectural proposals for the umbral McKay–Thompson series $H^{(\ell)}_{g,r}(\tau) = \sum_d c^{(\ell)}_{g,r}(d)q^d$ (cf. §§4,C) determine the $G^{(\ell)}$-modules $K^{(\ell)}_{r,d}$ up to isomorphism for $d > 0$, at least for those values of d for which we can identify all the Fourier coefficients $c^{(\ell)}_{g,r}(d)$. In this section we furnish tables of explicit decompositions into irreducible representations of $G^{(\ell)}$ for $K^{(\ell)}_{r,d}$, for the first few values of d. The coefficient $c^{(\ell)}_r(d)$ of $H^{(\ell)}_r = H^{(\ell)}_{e,r}$ is non-zero only when $d = n - r^2/4\ell$ for some integer $n \geq 0$. For each of the tables in this section the rows are labelled by the values $4\ell d$, so that the entry in row m and column χ_i indicates the multiplicity of the irreducible representation of $G^{(\ell)}$ with character χ_i (in the notation of the character tables of §B.1) appearing in the $G^{(\ell)}$-module $K^{(\ell)}_{r,m/4\ell}$. One can observe that these tables support Conjectures 5.1, 5.11 and 5.12, and also give evidence in support of the hypothesis that $K^{(\ell)}_{r,d}$ has a decomposition into irreducible representations that factor through $\bar{G}^{(\ell)}$ when r is odd, and has a decomposition into faithful irreducible representations of $G^{(\ell)}$ when r is even.

D.1 Lambency Two

Table 48: Decomposition of $K_1^{(2)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
-1	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	2	2	2	2
63	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	2	0	0	2	2	2	4	2	2	6
71	0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	2	0	2	2	2	4	4	4	8	8	10
79	0	0	0	0	2	2	0	2	2	0	0	2	2	2	2	2	4	4	4	6	6	8	12	10	10	24
87	0	0	0	0	0	0	0	0	0	4	4	4	4	6	4	4	2	8	10	8	14	12	22	24	26	40
95	0	2	0	0	2	2	2	4	4	6	6	8	8	4	8	8	12	12	12	18	26	30	40	38	40	80
103	0	0	2	2	2	2	4	2	6	10	10	14	14	18	14	14	16	26	30	28	44	44	70	80	84	136
111	0	0	0	0	8	8	4	6	14	16	16	24	24	22	24	24	34	38	46	58	80	86	128	126	132	254
119	0	0	2	2	8	8	12	8	18	38	38	40	40	46	44	44	46	78	86	88	138	144	218	238	246	424
127	0	2	2	2	18	18	18	22	36	50	50	72	72	68	72	72	100	122	140	170	232	252	378	382	400	742
135	0	2	8	8	25	25	30	26	54	94	94	116	116	130	124	124	140	212	246	262	392	410	630	670	704	1222
143	0	6	6	6	50	50	50	58	100	148	148	194	194	192	202	202	256	342	388	454	654	704	1044	1074	1120	2058
151	0	4	18	18	68	68	80	72	150	252	252	318	318	346	332	332	394	582	664	722	1062	1116	1702	1800	1880	3320
159	0	14	20	20	126	126	128	138	254	390	390	516	516	520	536	536	676	904	1036	1196	1716	1836	2764	2846	2980	5408
167	2	20	40	40	182	182	214	200	396	652	652	814	814	872	860	860	1020	1476	1684	1862	2742	2902	4384	4622	4828	8572
175	2	32	55	55	314	314	328	346	640	988	988	1298	1298	1336	1348	1348	1686	2302	2630	3000	4324	4616	6950	7204	7532	13620
183	2	40	98	98	460	460	512	496	972	1590	1590	2020	2020	2144	2118	2118	2546	3638	4162	4624	6768	7166	10856	11376	11898	21204

D.2 Lambency Three

Table 49: Decomposition of $K_1^{(3)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	2	0	2	0	0	2
59	0	0	0	0	0	2	2	0	0	0	2	2	2	2	2
71	0	0	2	0	0	0	0	2	2	2	2	2	4	4	6
83	0	0	0	2	2	4	4	2	2	2	4	6	6	8	10
95	0	0	2	0	0	4	4	8	8	6	6	10	12	14	18
107	0	2	2	4	4	8	12	8	8	8	14	16	22	28	30

Table 50: Decomposition of $K_2^{(3)}$

	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
8	1	1	0	0	0	0	0	0	0	0	0
20	0	0	0	0	1	1	0	0	0	0	0
32	0	0	0	0	0	0	1	1	0	0	0
44	0	0	0	0	0	0	0	0	2	1	1
56	0	0	0	2	0	0	2	2	0	2	2
68	0	0	2	0	2	2	2	2	4	4	4
80	2	2	0	0	1	1	6	6	4	8	8
92	0	0	2	4	6	6	8	8	12	14	14
104	2	2	0	4	6	6	20	20	16	24	24
116	2	2	6	8	12	12	26	26	36	44	44

D.3 Lambency Four

Table 51: Decomposition of $K_1^{(4)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
-1	-2	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	2	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	2
47	0	0	0	0	0	2	0	0	2	2	0
63	0	1	1	0	2	0	0	2	2	2	4
79	0	0	0	2	2	2	4	0	4	6	4
95	0	2	2	2	2	4	2	4	6	8	12
111	2	2	2	6	6	6	8	6	10	18	14
127	0	4	4	6	10	10	6	12	18	26	30
143	2	6	6	14	14	18	18	10	32	46	40
159	4	10	10	18	24	26	20	26	44	68	76

Table 52: Decomposition of $K_3^{(4)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
7	0	1	1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	2	0	0
39	0	0	0	0	2	0	0	0	0	2	0
55	0	0	0	2	0	0	0	2	2	2	2
71	0	2	2	0	2	2	0	0	2	4	4
87	2	0	0	2	2	2	4	4	6	6	8
103	0	2	2	2	6	6	4	2	6	14	12
119	2	2	2	8	4	8	6	8	18	20	22
135	2	8	8	8	14	14	12	10	20	36	34
151	4	6	6	18	16	20	20	22	42	54	56

Table 53: Decomposition of $K_2^{(4)}$

	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
12	0	1	1	0	0
28	0	0	0	1	1
44	2	0	0	2	2
60	0	2	2	4	4
76	2	2	2	8	8
92	6	4	4	14	14
108	6	9	9	24	24
124	14	14	14	40	40
140	24	20	20	66	66
156	32	36	36	104	104

D.4 Lambency Five

Table 54: Decomposition of $K_1^{(5)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
-1	-2	0	0	0	0	0	0
19	0	0	2	0	0	0	0
39	0	0	0	2	0	2	0
59	0	0	2	0	2	2	2
79	0	2	2	4	2	2	4
99	2	0	6	2	6	4	6
119	0	2	6	8	8	10	10
139	4	2	14	10	14	12	16
159	2	6	14	20	20	22	26
179	8	4	28	22	36	32	40

Table 55: Decomposition of $K_3^{(5)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
11	0	0	0	2	0	0	0
31	0	0	0	0	2	0	2
51	0	0	2	2	2	2	2
71	2	0	4	2	4	4	4
91	0	2	4	6	6	8	8
111	2	2	10	8	12	10	14
131	4	4	14	16	18	18	22
151	8	4	24	22	30	30	34
171	8	10	34	38	44	46	54
191	14	14	58	52	68	64	82

Table 56: Decomposition of $K_2^{(5)}$

	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}
16	0	0	0	0	1	1	0
36	0	0	1	1	1	1	2
56	2	2	2	2	2	2	2
76	0	0	4	4	4	4	6
96	2	2	6	6	8	8	8
116	2	2	10	10	12	12	18
136	4	4	16	16	22	22	22
156	6	6	26	26	32	32	42
176	12	12	40	40	50	50	56
196	13	13	60	60	74	74	94

Table 57: Decomposition of $K_4^{(5)}$

	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}
4	1	1	0	0	0	0	0
24	0	0	0	0	0	0	2
44	0	0	0	0	2	2	0
64	0	0	2	2	1	1	4
84	2	2	2	2	4	4	2
104	0	0	4	4	6	6	10
124	4	4	8	8	10	10	8
144	1	1	13	13	14	14	22
164	6	6	18	18	26	26	24
184	6	6	30	30	34	34	50

D.5 Lambency Seven

Table 58: Decomposition of $K_1^{(7)}$

	χ_1	χ_2	χ_3	χ_4
-1	−2	0	0	0
27	2	1	1	0
55	0	0	0	2
83	0	2	2	2
111	2	0	0	6
139	4	4	4	6
167	2	2	2	12
195	8	6	6	16
223	6	6	6	26
251	12	14	14	30

Table 59: Decomposition of $K_3^{(7)}$

	χ_1	χ_2	χ_3	χ_4
19	0	0	0	2
47	2	2	2	2
75	2	1	1	6
103	4	4	4	8
131	2	4	4	16
159	8	8	8	22
187	10	10	10	34
215	16	18	18	46
243	22	21	21	70
271	34	34	34	94

Table 60: Decomposition of $K_5^{(7)}$

	χ_1	χ_2	χ_3	χ_4
3	0	1	1	0
31	0	0	0	2
59	2	0	0	4
87	0	2	2	6
115	4	4	4	8
143	6	4	4	14
171	6	8	8	20
199	10	10	10	32
227	18	16	16	44
255	18	20	20	64

Table 61: Decomposition of $K_2^{(7)}$

	χ_5	χ_6	χ_7
24	2	0	0
52	2	2	2
80	2	4	4
108	6	5	5
136	8	8	8
164	12	14	14
192	20	17	17
220	28	28	28
248	36	40	40
276	56	54	54

Table 62: Decomposition of $K_4^{(7)}$

	χ_5	χ_6	χ_7
12	0	1	1
40	2	2	2
68	4	2	2
96	6	6	6
124	8	8	8
152	14	14	14
180	18	20	20
208	30	30	30
236	42	40	40
264	60	60	60

Table 63: Decomposition of $K_6^{(7)}$

	χ_5	χ_6	χ_7
20	2	0	0
48	0	1	1
76	2	2	2
104	2	2	2
132	4	6	6
160	6	6	6
188	12	10	10
216	12	14	14
244	22	22	22
272	28	26	26

D.6 Lambency Thirteen

Table 64: Decomposition of $K_1^{(13)}$

	χ_1	χ_2
-1	-2	0
51	2	0
103	0	2
155	0	0
207	0	2
259	2	0
311	2	2
363	4	2
415	2	4
467	6	2

Table 65: Decomposition of $K_3^{(13)}$

	χ_1	χ_2
43	0	2
95	2	0
147	2	2
199	4	2
251	2	6
303	6	4
355	6	8
407	10	8
459	10	12
511	14	12

Table 66: Decomposition of $K_5^{(13)}$

	χ_1	χ_2
27	2	0
79	2	2
131	4	2
183	2	4
235	6	4
287	6	8
339	8	8
391	10	12
443	16	14
495	16	20

Table 67: Decomposition of $K_2^{(13)}$

	χ_3	χ_4
48	0	0
100	1	1
152	2	2
204	2	2
256	3	3
308	4	4
360	4	4
412	6	6
464	8	8
516	10	10

Table 68: Decomposition of $K_4^{(13)}$

	χ_3	χ_4
36	1	1
88	2	2
140	2	2
192	4	4
244	4	4
296	6	6
348	8	8
400	11	11
452	12	12
504	18	18

Table 69: Decomposition of $K_6^{(13)}$

	χ_3	χ_4
16	1	1
68	2	2
120	2	2
172	4	4
224	4	4
276	8	8
328	8	8
380	12	12
432	14	14
484	19	19

Table 70: Decomposition of $K_7^{(13)}$

	χ_1	χ_2
3	0	2
55	2	0
107	2	2
159	4	4
211	4	6
263	6	6
315	8	8
367	12	10
419	12	14
471	18	16

Table 71: Decomposition of $K_9^{(13)}$

	χ_1	χ_2
23	0	2
75	2	2
127	2	2
179	4	2
231	4	4
283	6	6
335	6	8
387	10	10
439	12	14
491	16	14

Table 72: Decomposition of $K_{11}^{(13)}$

	χ_1	χ_2
35	2	0
87	2	0
139	0	2
191	2	2
243	2	2
295	4	4
347	4	6
399	6	4
451	8	8
503	10	10

Table 73: Decomposition of $K_8^{(13)}$

	χ_3	χ_4
40	2	2
92	2	2
144	3	3
196	3	3
248	6	6
300	6	6
352	10	10
404	12	12
456	16	16
508	18	18

Table 74: Decomposition of $K_{10}^{(13)}$

	χ_3	χ_4
4	1	1
56	0	0
108	2	2
160	2	2
212	4	4
264	4	4
316	6	6
368	6	6
420	10	10
472	10	10

Table 75: Decomposition of $K_{12}^{(13)}$

	χ_3	χ_4
12	0	0
64	1	1
116	0	0
168	2	2
220	0	0
272	2	2
324	1	1
376	4	4
428	2	2
480	6	6

References

[1] J. H. Conway and S. P. Norton, "Monstrous Moonshine," *Bull. London Math. Soc.* **11** (1979) 308 339.

[2] J. G. Thompson, "Some numerology between the Fischer-Griess Monster and the elliptic modular function," *Bull. London Math. Soc.* **11** (1979) no. 3, 352–353.

[3] J. G. Thompson, "Finite groups and modular functions," *Bull. London Math. Soc.* **11** (1979) no. 3, 347–351.

[4] P. Fong, "Characters arising in the Monster-modular connection," in *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, vol. 37 of *Proc. Sympos. Pure Math.*, pp. 557–559. Amer. Math. Soc., Providence, R.I., 1980.

[5] S. D. Smith, "On the head characters of the Monster simple group," in *Finite groups—coming of age (Montreal, Que., 1982)*, vol. 45 of *Contemp. Math.*, pp. 303–313. Amer. Math. Soc., Providence, RI, 1985. http://dx.doi.org/10.1090/conm/045/822245.

[6] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A natural representation of the Fischer-Griess Monster with the modular function J as character," *Proc. Nat. Acad. Sci. U.S.A.* **81** (1984) no. 10, Phys. Sci., 3256–3260.

[7] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A moonshine module for the Monster," in *Vertex operators in mathematics and physics (Berkeley, Calif., 1983)*, vol. 3 of *Math. Sci. Res. Inst. Publ.*, pp. 231–273. Springer, New York, 1985.

[8] I. B. Frenkel and V. G. Kac, "Basic representations of affine Lie algebras and dual resonance models," *Invent. Math.* **62** (1980/81) no. 1, 23–66.

[9] G. Segal, "Unitary representations of some infinite-dimensional groups," *Comm. Math. Phys.* **80** (1981) no. 3, 301–342. http://projecteuclid.org/getRecord?id=euclid.cmp/1103919978.

[10] R. L. Griess, Jr., "The friendly giant," *Invent. Math.* **69** (1982) no. 1, 1–102.

[11] R. Borcherds, "Vertex algebras, Kac-Moody algebras, and the Monster," *Proceedings of the National Academy of Sciences, U.S.A.* **83** (1986) no. 10, 3068–3071.

[12] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, vol. 134 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[13] R. Borcherds, "Generalized Kac-Moody algebras," *J. Algebra* **115** (1988) no. 2, 501–512. http://dx.doi.org/10.1016/0021-8693(88)90275-X.

[14] R. E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras," *Invent. Math.* **109, No.2** (1992) 405–444.

[15] T. Eguchi, H. Ooguri, and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," *Exper.Math.* **20** (2011) 91–96, arXiv:1004.0956 [hep-th].

[16] T. Eguchi and A. Taormina, "Unitary representations of the $N = 4$ superconformal algebra," *Phys. Lett. B* **196** (1987) no. 1, 75–81. http://dx.doi.org/10.1016/0370-2693(87)91679-0.

[17] T. Eguchi and A. Taormina, "Character formulas for the $N = 4$ superconformal algebra," *Phys. Lett. B* **200** (1988) no. 3, 315–322. http://dx.doi.org/10.1016/0370-2693(88)90778-2.

[18] V. G. Kac and M. Wakimoto, "Quantum reduction and representation theory of superconformal algebras," *Adv. Math.* **185** (2004) no. 2, 400–458. http://dx.doi.org/10.1016/j.aim.2003.12.005.

[19] M. C. N. Cheng, "$K3$ Surfaces, $N = 4$ Dyons, and the Mathieu Group M_{24}," 1005.5415.

[20] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu twining characters for K3," *JHEP* **1009** (2010) 058, arXiv:1006.0221 [hep-th]. 19 pages.

[21] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," *JHEP* **1010** (2010) 062, arXiv:1008.3778 [hep-th].

[22] T. Eguchi and K. Hikami, "Note on Twisted Elliptic Genus of K3 Surface," *Phys.Lett.* **B694** (2011) 446–455, arXiv:1008.4924 [hep-th].

[23] T. Gannon, "Much ado about Mathieu," arXiv:1211.5531 [math.RT].

[24] M. C. N. Cheng and J. F. R. Duncan, "On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine," *Commun. Number Theory Phys.* **6** (2012) no. 3, , 1110.3859.

[25] J. F. R. Duncan and I. B. Frenkel, "Rademacher sums, moonshine and gravity," *Commun. Number Theory Phys.* **5** (2011) no. 4, 1–128.

[26] H. Rademacher, "The Fourier Series and the Functional Equation of the Absolute Modular Invariant J(τ)," *Amer. J. Math.* **61** (1939) no. 1, 237–248.

[27] E. Witten, "Three-Dimensional Gravity Revisited," 0706.3359.

[28] A. Maloney and E. Witten, "Quantum gravity partition functions in three dimensions," *J. High Energy Phys.* (2010) no. 2, 029, 58. http://dx.doi.org/10.1007/JHEP02(2010)029.

[29] W. Li, W. Song, and A. Strominger, "Chiral gravity in three dimensions," *J. High Energy Phys.* (2008) no. 4, 082, 15. http://dx.doi.org/10.1088/1126-6708/2008/04/082.

[30] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, hep-th/9711200.

[31] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys.Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[32] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[33] R. Dijkgraaf, J. Maldacena, G. Moore, and E. Verlinde, "A black hole farey tail," hep-th/0005003.

[34] G. W. Moore, "Les Houches Lectures on Strings and Arithmetic," hep-th/0401049.

[35] J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot, and E. Verlinde, "A farey tail for attractor black holes," *JHEP* **0611:024,2006** (JHEP0611:024,2006) , hep-th/0608059.

[36] P. Kraus and F. Larsen, "Partition functions and elliptic genera from supergravity," *JHEP* **0701:002,2007** (JHEP 0701:002,2007) , hep-th/0607138.

[37] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos," *JHEP* **1111** (2011) 129, arXiv:hep-th/0702146 [HEP-TH]. 149 pages, 21 figures.

[38] A. Sen, "Black hole entropy function, attractors and precision counting of microstates," *Gen.Rel.Grav.* **40:2249-2431,2008** (Aug., 2007) , 0708.1270.

[39] A. Dabholkar, J. Gomes, and S. Murthy, "Localization and Exact Holography," arXiv:1111.1161 [hep-th].

[40] J. Manschot and G. W. Moore, "A modern fareytail," *Commun.Num.Theor.Phys.* **4,** (Dec., 2007) 103–159,2010, 0712.0573.

[41] M. C. N. Cheng and J. F. R. Duncan, "The largest Mathieu group and (mock) automorphic forms," in *String-Math 2011*, vol. 85 of *Proc. Sympos. Pure Math.*, pp. 53–82. Amer. Math. Soc., Providence, RI, 2012. http://dx.doi.org/10.1090/pspum/085/1374.

[42] J. S. Ellenberg, "On the error term in Duke's estimate for the average special value of L-functions," *Canad. Math. Bull.* **48** (2005) no. 4, 535–546. http://dx.doi.org/10.4153/CMB-2005-049-8.

[43] V. Gritsenko, "Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms," *Algebra i Analiz* **11** (1999) no. 5, 100–125.

[44] S. Zwegers, *Mock Theta Functions*. PhD thesis, Utrecht University, 2002.

[45] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," arXiv:1208.4074 [hep-th].

[46] K. Bringmann and K. Ono, "The $f(q)$ mock theta function conjecture and partition ranks," *Invent. Math.* **165** (2006) no. 2, 243–266. http://dx.doi.org/10.1007/s00222-005-0493-5.

[47] K. Bringmann and K. Ono, "Dyson's ranks and Maass forms," *Ann. of Math. (2)* **171** (2010) no. 1, 419–449. http://dx.doi.org/10.4007/annals.2010.171.419.

[48] T. Eguchi and K. Hikami, "Superconformal algebras and mock theta functions," *J.Phys.A* **42:304010,2009** (Dec., 2008) , 0812.1151.

[49] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions 2. Rademacher Expansion for K3 Surface," *Communications in Number Theory and Physics* **3,** (Apr., 2009) 531–554, 0904.0911.

[50] C. Vafa and E. Witten, "A strong coupling test of S-duality," *Nuclear Phys. B* **431** (1994) no. 1-2, 3–77. http://dx.doi.org/10.1016/0550-3213(94)90097-3.

[51] L. Göttsche and D. Zagier, "Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_+ = 1$," *Selecta Math. (N.S.)* **4** (1998) no. 1, 69–115. http://dx.doi.org/10.1007/s000290050025.

[52] J. Troost, "The non-compact elliptic genus: mock or modular," *JHEP* **1006** (2010) 104. arXiv:1004.3649 [hep-th].

[53] J. Manschot, "The Betti numbers of the moduli space of stable sheaves of rank 3 on $\mathbb{P}^2$," *Lett. Math. Phys.* **98** (2011) no. 1, 65–78. http://dx.doi.org/10.1007/s11005-011-0490-0.

[54] R. Lawrence and D. Zagier, "Modular forms and quantum invariants of 3-manifolds," *Asian J. Math.* **3** (1999) no. 1, 93–107. Sir Michael Atiyah: a great mathematician of the twentieth century.

[55] D. Zagier, "Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)," *Astérisque* (2009) no. 326, Exp. No. 986, vii–viii, 143–164 (2010). Séminaire Bourbaki. Vol. 2007/2008.

[56] A. Folsom, "What is ... a mock modular form?," *Notices Amer. Math. Soc.* **57** (2010) no. 11, 1441–1443.

[57] K. Ono, "Unearthing the visions of a master: harmonic Maass forms and number theory," in *Current developments in mathematics, 2008*, pp. 347–454. Int. Press, Somerville, MA, 2009.

[58] S. Ramanujan, *The lost notebook and other unpublished papers*. Springer-Verlag, Berlin, 1988. With an introduction by George E. Andrews.

[59] B. Gordon and R. J. McIntosh, "Some eighth order mock theta functions," *J. London Math. Soc. (2)* **62** (2000) no. 2, 321–335. http://dx.doi.org/10.1112/S0024610700008735.

[60] M. Eichler and D. Zagier, *The theory of Jacobi forms*. Birkhäuser, 1985.

[61] V. Gritsenko, "Complex vector bundles and Jacobi forms," *Sūrikaisekikenkyūsho Kōkyūroku* (1999) no. 1103, 71–85. Automorphic forms and L-functions (Kyoto, 1999).

[62] N.-P. Skoruppa, *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 159. Universität Bonn Mathematisches Institut, Bonn, 1985. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1984.

[63] A. W. Knapp, *Elliptic curves*, vol. 40 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1992.

[64] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971. Kanô Memorial Lectures, No. 1.

[65] B. J. Birch and H. P. F. Swinnerton-Dyer, "Notes on elliptic curves. II," *J. Reine Angew. Math.* **218** (1965) 79–108.

[66] A. Wiles, "The Birch and Swinnerton-Dyer conjecture," in *The millennium prize problems*, pp. 31–41. Clay Math. Inst., Cambridge, MA, 2006.

[67] J. H. Bruinier and J. Funke, "On two geometric theta lifts," *Duke Math. J.* **125** (2004) no. 1, 45–90. http://dx.doi.org/10.1215/S0012-7094-04-12513-8.

[68] N.-P. Skoruppa, "Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms," *Invent. Math.* **102** (1990) no. 3, 501–520. http://dx.doi.org/10.1007/BF01233438.

[69] N.-P. Skoruppa, "Heegner cycles, modular forms and Jacobi forms," *Sém. Théor. Nombres Bordeaux (2)* **3** (1991) no. 1, 93–116. http://jtnb.cedram.org/item?id=JTNB_1991__3_1_93_0.

[70] W. Duke, "The critical order of vanishing of automorphic L-functions with large level," *Invent. Math.* **119** (1995) no. 1, 165–174. http://dx.doi.org/10.1007/BF01245178.

[71] W. Stein, "The Modular Forms Database,". http://wstein.org/Tables.

[72] G. Hoehn, "Selbstduale vertexoperatorsuperalgebren und das babymonster (self-dual vertex operator super algebras and the baby monster)," *Bonner Mathematische Schriften, Vol.* **286,** (June, 2007) 1–85,Bonn1996, 0706.0236.

[73] M. R. Gaberdiel, S. Gukov, C. A. Keller, G. W. Moore, and H. Ooguri, "Extremal $N = (2, 2)$ 2D conformal field theories and constraints of modularity," *Commun. Number Theory Phys.* **2** (2008) no. 4, 743–801.

[74] A. Maloney, W. Song, and A. Strominger, "Chiral Gravity, Log Gravity and Extremal CFT," *Phys.Rev.* **D81** (2010) 064007, arXiv:0903.4573 [hep-th].

[75] S. Carlip, S. Deser, A. Waldron, and D. Wise, "Topologically Massive AdS Gravity," *Phys.Lett.* **B666** (2008) 272–276, arXiv:0807.0486 [hep-th].

[76] S. Carlip, S. Deser, A. Waldron, and D. Wise, "Cosmological Topologically Massive Gravitons and Photons," *Class.Quant.Grav.* **26** (2009) 075008, arXiv:0803.3998 [hep-th].

[77] G. Giribet, M. Kleban, and M. Porrati, "Topologically Massive Gravity at the Chiral Point is Not Chiral," *JHEP* **0810** (2008) 045, arXiv:0807.4703 [hep-th].

[78] D. Grumiller, R. Jackiw, and N. Johansson, "Canonical analysis of cosmological topologically massive gravity at the chiral point," arXiv:0806.4185 [hep-th]. Published in 'Fundamental Interactions: A Memorial Volume for Wolfgang Kummer,' Editors: Daniel Grumiller, Anton Rebhan and Dimitri Vassilevich, World Scientific, 2010, pp.363-374.

[79] A. Strominger, "A Simple Proof of the Chiral Gravity Conjecture," arXiv:0808.0506 [hep-th].

[80] S. Carlip, "The Constraint Algebra of Topologically Massive AdS Gravity," *JHEP* **0810** (2008) 078, arXiv:0807.4152 [hep-th].

[81] W. Li, W. Song, and A. Strominger, "Comment on 'Cosmological Topological Massive Gravitons and Photons'," arXiv:0805.3101 [hep-th].

[82] Y. S. Myung, "Logarithmic conformal field theory approach to topologically massive gravity," *Phys.Lett.* **B670** (2008) 220–223, arXiv:0808.1942 [hep-th].

[83] I. Sachs and S. N. Solodukhin, "Quasi-Normal Modes in Topologically Massive Gravity," *JHEP* **0808** (2008) 003, arXiv:0806.1788 [hep-th].

[84] G. Gibbons, C. Pope, and E. Sezgin, "The General Supersymmetric Solution of Topologically Massive Supergravity," *Class.Quant.Grav.* **25** (2008) 205005, arXiv:0807.2613 [hep-th].

[85] K. Skenderis, M. Taylor, and B. C. van Rees, "Topologically Massive Gravity and the AdS/CFT Correspondence," *JHEP* **0909** (2009) 045, arXiv:0906.4926 [hep-th].

[86] A. J. Feingold and I. B. Frenkel, "A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus 2," *J. Math. Ann.* **263** (1983) 87–144.

[87] V. A. Gritsenko and V. V. Nikulin, "Automorphic forms and Lorentzian Kac-Moody algebras. II," *Internat. J. Math.* **9** (1998) no. 2, 201–275. http://dx.doi.org/10.1142/S0129167X98000117.

[88] V. A. Gritsenko and V. V. Nikulin, "Automorphic forms and Lorentzian Kac-Moody algebras. I," *Internat. J. Math.* **9** (1998) no. 2, 153–199. http://dx.doi.org/10.1142/S0129167X98000105.

[89] V. A. Gritsenko and V. V. Nikulin, "Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras," *Amer. J. Math.* **119** (1997) no. 1, 181–224. http://muse.jhu.edu/journals/american_journal_of_mathematics/v119/119.1gritsenko.pdf.

[90] J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.* Oxford: Clarendon Press, 1985.

[91] V. Pless, "On the uniqueness of the Golay codes," *J. Combinatorial Theory* **5** (1968) 215–228.

[92] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups.* Springer-Verlag, 1999.

[93] J. McKay, "Graphs, singularities, and finite groups," in *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, vol. 37 of *Proc. Sympos. Pure Math.*, pp. 183–186. Amer. Math. Soc., Providence, R.I., 1980.

[94] P. Slodowy, *Simple singularities and simple algebraic groups*, vol. 815 of *Lecture Notes in Mathematics.* Springer, Berlin, 1980.

[95] G. Gonzalez-Sprinberg and J.-L. Verdier, "Construction géométrique de la correspondance de McKay," *Ann. Sci. École Norm. Sup. (4)* **16** (1983) no. 3, 409–449 (1984). http://www.numdam.org/item?id=ASENS_1983_4_16_3_409_0.

[96] J. H. Conway, "A simple construction for the Fischer-Griess monster group," *Invent. Math.* **79** (1985) no. 3, 513–540.

[97] G. Glauberman and S. P. Norton, "On McKay's connection between the affine E_8 diagram and the Monster," in *Proceedings on Moonshine and related topics (Montréal, QC, 1999)*, vol. 30 of *CRM Proc. Lecture Notes*, pp. 37–42. Amer. Math. Soc., Providence, RI, 2001.

[98] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2005. (http://www.gap-system.org).

[99] B. Kostant, "The graph of the truncated icosahedron and the last letter of Galois," *Notices Amer. Math. Soc.* **42** (1995) no. 9, 959–968.

[100] B. Kostant, "Structure of the truncated icosahedron (e.g. Fullerene or C_{60}, viral coatings) and a 60-element conjugacy class in $\mathrm{PSL}(2,11)$," *Selecta Math. (N.S.)* **1** (1995) no. 1, 163–195. http://dx.doi.org/10.1007/BF01614076.

[101] C. J. Cummins and J. F. Duncan, "An E_8 Correspondence for Multiplicative Eta-Products," *Canad. Math. Bull.* **55** (2012) no. 1, 67–72. http://dx.doi.org/10.4153/CMB-2011-068-9.

[102] J. F. Duncan, "Arithmetic groups and the affine E_8 Dynkin diagram," in *Groups and symmetries*, vol. 47 of *CRM Proc. Lecture Notes*, pp. 135–163. Amer. Math. Soc., Providence, RI, 2009.

[103] S. Sakuma, "6-transposition property of τ-involutions of vertex operator algebras," *Int. Math. Res. Not. IMRN* (2007) no. 9, Art. ID rnm 030, 19. http://dx.doi.org/10.1093/imrn/rnm030.

[104] C. H. Lam, H. Yamada, and H. Yamauchi, "Vertex operator algebras, extended E_8 diagram, and McKay's observation on the Monster simple group," *Trans. Amer. Math. Soc.* **359** (2007) no. 9, 4107–4123 (electronic). http://dx.doi.org/10.1090/S0002-9947-07-04002-0.

[105] C. H. Lam, H. Yamada, and H. Yamauchi, "McKay's observation and vertex operator algebras generated by two conformal vectors of central charge 1/2," *IMRP Int. Math. Res. Pap.* (2005) no. 3, 117–181.

[106] G. Höhn, C. H. Lam, and H. Yamauchi, "McKay's E_7 observation on the Baby Monster," *Int. Math. Res. Not. IMRN* (2012) no. 1, 166–212.

[107] G. Höhn, C. H. Lam, and H. Yamauchi, "McKay's E_6 observation on the largest Fischer group," *Comm. Math. Phys.* **310** (2012) no. 2, 329–365. http://dx.doi.org/10.1007/s00220-011-1413-8.

[108] T. Eguchi and K. Hikami, "Twisted elliptic genus for $K3$ and Borcherds product," 1112.5928.

[109] S. Govindarajan, "Brewing moonshine for mathieu," 1012.5732.

[110] R. Volpato, "Mathieu Moonshine and symmetries of K3 sigma models," arXiv:1201.6172 [hep-th].

[111] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Symmetries of $K3$ sigma models," 1106.4315.

[112] A. Taormina and K. Wendland, "The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}," arXiv:1107.3834 [hep-th].

[113] B. Gordon and R. J. McIntosh, "A survey of classical mock theta functions," in *Partitions, q-Series, and Modular Forms*, K. Alladi and F. Garvan, eds., vol. 23 of *Developments in Mathematics*, pp. 95–144. Springer New York, 2012. http://dx.doi.org/10.1007/978-1-4614-0028-8$_9$. 10.1007/978-1-4614-0028-8_9.

[114] Y. Zhu, "Modular invariance of characters of vertex operator algebras," *Journal of the American Mathematical Society* **9** (1996) no. 1, 237–302.

[115] C. Dong, H. Li, and G. Mason, "Modular invariance of trace functions in orbifold theory and generalized Moonshine," *Communications in Mathematical Physics* **214** (2000) 1–56, q-alg/9703016.

[116] M. Benard, "Schur indexes of sporadic simple groups," *J. Algebra* **58** (1979) no. 2, 508–522. http://dx.doi.org/10.1016/0021-8693(79)90177-7.

[117] R. S. Margolin, "Representations of M_{12}," *J. Algebra* **156** (1993) no. 2, 362–369. http://dx.doi.org/10.1006/jabr.1993.1078.

[118] S. Mukai, "Finite groups of automorphisms of $K3$ surfaces and the Mathieu group," *Invent. Math.* **94** (1988) no. 1, 183–221. http://dx.doi.org/10.1007/BF01394352.

[119] S. Kondō, "Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces," *Duke Math. J.* **92** (1998) no. 3, 593–603.

http://dx.doi.org/10.1215/S0012-7094-98-09217-1. With an appendix by Shigeru Mukai.

[120] V. A. Gritsenko and V. V. Nikulin, "$K3$ surfaces, Lorentzian Kac-Moody algebras and mirror symmetry," *Math. Res. Lett.* **3** (1996) no. 2, 211–229.

[121] V. A. Gritsenko and V. V. Nikulin, "The arithmetic mirror symmetry and Calabi-Yau manifolds," *Comm. Math. Phys.* **210** (2000) no. 1, 1–11. http://dx.doi.org/10.1007/s002200050769.

[122] I. V. Dolgachev, "Mirror symmetry for lattice polarized $K3$ surfaces," *J. Math. Sci.* **81** (1996) no. 3, 2599–2630. http://dx.doi.org/10.1007/BF02362332. Algebraic geometry, 4.

[123] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "Counting Dyons in N=4 String Theory," *Nucl.Phys.B* **484:543-561,1997** (Nucl.Phys.B484:543-561,1997) , hep-th/9607026.

[124] A. Sen, "Walls of Marginal Stability and Dyon Spectrum in N=4 Supersymmetric String Theories," *JHEP* **0705:039,2007** (JHEP 0705:039,2007) , hep-th/0702141.

[125] M. C. N. Cheng and E. Verlinde, "Dying Dyons Don't Count," *JHEP* **09** (2007) 070, arXiv:0706.2363 [hep-th].

[126] M. C. Cheng and E. P. Verlinde, "Wall Crossing, Discrete Attractor Flow, and Borcherds Algebra," *SIGMA* **4** (2008) 068, arXiv:0806.2337 [hep-th].

[127] A. Dabholkar, D. Gaiotto, and S. Nampuri, "Comments on the spectrum of CHL dyons," *JHEP* **01** (2008) 023, arXiv:hep-th/0702150.

[128] A. Dabholkar, "Cargese lectures on black holes, dyons, and modular forms," *Nucl. Phys. Proc. Suppl.* **171** (2007) 2–15.

[129] S. Kachru and C. Vafa, "Exact results for N=2 compactifications of heterotic strings," *Nucl.Phys.* **B450** (1995) 69–89, arXiv:hep-th/9505105 [hep-th].

[130] S. Ferrara, J. A. Harvey, A. Strominger, and C. Vafa, "Second quantized mirror symmetry," *Phys.Lett.* **B361** (1995) 59–65, arXiv:hep-th/9505162 [hep-th].

[131] J. A. Harvey and G. W. Moore, "Algebras, BPS states, and strings," *Nucl.Phys.* **B463** (1996) 315–368, arXiv:hep-th/9510182 [hep-th].

[132] R. E. Borcherds, "Automorphic forms with singularities on Grassmannians," *Invent. Math.* **132** (1998) no. 3, 491–562. http://dx.doi.org/10.1007/s002220050232.

[133] M. Kontsevich, "Product formulas for modular forms on O$(2, n)$ (after R. Borcherds)," *Astérisque* (1997) no. 245, Exp. No. 821, 3, 41–56. Séminaire Bourbaki, Vol. 1996/97.

[134] G. Lopes Cardoso, "Perturbative gravitational couplings and Siegel modular forms in D = 4, N=2 string models," *Nucl.Phys.Proc.Suppl.* **56B** (1997) 94–101, arXiv:hep-th/9612200 [hep-th].

[135] G. Lopes Cardoso, G. Curio, and D. Lust, "Perturbative couplings and modular forms in N=2 string models with a Wilson line," *Nucl.Phys.* **B491** (1997) 147–183, arXiv:hep-th/9608154 [hep-th].

[136] S. Govindarajan, "BKM Lie superalgebras from counting twisted CHL dyons," *JHEP* **1105** (2011) 089, arXiv:1006.3472 [hep-th]. Dedicated to the memories of Jaydeep Majumder and Alok Kumar.

[137] http://modi.countnumber.de/index.php?chap=ell.newforms/ell.newforms.html.

[138] C. Itzykson, ed., *From Number Theory to Physics.* Springer, 1992.

[139] H. Aoki and T. Ibukiyama, "Simple graded rings of Siegel modular forms, differential operators and Borcherds products," *Int. J. Math.* **16, No. 3** (2005) 249–279.

arXiv:0904.0911v2 [math-ph] 15 Sep 2009

SUPERCONFORMAL ALGEBRAS AND MOCK THETA FUNCTIONS 2. RADEMACHER EXPANSION FOR K3 SURFACE

TOHRU EGUCHI AND KAZUHIRO HIKAMI

Abstract. The elliptic genera of the K3 surfaces, both compact and non-compact cases, are studied by using the theory of mock theta functions. We decompose the elliptic genus in terms of the $\mathcal{N}=4$ superconformal characters at level-1, and present an exact formula for the coefficients of the massive (non-BPS) representations using Poincaré–Maass series.

1. Introduction

Studies of an asymptotic behaviour of $p(n)$, which is the number of partitions of n defined by

$$\prod_{n=1}^{\infty}\frac{1}{1-q^n}=\sum_{n=0}^{\infty}p(n)\,q^n,$$

were initiated by Hardy and Ramanujan [22]. The generating function in the left hand side is essentially the inverse of the Dedekind η function, and it is well-known that its asymptotic behavior is given by

$$p(n)\sim\frac{1}{4\,n\,\sqrt{3}}\,\mathrm{e}^{\pi\sqrt{\frac{2\,n}{3}}}.$$

An exact asymptotic expansion was later derived by Rademacher by use of the circle method (*e.g.* [41, Chapter 14]) as

$$p(n)=\frac{\pi}{(24\,n-1)^{\frac{3}{4}}}\sum_{c=1}^{\infty}\frac{1}{\sqrt{12\,c}}\,I_{\frac{3}{2}}\left(\frac{\pi}{6\,c}\sqrt{24\,n-1}\right)\sum_{\substack{d \mod 24\,c\\ d^2=-24\,n+1 \mod 24\,c}}\left(\frac{12}{d}\right)\mathrm{e}^{\frac{d}{6\,c}\,\pi\,\mathrm{i}},$$

Here $I_{3/2}$ denotes the modified Bessel-function and we have introduced the Legendre symbol $\left(\frac{12}{\bullet}\right)$.

The number of partition $p(n)$ has received wide interests [2], and has played an important role in various areas of mathematics and physics. One of the generalizations of $p(n)$ is given by the Ramanujan mock theta function [11] (see also Refs. 3, 21 for a review),

$$\begin{aligned}f(q)&=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2\,(1+q^2)^2\cdots(1+q^n)^2}\\&=1+\sum_{n=1}^{\infty}\alpha(n)\,q^n.\end{aligned}$$

Date: April 6, 2009. Revised on September 14, 2009.

Asymptotic behavior of $\alpha(n)$ was written in Ramanujan's last letter to Hardy as

$$\alpha(n) \sim \frac{(-1)^{n-1}}{2\sqrt{n-\frac{1}{24}}}\, \mathrm{e}^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}},$$

which was proved by Dragonette [10]. This asymptotics was later improved by Andrews [1], and an exact formula for $\alpha(n)$ called the Andrews–Dragonette identity was conjectured. Bringmann and Ono proved the conjecture using the work of Zwegers [4] on mock theta functions.

In our previous paper [12], we have shown that the theory of mock theta functions is useful in studying the elliptic genera of hyperKähler manifolds in terms of the representations of $\mathcal{N}=4$ superconformal algebra. We have used the results of Zwegers [47] that mock theta function is a holomorphic part of the harmonic Maass form with weight-1/2 and has a weight-3/2 (vector) modular form as its "shadow" (see *e.g.* Refs. 39, 46 for reviews). As one of the applications of this structure behind the mock theta functions, we employ the method of Bringmann and Ono [4, 5] (see also Ref. 6) in this paper to derive an exact formula for the Fourier coefficients of the elliptic genus for the K3 surface which counts the number of non-BPS (massive) representations. Analogous computations of Fourier coefficients of the partition function of three-dimensional gravity using the Poincaré series were discussed in Refs. 9, 33, 34, 36. In these papers Jacobi forms are considered instead of mock theta functions.

We shall also give an exact formula for the number of non-BPS representations for the ALE space, which is a degenerate limit of the K3 surface. The ALE spaces, or the asymptotically locally Euclidean spaces, are hyperKähler 4-manifolds, and are constructed from resolutions of the Kleinian singularities.

We also would like to point out that the non-holomorphic partner of the level-1 superconformal characters is connected to the Witten–Reshetikhin–Turaev (WRT) invariant for 3-manifold associated with the D_4-type singularity.

This paper is organized as follows. In section 2 we recall a relationship between the $\mathcal{N}=4$ superconformal characters and the mock theta functions studied in Ref. 12. We briefly review the elliptic genus of the K3 surfaces in section 3. We show how to decompose the elliptic genus in terms of the superconformal characters. In section 4, we introduce the Poincaré series, whose holomorphic part has the same Fourier coefficients as the number of non-BPS representations. Following Bringmann and Ono, we compute the Fourier expansion of the Poincaré series, and give an exact asymptotic formula. We numerically compute the asymptotic expansion to confirm the validity of our analytic expressions. In section 5 we recall a fact that a limiting value of the non-holomorphic part of the harmonic Maass form is related to the SU(2) WRT invariant for the Seifert manifold $M(2,2,2)$. The last section is devoted to concluding remarks.

2. The $\mathcal{N}=4$ Superconformal Algebras and Mock Theta Functions

The $\mathcal{N}=4$ superconformal algebra is generated by the energy-momentum tensor, 4 supercurrents, and a triplet of currents which constitute the affine Lie algebra $SU(2)_k$. The central charge is quantized to $c=6\,k$, and the unitary highest weight state is labeled by the conformal weight h and the isospin ℓ;

$$L_0\ |\Omega\rangle = h\ |\Omega\rangle\,,$$

$$T_0^3\ |\Omega\rangle = \ell\ |\Omega\rangle\,.$$

where $h \geq k/4$ and $0 \leq \ell \leq k/2$ in the Ramond sector. The character of a representation is given by

$$\mathrm{ch}_{k,h,\ell}(z;\tau) = \mathrm{Tr}_{\mathcal{H}}\left(\mathrm{e}^{2\,\pi\,\mathrm{i}\,z\,T_0^3}\,q^{L_0-\frac{c}{24}}\right), \tag{2.1}$$

where $q=\mathrm{e}^{2\,\pi\,\mathrm{i}\,\tau}$ with $\tau\in\mathbb{H}$, and $\mathcal{H}$ denotes the Hilbert space of the representation.

There exist two types of representations in the $\mathcal{N}=4$ superconformal algebra [17, 18]: massless (BPS) and massive (non-BPS) representations. In the Ramond sector, their character formulas are given as follows;

- massless representations ($h=\frac{k}{4}$, and $\ell=0,\frac{1}{2},\ldots,\frac{k}{2}$),

$$\mathrm{ch}^R_{k,\frac{k}{4},\ell}(z;\tau) = \frac{\mathrm{i}}{\theta_{11}(2z;\tau)}\cdot\frac{\left[\theta_{10}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3}\sum_{\varepsilon=\pm1}\sum_{m\in\mathbb{Z}}\varepsilon\,\frac{\mathrm{e}^{4\pi\mathrm{i}\varepsilon((k+1)m+\ell)z}}{\left(1+\mathrm{e}^{-2\pi\mathrm{i}\varepsilon z}\,q^{-m}\right)^2}\,q^{(k+1)m^2+2\ell m}. \tag{2.2}$$

  See Appendix for definitions of the Jacobi theta functions.
- massive representations ($h>\frac{k}{4}$ and $\ell=\frac{1}{2},1,\ldots,\frac{k}{2}$),

$$\mathrm{ch}^R_{k,h,\ell}(z;\tau) = q^{h-\frac{\ell^2}{k+1}-\frac{k}{4}}\,\frac{\left[\theta_{10}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3}\,\chi_{k-1,\ell-\frac{1}{2}}(z;\tau), \tag{2.3}$$

  where $\chi_{k,\ell}(z;\tau)$ denotes the affine SU(2) character

$$\chi_{k,\ell}(z;\tau) = \frac{\vartheta_{k+2,2\ell+1}-\vartheta_{k+2,-2\ell-1}}{\vartheta_{2,1}-\vartheta_{2,-1}}(z;\tau),$$

  with the theta series defined by

$$\vartheta_{P,a}(z;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2Pn+a)^2}{4P}}\,\mathrm{e}^{2\pi\mathrm{i}z(2Pn+a)}.$$

Characters in other sectors are related to each other by the spectral flow;

$$\begin{array}{ccc} R: z+\frac{\tau}{2} & \leftrightarrow & \widetilde{R}: z+\frac{1+\tau}{2} \\ \updownarrow & & \updownarrow \\ NS: z & \leftrightarrow & \widetilde{NS}: z+\frac{1}{2} \end{array}$$

Hereafter we study the theory at level $k=1$ ($c=6$) in the $\widetilde{R}$-sector, where the massless character is given by

$$\mathrm{ch}^{\widetilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau)=\frac{\mathrm{i}}{\theta_{11}(2z;\tau)}\cdot\frac{\left[\theta_{11}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3}\sum_{m\in\mathbb{Z}}q^{2m^2}\,\mathrm{e}^{8\pi\mathrm{i}mz}\,\frac{1+\mathrm{e}^{2\pi\mathrm{i}z}\,q^m}{1-\mathrm{e}^{2\pi\mathrm{i}z}\,q^m}. \tag{2.4}$$

It is known that this formula may be rewritten as [19]

$$\mathrm{ch}^{\widetilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau)=\frac{\left[\theta_{11}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3}\,\mu(z;\tau), \tag{2.5}$$

where $\mu(z;\tau)$ is a Lerch sum defined by

$$\mu(z;\tau)=\frac{\mathrm{i}\,\mathrm{e}^{\pi\mathrm{i}z}}{\theta_{11}(z;\tau)}\sum_{n\in\mathbb{Z}}(-1)^n\,\frac{q^{\frac{1}{2}n(n+1)}\,\mathrm{e}^{2\pi\mathrm{i}nz}}{1-q^n\,\mathrm{e}^{2\pi\mathrm{i}z}}. \tag{2.6}$$

Note that the massless characters fulfill an identity

$$\begin{aligned}\mathrm{ch}^{\widetilde{R}}_{k=1,h=\frac{1}{4},\ell=\frac{1}{2}}(z;\tau)+2\,\mathrm{ch}^{\widetilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau)&=q^{-\frac{1}{8}}\,\frac{\left[\theta_{11}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3}\\&=\lim_{h\searrow\frac{1}{4}}\mathrm{ch}^{\widetilde{R}}_{k=1,h,\ell=\frac{1}{2}}(z;\tau),\end{aligned} \tag{2.7}$$

which shows that the non-BPS representation decomposes into a sum of BPS representations at the unitarity bound.

As shown in Ref. 47, we can complete the Lerch sum $\mu(z;\tau)$ to a Jacobi-like form $\widehat{\mu}(z;\tau)$ as

$$\widehat{\mu}(z;\tau)=\mu(z;\tau)-\frac{1}{2}\,R(\tau). \tag{2.8}$$

Here $R(\tau)$ denotes a non-holomorphic function defined by

$$R(\tau)=\sum_{n\in\mathbb{Z}}(-1)^n\left[\mathrm{sgn}\left(n+\frac{1}{2}\right)-E\left(\left(n+\frac{1}{2}\right)\sqrt{2\,\Im\tau}\right)\right]q^{-\frac{1}{2}\left(n+\frac{1}{2}\right)^2}. \tag{2.9}$$

$E(z)$ denotes the error function given by

$$E(z)=2\int_0^z\mathrm{e}^{-\pi u^2}\mathrm{d}u=1-\mathrm{erfc}\left(\sqrt{\pi}\,z\right). \tag{2.10}$$

The function $\widehat{\mu}(z;\tau)$ transforms like a Jacobi form [20] as follows;

$$\begin{aligned}\widehat{\mu}(z;\tau)&=-\sqrt{\frac{\mathrm{i}}{\tau}}\,\widehat{\mu}\left(\frac{z}{\tau};-\frac{1}{\tau}\right),\\ \widehat{\mu}(z;\tau+1)&=\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\,\widehat{\mu}(z;\tau),\\ \widehat{\mu}(z+1;\tau)&=\widehat{\mu}(z+\tau;\tau)=\widehat{\mu}(z;\tau).\end{aligned} \tag{2.11}$$

One can conclude from Ref. 47 that the non-holomorphic function (2.9) is a period integral of the third power of the Dedekind η-function,

$$R(\tau)=\frac{1}{\sqrt{\mathrm{i}}}\int_{-\overline{\tau}}^{\mathrm{i}\infty}\frac{\left[\eta(x)\right]^3}{\sqrt{x+\tau}}\,\mathrm{d}x. \tag{2.12}$$

Then the function $\widehat{\mu}(z;\tau)$ satisfies

$$\frac{\partial}{\partial\overline{\tau}}\widehat{\mu}(z;\tau)=\frac{\mathrm{i}}{2}\,\frac{[\eta(-\overline{\tau})]^3}{\sqrt{2\,\Im\tau}}, \tag{2.13}$$

and

$$(\Im\tau)^{\frac{3}{2}}\,\frac{\partial}{\partial\tau}\sqrt{\Im\tau}\frac{\partial}{\partial\overline{\tau}}\,\widehat{\mu}(z;\tau)=0. \tag{2.14}$$

Here the derivatives in $\tau=u+\mathrm{i}\,v$ are defined by

$$\frac{\partial}{\partial\tau}=\frac{1}{2}\left(\frac{\partial}{\partial u}-\mathrm{i}\,\frac{\partial}{\partial v}\right),\qquad\qquad \frac{\partial}{\partial\overline{\tau}}=\frac{1}{2}\left(\frac{\partial}{\partial u}+\mathrm{i}\,\frac{\partial}{\partial v}\right).$$

Thus the function $\widehat{\mu}(z;\tau)$ is a harmonic Maass form with weight 1/2 and has $[\eta(\tau)]^3$ as its "shadow" according to the terminology of Zagier [46]. Similar q-series was studied in recent work on the Donaldson invariant [32]. We note that the above differential equation (2.14) reduces to

$$\Delta_{\frac{1}{2}}\,\widehat{\mu}(z;\tau)=0, \tag{2.15}$$

where Δ_k is the hyperbolic Laplacian of weight k,

$$\Delta_k=-v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+\mathrm{i}\,k\,v\left(\frac{\partial}{\partial u}+\mathrm{i}\,\frac{\partial}{\partial v}\right). \tag{2.16}$$

3. Elliptic Genus of K3 Surface and Harmonic Maass Form

3.1. **Decomposition of Elliptic Genus into $\mathcal{N}$=4 Characters.** The elliptic genus of the Calabi-Yau manifold X with complex dimension $c/3$ is identified as [44]

$$Z_X(z;\tau)=\mathrm{Tr}_{\mathcal{H}^R\otimes\mathcal{H}^R}(-1)^F\,\mathrm{e}^{2\,\pi\,\mathrm{i}\,T_0^3\,z}\,q^{L_0-\frac{c}{24}}\,\overline{q}^{\overline{L}_0-\frac{c}{24}}, \tag{3.1}$$

where $(-1)^F=\mathrm{e}^{\pi\,\mathrm{i}\left(T_0^3-\overline{T}_0^3\right)}$, and $\mathcal{H}^R$ is the Hilbert space on the Ramond sector. In the case of K3 surface, it is known that [14, 30]

$$Z_{K3}(z;\tau)=8\left[\left(\frac{\theta_{10}(z;\tau)}{\theta_{10}(0;\tau)}\right)^2+\left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2+\left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2\right]. \tag{3.2}$$

To rewrite the elliptic genus in terms of the level-1 characters we introduce the function $J(z;w;\tau)$ by

$$\begin{aligned}J(z;w;\tau)&=\frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3}\,\left(\widehat{\mu}(z;\tau)-\widehat{\mu}(w;\tau)\right)\\&=\frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3}\,\left(\mu(z;\tau)-\mu(w;\tau)\right)\\&=\mathrm{ch}^{\widetilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau)-\frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3}\,\mu(w;\tau).\end{aligned} \tag{3.3}$$

Note that

$$J(z;z;\tau)=0. \tag{3.4}$$

It was shown [12] that $J(z;w;\tau)$ behaves like a 2-variable Jacobi form [20] under the modular transformation;

$$\begin{aligned} J(z;w;\tau) &= \mathrm{e}^{-2\pi\mathrm{i}\frac{z^2}{\tau}}\, J\left(\frac{z}{\tau};\frac{w}{\tau};-\frac{1}{\tau}\right), \\ J(z+1;w;\tau) &= J(z;w;\tau+1) = J(z;w+\tau;\tau) = J(z;w;\tau), \\ J(z+\tau;w;\tau) &= q^{-1}\,\mathrm{e}^{-4\pi\mathrm{i}z}\, J(z;w;\tau). \end{aligned} \tag{3.5}$$

These modular properties together with (3.4) show that $J(z;w;\tau)$ with w at the half-periods $\left\{\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2}\right\}$ is given by Jacobi theta functions as follows;

$$\begin{aligned} J\left(z;\frac{1}{2};\tau\right) &= \left(\frac{\theta_{10}(z;\tau)}{\theta_{10}(0;\tau)}\right)^2, \\ J\left(z;\frac{1+\tau}{2};\tau\right) &= \left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2, \\ J\left(z;\frac{\tau}{2};\tau\right) &= \left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2. \end{aligned} \tag{3.6}$$

Note that the Lerch sums introduced in Ref. 19 are proportional to $\mu(w;\tau)$ at $w=\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2}$, respectively;

$$\begin{aligned} h_2(\tau) &\equiv \frac{\mu\left(\frac{1}{2};\tau\right)}{\eta(\tau)} = \frac{1}{\eta(\tau)\,\theta_{10}(0;\tau)}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{1}{2}n(n+1)}}{1+q^n}, \\ h_3(\tau) &\equiv \frac{\mu\left(\frac{1+\tau}{2};\tau\right)}{\eta(\tau)} = \frac{1}{\eta(\tau)\,\theta_{00}(0;\tau)}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{1}{2}n^2-\frac{1}{8}}}{1+q^{n-\frac{1}{2}}}, \\ h_4(\tau) &\equiv \frac{\mu\left(\frac{\tau}{2};\tau\right)}{\eta(\tau)} = \frac{1}{\eta(\tau)\,\theta_{01}(0;\tau)}\sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{\frac{1}{2}n^2-\frac{1}{8}}}{1-q^{n-\frac{1}{2}}}. \end{aligned} \tag{3.7}$$

We thus find that, using (3.6), the elliptic genus (3.2) is written as

$$Z_{K3}(z;\tau) = 24\ \mathrm{ch}^{\widetilde{R}}_{k=1,h=\frac{1}{4},\ell=0}(z;\tau) - 8\left[\frac{\theta_{11}(z;\tau)}{\eta(\tau)}\right]^2\sum_{a=2,3,4}h_a(\tau).$$

For later convenience let us define

$$\begin{aligned} \Sigma(\tau) &\equiv 8\sum_{w\in\left\{\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2}\right\}}\mu(w;\tau) \\ &= 8\,\eta(\tau)\sum_{a=2,3,4}h_a(\tau) \\ &= q^{-\frac{1}{8}}\left(2-\sum_{n=1}^{\infty}A_n\,q^n\right). \end{aligned} \tag{3.8}$$

Here coefficients A_n are positive integers, and some of them are computed as follows;

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
A_n	90	462	1540	4554	11592	27830	61686	131100	265650	521136	$\cdots$

(3.9)

Using the massive character (2.3) ($k=1, \ell=1/2$) and the identity (2.7) we conclude that the elliptic genus (3.2) for K3 surface is decomposed into a sum of superconformal characters as [14]

$$Z_{K3}(z;\tau) = 20\ \mathrm{ch}^{\widetilde{R}}_{1,\frac{1}{4},0}(z;\tau) - 2\ \mathrm{ch}^{\widetilde{R}}_{1,\frac{1}{4},\frac{1}{2}}(z;\tau) + \sum_{n=1}^{\infty} A_n\ \mathrm{ch}^{\widetilde{R}}_{1,n+\frac{1}{4},\frac{1}{2}}(z;\tau). \tag{3.10}$$

Here the first two terms are isospin $\ell=0$ and $1/2$ massless representations, and the last term denotes an infinite sum of massive representations.

3.2. **Elliptic Genus of ALE space.** The isospin-$\frac{1}{2}$ term in (3.10) comes from $\mu\left(\frac{1}{2};\tau\right)$ and corresponds to the identity representation in the NS sector. It describes the gravity multiplet in string compactification on $K3$ surface. When we decompactify K3 into an ALE space, we decouple gravity. Thus we may drop $\mu\left(\frac{1}{2};\tau\right)$ from the elliptic genus and consider [15, 16]

$$Z_{K3,\mathrm{decompactified}}(z;\tau) = 8\left[\left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2 + \left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2\right]. \tag{3.11}$$

When we define

$$\begin{aligned}\Sigma^{\circ}(\tau) &\equiv 8\,\mu\left(\frac{1}{2};\tau\right)\\ &= q^{-\frac{1}{8}}\left(2-\sum_{n=1}^{\infty} A^{\circ}_n\, q^n\right),\end{aligned} \tag{3.12}$$

we have a character decomposition of the elliptic genus as

$$Z_{K3,\mathrm{decompactified}}(z;\tau) = 16\ \mathrm{ch}^{\widetilde{R}}_{1,\frac{1}{4},0}(z;\tau) + \sum_{n=1}^{\infty}(A_n - A^{\circ}_n)\ \mathrm{ch}^{\widetilde{R}}_{1,n+\frac{1}{4},\frac{1}{2}}(z;\tau). \tag{3.13}$$

Here coefficients A°_n are integers, and some of them are computed from (2.6) as follows;

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
A°_n	-6	14	-28	42	-56	86	-138	188	-238	336	$\cdots$

(3.14)

It is known that the K3 surface may be decomposed into a sum of 16 A_1 spaces [40], and the elliptic genus of A_1 space is proposed as [15];

$$\begin{aligned}Z_{A_1}(z;\tau) &= \frac{1}{2}\left[\left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2 + \left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2\right]\\ &= \mathrm{ch}^{\widetilde{R}}_{1,\frac{1}{4},0}(z;\tau) + \frac{1}{16}\sum_{n=1}^{\infty}(A_n - A^{\circ}_n)\ \mathrm{ch}^{\widetilde{R}}_{1,n+\frac{1}{4},\frac{1}{2}}(z;\tau).\end{aligned} \tag{3.15}$$

Note that $(A_n - A^{\circ}_n)/16$ are all positive integers.

4. Poincaré–Maass Series and Character Decomposition

4.1. **Multiplier System for Harmonic Maass Forms.** As the completion of $\Sigma(\tau)$ defined in (3.8), we define

$$\widehat{\Sigma}(\tau) = 8 \sum_{w \in \left\{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\right\}} \widehat{\mu}(w;\tau), \tag{4.1}$$

which transforms as

$$\widehat{\Sigma}\left(-\frac{1}{\tau}\right) = -\sqrt{\frac{\tau}{\mathrm{i}}}\,\widehat{\Sigma}(\tau), \tag{4.2}$$

$$\widehat{\Sigma}(\tau+1) = \mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\,\widehat{\Sigma}(\tau). \tag{4.3}$$

One finds in (4.2) and (4.3) that the multiplier system for Σ is a complex conjugate to that of $[\eta(\tau)]^3$. We recall a transformation formula of the Dedekind η-function (*e.g.* [41, Chapter 9]);

$$\eta\left(\gamma(\tau)\right) = \mathrm{i}^{-\frac{1}{2}}\,\mathrm{e}^{\frac{a+d}{12\,c}\,\pi\,\mathrm{i} - s(d,c)\,\pi\,\mathrm{i}}\,\sqrt{c\,\tau+d}\,\eta(\tau). \tag{4.4}$$

Here $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$ with $c>0$, and $s(d,c)$ is the Dedekind sum defined by

$$s(d,c) = \sum_{k \mod c} \left(\!\!\left(\frac{k}{c}\right)\!\!\right)\left(\!\!\left(\frac{k\,d}{c}\right)\!\!\right),$$

where

$$(\!(x)\!) = \begin{cases} x - \lfloor x \rfloor - \dfrac{1}{2}, & \text{for } x \in \mathbb{R}\setminus\mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

We thus conclude that the completion (4.1) satisfies

$$\widehat{\Sigma}\left(\gamma(\tau)\right) = \mathrm{i}^{\frac{3}{2}}\,\mathrm{e}^{-\frac{a+d}{4c}\,\pi\,\mathrm{i} + 3\,s(d,c)\,\pi\,\mathrm{i}}\,\sqrt{c\,\tau+d}\;\widehat{\Sigma}(\tau). \tag{4.5}$$

4.2. **The Whittaker Function and the Poincaré–Maass Series.** The Whittaker functions [43], $M_{\alpha,\beta}(z)$ and $W_{\alpha,\beta}(z)$, are solutions of the second order differential equation

$$\left[\frac{\partial^2}{\partial z^2} + \left(-\frac{1}{4} + \frac{\alpha}{z} + \frac{\frac{1}{4}-\beta^2}{z^2}\right)\right] w(z) = 0. \tag{4.6}$$

We have

$$W_{\alpha,\beta}(z) = \frac{\Gamma(-2\,\beta)}{\Gamma\left(\frac{1}{2}-\alpha-\beta\right)}\,M_{\alpha,\beta}(z) + \frac{\Gamma(2\,\beta)}{\Gamma\left(\frac{1}{2}-\alpha+\beta\right)}\,M_{\alpha,-\beta}(z),$$

and

$$M_{\alpha,\beta}(z) \underset{\Re z\to+\infty}{\sim} \frac{\Gamma(1+2\,\beta)}{\Gamma\left(\frac{1}{2}-\alpha+\beta\right)}\,\mathrm{e}^{\frac{z}{2}}\,z^{-\alpha},$$

$$W_{\alpha,\beta}(z) \underset{\Re z\to+\infty}{\sim} \mathrm{e}^{-\frac{z}{2}}\,z^{\alpha}.$$

Following Ref. 6, we define functions $\mathcal{M}_s^k(v)$ and $\mathcal{W}_s^k(v)$ in terms of the Whittaker functions as

$$\begin{aligned}\mathcal{M}_s^k(v) &= |v|^{-\frac{k}{2}}\ M_{\frac{k}{2}\,\mathrm{sgn}(v),s-\frac{1}{2}}\left(|v|\right),\\ \mathcal{W}_s^k(v) &= |v|^{-\frac{k}{2}}\ W_{\frac{k}{2}\,\mathrm{sgn}(v),s-\frac{1}{2}}\left(|v|\right).\end{aligned} \tag{4.7}$$

Some of them are given as follows;

$$\mathcal{W}_{\frac{3}{4}}^{\frac{1}{2}}(v) = \mathrm{e}^{-\frac{v}{2}},$$

$$\mathcal{W}_{\frac{3}{4}}^{\frac{1}{2}}(-v) = \sqrt{\pi}\left(1 - E\left(\sqrt{\frac{v}{\pi}}\right)\right)\mathrm{e}^{\frac{v}{2}},$$

$$\mathcal{M}_{\frac{3}{4}}^{\frac{1}{2}}(-v) = \frac{\sqrt{\pi}}{2}\,E\left(\sqrt{\frac{v}{\pi}}\right)\mathrm{e}^{\frac{v}{2}},$$

where we assume $v > 0$, and the error function $E(z)$ is defined in (2.10).

We define the function $\varphi_{-h,s}^k(\tau)$ for $h > 0$ by

$$\varphi_{-h,s}^k(\tau) = \mathcal{M}_s^k\left(-4\,\pi\,h\,\Im(\tau)\right)\,\mathrm{e}^{-2\,\pi\,\mathrm{i}\,h\,\Re(\tau)}. \tag{4.8}$$

Then it becomes an eigenfunction of the second order differential equation;

$$\Delta_k\,\varphi_{-h,s}^k(\tau) = \left[s\,(1-s) + \frac{k}{2}\left(\frac{k}{2}-1\right)\right]\varphi_{-h,s}^k(\tau), \tag{4.9}$$

where Δ_k is the Laplacian defined in (2.16). Note that at $\Im\tau \to +\infty$

$$\varphi_{-h,s}^k(\tau) \sim \frac{\Gamma(2\,s)}{\Gamma\left(\frac{k}{2}+s\right)}\,q^{-h}.$$

By using the function $\varphi_{-h,s}^k(\tau)$, we introduce the Poincaré–Maass series [6] as

$$P_s(\tau) = \frac{2}{\sqrt{\pi}}\sum_{\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma(1)}\left[\chi(\gamma)\right]^{-1}\frac{1}{\sqrt{c\,\tau+d}}\,\varphi_{-\frac{1}{8},s}^{\frac{1}{2}}\left(\gamma(\tau)\right), \tag{4.10}$$

where the multiplier system is chosen to be

$$\chi(\gamma) = \begin{cases}\mathrm{i}^{\frac{3}{2}}\,\mathrm{e}^{-\frac{a+d}{4c}\,\pi\,\mathrm{i}+3\,\pi\,\mathrm{i}\,s(d,c)}, & \text{for } c > 0,\\ \mathrm{e}^{-\frac{b}{4}\,\pi\,\mathrm{i}}, & \text{for } c = 0 \text{ and } d = 1.\end{cases} \tag{4.11}$$

We mean that $\Gamma(1) = SL(2;\mathbb{Z})$, and Γ_∞ is the stabilizer of ∞,

$$\Gamma_\infty = \left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\,\middle|\,n\in\mathbb{Z}\right\}.$$

We see that the Poincaré–Maass series $P_s(\tau)$ transforms in the same way as $\widehat{\Sigma}(\tau)$ (4.2), (4.3)

$$P_s\left(\gamma(\tau)\right) = \chi(\gamma)\,\sqrt{c\,\tau+d}\,P_s(\tau), \tag{4.12}$$

and that, due to a commutativity of the Laplacian Δ_k (2.16) and the γ action, it is an eigenfunction of the Laplacian

$$\Delta_{\frac{1}{2}} P_s(\tau) = \left[s\,(1-s) - \frac{3}{16} \right] P_s(\tau). \tag{4.13}$$

At $\Im\tau \to +\infty$, the asymptotic behavior of the M-Whittaker function shows that the Poincaré–Maass series behaves exponentially as $\mathrm{e}^{\pi\,\Im(\tau)/4}$.

Note that $P_s(\tau)$ is annihilated by the Laplacian $\Delta_{\frac{1}{2}}$ like $\widehat{\mu}(z;\tau)$ (2.15) when we set $s = \frac{3}{4}$;

$$\Delta_{\frac{1}{2}}\, P_{\frac{3}{4}}(\tau) = 0. \tag{4.14}$$

These facts show that the Poincaré–Maass series $P_{\frac{3}{4}}(\tau)$ is the harmonic Maass form with weight 1/2.

We shall compute the Fourier coefficients of $P_s(\tau)$ in the form of Rademacher expansion. By using the standard method, we have

$$\begin{aligned} P_s(\tau) &= \frac{4}{\sqrt{\pi}}\, \varphi^{\frac{1}{2}}_{-\frac{1}{8},s}(\tau) \\ &\quad + \frac{4}{\sqrt{\pi}} \sum_{\substack{c\neq 0 \\ \gamma\in\Gamma_\infty\backslash\Gamma(1)/\Gamma_\infty}} \sum_{n\in\mathbb{Z}} \left[\chi\left(\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) \right]^{-1} \frac{1}{\sqrt{c\,(\tau+n)+d}}\, \varphi^{\frac{1}{2}}_{-\frac{1}{8},s}\left(\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} (\tau) \right). \end{aligned}$$

Using $\gamma(\tau) = \frac{a}{c} - \frac{1}{c\,(c\,\tau+d)}$, the second term up to an overall factor is rewritten as

$$\begin{aligned} &\sum_{\substack{c>0 \\ \gamma\in\Gamma_\infty\backslash\Gamma(1)/\Gamma_\infty}} \frac{1}{\sqrt{c}}\, [\chi(\gamma)]^{-1}\, \mathrm{e}^{-\frac{a}{4\,c}\,\pi\,\mathrm{i}} \\ &\quad \times \sum_{n\in\mathbb{Z}} \left[\chi\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) \right]^{-1} \frac{1}{\sqrt{\tau+n+\frac{d}{c}}}\, \mathcal{M}^{\frac{1}{2}}_s \left(-\frac{\pi}{2}\, \frac{\Im(\tau)}{c^2\, \left|\tau+n+\frac{d}{c}\right|^2} \right) \mathrm{e}^{\frac{1}{4\,c^2}\,\Re\left(\frac{1}{\tau+n+\frac{d}{c}}\right)\pi\mathrm{i}}. \end{aligned}$$

We then apply a Fourier transformation formula [6, 23]

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{\tau+n}}\, \mathcal{M}^{\frac{1}{2}}_s \left(-4\,\pi\,h\, \frac{\Im(\tau)}{c^2\, |\tau+n|^2} \right) \mathrm{e}^{2\,\pi\,\mathrm{i}\,h'\,n + 2\,\pi\,\mathrm{i}\,\frac{h}{c^2}\,\Re\left(\frac{1}{\tau+n}\right)} \\ &\qquad\qquad = \sum_{n\in\mathbb{Z}} a_n\,(\Im(\tau))\, \mathrm{e}^{2\,\pi\,\mathrm{i}\,(n-h')\,\Re(\tau)}, \end{aligned} \tag{4.15}$$

where the Fourier coefficients $a_n(v)$ are given as follows;

- for $n > h'$,

$$\begin{aligned} a_n(v) &= \frac{1}{\sqrt{\mathrm{i}}} \left(\frac{h}{4\,\pi\,c^2\,v} \right)^{\frac{1}{4}} \frac{\Gamma(2\,s)}{\Gamma\left(s+\frac{1}{4}\right)} \\ &\qquad \times \frac{2\,\pi}{\sqrt{n-h'}}\, W_{\frac{1}{4},s-\frac{1}{2}}\,(4\,\pi\,(n-h')\,v)\; I_{2s-1}\left(\frac{4\,\pi}{|c|}\, \sqrt{(n-h')\,h} \right), \end{aligned}$$

- for $n = h'$,

$$a_n(v) = \frac{1}{\sqrt{\mathrm{i}}} \frac{2^{\frac{3}{2}} \pi^{s+\frac{3}{4}} \Gamma(2\,s)}{(2\,s-1)\,\Gamma\left(s+\frac{1}{4}\right)\,\Gamma\left(s-\frac{1}{4}\right)} \frac{h^{s-\frac{1}{4}}}{|c|^{2s-\frac{1}{2}}\;v^{s-\frac{3}{4}}},$$

- for $n < h'$,

$$a_n(v) = \frac{1}{\sqrt{\mathrm{i}}} \left(\frac{h}{4\,\pi\,c^2\,v}\right)^{\frac{1}{4}} \frac{\Gamma(2\,s)}{\Gamma\left(s-\frac{1}{4}\right)} \times \frac{2\,\pi}{\sqrt{h'-n}} W_{-\frac{1}{4},s-\frac{1}{2}}\left(4\,\pi\,(h'-n)\,v\right) J_{2s-1}\left(\frac{4\,\pi}{|c|}\sqrt{(h'-n)\,h}\right).$$

Here the (modified) Bessel function, $I_\alpha(z)$ and $J_\alpha(z)$, satisfy

$$I_\alpha(z) \underset{z\to 0}{\sim} \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha, \quad I_\alpha(z) \underset{|z|\to\infty}{\sim} \frac{1}{\sqrt{2\,\pi\,z}}\,\mathrm{e}^z,$$
$$J_\alpha(z) \underset{z\to 0}{\sim} \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha, \quad J_\alpha(z) \underset{|z|\to\infty}{\sim} \sqrt{\frac{2}{\pi\,z}}\cos\left(z-\frac{\alpha}{2}\,\pi-\frac{1}{4}\,\pi\right).$$

As a result, we obtain the Fourier expansion of the Poincaré–Maass series as follows;

$$\begin{aligned} P_{\frac{3}{4}}(\tau) = {} & 2\,E\left(\sqrt{\frac{\Im(\tau)}{2}}\right) q^{-\frac{1}{8}} \\ & - \sum_{\substack{n\in\mathbb{Z}\\ n\geq 1}} \frac{4\,\pi}{(8\,n-1)^{\frac{1}{4}}} \left[\sum_{c>0} \frac{1}{c}\, I_{\frac{1}{2}}\left(\frac{\pi}{2\,c}\sqrt{8\,n-1}\right) \sum_{\substack{d \bmod c\\ (c,d)=1}} \mathrm{e}^{-3\,\pi\,\mathrm{i}\,s(d,c)+2\,\pi\,\mathrm{i}\,n\,\frac{d}{c}}\right] q^{n-\frac{1}{8}} \\ & - \sum_{\substack{n\in\mathbb{Z}\\ n\leq 0}} \frac{4\,\pi}{(1-8\,n)^{\frac{1}{4}}} \left[1 - E\left(\sqrt{\frac{(1-8\,n)\,\Im(\tau)}{2}}\right)\right] \\ & \times \left[\sum_{c>0} \frac{1}{c}\, J_{\frac{1}{2}}\left(\frac{\pi}{2\,c}\sqrt{1-8\,n}\right) \sum_{\substack{d \bmod c\\ (c,d)=1}} \mathrm{e}^{-3\,\pi\,\mathrm{i}\,s(d,c)+2\,\pi\,\mathrm{i}\,\frac{d}{c}\,n}\right] q^{n-\frac{1}{8}}. \end{aligned} \tag{4.16}$$

Proving the convergence of the above series in $c > 0$ is a somewhat difficult issue. Such Poincaré–Maass series appeared in the Andrews–Dragonette identity, and its convergence was proved by Bringmann and Ono [4, Section 4] by use of properties of the Kloosterman sums and Salié sums. In our case, we recall that the sum involved in (4.16) can be rewritten as

$$\sum_{\substack{d \bmod c\\ (c,d)=1}} \mathrm{e}^{-3\,\pi\,\mathrm{i}\,s(d,c)+2\,\pi\,\mathrm{i}\,\frac{d}{c}\,n} = -\frac{\mathrm{i}\sqrt{c}}{2} \sum_{\substack{k \bmod 4c\\ k^2=-8\,n+1 \bmod 8c}} \left(\frac{-4}{k}\right) \mathrm{e}^{\pi\,\mathrm{i}\,\frac{k}{2c}}. \tag{4.17}$$

The sum on the right hand side can be parameterized by use of binary quadratic forms of discriminant $-8\,n+1$, and the method of Bringmann and Ono shows how this description naturally forces sufficient cancellation to justify convergence. In section 4.4 we present

evidence for the convergence of the series by numerically computing the truncated series at finite values of c.

Our claim is that the Poincaré–Maass series (4.10) coincides with the completion (4.1);

$$\widehat{\Sigma}(\tau) = P_{\frac{3}{4}}(\tau). \tag{4.18}$$

Proof of this statement is analogous to that of the Andrews–Dragonette identity by Bringmann and Ono [4]. We first look at the non-holomorphic part of the Poincaré–Maass series $P_{\frac{3}{4}}(\tau)$. From the above expression (4.16), we obtain

$$\begin{aligned} &\mathrm{i}\sqrt{\Im\tau}\,\overline{\frac{\partial}{\partial\overline{\tau}}P_{\frac{3}{4}}(8\,\tau)} \\ &\quad= \sum_{n=0}^{\infty}\left[2\,\delta_{n,0} + (8\,n+1)^{\frac{1}{4}}\sum_{c>0}\frac{4\,\pi}{c}\,J_{\frac{1}{2}}\left(\frac{\pi}{2\,c}\sqrt{8\,n+1}\right)\sum_{\substack{d \bmod c\\(c,d)=1}}\mathrm{e}^{3\,\pi\,\mathrm{i}\,s(d,c)+2\,\pi\,\mathrm{i}\,\frac{d}{c}\,n}\right]q^{8\,n+1}. \end{aligned} \tag{4.19}$$

As was proved by Bruinier and Funke [7, Proposition 3.2], the left hand side is a weight-3/2 cusp form on $\Gamma_0(64)$ with a trivial character. On the other hand we have from (2.13) that

$$\begin{aligned} \mathrm{i}\,\sqrt{\Im\tau}\,\overline{\frac{\partial}{\partial\overline{\tau}}\widehat{\Sigma}(8\,\tau)} &= 24\,\left[\eta\,(8\,\tau)\right]^3 \\ &= 24\,\left[q - 3\,q^9 + 5\,q^{25} - 7\,q^{49} + 9\,q^{81} - 11\,q^{121} + 13\,q^{169} - \cdots\right], \end{aligned} \tag{4.20}$$

which is also a weight-3/2 cusp form on $\Gamma_0(64)$ with a trivial character. According to the dimension formulas for spaces of half-integral weight modular forms [8] (see also Ref. 38), the dimension of cusp form on $\Gamma_0(64)$ is 1. We thus see that $\widehat{\Sigma}(8\,\tau)$ is proportional to $P_{\frac{3}{4}}(8\,\tau)$. Next by comparing the coefficients of q^{-1}, we see that the holomorphic part of $P_{\frac{3}{4}}(8\,\tau)$ coincides with that of $\Sigma(8\,\tau)$ and hence we have the equality (4.18).

Finally, we obtain an exact asymptotic expansion for A_n (3.9) as

$$A_n = \frac{4\,\pi}{(8\,n-1)^{\frac{1}{4}}}\sum_{c=1}^{\infty}\frac{1}{c}\,I_{\frac{1}{2}}\left(\frac{\pi\sqrt{8\,n-1}}{2\,c}\right)\sum_{\substack{d \bmod c\\(c,d)=1}}\mathrm{e}^{-3\,\pi\,\mathrm{i}\,s(d,c)+2\,\pi\,\mathrm{i}\,\frac{d}{c}\,n}. \tag{4.21}$$

The dominating contribution comes from the term $c = 1$ in the above expression,

$$A_n \sim \frac{4\,\pi}{(8\,n-1)^{\frac{1}{4}}}\,I_{\frac{1}{2}}\left(\frac{\pi\sqrt{8\,n-1}}{2}\right). \tag{4.22}$$

Substituting an explicit form of the Bessel function,

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi\,x}}\,\sinh(x),$$

we have the Cardy type formula

$$\log A_n \sim 2\,\pi\sqrt{\frac{1}{2}\left(n - \frac{1}{8}\right)}. \tag{4.23}$$

4.3. **Non-Compact Case.** We shall next determine the Fourier coefficients A°_n (3.12) which are related to the number of non-BPS representations in the decompactified K3 surface. We set the completion of $\Sigma^\circ(\tau)$ to be

$$\widehat{\Sigma}^\circ(\tau) = 8\,\widehat{\mu}\left(\frac{1}{2};\tau\right). \tag{4.24}$$

Its modular transformation formulae can be deduced from (2.11). Recalling the transformation formulae for the Jacobi theta function $\theta_{10}(z;\tau)$ (*e.g.* [41, Chapter 10]), we see that for $\gamma \in \Gamma_0(2)$ with $c > 0$

$$\widehat{\Sigma}^\circ\left(\gamma(\tau)\right) = \mathrm{i}^{\frac{3}{2}}\,\mathrm{e}^{-\frac{a+d}{4c}\,\pi\,\mathrm{i}+3\,\pi\,\mathrm{i}\,s(d,c)}\,\sqrt{c\,\tau+d}\;\widehat{\Sigma}^\circ(\tau). \tag{4.25}$$

Using the same argument with the above, we conclude that it coincides with the Poincaré–Maass series for $\widehat{\Sigma}^\circ(\tau)$,

$$\widehat{\Sigma}^\circ(\tau) = \frac{2}{\sqrt{\pi}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(2)} [\chi(\gamma)]^{-1}\,\frac{1}{\sqrt{c\,\tau+d}}\,\varphi^{\frac{1}{2}}_{-\frac{1}{8},\frac{3}{4}}\left(\gamma(\tau)\right), \tag{4.26}$$

where the multiplier system $\chi(\gamma)$ is given in (4.11). Correspondingly the Fourier coefficients of $\widehat{\Sigma}^\circ(\tau)$ can be computed from the Poincaré–Maass series, and we obtain

$$A^\circ_n = \frac{4\,\pi}{(8\,n-1)^{\frac{1}{4}}} \sum_{\substack{c=1\\ 2|c}}^{\infty} \frac{1}{c}\,I_{\frac{1}{2}}\left(\frac{\pi\,\sqrt{8\,n-1}}{2\,c}\right) \sum_{\substack{d \mod c\\ (c,d)=1}} \mathrm{e}^{-3\,\pi\,\mathrm{i}\,s(d,c)+2\,\pi\,\mathrm{i}\,\frac{d}{c}\,n}. \tag{4.27}$$

Asymptotic behavior of coefficients is given as

$$A^\circ_n \sim (-1)^n\,\frac{2\,\pi}{(8\,n-1)^{\frac{1}{4}}}\,I_{\frac{1}{2}}\left(\frac{\pi\,\sqrt{8\,n-1}}{4}\right). \tag{4.28}$$

4.4. **Numerical Checks.** Now we present some results of numerical calculations and their comparison with exact results in order to confirm the convergence of the series (4.16) and asymptotic formulas. We have plotted the exact values of Fourier coefficients A_n together with the values of the asymptotic formula (4.22) in Fig. 1. We have also presented the exact values of A°_n and compared with the predictions given by modified Bessel function (4.28). We find very good agreements.

In the tables we present more detailed results. We have numerically computed the exact asymptotic series (4.21) of A_n by truncating the infinite sum $\sum_{c=1}^{\infty}$ by a finite number of terms. For comparison we also list the exact values of A_n;

n	exact	leading; (4.22)	sum of 5 terms	20 terms
2	462	453.018	462.026	462.427
5	11592	11662.495	11594.141	11592.421
20	126894174	126889894.140	126894174.078	126894173.718
30	9104078592	9104043456.138	9104078600.515	9104078592.403
40	342322413552	342322217629.135	342322413549.736	342322413551.574
45	1778826191324	1778826619936.736	1778826191295.658	1778826191322.367

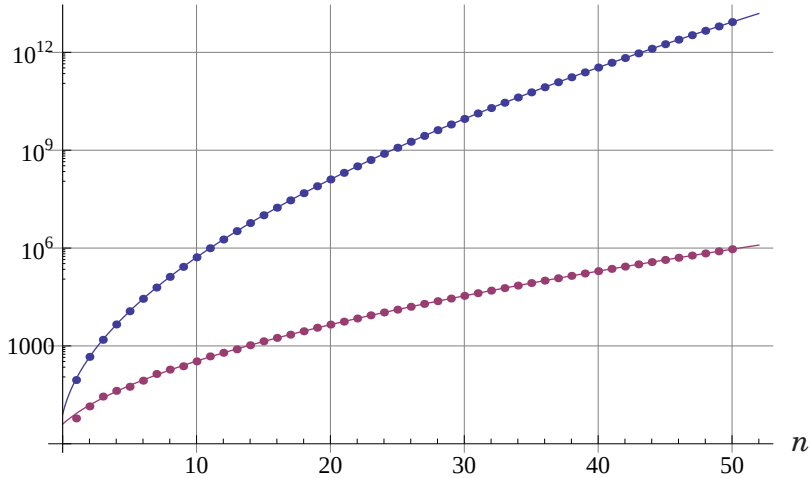


FIGURE 1. Exact values of A_n (3.9) and values of the asymptotic formula (4.22) are given by blue dots and blue line, respectively. We have also plotted exact absolute values $|A_n^\circ|$ (3.14) together with their asymptotic values (4.28) in red.

For the case of A_n°, we also present their exact values and values given by the truncation of the asymptotic expansion (4.27)

n	exact	leading; (4.28)	sum of 5 terms	10 terms
5	-56	-61.111	-56.544	-56.336
20	4510	4486.206	4511.303	4509.981
21	-5544	-5598.785	-5543.374	-5543.584
40	195888	195787.459	195888.432	195887.820
60	3772468	3772123.173	3772465.128	3772468.117
100	438370422	438366833.884	438370424.848	438370421.862

One sees that numerical results support the convergence of the series (4.16) the validity of our asymptotic formulae (4.21) and (4.27).

For its direct verification, we have numerically computed (4.19) to obtain

$$\begin{aligned}\mathrm{i}\sqrt{\Im\tau}\,\overline{\frac{\partial}{\partial\overline{\tau}}P_{\frac{3}{4}}(8\,\tau)} &= 23.851\,q - 72.0946\,q^9 + 0.320386\,q^{17} + 119.083\,q^{25}\\ &\quad + 0.295543\,q^{33} + 0.152477\,q^{41} - 166.728\,q^{49} - 0.587912\,q^{57}\\ &\quad - 0.0773375\,q^{65} - 0.652751\,q^{73} + 213.397\,q^{81} - 0.217745\,q^{89} + \cdots .\end{aligned}$$

Here we have truncated the sum over c after the first 800 terms. Though the convergence is slower than before, this agrees with (4.20).

5. Chern–Simons Theory

A key in our analysis is to complete the massless superconformal character $\mu(z;\tau)$ by adding the non-holomorphic partner $R(\tau)$ (2.9) so that the sum has a nice modular property.

Here we would like to point out that $R(\tau)$ has its own meaning as a topological quantum invariant related to a simple singularity.

The Witten invariant of 3-manifold M [45] is defined by the Chern–Simons path integral

$$Z_k(M) = \int \mathrm{e}^{2\,\pi\,\mathrm{i}\,k\ \mathrm{CS}(A)} \mathcal{D}A, \tag{5.1}$$

where the coupling constant $k \in \mathbb{Z}$ denotes the level, and A is a G-gauge connection on the trivial bundle over M. The Chern–Simons action $\mathrm{CS}(A)$ with gauge group G is

$$\mathrm{CS}(A) = \frac{1}{8\,\pi^2} \int\limits_M \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}\,A \wedge A \wedge A\right).$$

See Ref. 42 for mathematically rigorous definition of this quantum invariant (Witten–Reshetikhin–Turaev invariant) in terms of the quantum invariants of links to be surgered. When M is the Poincaré homology sphere, it was shown that the Witten invariant $Z_k(M)$ with gauge group $SU(2)$ can be regarded as a limiting value of the Eichler integral of a weight-3/2 modular form [31]. This correspondence has been checked for other Seifert manifolds [26, 27].

Let M be the Seifert manifold $M(2,2,2)$ [35], which is a spherical neighborhood around the isolated singularity of type-D_4;

$$M = \left\{x^2\,y + y^3 + z^2 = 0\right\} \cap S^5,$$

where S^5 denotes a sufficiently small 5-sphere around the origin. It was shown [27] that the Witten invariant $Z_k(M)$ with SU(2) gauge group is given by (see also Refs. 25, 28)

$$Z_k(M) = \frac{1}{2\,\mathrm{i}} \sqrt{\frac{2}{k+2}}\, \mathrm{e}^{-\frac{1}{k+2}\,\pi\,\mathrm{i}} \left[1 - \mathrm{e}^{-\frac{1}{4(k+2)}\pi\mathrm{i}}\, R\left(-\frac{1}{k+2}\right)\right]. \tag{5.2}$$

Here $R(\tau)$ is the period integral of the third power of the Dedekind η-function (2.12), and we have a limiting value in $\tau \to -\frac{1}{k+2}$. It was shown that the topological invariants, such as the Chern–Simons invariant, the Reidemeister torsion, and the Ohtsuki invariant, of M are given by the asymptotic expansion of $Z_k(M)$ in $k \to \infty$.

It should also be noted that the non-holomorphic partner of the massless character $R(\tau)$ is related to the knot invariant for the torus link $T_{2,4}$ [24]

$$\langle T_{2,4} \rangle_N = N\,\mathrm{e}^{-\frac{1}{4N}\,\pi\,\mathrm{i}}\, R\left(-\frac{1}{N}\right). \tag{5.3}$$

Here $\langle L \rangle_N$ denotes a specific value of the N-colored Jones polynomial for link L at $q = \mathrm{e}^{2\pi\mathrm{i}/N}$: this quantity receives renewed interests from the viewpoint of the volume conjecture raised by Kashaev [29], and H. Murakami and J. Murakami [37].

6. Concluding Remarks

As an application of our previous result [12] that the coefficients of massive characters of the elliptic genera are the holomorphic part of harmonic Maass form, we have obtained their Rademacher-type expansion by computing the Fourier coefficients of the Poincaré–Maass series.

We note that in the elliptic genus the right-moving sector is fixed to Ramond ground state and thus the non-BPS states in the left-moving sector are actually the overall half-BPS states (BPS (non-BPS) in the right (left) moving sector). It is known that asymptotic increase of the number of half-BPS states is related to the entropy of supersymmetric systems. In fact the multiplicity factor A_n behaves like an exponential (4.23) and we may identity

$$S = 2\,\pi\,\sqrt{\frac{1}{2}\,n} \tag{6.1}$$

as the entropy of K3 surface.

Our methods are applicable to higher level superconformal algebras and higher-dimensional hyperKähler manifolds. In general hyperKähler manifolds with complex $2k$-dimensions we find an entropy [13]

$$S = 2\,\pi\,\sqrt{\frac{k^2}{k+1}\,n}. \tag{6.2}$$

If we consider the case of symmetric product of K3 surfaces $K3^{[k]}$, the above entropy reproduces the black hole entropy of string theory compactified on K3 surface at large k.

Acknowledgments

One of the authors (KH) would like to thank M. Kaneko and in particular K. Ono for his useful communications on the issue of convergence of Poincaré series. This work is supported in part by Grant-in-Aid from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Appendix A. Jacobi Theta Functions

The Jacobi theta functions are defined by

$$\theta_{11}(z;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\,\mathrm{e}^{2\pi\mathrm{i}\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)} = \theta_1(z;\tau),$$

$$\theta_{10}(z;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\,\mathrm{e}^{2\pi\mathrm{i}\left(n+\frac{1}{2}\right)z} = \theta_2(z;\tau),$$

$$\theta_{00}(z;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}\,\mathrm{e}^{2\pi\mathrm{i}nz} = \theta_3(z;\tau),$$

$$\theta_{01}(z;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}\, \mathrm{e}^{2\pi\mathrm{i}n\left(z+\frac{1}{2}\right)} = \theta_4(z;\tau),$$

where we have also shown the relation to the conventional notations.

References

[1] G. E. Andrews, *On the theorems of Watson and Dragonette for Ramanujan's mock theta functions*, Amer. J. Math. **88**, 454–490 (1966).

[2] ———, *The Theory of Partitions*, Addison-Wesley, London, 1976.

[3] ———, *Mock theta functions*, in L. Ehrenpreis and R. C. Gunning, eds., *Theta Functions — Bowdoin 1987*, vol. **49** (part 2) of *Proc. Symp. Pure Math.*, pp. 283–298, Amer. Math. Soc., Providence, 1989.

[4] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165**, 243–266 (2006).

[5] ———, *Coefficients of harmonic Maass forms*, preprint (2008).

[6] J. H. Bruinier, *Borcherds Products on $O(2,\ell)$ and Chern Classes of Heegner Divisors*, vol. **1780** of *Lecture Notes in Mathematics*, Springer, Berlin, 2002.

[7] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125**, 45–90 (2004).

[8] H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, in J.-P. Serre and D. Zagier, eds., *Modular Functions of One Variable VI*, vol. **627** of *Lecture Notes in Mathematics*, pp. 69–78, Springer, Berlin, 1977.

[9] R. Dijkgraaf, J. Maldacena, G. Moore, and E. Verlinde, *A black hole Farey tail*, `hep-th/0005003v3` (2007).

[10] L. Dragonette, *Some asymptotic formulae for the mock theta series of Ramanujan*, Trans. Amer. Math. Soc. **72**, 474–500 (1952).

[11] F. J. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8**, 10–15 (1944).

[12] T. Eguchi and K. Hikami, *Superconformal algebras and mock theta functions*, J. Phys. A: Math. Theor. **42**, 304010 (2009), 23 pages.

[13] ———, preprint in preparation.

[14] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with SU(n) holonomy*, Nucl. Phys. B **315**, 193–221 (1989).

[15] T. Eguchi, Y. Sugawara, and A. Taormina, *Liouville field, modular forms and elliptic genera*, JHEP **2007**, 119 (2007), 21 pages.

[16] ———, *Modular forms and elliptic genera for ALE spaces*, `arXiv:0803.0377` (2008).

[17] T. Eguchi and A. Taormina, *Unitary representations of the $N = 4$ superconformal algebra*, Phys. Lett. B **196**, 75–81 (1986).

[18] ———, *Character formulas for the $N = 4$ superconformal algebra*, Phys. Lett. B **200**, 315–322 (1988).

[19] ———, *On the unitary representations of $N = 2$ and $N = 4$ superconformal algebras*, Phys. Lett. B **210**, 125–132 (1988).

[20] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, vol. **55** of *Progress in Mathematics*, Birkhäuser, Boston, 1985.

[21] B. Gordon and R. J. McIntosh, *A survey of classical mock theta functions*, preprint (2009).

[22] G. H. Hardy and S. Ramanujan, *Asymptotic formulæ in combinatory analysis*, Proc. London Math. Soc. **2**, 75–115 (1918).

[23] D. A. Hejhal, *The Selberg Trace Formula for PSL($2, \mathbb{R}$) Vol. 2*, vol. **1001** of *Lecture Notes in Mathematics*, Springer, Berlin, 1983.

[24] K. Hikami, *Quantum invariant for torus link and modular forms*, Commun. Math. Phys. **246**, 403–426 (2004).

[25] ———, *Mock (false) theta functions as quantum invariants*, Regular & Chaotic Dyn. **10**, 509–530 (2005).

[26] ———, *On the quantum invariant for the Brieskorn homology spheres*, Int. J. Math. **16**, 661–685 (2005).

[27] ———, *On the quantum invariant for the spherical Seifert manifold*, Commun. Math. Phys. **268**, 285–319 (2006).

[28] ———, *Transformation formula of the "2nd" order mock theta function*, Lett. Math. Phys. **75**, 93–98 (2006).

[29] R. M. Kashaev, *The hyperbolic volume of knots from quantum dilogarithm*, Lett. Math. Phys. **39**, 269–275 (1997).

[30] T. Kawai, Y. Yamada, and S.-K. Yang, *Elliptic genera and $N = 2$ superconformal field theory*, Nucl. Phys. B **414**, 191–212 (1994).

[31] R. Lawrence and D. Zagier, *Modular forms and quantum invariants of 3-manifolds*, Asian J. Math. **3**, 93–107 (1999).

[32] A. Malmendier and K. Ono, *SO(3)-Donaldson invariants of $\mathbb{C}\mathrm{P}^2$ and mock theta functions*, `arXiv:0808.1442` (2008).

[33] A. Maloney and E. Witten, *Quantum gravity partition function in three dimensions*, `arXiv:0712.0155` (2007).

[34] J. Manschot and G. W. Moore, *A modern Farey tail*, `arXiv:0712.0573` (2007).

[35] J. Milnor, *On the 3-dimensional Brieskorn manifolds $M(p, q, r)$*, in L. P. Neuwirth, ed., *Knots, Groups, and 3-Manifolds*, pp. 175–225, Princeton Univ. Press, 1975, papers Dedicated to the Memory of R. H. Fox.

[36] G. Moore, *Strings and arithmetic*, in P. Cartier, B. Julia, and P. Moussa, eds., *Frontiers in Number Theory, Physics and Geometry II*, pp. 303–359, Springer, Berlin, 2006.

[37] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta. Math. **186**, 85–104 (2001).

[38] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, no. **102** in CBMS Regional Conference Series in Math., Amer. Math. Soc., Providence, 2004.

[39] ———, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Proceedings of the 2008 Harvard-MIT Current Developments in Mathematics Conference, Intl. Press, Boston, in press.

[40] D. N. Page, *A physical picture of the K3 gravitational instanton*, Phys. Lett. B **80**, 55–57 (1978).

[41] H. Rademacher, *Topics in Analytic Number Theory*, vol. **169** of *Die Grundlehren der mathematischen Wissenschaften*, Springer, New York, 1973.

[42] N. Yu. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103**, 547–597 (1991).

[43] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge, 1927, 4th ed.

[44] E. Witten, *Elliptic genera and quantum field theory*, Commun. Math. Phys. **109**, 525–536 (1987).

[45] ———, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121**, 351–399 (1989).

[46] D. Zagier, *Ramanujan's mock theta functions and their applications [d'après Zwegers and Bringmann–Ono]*, Séminaire Bourbaki **986** (2006–2007).

[47] S. P. Zwegers, *Mock Theta Functions*, Ph.D. thesis, Universiteit Utrecht (2002).

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606–8502, Japan

E-mail address: eguchi@yukawa.kyoto-u.ac.jp

Department of Mathematics, Naruto University of Education, Tokushima 772-8502, Japan.

E-mail address: KHikami@gmail.com

arXiv:1503.01472v2 [math.RT] 26 Nov 2015

Proof of the Umbral Moonshine Conjecture

John F. R. Duncan, Michael J. Griffin and Ken Ono

2015 November 25

Abstract

The Umbral Moonshine Conjectures assert that there are infinite-dimensional graded modules, for prescribed finite groups, whose McKay-Thompson series are certain distinguished mock modular forms. Gannon has proved this for the special case involving the largest sporadic simple Mathieu group. Here we establish the existence of the umbral moonshine modules in the remaining 22 cases.

* *MSC2010:* 11F22, 11F37.

*The authors thank the NSF for its support. The first author also thanks the Simons Foundation (#316779), and the third author thanks the A. G. Candler Fund. The authors thank Miranda Cheng, Jeff Harvey, Michael Somos and the anonymous referee for helpful comments and corrections.

1 Introduction and Statement of Results

Monstrous moonshine relates distinguished modular functions to the representation theory of the Monster, $\mathbb{M}$, the largest sporadic simple group. This theory was inspired by the famous observations of McKay and Thompson in the late 1970s [18,51] that

$$196884 = 1 + 196883,$$
$$21493760 = 1 + 196883 + 21296876.$$

The left hand sides here are familiar as coefficients of Klein's modular function (note $q := e^{2\pi i\tau}$),

$$J(\tau) = \sum_{n=-1}^{\infty} c(n)q^n := j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots.$$

The sums on the right hand sides involve the first three numbers arising as dimensions of irreducible representations of $\mathbb{M}$,

$$1,\ 196883,\ 21296876,\ 842609326,\ \dots,\ 258823477531055064045234375.$$

Thompson conjectured that there is a graded infinite-dimensional $\mathbb{M}$-module

$$V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural,$$

satisfying $\dim(V_n^\natural) = c(n)$. For $g \in \mathbb{M}$, he also suggested [50] to consider the graded-trace functions

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|V_n^\natural)q^n,$$

now known as the *McKay-Thompson series*, that arise from the conjectured $\mathbb{M}$-module $V^\natural$. Using the character table for $\mathbb{M}$, it was observed [18,50] that the first few coefficients of each $T_g(\tau)$ coincide with those of a generator for the function field of a discrete group $\Gamma_g < SL_2(\mathbb{R})$, leading Conway and Norton [18] to their famous *Monstrous Moonshine Conjecture*: This is the claim that for each $g \in \mathbb{M}$ there is a specific *genus zero* group Γ_g such that $T_g(\tau)$ is the unique normalized *hauptmodul* for Γ_g, i.e., the unique Γ_g-invariant holomorphic function on $\mathbb{H}$ which satisfies $T_g(\tau) = q^{-1} + O(q)$ as $\Im(\tau) \to \infty$, and remains bounded near any cusp not equivalent to the infinite one.

In a series of ground-breaking works, Borcherds introduced vertex algebras [2], and generalized Kac–Moody Lie algebras [3,4], and used these notions to prove [5] the Monstrous Moonshine Conjecture of Conway and Norton. He confirmed the conjecture for the module $V^\natural$ constructed by Frenkel, Lepowsky, and Meurman [30–32] in the early 1980s. These results provide much more than the predictions of monstrous moonshine. The $\mathbb{M}$-module $V^\natural$ is a vertex operator algebra, one whose automorphism group is precisely $\mathbb{M}$. The construction of Frenkel, Lepowsky and Meurman can be regarded as one of the first examples of an *orbifold conformal field theory*. (Cf. [23].) Here the orbifold in question is the quotient $\left(\mathbb{R}^{24}/\Lambda_{24}\right)/(\mathbb{Z}/2\mathbb{Z})$, of the 24-dimensional torus $\Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{R}/\Lambda_{24} \simeq \mathbb{R}^{24}/\Lambda_{24}$ by the Kummer involution $x \mapsto -x$, where Λ_{24} denotes the Leech lattice.

We refer to [24,32,35,36] for more on monstrous moonshine.

In 2010, Eguchi, Ooguri, and Tachikawa reignited moonshine with their observation [28] that dimensions of some representations of M_{24}, the largest sporadic simple Mathieu group (cf. e.g. [20, 21]), are multiplicities of superconformal algebra characters in the K3 elliptic genus. This observation suggested a manifestation of moonshine for M_{24}: Namely, there should be an infinite-dimensional graded M_{24}-module whose McKay-Thompson series are holomorphic parts of *harmonic Maass forms*, the so-called *mock modular forms*. (See [45, 54, 55] for introductory accounts of the theory of mock modular forms.)

Following the work of Cheng [10], Eguchi and Hikami [27], and Gaberdiel, Hohenegger, and Volpato [33, 34], Gannon established the existence of this infinite-dimensional graded M_{24}-module in [37].

It is natural to seek a general mathematical and physical setting for these results. Here we consider the mathematical setting, which develops from the close relationship between the monster group $\mathbb{M}$ and the Leech lattice Λ_{24}. Recall (cf. e.g. [20]) that the Leech lattice is even, unimodular, and positive-definite of rank 24. It turns out that M_{24} is closely related to another such lattice. Such observations led Cheng, Duncan and Harvey to further instances of moonshine within the setting of even unimodular positive-definite lattices of rank 24. In this way they arrived at the *Umbral Moonshine Conjectures* (cf. §5 of [15], §6 of [16], and §2 of [17]), predicting the existence of 22 further, graded infinite-dimensional modules, relating certain finite groups to distinguished mock modular forms.

To explain this prediction in more detail we recall Niemeier's result [43] that there are 24 (up to isomorphism) even unimodular positive-definite lattices of rank 24. The Leech lattice is the unique one with no root vectors (i.e. lattice vectors with norm-square 2), while the other 23 have root systems with full rank, 24. These *Niemeier root systems* are unions of simple simply-laced root systems with the same Coxeter numbers, and are given explicitly as

$$\begin{gathered} A_1^{24},\ A_2^{12},\ A_3^8,\ A_4^6,\ A_6^4,\ A_{12}^2, \\ A_5^4D_4,\ A_7^2D_5^2,\ A_8^3,\ A_9^2D_6,\ A_{11}D_7E_6,\ A_{15}D_9,\ A_{17}E_7,\ A_{24}, \\ D_4^6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2,\ D_{12}^2,\ D_{16}E_8,\ D_{24},\ E_6^4,\ E_8^3, \end{gathered} \tag{1.1}$$

in terms of the standard ADE notation. (Cf. e.g. [20] or [39] for more on root systems.)

For each Niemeier root system X let N^X denote the corresponding unimodular lattice, let W^X denote the (normal) subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in roots, and define the *umbral group* of X by setting

$$G^X := \mathrm{Aut}(N^X)/W^X. \tag{1.2}$$

(See §A.1 for explicit descriptions of the groups G^X.)

Let m^X denote the Coxeter number of any simple component of X. An association of distinguished $2m^X$-vector-valued mock modular forms $H_g^X(\tau) = (H_{g,r}^X(\tau))$ to elements $g \in G^X$ is described and analyzed in [15–17].

For $X = A_1^{24}$ we have $G^X \simeq M_{24}$ and $m^X = 2$, and the functions $H_{g,1}^X(\tau)$ are precisely the mock modular forms assigned to elements $g \in M_{24}$ in the works [10, 27, 33, 34] mentioned above. Generalizing the M_{24} moonshine initiated by Eguchi, Ooguri and Tachikawa, we have the following conjecture of Cheng, Duncan and Harvey (cf. §2 of [17] or §9.3 of [24]).

Conjecture (Umbral Moonshine Modules)**.** Let X be a Niemeier root system X and set $m := m^X$. There is a naturally defined bi-graded infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \in I^X} \bigoplus_{\substack{D \in \mathbb{Z},\ D \leq 0, \\ D = r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m} \tag{1.3}$$

such that the vector-valued mock modular form $H^X_g = (H^X_{g,r})$ is a McKay-Thompson series for $\check{K}^X$ related[1] to the graded trace of g on $\check{K}^X$ by

$$H^X_{g,r}(\tau) = -2q^{-1/4m}\delta_{r,1} + \sum_{\substack{D \in \mathbb{Z},\ D \leq 0, \\ D = r^2 \pmod{4m}}} \mathrm{tr}(g|\check{K}^X_{r,-D/4m})q^{-D/4m} \tag{1.4}$$

for $r \in I^X$.

In (1.3) and (1.4) the set $I^X \subset \mathbb{Z}/2m\mathbb{Z}$ is defined in the following way. If X has an A-type component then $I^X := \{1, 2, 3, \ldots, m-1\}$. If X has no A-type component but does have a D-type component then $m = 2 \mod 4$, and $I^X := \{1, 3, 5, \ldots, m/2\}$. The remaining cases are $X = E_6^4$ and $X = E_8^3$. In the former of these, $I^X := \{1, 4, 5\}$, and in the latter case $I^X := \{1, 7\}$.

Remark. The functions $H^X_g(\tau)$ are defined explicitly in §B.3. An alternative description in terms of Rademacher sums is given in §B.4.

Here we prove the following theorem.

Theorem 1.1. *The umbral moonshine modules exist.*

Two remarks.

1) Theorem 1.1 for $X = A_1^{24}$ is the main result of Gannon's work [37].

2) The vector-valued mock modular forms $H^X = (H^X_{g,r})$ have "minimal" *principal parts*. This minimality is analogous to the fact that the original McKay-Thompson series $T_g(\tau)$ for the Monster are hauptmoduln, and plays an important role in our proof.

Example. Many of Ramanujan's mock theta functions [46] are components of the vector-valued umbral McKay-Thompson series $H^X_g = (H^X_{g,r})$. For example, consider the root system $X = A_2^{12}$, whose umbral group is a double cover $2.M_{12}$ of the sporadic simple Mathieu group M_{12}. In terms

[1] In the statement of Conjecture 6.1 of [16] the function $H^X_{g,r}$ in (1.4) is replaced with $3H^X_{g,r}$ in the case that $X = A_8^3$. This is now known to be an error, arising from a misspecification of some of the functions H^X_g for $X = A_8^3$. Our treatment of the case $X = A_8^3$ in this work reflects the corrected specification of the corresponding H^X_g which is described and discussed in detail in [17].

of Ramanujan's 3rd order mock theta functions

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$
$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$
$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$
$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2},$$
$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})},$$

we have that

$$H^X_{2B,1}(\tau) = H^X_{2C,1}(\tau) = H^X_{4C,1}(\tau) = -2q^{-\frac{1}{12}} \cdot f(q^2),$$
$$H^X_{6C,1}(\tau) = H^X_{6D,1}(\tau) = -2q^{-\frac{1}{12}} \cdot \chi(q^2),$$
$$H^X_{8C,1}(\tau) = H^X_{8D,1}(\tau) = -2q^{-\frac{1}{12}} \cdot \phi(-q^2),$$
$$H^X_{2B,2}(\tau) = -H^X_{2C,2}(\tau) = -4q^{\frac{2}{3}} \cdot \omega(-q),$$
$$H^X_{6C,2}(\tau) = -H^X_{6D,2}(\tau) = 2q^{\frac{2}{3}} \cdot \rho(-q).$$

See §5.4 of [16] for more coincidences between umbral McKay-Thompson series and mock theta functions identified by Ramanujan almost a hundred years ago.

Our proof of Theorem 1.1 involves the explicit determination of each G^X-module $\check{K}^X$ by computing the multiplicity of each irreducible component for each homogeneous subspace. It guarantees the existence and uniqueness of a $\check{K}^X$ which is compatible with the representation theory of G^X and the Fourier expansions of the vector-valued mock modular forms $H^X_g(\tau) = (H^X_{g,r}(\tau))$.

At first glance our methods do not appear to shed light on any deeper algebraic properties of the $\check{K}^X$, such as might correspond to the vertex operator algebra structure on $V^\natural$, or the monster Lie algebra introduced by Borcherds in [5]. However, we do determine, and utilize, specific recursion relations for the coefficients of the umbral McKay-Thompson series which are analogous to the replicability properties of monstrous moonshine formulated by Conway and Norton in §8 of [18] (cf. also [1]). More specifically, we use recent work [41] of Imamoğlu, Raum and Richter, as generalized [42] by Mertens, to obtain such recursions. These results are based on the process of *holomorphic projection*.

Theorem 1.2. *For each $g \in G^X$ and $0 < r < m$, the mock modular form $H^X_{g,r}(\tau)$ is replicable in the mock modular sense.*

A key step in Borcherds' proof [5] of the monstrous moonshine conjecture is the reformulation of replicability in Lie theoretic terms. We may speculate that the *mock modular replicability* utilized

in this work will ultimately admit an analogous algebraic interpretation. Such a result remains an important goal for future work.

In the statement of Theorem 1.2, replicable means that there are explicit recursion relations for the coefficients of the vector-valued mock modular form in question. For example, we recall the recurrence formula for Ramanujan's third order mock theta function $f(q) = \sum_{n=0}^{\infty} c_f(n)q^n$ that was obtained recently by Imamoğlu, Raum and Richter [41]. If $n \in \mathbb{Q}$, then let

$$\sigma_1(n) := \begin{cases} \sum_{d|n} d & \text{if } n \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{sgn}^+(n) := \begin{cases} \mathrm{sgn}(n) & \text{if } n \neq 0, \\ 1 & \text{if } n = 0, \end{cases}$$

and then define

$$d(N, \widetilde{N}, t, \widetilde{t}) := \mathrm{sgn}^+(N) \cdot \mathrm{sgn}^+(\widetilde{N}) \cdot \left(|N + t| - |\widetilde{N} + \widetilde{t}| \right).$$

Then for positive integers n, we have that

$$\sum_{\substack{m \in \mathbb{Z} \\ 3m^2+m \leq 2n}} \left(m + \frac{1}{6}\right) c_f\left(n - \frac{3}{2}m^2 - \frac{1}{2}m\right) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{2}\right) - 2 \sum_{\substack{a,b \in \mathbb{Z} \\ 2n = ab}} d\left(N, \widetilde{N}, \frac{1}{6}, \frac{1}{6}\right),$$

where $N := \frac{1}{6}(-3a + b - 1)$ and $\widetilde{N} := \frac{1}{6}(3a + b - 1)$, and the sum is over integers a, b for which $N, \widetilde{N} \in \mathbb{Z}$. This is easily seen to be a recurrence relation for the coefficients $c_f(n)$. The replicability formulas for all of the $H^X_{g,r}(\tau)$ are similar (although some of these relations are slightly more complicated and involve the coefficients of weight 2 cusp forms).

It is important to emphasize that, despite the progress which is represented by our main results, Theorems 1.1 and 1.2, the following important question remains open in general.

Question. *Is there a "natural" construction of $\check{K}^X$? Is $\check{K}^X$ equipped with a deeper algebra structure as in the case of the monster module $V^\natural$ of Frenkel, Lepowsky and Meurman?*

We remark that this question has been answered positively, recently, in one special case: A vertex operator algebra structure underlying the umbral moonshine module $\check{K}^X$ for $X = E_8^3$ has been described explicitly in [25]. See also [14, 26], where the problem of constructing algebraic structures that illuminate the umbral moonshine observations is addressed from a different point of view.

The proof of Theorem 1.1 is not difficult. It is essentially a collection of tedious calculations. We use the theory of mock modular forms and the character table for each G^X (cf. §A.2) to solve for the multiplicities of the irreducible G^X-module constituents of each homogeneous subspace in the alleged G^X-module $\check{K}^X$. To prove Theorem 1.1 it suffices to prove that these multiplicities are non-negative integers. To prove Theorem 1.2 we apply recent work [42] of Mertens on the holomorphic projection of weight $\frac{1}{2}$ mock modular forms, which generalizes earlier work [41] of Imamoğlu, Raum and Richter.

In §2 we recall the facts about mock modular forms that we require, and we prove Theorem 1.2. We prove Theorem 1.1 in §3. The appendices furnish all the data that our method requires. In particular, the umbral groups G^X are described in detail in §A, and explicit definitions for the mock modular forms $H_g^X(\tau)$ are given in §B.

2 Harmonic Maass forms and Mock modular forms

Here we recall some very basic facts about harmonic Maass forms as developed by Bruinier and Funke [9] (see also [45]).

We begin by briefly recalling the definition of a *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier ν (a generalization of the notion of a Nebentypus). If $\tau = x + iy$ with x and y real, we define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{2.1}$$

Suppose Γ is a subgroup of finite index in $SL_2(\mathbb{Z})$ and $k \in \frac{1}{2}\mathbb{Z}$. Then a function $F(\tau)$ which is real-analytic on the upper half of the complex plane is a *harmonic Maass form* of weight k on Γ with multiplier ν if:

(a) The function $F(\tau)$ satisfies the weight k modular transformation,

$$F(\tau)|_k\gamma = \nu(\gamma)F(\tau)$$

for every matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where $F(\tau)|_k\gamma := F(\gamma\tau)(c\tau + d)^{-k}$, and if $k \in \mathbb{Z} + \frac{1}{2}$, the square root is taken to be the principal branch.

(b) We have that $\Delta_k F(\tau) = 0$,

(c) There is a polynomial $P_F(q^{-1})$ and a constant $c > 0$ such that $F(\tau) - P_F(e^{-2\pi i\tau}) = O(e^{-cy})$ as $\tau \to i\infty$. Analogous conditions are required at each cusp of Γ.

We denote the $\mathbb{C}$-vector space of harmonic Maass forms of a given weight k, group Γ and multiplier ν by $H_k(\Gamma, \nu)$. If no multiplier is specified, we will take

$$\nu_0(\gamma) := \left(\left(\frac{c}{d}\right) \sqrt{\left(\frac{-1}{d}\right)}^{-1} \right)^{2k},$$

where $\left(\frac{*}{d}\right)$ is the Kronecker symbol.

2.1 Main properties

The Fourier expansion of a harmonic Maass form F (see Proposition 3.2 of [9]) splits into two components. As before, we let $q := e^{2\pi i\tau}$.

Lemma 2.1. *If $F(\tau)$ is a harmonic Maass form of weight $2-k$ for Γ where $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, then*

$$F(\tau) = F^+(\tau) + F^-(\tau),$$

where F^+ is the holomorphic part of F, given by

$$F^+(\tau) = \sum_{n \gg -\infty} c_F^+(n) q^n$$

where the sum admits only finitely many non-zero terms with $n < 0$, and F^- is the nonholomorphic part, given by

$$F^-(\tau) = \sum_{n<0} c_F^-(n) \Gamma(k-1, 4\pi y|n|) q^n.$$

Here $\Gamma(s,z)$ is the upper incomplete gamma function.

The holomorphic part of a harmonic Maass form is called a *mock modular form.* We denote the space of harmonic Maass forms of weight $2-k$ for Γ and multiplier ν by $H_k(\Gamma,\nu)$. Similarly, we denote the corresponding subspace of holomorphic modular forms by $M_k(\Gamma,\nu)$, and the space of cusp forms by $S_k(\Gamma,\nu)$. The differential operator $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{\tau}}}$ (see [9]) defines a surjective map

$$\xi_{2-k} : H_{2-k}(\Gamma,\nu) \to S_k(\Gamma,\overline{\nu})$$

onto the space of weight k cusp forms for the same group but conjugate multiplier. The *shadow* of a Maass form $f(\tau) \in H_{2-k}(\Gamma,\nu)$ is the cusp form $g(\tau) \in S_k(\Gamma,\overline{\nu})$ (defined, for now, only up to scale) such that $\xi_{2-k} f(\tau) = \frac{g}{||g||}$, where $|| \bullet ||$ denotes the usual Petersson norm.

2.2 Holomorphic projection of weight $\frac{1}{2}$ mock modular forms

As noted above, the modular transformations of a weight $\frac{1}{2}$ harmonic Maass form may be simplified by multiplying by its shadow to obtain a weight 2 nonholomorphic modular form. One can use the theory of holomorphic projections to obtain explicit identities relating these nonholomorphic modular forms to classical quasimodular forms. In this way, we may essentially reduce many questions about the coefficients of weight $\frac{1}{2}$ mock modular forms to questions about weight 2 holomorphic modular forms. The following theorem is a special case of a more general theorem due to Mertens (cf. Theorem 6.3 of [42]). See also [41].

Theorem 2.2 (Mertens)**.** *Suppose $g(\tau)$ and $h(\tau)$ are both theta functions of weight $\frac{3}{2}$ contained in $S_{\frac{3}{2}}(\Gamma,\nu_g)$ and $S_{\frac{3}{2}}(\Gamma,\nu_h)$ respectively, with Fourier expansions*

$$g(\tau) := \sum_{i=1}^{s} \sum_{n\in\mathbb{Z}} n\chi_i(n) q^{n^2},$$

$$h(\tau) := \sum_{j=1}^{t} \sum_{n\in\mathbb{Z}} n\psi_j(n) q^{n^2},$$

where each χ_i and ψ_i is a Dirichlet character. Moreover, suppose $h(\tau)$ is the shadow of a weight $\frac{1}{2}$ harmonic Maass form $f(\tau) \in H_{\frac{1}{2}}(\Gamma, \overline{\nu_h})$. Define the function

$$D^{f,g}(\tau) := 2 \sum_{r=1}^{\infty} \sum_{\chi_i, \psi_j} \sum_{\substack{m,n \in \mathbb{Z}^+ \\ m^2 - n^2 = r}} \chi_i(m) \overline{\psi_j(n)} (m-n) q^r.$$

If $f(\tau)g(\tau)$ has no singularity at any cusp, then $f^+(\tau)g(\tau) + D^{f,g}(\tau)$ is a weight 2 quasimodular form. In other words, it lies in the space $\mathbb{C}E_2(\tau) \oplus M_2(\Gamma, \nu_g \overline{\nu_h})$, where $E_2(\tau)$ is the quasimodular Eisenstein series $E_2(\tau) := 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1-q^n}$.

Two Remarks.
1) These identities give recurrence relations for the weight $\frac{1}{2}$ mock modular form f^+ in terms of the weight 2 quasimodular form which equals $f^+(\tau)g(\tau) + D^{f,g}(\tau)$. The example after Theorem 1.2 for Ramanujan's third order mock theta function f is an explicit example of such a relation.

2) Theorem 2.2 extends to vector-valued mock modular forms in a natural way.

Proof of Theorem 1.2. Fix a Niemeier lattice and its root system X, and let $M = m^X$ denote its Coxeter number. Each $H^X_{g,r}(\tau)$ is the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass form $\widehat{H}^X_{g,r}(\tau)$. To simplify the exposition in the following section, we will emphasize the case that the root system X is of pure A-type. If the root system X is of pure A-type, the shadow function $S^X_{g,r}(\tau)$ is given by $\hat{\chi}^{X_A}_{g,r} S_{M,r}(\tau)$ (see §B.2), where

$$S_{M,r}(\tau) = \sum_{\substack{n \in \mathbb{Z} \\ n \equiv r \pmod{2M}}} n \, q^{\frac{n^2}{4M}},$$

and $\hat{\chi}^{X_A}_{g,r} = \chi^{X_A}_g$ or $\bar{\chi}^{X_A}_g$ depending on the parity of r is the twisted Euler character given in the appropriate table in §A.3, a character of G^X. (If X is not of pure A-type, then the shadow function $S^X_{g,r}(\tau)$ is a linear combination of similar functions as described in §B.2.)

Given X and g, the symbol $n_g | h_g$ given in the corresponding table in §A.3 defines the modularity for the vector-valued function $(\widehat{H}^X_{g,r}(\tau))$. In particular, if the shadow $(S^X_{g,r}(\tau))$ is nonzero, and if for $\gamma \in \Gamma_0(n_g)$ we have that

$$(S^X_{g,r}(\tau))|_{3/2}\gamma = \sigma_{g,\gamma}(S^X_{g,r}(\tau)),$$

then

$$(\widehat{H}^X_{g,r}(\tau))|_{1/2}\gamma = \overline{\sigma_{g,\gamma}}(\widehat{H}^X_{g,r}(\tau)).$$

Here, for $\gamma \in \Gamma_0(n_g)$, we have $\sigma_{g,\gamma} = \nu_g(\gamma)\sigma_{e,\gamma}$ where $\nu_g(\gamma)$ is a multiplier which is trivial on $\Gamma_0(n_g h_g)$. This identity holds even in the case that the shadow $S^X_{g,r}$ vanishes.

The vector-valued function $(H^X_{g,r}(\tau))$ has poles only at the infinite cusp of $\Gamma_0(n_g)$, and only at the component $H^X_{g,r}(\tau)$ where $r = 1$ if X has pure A-type, or at components where $r^2 \equiv 1 \pmod{4M}$ otherwise. These poles may only have order $\frac{1}{4M}$. This implies that the function $(\widehat{H}^X_{g,r}(\tau) S^X_{g,r}(\tau))$ has no pole at any cusp, and is therefore a candidate for an application of Theorem 2.2.

The modular transformation of $S_{M,r}(\tau)$ implies that

$$(\sigma_{e,S})^2 = (\sigma_{e,T})^{4M} = \mathbf{I}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\mathbf{I}$ is the identity matrix. Therefore $S^X_{M,r}(\tau)$, viewed as a scalar-valued modular function, is modular on $\Gamma(4M)$, and so $(\widehat{H}^X_{g,r}(\tau)S^X_{g,r}(\tau))$ is a weight 2 nonholomorphic scalar-valued modular form for the group $\Gamma(4M)\cap\Gamma_0(n_g)$ with trivial multiplier.

Applying Theorem 2.2, we obtain a function $F^X_{g,r}(\tau)$—call it the holomorphic projection of $\widehat{H^X_{g,r}}(\tau)S^X_{e,r}(\tau)$—which is a weight 2 quasimodular form on $\Gamma(4M)\cap\Gamma_0(n_g)$. In the case that $S^X_{g,r}(\tau)$ is zero, we substitute $S^X_{e,r}(\tau)$ in its place to obtain a function $\widetilde{F}^X_{g,r}(\tau) = H^X_{g,r}(\tau)S^X_{e,r}(\tau)$ which is a weight 2 holomorphic scalar-valued modular form for the group $\Gamma(4M)\cap\Gamma_0(n_g)$ with multiplier ν_g (alternatively, modular for the group $\Gamma(4M)\cap\Gamma_0(n_gh_g)$ with trivial multiplier).

The function $F^X_{g,r}(\tau)$ may be determined explicitly as the sum of Eisenstein series and cusp forms on $\Gamma(4M)\cap\Gamma_0(n_gh_g)$ using the standard arguments from the theory of holomorphic modular forms (i.e. the "first few" coefficients determine such a form). Therefore, we have the identity

$$F^X_{g,r}(\tau) = H^X_{g,r}(\tau)\cdot S^X_{g,r}(\tau) + D^X_{g,r}(\tau), \tag{2.2}$$

where the function $D^X_{g,r}(\tau)$ is the correction term arising in Theorem 2.2. If X has pure A-type, then

$$D^X_{g,r}(\tau) = (\hat{\chi}^{X_A}_{g,r})^2 \sum_{N=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z}_+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n)q^{\frac{N}{4M}}, \tag{2.3}$$

where

$$\phi_r(\ell) = \begin{cases} \pm 1 & \text{if } \ell \equiv \pm r \pmod{2M} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} A^X_{g,r}(n)q^{n-\frac{D}{4M}}$ where $0 < D < 4M$ and $D \equiv r^2 \pmod{4M}$, and $F^X_{g,r}(\tau) = \sum_{N=0}^{\infty} B^X_{g,r}(n)q^n$. Then by Theorem 2.2, we find that

$$B^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv r \pmod{2M}}} m\cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) + (\hat{\chi}^{X_A}_{g,r})^2 \sum_{\substack{m,n\in\mathbb{Z}^+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n). \tag{2.4}$$

The function $F^X_{g,r}(\tau)$ may be found in the following manner. Using the explicit prescriptions for $H^X_{g,r}(\tau)$ given in §B.3 and (2.2) above, we may calculate the first several coefficients of each component. The Eisenstein component is determined by the constant terms at cusps. Since $D^X_{g,r}(\tau)$ (and the corresponding correction terms at other cusps) has no constant term, these are the same as the constant terms of $\widehat{H^X_{g,r}}(\tau)S^X_{g,r}(\tau)$, which are determined by the poles of $\widehat{H^X_{g,r}}$. Call this Eisenstein

component $E^X_{g,r}(\tau)$. The cuspidal component can be found by matching the initial coefficients of $F^X_{g,r}(\tau) - E^X_{g,r}(\tau)$.

Once the coefficients $B^X_{g,r}(n)$ are known, equation (2.4) provides a recursion relation which may be used to calculate the coefficients of $H^X_{g,r}(\tau)$. If the shadows $S^X_{g,r}(\tau)$ are zero, then we may apply a similar procedure in order to determine $\widetilde{F}^X_{g,r}(\tau)$. For example, suppose $\widetilde{F}^X_{g,r}(\tau) = \sum_{N=0}^{\infty} \widetilde{B}^X_{g,r}(n)q^n$, and X has pure A-type. Then we find that the coefficients $\widetilde{B}^X_{g,r}(N)$ satisfy

$$\widetilde{B}^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv r \pmod{2M}}} m \cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) \tag{2.5}$$

Proceeding in this way we obtain the claimed results. □

3 Proof of Theorem 1.1

Here we prove Theorem 1.1. The idea is as follows. For each Niemeier root system X we begin with the vector-valued mock modular forms $(H^X_g(\tau))$ for $g \in G^X$. We use their q-expansions to solve for the q-series whose coefficients are the alleged multiplicities of the irreducible components of the alleged infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \pmod{2m}} \bigoplus_{\substack{D\in\mathbb{Z},\, D\leq 0, \\ D=r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m}.$$

These q-series turn out to be mock modular forms. The proof requires that we establish that these mock modular forms have non-negative integer coefficients.

Proof of Theorem 1.1. As in the previous section, we fix a root system X and set $M := m^X$, and we emphasize the case when X is of pure A-type.

The umbral moonshine conjecture asserts that

$$H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} \sum_{\chi} m^X_{\chi,r}(n)\chi(g)q^{n-\frac{r^2}{4M}} \tag{3.1}$$

where the second sum is over the irreducible characters of G^X. Here we have rewritten the traces of the graded components $\check{K}^X_{r,n-r^2/4M}$ in 1.4 in terms of the values of the irreducible characters of G^X, where the $m^X_{\chi,r}(n)$ are the corresponding multiplicities. Naturally, if such a $\check{K}^X$ exists, these multiplicities must be non-negative integers for $n > 0$. Similarly, if the mock modular forms $H^X_{g,r}(\tau)$ can be expressed as in 3.1 with $m^X_{\chi,r}(n)$ non-negative integers, then we may construct the umbral moonshine module $\check{K}^X$ explicitly with $\check{K}^X_{r,n-r^2/4M}$ defined as the direct sum of irreducible components with the given multiplicities $m^X_{\chi,r}(n)$.

Let

$$H^X_{\chi,r}(\tau) := \frac{1}{|G^X|} \sum_g \overline{\chi(g)} H^X_{g,r}(\tau). \tag{3.2}$$

It turns out that the coefficients of $H^X_{\chi,r}(\tau)$ are precisely the multiplicities $m^X_{\chi,r}(n)$ required so that 3.1 holds: if

$$H^X_{\chi,r}(\tau) = \sum_{n=0}^{\infty} m^X_{\chi,r}(n) q^{n-\frac{r^2}{4M}}, \tag{3.3}$$

then

$$H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} \sum_{\chi} m^X_{\chi,r}(n)\chi(g) q^{n-\frac{r^2}{4M}}.$$

Thus the umbral moonshine conjecture is true if and only if the Fourier coefficients of $H^X_{\chi,r}(\tau)$ are non-negative integers.

To see this fact, we recall the orthogonality of characters. For irreducible characters χ_i and χ_j,

$$\frac{1}{|G^X|} \sum_{g\in G^X} \overline{\chi_i(g)}\chi_j(g) = \begin{cases} 1 & \text{if } \chi_i = \chi_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

We also have the relation for g and $h \in G^X$,

$$\sum_{\chi} \overline{\chi_i(g)}\chi_i(h) = \begin{cases} |C_{G^X}(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Here $|C_{G^X}(g)|$ is the order of the centralizer of g in G^X. Since the order of the centralizer times the order of the conjugacy class of an element is the order of the group, (3.2) and (3.5) together imply the relation

$$H^X_{g,r}(\tau) = \sum_{\chi} \chi(g) H^X_{\chi,r}(\tau),$$

which in turn implies 3.3.

We have reduced the theorem to proving that the coefficients of certain weight $\frac{1}{2}$ mock modular forms are all non-negative integers. For holomorphic modular forms we may answer questions of this type by making use of Sturm's theorem [49] (see also Theorem 2.58 of [44]). This theorem provides a bound B associated to a space of modular forms such that if the first B coefficients of a modular form $f(\tau)$ are integral, then all of the coefficients of $f(\tau)$ are integral. This bound reduces many questions about the Fourier coefficients of modular forms to finite calculations.

Sturm's theorem relies on the finite dimensionality of certain spaces of modular forms, and so it can not be applied directly to spaces of mock modular forms. However, by making use of holomorphic projection we can adapt Sturm's theorem to this setting.

Let $\widehat{H^X_{\chi,r}}(\tau)$ be defined as above. Recall that the transformation matrix for the vector-valued function $\widehat{H^X_{g,r}}(\tau))$ is $\overline{\sigma_{g,\gamma}}$, the conjugate of the transformation matrix for $(S^X_{e,r}(\tau))$ when $\gamma \in \Gamma_0(n_g h_g)$, and $\sigma_{g,\gamma}$ is the identity for $\gamma \in \Gamma(4M)$. Therefore if

$$N^X_{\chi} := \operatorname{lcm}\{n_g h_g \mid g \in G, \chi(g) \neq 0\},$$

then the scalar-valued functions $\widehat{H^X_{\chi,r}}(\tau)$ are modular on $\Gamma(4M) \cap \Gamma_0(N^X_{\chi})$.

Let

$$A_{\chi,r}(\tau) := H^X_{\chi,r}(\tau) S^X_{e,1}(\tau),$$

and let $\tilde{A}_{\chi,r}(\tau)$ be the holomorphic projection of $A_{\chi,r}(\tau)$. Suppose that $H^X_{\chi,r}(\tau)$ has integral coefficients up to some bound B. Formulas for the shadow functions (cf. §B.2) show that the leading coefficient of $S^X_{e,1}(\tau)$ is 1 and has integral coefficients. This implies that the function

$$A_{\chi,r}(\tau) := H^X_{\chi,r}(\tau) S^X_{e,1}(\tau)$$

also has integral coefficients up to the bound B. The shadow of $H^X_{\chi,r}(\tau)$ is given by

$$S^X_{\chi,r}(\tau) := \frac{1}{|G^X|} \sum_g \overline{\chi(g)} S^X_{g,r}(\tau).$$

If X is pure A-type, then $S^X_{g,r}(\tau) = \chi^{X_A}_{g,r} S_{M,r}(\tau) = (\chi'(g) + \chi''(g)) S_{M,r}(\tau)$ for some irreducible characters χ' and χ'', according to §A.3 and §B.2. Therefore,

$$S^X_{\chi,r}(\tau) = \begin{cases} S_{M,r}(\tau) & \text{if } \chi = \chi' \text{ or } \chi'', \\ 0 & \text{otherwise.} \end{cases}$$

When X is not of pure A-type the shadow is some sum of such functions, but in every case has integer coefficients, and so, applying Theorem 2.2 to $A_{\chi,r}(\tau)$, we find that $\tilde{A}_{\chi,r}(\tau)$ also has integer coefficients up to the bound B. In particular, since $\tilde{A}_{\chi,r}(\tau)$ is modular on $\Gamma(4M) \cap \Gamma_0(N^X_\chi)$, then if B is at least the Sturm bound for this group we have that every coefficient of $\tilde{A}_{\chi,r}(\tau)$ is integral. Since the leading coefficient of $S^X_{e,1}(\tau)$ is 1, we may reverse this argument and obtain that every coefficient of $H^X_{\chi,r}(\tau)$ is integral. Therefore, in order to check that $H^X_{\chi,r}(\tau)$ has only integer coefficients, it suffices to check up to the Sturm bound for $\Gamma(4M) \cap \Gamma_0(N_\chi)$. These calculations were carried out using the `sage` mathematical software [47].

The calculations and argument given above shows that the multiplicities $m^X_{\chi,r}(n)$ are all integers. To complete the proof, it suffices to check that they are also are non-negative. The proof of this claim follows easily by modifying step-by-step the argument in Gannon's proof of non-negativity in the M_{24} case [37] (i.e. $X = A_1^{24}$). Here we describe how this is done.

Expressions for the alleged McKay-Thompson series $H^X_{g,r}(\tau)$ in terms of Rademacher sums and unary theta functions are given in §B.4. Exact formulas are known for all the coefficients of Rademacher sums because they are defined by averaging the special function $\mathrm{r}^{[\alpha]}_{1/2}(\gamma, \tau)$ (see (B.114)) over cosets of a specific modular group modulo Γ_∞, the subgroup of translations. Therefore, Rademacher sums are standard Maass-Poincaré series, and as a result we have formulas for each of their coefficients as convergent infinite sums of Kloosterman-type sums weighted by values of the $I_{1/2}$ modified Bessel function. (For example, see [8] or [53] for the general theory, and [12] for the specific case that $X = A_1^{24}$.) More importantly, this means also that the generating function for the multiplicities $m^X_{\chi,r}(n)$ is a weight $\frac{1}{2}$ harmonic Maass form, which in turn means that exact formulas (modulo the unary theta functions) are also available in similar terms. For positive integers n, this then means that (cf. Theorem 1.1 of [8])

$$m^X_{\chi,r}(n) = \sum_\rho \sum_{m<0} \frac{a^X_\rho(m)}{n^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{K^X_\rho(m,n,c)}{c} \cdot \mathbb{I}^X \left(\frac{4\pi\sqrt{|nm|}}{c} \right), \tag{3.6}$$

where the sums are over the cusps ρ of the group $\Gamma_0(N^X_g)$, and finitely many explicit negative rational numbers m. The constants $a^X_\rho(m)$ are essentially the coefficients which describe the generating function in terms of Maass-Poincaré series. Here $\mathbb{I}$ is a suitable normalization and change of variable for the standard $I_{1/2}$ modified Bessel-function.

The Kloosterman-type sums $K_\rho^X(m,n,c)$ are well known to be related to Salié-type sums (for example see Proposition 5 of [40]). These Salié-type sums are of the form

$$S_\rho^X(m,n,c) = \sum_{\substack{x \pmod c \\ x^2 \equiv -D(m,n) \pmod c}} \epsilon_\rho^X(m,n) \cdot e\left(\frac{\beta^X x}{c}\right),$$

where $\epsilon_\rho^X(m,n)$ is a root of unity, $-D(m,n)$ is a discriminant of a positive definite binary quadratic form, and β^X is a nonzero positive rational number.

These Salié sums may then be estimated using the equidistribution of CM points with discriminant $-D(m,n)$. This process was first introduced by Hooley [38], and it was first applied to the coefficients of weight $\frac{1}{2}$ mock modular forms by Bringmann and Ono [7]. Gannon explains how to make effective the estimates for sums of this shape in §4 of [37], thereby reducing the proof of the M_{24} case of umbral moonshine to a finite calculation. In particular, in equations (4.6-4.10) of [37] Gannon shows how to bound coefficients of the form (3.6) in terms of the Selberg–Kloosterman zeta function, which is bounded in turn in his proof of Theorem 3 of [37]. We follow Gannon's proof *mutatis mutandis*. We find, for most root systems, that the coefficients of each multiplicity generating function are positive beyond the 390th coefficient. In the worst case, for the root system $X = E_8^3$, we find that the coefficients are positive beyond the 1780th coefficient. Moreover, the coefficients exhibit subexponential growth. A finite computer calculation in `sage` has verified the non-negativity of the finitely many remaining coefficients. □

Remark. It turns out that the estimates required for proving non-negativity are the worst for the $X = E_8^3$ case which required 1780 coefficients.

A The Umbral Groups

In this section we present the facts about the umbral groups that we have used in establishing the main results of this paper. We recall (from [16]) their construction in terms of Niemeier root systems in §A.1, and we reproduce their character tables (appearing also in [16]) in §A.2. Note that we use the abbreviations $a_n := \sqrt{-n}$ and $b_n := (-1+\sqrt{-n})/2$ in the tables of §A.2.

The root system description of the umbral groups (cf. §A.1) gives rise to certain characters called *twisted Euler characters* which we recall (from [16]) in §A.3. The data appearing in §A.3 plays an important role in §B.2, where we use it to describe the shadows S_g^X of the umbral McKay-Thompson series H_g^X explicitly.

A.1 Construction

As mentioned in §1, there are exactly 24 self-dual even positive-definite lattices of rank 24 up to isomorphism, according to the classification of Niemeier [43] (cf. also [19,52]). Such a lattice L is determined up to isomorphism by its *root system* $L_2 := \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$. The unique example without roots is the Leech lattice. We refer to the remaining 23 as the *Niemeier lattices*, and we call a root system X a *Niemeier root system* if it occurs as the root system of a Niemeier lattice.

The simple components of Niemeier root systems are root systems of ADE type, and it turns out that the simple components of a Niemeier root system X all have the same Coxeter number. Define m^X to be the Coxeter number of any simple component of X, and call this the *Coxeter number* of X.

For X a Niemeier root system write N^X for the corresponding Niemeier lattice. The *umbral group* attached to X is defined by setting

$$G^X := \mathrm{Aut}(N^X)/W^X \tag{A.1}$$

where W^X is the normal subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in root vectors.

Observe that G^X acts as permutations on the simple components of X. In general this action is not faithful, so define $\overline{G}^X$ to be the quotient of G^X by its kernel. It turns out that the level of the mock modular form H_g^X attached to $g \in G^X$ is given by the order, denoted n_g, of the image of g in $\bar{G}^X$. (Cf. §A.3 for the values n_g.)

The Niemeier root systems and their corresponding umbral groups are described in Table 1. The root systems are given in terms of their simple components of ADE type. Here $D_{10}E_7^2$, for example, means the direct sum of one copy of the D_{10} root system and two copies of the E_7 root system. The symbol ℓ is called the *lambency* of X, and the Coxeter number m^X appears as the first summand of ℓ.

In the descriptions of the umbral groups G^X, and their permutation group quotients $\bar{G}^X$, we write M_{24} and M_{12} for the sporadic simple groups of Mathieu which act quintuply transitively on 24 and 12 points, respectively. (Cf. e.g. [21].) We write $GL_n(q)$ for the general linear group of a vector space of dimension n over a field with q elements, and $SL_n(q)$ is the subgroup of linear transformations with determinant 1, &c. The symbols $AGL_3(2)$ denote the *affine general linear group*, obtained by adjoining translations to $GL_3(2)$. We write Dih_n for the dihedral group of order $2n$, and Sym_n denotes the symmetric group on n symbols. We use n as a shorthand for a cyclic group of order n.

Table 1: The Umbral Groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4 D_4$	A_6^4	$A_7^2 D_5^2$
ℓ	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2 D_6$	$A_{11} D_7 E_6$	A_{12}^2	$A_{15} D_9$	$A_{17} E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10} E_7^2$	D_{12}^2	$D_{16} E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

We also use the notational convention of writing $A.B$ to denote the middle term in a short exact sequence $1 \to A \to A.B \to B \to 1$. This introduces some ambiguity which is nonetheless easily navigated in practice. For example, $2.M_{12}$ is the unique (up to isomorphism) double cover of M_{12} which is not $2 \times M_{12}$. The group $AGL_3(2)$ naturally embeds in $GL_4(2)$, which in turn admits a unique (up to isomorphism) double cover $2.GL_4(2)$ which is not a direct product. The group we denote $2.AGL_3(2)$ is the preimage of $AGL_3(2) < GL_4(2)$ in $2.GL_4(2)$ under the natural projection.

A.2 Character Tables

Table 2: Character table of $G^X \simeq M_{24}$, $X = A_1^{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
χ_3	$\circ$	45	-3	5	0	3	-3	1	1	0	0	-1	b_7	$\overline{b_7}$	-1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	-1	-1
χ_4	$\circ$	45	-3	5	0	3	-3	1	1	0	0	-1	$\overline{b_7}$	b_7	-1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	-1	-1
χ_5	$\circ$	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	$\circ$	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	$+$	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	$+$	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
χ_9	$+$	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	$\circ$	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	$\circ$	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	$\circ$	990	-18	-10	0	3	6	2	-2	0	0	-1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	$\circ$	990	-18	-10	0	3	6	2	-2	0	0	-1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	$+$	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
χ_{15}	$\circ$	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2b_7$	$2\overline{b_7}$	-1	0	1	0	-1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	$\circ$	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\overline{b_7}$	$2b_7$	-1	0	1	0	-1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	$+$	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0
χ_{18}	$+$	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0
χ_{19}	$+$	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	$+$	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
χ_{21}	$+$	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0
χ_{22}	$+$	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1
χ_{23}	$+$	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	$+$	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
χ_{25}	$+$	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0
χ_{26}	$+$	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

Table 3: Character table of $G^X \simeq 2.M_{12}$, $X = A_2^{12}$

$[g]$	FS	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^2]$		1A	1A	2A	1A	1A	3A	3A	3B	3B	2B	2B	5A	5A	6B	3A	3A	4B	4B	4C	4C	10A	10A	11B	11B	11A	11A
$[g^3]$		1A	2A	4A	2B	2C	1A	2A	1A	2A	4B	4C	5A	10A	4A	2B	2C	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^5]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	1A	2A	12A	6C	6D	8B	8A	8D	8C	4A	4A	11A	22A	11B	22B
$[g^{11}]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20B	20A	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	11	11	-1	3	3	2	2	-1	-1	-1	3	1	1	-1	0	0	-1	-1	1	1	-1	-1	0	0	0	0
χ_3	+	11	11	-1	3	3	2	2	-1	-1	3	-1	1	1	-1	0	0	1	1	-1	-1	-1	-1	0	0	0	0
χ_4	∘	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	b_{11}	b_{11}	$\overline{b_{11}}$	$\overline{b_{11}}$
χ_5	∘	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	$\overline{b_{11}}$	$\overline{b_{11}}$	b_{11}	b_{11}
χ_6	+	45	45	5	-3	-3	0	0	3	3	1	1	0	0	-1	0	0	-1	-1	-1	-1	0	0	1	1	1	1
χ_7	+	54	54	6	6	6	0	0	0	0	2	2	-1	-1	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
χ_8	+	55	55	-5	7	7	1	1	1	1	-1	-1	0	0	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
χ_9	+	55	55	-5	-1	-1	1	1	1	1	3	-1	0	0	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	+	55	55	-5	-1	-1	1	1	1	1	-1	3	0	0	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
χ_{11}	+	66	66	6	2	2	3	3	0	0	-2	-2	1	1	0	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{12}	+	99	99	-1	3	3	0	0	3	3	-1	-1	-1	-1	-1	0	0	1	1	1	1	-1	-1	0	0	0	0
χ_{13}	+	120	120	0	-8	-8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	-1	-1	-1
χ_{14}	+	144	144	4	0	0	0	0	-3	-3	0	0	-1	-1	1	0	0	0	0	0	0	-1	-1	1	1	1	1
χ_{15}	+	176	176	-4	0	0	-4	-4	-1	-1	0	0	1	1	-1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	∘	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	a_2	$\overline{a_2}$	a_2	$\overline{a_2}$	0	0	-1	1	-1	1
χ_{17}	∘	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	$\overline{a_2}$	a_2	$\overline{a_2}$	a_2	0	0	-1	1	-1	1
χ_{18}	+	12	-12	0	4	-4	3	-3	0	0	0	0	2	-2	0	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{19}	−	32	-32	0	0	0	-4	4	2	-2	0	0	2	-2	0	0	0	0	0	0	0	0	0	-1	1	-1	1
χ_{20}	∘	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	a_5	$\overline{a_5}$	0	0	0	0
χ_{21}	∘	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	$\overline{a_5}$	a_5	0	0	0	0
χ_{22}	∘	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	a_2	$\overline{a_2}$	$\overline{a_2}$	a_2	0	0	0	0	0	0
χ_{23}	∘	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	$\overline{a_2}$	a_2	a_2	$\overline{a_2}$	0	0	0	0	0	0
χ_{24}	+	120	-120	0	8	-8	3	-3	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	-1	1	-1	1
χ_{25}	∘	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-b_{11}$	b_{11}	$-\overline{b_{11}}$	$\overline{b_{11}}$
χ_{26}	∘	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\overline{b_{11}}$	$\overline{b_{11}}$	$-b_{11}$	b_{11}

Table 4: Character table of $G^X \simeq 2.AGL_3(2)$, $X = A_3^8$

$[g]$	FS	1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	7A	14A	7B	14B
$[g^2]$		1A	1A	1A	2A	2B	1A	3A	3A	3A	3A	4A	2C	7A	7A	7B	7B
$[g^3]$		1A	2A	2B	4A	4B	2C	1A	2A	2B	2B	8A	4C	7B	14B	7A	14A
$[g^7]$		1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	b_7	b_7	$\overline{b_7}$	$\overline{b_7}$
χ_3	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	$\overline{b_7}$	$\overline{b_7}$	b_7	b_7
χ_4	+	6	6	6	2	2	2	0	0	0	0	0	0	−1	−1	−1	−1
χ_5	+	7	7	7	−1	−1	−1	1	1	1	1	−1	−1	0	0	0	0
χ_6	+	8	8	8	0	0	0	−1	−1	−1	−1	0	0	1	1	1	1
χ_7	+	7	7	−1	3	−1	−1	1	1	−1	−1	1	−1	0	0	0	0
χ_8	+	7	7	−1	−1	−1	3	1	1	−1	−1	−1	1	0	0	0	0
χ_9	+	14	14	−2	2	−2	2	−1	−1	1	1	0	0	0	0	0	0
χ_{10}	+	21	21	−3	1	1	−3	0	0	0	0	−1	1	0	0	0	0
χ_{11}	+	21	21	−3	−3	1	1	0	0	0	0	1	−1	0	0	0	0
χ_{12}	+	8	−8	0	0	0	0	2	−2	0	0	0	0	1	−1	1	−1
χ_{13}	∘	8	−8	0	0	0	0	−1	1	a_3	$\overline{a_3}$	0	0	1	−1	1	−1
χ_{14}	∘	8	−8	0	0	0	0	−1	1	$\overline{a_3}$	a_3	0	0	1	−1	1	−1
χ_{15}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	$\overline{b_7}$	$-\overline{b_7}$	b_7	$-b_7$
χ_{16}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	b_7	$-b_7$	$\overline{b_7}$	$-\overline{b_7}$

Table 5: Character table of $G^X \simeq GL_2(5)/2$, $X = A_4^6$

$[g]$	FS	1A	2A	2B	2C	3A	6A	5A	10A	4A	4B	4C	4D	12A	12B
$[g^2]$		1A	1A	1A	1A	3A	3A	5A	5A	2A	2A	2C	2C	6A	6A
$[g^3]$		1A	2A	2B	2C	1A	2A	5A	10A	4B	4A	4D	4C	4B	4A
$[g^5]$		1A	2A	2B	2C	3A	6A	1A	2A	4A	4B	4C	4D	12A	12B
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_3	+	4	4	0	0	1	1	−1	−1	2	2	0	0	−1	−1
χ_4	+	4	4	0	0	1	1	−1	−1	−2	−2	0	0	1	1
χ_5	+	5	5	1	1	−1	−1	0	0	1	1	−1	−1	1	1
χ_6	+	5	5	1	1	−1	−1	0	0	−1	−1	1	1	−1	−1
χ_7	+	6	6	−2	−2	0	0	1	1	0	0	0	0	0	0
χ_8	∘	1	−1	1	−1	1	−1	1	−1	a_1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$
χ_9	∘	1	−1	1	−1	1	−1	1	−1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$	a_1
χ_{10}	∘	4	−4	0	0	1	−1	−1	1	$2a_1$	$-2a_1$	0	0	$-a_1$	a_1
χ_{11}	∘	4	−4	0	0	1	−1	−1	1	$-2a_1$	$2a_1$	0	0	a_1	$-a_1$
χ_{12}	∘	5	−5	1	−1	−1	1	0	0	a_1	$-a_1$	$-a_1$	a_1	a_1	$-a_1$
χ_{13}	∘	5	−5	1	−1	−1	1	0	0	$-a_1$	a_1	a_1	$-a_1$	$-a_1$	a_1
χ_{14}	+	6	−6	−2	2	0	0	1	−1	0	0	0	0	0	0

Table 6: Character table of $G^X \simeq GL_2(3)$, $X \in \{A_5^4 D_4, E_6^4\}$

$[g]$	FS	1A	2A	2B	4A	3A	6A	8A	8B
$[g^2]$		1A	1A	1A	2A	3A	3A	4A	4A
$[g^3]$		1A	2A	2B	4A	1A	2A	8A	8B
χ_1	$+$	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	-1	1	1	1	-1	-1
χ_3	$+$	2	2	0	2	-1	-1	0	0
χ_4	$+$	3	3	-1	-1	0	0	1	1
χ_5	$+$	3	3	1	-1	0	0	-1	-1
χ_6	$\circ$	2	-2	0	0	-1	1	a_2	$\overline{a_2}$
χ_7	$\circ$	2	-2	0	0	-1	1	$\overline{a_2}$	a_2
χ_8	$+$	4	-4	0	0	1	-1	0	0

Table 7: Character table of $G^X \simeq 3.Sym_6$, $X = D_4^6$

$[g]$	FS	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	15B	2B	2C	4B	6B	6C
$[g^2]$		1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	15B	1A	1A	2A	3B	3C
$[g^3]$		1A	1A	2A	2A	1A	1A	4A	4A	5A	5A	5A	2B	2C	4B	2B	2C
$[g^5]$		1A	3A	2A	6A	3B	3C	4A	12A	1A	3A	3A	2B	2C	4B	6B	6C
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_3	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	3	-1	1	0	-1
χ_4	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	-3	1	-1	0	1
χ_5	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	-1	3	1	-1	0
χ_6	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	1	-3	-1	1	0
χ_7	$+$	16	16	0	0	-2	-2	0	0	1	1	1	0	0	0	0	0
χ_8	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	3	3	-1	0	0
χ_9	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	-3	-3	1	0	0
χ_{10}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	2	-2	0	-1	1
χ_{11}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	-2	2	0	1	-1
χ_{12}	$\circ$	6	-3	-2	1	0	0	2	-1	1	b_{15}	$\overline{b_{15}}$	0	0	0	0	0
χ_{13}	$\circ$	6	-3	-2	1	0	0	2	-1	1	$\overline{b_{15}}$	b_{15}	0	0	0	0	0
χ_{14}	$+$	12	-6	4	-2	0	0	0	0	2	-1	-1	0	0	0	0	0
χ_{15}	$+$	18	-9	2	-1	0	0	2	-1	-2	1	1	0	0	0	0	0
χ_{16}	$+$	30	-15	-2	1	0	0	-2	1	0	0	0	0	0	0	0	0

Table 8: Character table of $G^X \simeq SL_2(3)$, $X = A_6^4$

$[g]$	FS	1A	2A	4A	3A	6A	3B	6B
$[g^2]$		1A	1A	2A	3B	3A	3A	3B
$[g^3]$		1A	2A	4A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1
χ_2	$\circ$	1	1	1	b_3	$\overline{b_3}$	$\overline{b_3}$	b_3
χ_3	$\circ$	1	1	1	$\overline{b_3}$	b_3	b_3	$\overline{b_3}$
χ_4	$+$	3	3	-1	0	0	0	0
χ_5	$-$	2	-2	0	-1	1	-1	1
χ_6	$\circ$	2	-2	0	$-\overline{b_3}$	b_3	$-b_3$	$\overline{b_3}$
χ_7	$\circ$	2	-2	0	$-b_3$	$\overline{b_3}$	$-\overline{b_3}$	b_3

Table 9: Character table of $G^X \simeq Dih_4$, $X = A_7^2 D_5^2$

$[g]$	FS	1A	2A	2B	2C	4A
$[g^2]$		1A	1A	1A	1A	2A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1
χ_3	$+$	1	1	-1	1	-1
χ_4	$+$	1	1	1	-1	-1
χ_5	$+$	2	-2	0	0	0

Table 10: Character table of $G^X \simeq Dih_6$, $X = A_8^3$

$[g]$	FS	1A	2A	2B	2C	3A	6A
$[g^2]$		1A	1A	1A	1A	3A	3A
$[g^3]$		1A	2A	2B	2C	1A	2A
χ_1	$+$	1	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1	1
χ_3	$+$	2	2	0	0	-1	-1
χ_4	$+$	1	-1	-1	1	1	-1
χ_5	$+$	1	-1	1	-1	1	-1
χ_6	$+$	2	-2	0	0	-1	1

Table 11: Character table of $G^X \simeq 4$, for $X \in \{A_9^2 D_6, A_{12}^2\}$

$[g]$	FS	1A	2A	4A	4B
$[g^2]$		1A	1A	2A	2A
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	$\circ$	1	-1	a_1	$\overline{a_1}$
χ_4	$\circ$	1	-1	$\overline{a_1}$	a_1

Table 12: Character table of $G^X \simeq PGL_2(3) \simeq Sym_4$, $X = D_6^4$

$[g]$	FS	1A	2A	3A	2B	4A
$[g^2]$		1A	1A	3A	1A	2A
$[g^3]$		1A	2A	1A	2B	4A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	1	-1	-1
χ_3	$+$	2	2	-1	0	0
χ_4	$+$	3	-1	0	1	-1
χ_5	$+$	3	-1	0	-1	1

Table 13: Character table of $G^X \simeq 2$, for $X \in \{A_{11} D_7 E_6, A_{15} D_9, A_{17} E_7, A_{24}, D_{10} E_7^2, D_{12}^2\}$

$[g]$	FS	1A	2A
$[g^2]$		1A	1A
χ_1	$+$	1	1
χ_2	$+$	1	-1

Table 14: Character table of $G^X \simeq Sym_3$, $X \in \{D_8^3, E_8^3\}$

$[g]$	FS	1A	2A	3A
$[g^2]$		1A	1A	3A
$[g^3]$		1A	2A	1A
χ_1	$+$	1	1	1
χ_2	$+$	1	-1	1
χ_3	$+$	2	0	-1

A.3 Twisted Euler Characters

In this section we reproduce certain characters—the *twisted Euler characters*—which are attached to each group G^X, via its action on the root system X. (Their construction is described in detail in §2.4 of [16].)

To interpret the tables, write X_A for the (possibly empty) union of type A components of X, and interpret X_D and X_E similarly, so that if $m = m^X$ Then $X = A^d_{m-1}$ for some d, and $X = X_A \cup X_D \cup X_E$, for example. Then $g \mapsto \bar{\chi}_g^{X_A}$ denotes the character of the permutation representation attached to the action of $\bar{G}^X$ on the simple components of X_A. The characters $g \mapsto \bar{\chi}_g^{X_D}$ and $g \mapsto \bar{\chi}_g^{X_E}$ are defined similarly. The characters $\chi_g^{X_A}$, $\chi_g^{X_D}$, $\chi_g^{X_E}$ and $\check{\chi}_g^{X_D}$ incorporate outer automorphisms of simple root systems induced by the action G^X on X. We refer to §2.4 of [16] for full details of the construction. For the purposes of this work, it suffices to have the explicit descriptions in the tables in this section. The twisted Euler characters presented here will be used to specify the umbral shadow functions in §B.2.

The twisted Euler character tables also attach integers n_g and h_g to each $g \in G^X$. By definition, n_g is the order of the image of $g \in G^X$ in $\bar{G}^X$ (cf. §A.1). The integer h_g may be defined by setting $h_g := N_g/n_g$ where N_g is the product of the shortest and longest cycle lengths appearing in the cycle shape attached to g by the action of G^X on a (suitable) set of simple roots for X.

Table 15: Twisted Euler characters at $\ell = 2$, $X = A_1^{24}$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
$n_g\|h_g$	1|1	2|1	2|2	3|1	3|3	4|2	4|1	4|4	5|1	6|1	6|6
$\bar{\chi}_g^{X_A}$	24	8	0	6	0	0	4	0	4	2	0

$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
$n_g\|h_g$	7|1	8|1	10|2	11|1	12|2	12|12	14|1	15|1	21|3	23|1
$\bar{\chi}_g^{X_A}$	3	2	0	2	0	0	1	1	0	1

Table 16: Twisted Euler characters at $\ell = 3$, $X = A_2^{12}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
$n_g\|h_g$	1|1	1|4	2|8	2|1	2|2	3|1	3|4	3|3	3|12	4|2	4|1	5|1	5|4	6|24	6|1	6|2	8|4	8|1	10|8	11|1	11|4
$\bar{\chi}_g^{X_A}$	12	12	0	4	4	3	3	0	0	0	4	2	2	0	1	1	0	2	0	1	1
$\chi_g^{X_A}$	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	1	−1

Table 17: Twisted Euler characters at $\ell = 4$, $X = A_3^8$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$n_g\|h_g$	1|1	1|2	2|2	2|4	4|4	2|1	3|1	3|2	6|2	4|8	4|1	7|1	7|2
$\bar{\chi}_g^{X_A}$	8	8	0	0	0	4	2	2	0	0	2	1	1
$\chi_g^{X_A}$	8	−8	0	0	0	0	2	−2	0	0	0	1	−1

Table 18: Twisted Euler characters at $\ell = 5$, $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$n_g\|h_g$	1|1	1|4	2|2	2|1	3|3	3|12	5|1	5|4	2|8	4|1	6|24
$\bar{\chi}_g^{X_A}$	6	6	2	2	0	0	1	1	0	2	0
$\chi_g^{X_A}$	6	−6	−2	2	0	0	1	−1	0	0	0

Table 19: Twisted Euler characters at $\ell = 6$, $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	1|1	1|2	2|1	2|2	3|1	3|2	4|2
$\bar{\chi}_g^{X_A}$	4	4	2	0	1	1	0
$\chi_g^{X_A}$	4	−4	0	0	1	−1	0
$\bar{\chi}_g^{X_D}$	1	1	1	1	1	1	1
$\chi_g^{X_D}$	1	1	−1	1	1	1	−1
$\check{\chi}_g^{X_D}$	2	2	0	2	−1	−1	0

Table 20: Twisted Euler characters at $\ell = 6 + 3$, $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	6C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
$n_g\|h_g$	1|1	1|3	2|1	2|3	3|1	3|3	4|2	4|6	5|1	5|3	2|1	2|2	4|1	6|1	6|6
$\bar{\chi}_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	4	0	2	1	0
$\chi_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	−4	0	−2	−1	0
$\check{\chi}_g^{X_D}$	12	−6	4	−2	0	0	0	0	2	−1	0	0	0	0	0

Table 21: Twisted Euler characters at $\ell = 7$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
$n_g \vert h_g$	1|1	1|4	2|8	3|1	3|4
$\bar{\chi}_g^{X_A}$	4	4	0	1	1
$\chi_g^{X_A}$	4	-4	0	1	-1

Table 22: Twisted Euler characters at $\ell = 8$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2B	2C	4A
$n_g \vert h_g$	1|1	1|2	2|1	2|1	2|4
$\bar{\chi}_g^{X_A}$	2	2	0	2	0
$\chi_g^{X_A}$	2	-2	0	0	0
$\bar{\chi}_g^{X_D}$	2	2	2	0	0
$\chi_g^{X_D}$	2	-2	0	0	0

Table 23: Twisted Euler characters at $\ell = 9$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
$n_g \vert h_g$	1|1	1|4	2|1	2|2	3|3	3|12
$\bar{\chi}_g^{X_A}$	3	3	1	1	0	0
$\chi_g^{X_A}$	3	-3	1	-1	0	0

Table 24: Twisted Euler characters at $\ell = 10$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
$n_g \vert h_g$	1|1	1|2	2|2
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0
$\bar{\chi}_g^{X_D}$	1	1	1
$\chi_g^{X_D}$	1	1	-1

Table 25: Twisted Euler characters at $\ell = 10 + 5$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
$n_g \vert h_g$	$1\vert 1$	$2\vert 2$	$3\vert 1$	$2\vert 1$	$4\vert 4$
$\bar{\chi}_g^{X_D}$	4	0	1	2	0
$\chi_g^{X_D}$	4	0	1	-2	0

Table 26: Twisted Euler characters at $\ell = 12$, $X = A_{11}D_7E_6$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\chi}_g^{X_E}$	1	1
$\chi_g^{X_E}$	1	-1

Table 27: Twisted Euler characters at $\ell = 12 + 4$, $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$	$2\vert 1$	$2\vert 4$	$3\vert 1$	$3\vert 2$	$4\vert 8$
$\bar{\chi}_g^{X_E}$	4	4	2	0	1	1	0
$\chi_g^{X_E}$	4	-4	0	0	1	-1	0

Table 28: Twisted Euler characters at $\ell = 13$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
$n_g \vert h_g$	$1\vert 1$	$1\vert 4$	$2\vert 8$
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0

Table 29: Twisted Euler characters at $\ell = 14 + 7$, $X = D_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$	$3\vert 3$
$\bar{\chi}_g^{X_D}$	3	1	0
$\chi_g^{X_D}$	3	1	0

Table 30: Twisted Euler characters at $\ell = 16$, $X = A_{15}D_9$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1

Table 31: Twisted Euler characters at $\ell = 18$, $X = A_{17}E_7$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_E}$	1	1

Table 32: Twisted Euler characters at $\ell = 18 + 9$, $X = D_{10}E_7^2$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\chi}_g^{X_E}$	2	0

Table 33: Twisted Euler characters at $\ell = 22 + 11$, $X = D_{12}^2$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$2\vert 2$
$\bar{\chi}_g^{X_D}$	2	0
$\chi_g^{X_D}$	2	0

Table 34: Twisted Euler characters at $\ell = 25$, $X = A_{24}$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 4$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1

Table 35: Twisted Euler characters at $\ell = 30 + 6, 10, 15$, $X = E_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$	$3\vert 3$
$\bar{\chi}_g^{X_E}$	3	1	0

B The Umbral McKay-Thompson Series

In this section we describe the umbral McKay-Thompson series in complete detail. In particular, we present explicit formulas for all the McKay-Thompson series attached to elements of the umbral groups by umbral moonshine in §B.3. Most of these expressions appeared first in [15,16], but some appear for the first time in this work.

In order to facilitate explicit formulations we recall certain standard functions in §B.1. We then, using the twisted Euler characters of §A.3, explicitly describe the shadow functions of umbral moonshine in §B.2. The umbral McKay–Thompson series defined in §B.3 may also be described in terms of Rademacher sums, according to the results of [17]. We present this description in §B.4.

B.1 Special Functions

Throughout this section we assume $q := e^{2\pi i\tau}$, and $u := e^{2\pi iz}$, where $\tau, z \in \mathbb{C}$ with $\mathrm{Im}\ \tau > 0$. The Dedekind eta function is $\eta(\tau) := q^{1/24}\prod_{n>0}(1-q^n)$, where . Write $\Lambda_M(\tau)$ for the function

$$\Lambda_M(\tau) := Mq\frac{\mathrm{d}}{\mathrm{d}q}\left(\log\frac{\eta(M\tau)}{\eta(\tau)}\right) = \frac{M(M-1)}{24} + M\sum_{k>0}\sum_{d|k} d\left(q^k - Mq^{Mk}\right),$$

which is a modular form of weight two for $\Gamma_0(N)$ if $M|N$.

Define the Jacobi theta function $\theta_1(\tau, z)$ by setting

$$\theta_1(\tau, z) := iq^{1/8}u^{-1/2}\sum_{n\in\mathbb{Z}}(-1)^n u^n q^{n(n-1)/2}. \tag{B.1}$$

According to the Jacobi triple product identity we have

$$\theta_1(\tau, z) = -iq^{1/8}u^{1/2}\prod_{n>0}(1-u^{-1}q^{n-1})(1-uq^n)(1-q^n). \tag{B.2}$$

The other Jacobi theta functions are

$$\begin{aligned}
\theta_2(\tau, z) &:= q^{1/8}u^{1/2}\prod_{n>0}(1+u^{-1}q^{n-1})(1+uq^n)(1-q^n),\\
\theta_3(\tau, z) &:= \prod_{n>0}(1+u^{-1}q^{n-1/2})(1+uq^{n-1/2})(1-q^n),\\
\theta_4(\tau, z) &:= \prod_{n>0}(1-u^{-1}q^{n-1/2})(1-uq^{n-1/2})(1-q^n).
\end{aligned} \tag{B.3}$$

Define $\Psi_{1,1}$ and $\Psi_{1,-1/2}$ by setting

$$\begin{aligned}
\Psi_{1,1}(\tau, z) &:= -i\frac{\theta_1(\tau, 2z)\eta(\tau)^3}{\theta_1(\tau, z)^2},\\
\Psi_{1,-1/2}(\tau, z) &:= -i\frac{\eta(\tau)^3}{\theta_1(\tau, z)}.
\end{aligned} \tag{B.4}$$

These are meromorphic Jacobi forms of weight one, with indexes 1 and $-1/2$, respectively. Here, the term meromorphic refers to the presence of simple poles in the functions $z \mapsto \Psi_{1,*}(\tau, z)$, for fixed $\tau \in \mathbb{H}$, at lattice points $z \in \mathbb{Z}\tau + \mathbb{Z}$. (Cf. §8 of [22].)

From §5 of [29] we recall the *index m theta functions*, for $m \in \mathbb{Z}$, defined by setting

$$\theta_{m,r}(\tau, z) := \sum_{k\in\mathbb{Z}} u^{2mk+r} q^{(2mk+r)^2/4m}, \tag{B.5}$$

where $r \in \mathbb{Z}$. Evidently, $\theta_{m,r}$ only depends on $r \mod 2m$. We set $S_{m,r}(\tau) := \frac{1}{2\pi i}\, \partial_z \theta_{m,r}(\tau, z)|_{z=0}$, so that

$$S_{m,r}(\tau) = \sum_{k\in\mathbb{Z}} (2mk + r) q^{(2mk+r)^2/4m}. \tag{B.6}$$

For a m a positive integer define

$$\mu_{m,0}(\tau, z) = \sum_{k\in\mathbb{Z}} u^{2km} q^{mk^2} \frac{uq^k + 1}{uq^k - 1} = \frac{u+1}{u-1} + O(q), \tag{B.7}$$

and observe that we recover $\Psi_{1,1}$ upon specializing (B.7) to $m = 1$. Observe also that

$$\mu_{m,0}(\tau, z + 1/2) = \sum_{k\in\mathbb{Z}} u^{2km} q^{mk^2} \frac{uq^k - 1}{uq^k + 1} = \frac{u-1}{u+1} + O(q). \tag{B.8}$$

Define the even and odd parts of $\mu_{m,0}$ by setting

$$\mu^k_{m,0}(\tau, z) := \frac{1}{2}(\mu_{m,0}(\tau, z) + (-1)^k \mu_{m,0}(\tau, z + 1/2)) \tag{B.9}$$

for $k \mod 2$.

For $m, r \in \mathbb{Z} + \frac{1}{2}$ with $m > 0$ define *half-integral index theta functions*

$$\theta_{m,r}(\tau, z) := \sum_{k\in\mathbb{Z}} e(mk + r/2) u^{2mk+r} q^{(2mk+r)^2/4m}, \tag{B.10}$$

and define also $S_{m,r}(\tau) := \frac{1}{2\pi i}\partial_z \theta_{m,r}(\tau, z)|_{z=0}$, so that

$$S_{m,r}(\tau) = \sum_{k\in\mathbb{Z}} e(mk + r/2)(2mk + r) q^{(2mk+r)^2/4m}. \tag{B.11}$$

As in the integral index case, $\theta_{m,r}$ depends only on $r \mod 2m$. We recover $-\theta_1$ upon specializing $\theta_{m,r}$ to $m = r = 1/2$.

For $m \in \mathbb{Z} + 1/2$, $m > 0$, define

$$\mu_{m,0}(\tau, z) := i \sum_{k\in\mathbb{Z}} (-1)^k u^{2mk+1/2} q^{mk^2+k/2} \frac{1}{1 - uq^k} = \frac{-iu^{1/2}}{y - 1} + O(q). \tag{B.12}$$

Given $\alpha \in \mathbb{Q}$ write $[\alpha]$ for the operator on q-series (in rational, possibility negative powers of q) that eliminates exponents not contained in $\mathbb{Z} + \alpha$, so that if $f = \sum_{\beta\in\mathbb{Q}} c(\beta) q^\beta$ then

$$[\alpha] f := \sum_{n\in\mathbb{Z}} c(n + \alpha) q^{n+\alpha} \tag{B.13}$$

B.2 Shadows

Let X be a Niemeier root system and let $m = m^X$ be the Coxeter number of X. For $g \in G^X$ we define the associated *shadow function* $S_g^X = (S_{g,r}^X)$ by setting

$$S_g^X := S_g^{X_A} + S_g^{X_D} + S_g^{X_E} \tag{B.14}$$

where the $S_g^{X_A}$, &c., are defined in the following way, in terms of the twisted Euler characters $\chi_g^{X_A}$, &c. given in §A.3, and the unary theta series $S_{m,r}$ (cf. (B.6)).

Note that if $m = m^X$ then $S_{g,r}^X = S_{g,r+2m}^X = -S_{g,-r}^X$ for all $g \in G^X$, so we need specify the $S_{g,r}^{X_A}$, &c., only for $0 < r < m$.

If $X_A = \emptyset$ then $S_g^{X_A} := 0$. Otherwise, we define $S_{g,r}^{X_A}$ for $0 < r < m$ by setting

$$S_{g,r}^{X_A} := \begin{cases} \chi_g^{X_A} S_{m,r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_A} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.15}$$

If $X_D = \emptyset$ then $S_g^{X_D} := 0$. If $X_D \neq \emptyset$ then m is even and $m \geq 6$. If $m = 6$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{6,r} + \chi_g^{X_D} S_{6,6-r} & \text{if } r = 1, 5 \mod 6, \\ \check{\chi}_g^{X_D} S_{6,r} & \text{if } r = 3 \mod 6. \end{cases} \tag{B.16}$$

If $m > 6$ and $m = 2 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} + \chi_g^{X_D} S_{m,m-r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.17}$$

If $m > 6$ and $m = 0 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} \chi_g^{X_D} S_{m,m-r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.18}$$

If $X_E = \emptyset$ then $S_g^{X_E} := 0$. Otherwise, m is 12 or 18 or 30. In case $m = 12$ define $S_{g,r}^{X_E}$ for $0 < r < 12$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{12,1} + S_{12,7}) & \text{if } r \in \{1, 7\}, \\ \bar{\chi}_g^{X_E}(S_{12,5} + S_{12,11}) & \text{if } r \in \{5, 11\}, \\ \chi_g^{X_E}(S_{12,4} + S_{12,8}) & \text{if } r \in \{4, 8\}, \\ 0 & \text{else.} \end{cases} \tag{B.19}$$

In case $m = 18$ define $S_{g,r}^{X_E}$ for $0 < r < 18$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{18,r} + S_{18,18-r}) & \text{if } r \in \{1, 5, 7, 11, 13, 17\}, \\ \bar{\chi}_g^{X_E} S_{18,9} & \text{if } r \in \{3, 15\}, \\ \bar{\chi}_g^{X_E}(S_{18,3} + S_{18,9} + S_{18,15}) & \text{if } r = 9, \\ 0 & \text{else.} \end{cases} \tag{B.20}$$

In case $m = 30$ define $S^{X_E}_{g,r}$ for $0 < r < 30$ by setting

$$S^{X_E}_{g,r} = \begin{cases} \bar{\chi}^{X_E}_g(S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29}) & \text{if } r \in \{1, 11, 19, 29\}, \\ \bar{\chi}^{X_E}_g(S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23}) & \text{if } r \in \{7, 13, 17, 23\}, \\ 0 & \text{else.} \end{cases} \tag{B.21}$$

B.3 Explicit Prescriptions

Here we give explicit expressions for all the umbral McKay-Thompson series H^X_g. Most of these appeared first in [15,16]. The expressions in §§B.3.3, B.3.4, B.3.7, B.3.14 are taken from [26]. The expressions in §§B.3.11, B.3.15, B.3.19, B.3.23 are taken from [14]. The expressions for H^X_g with $X = E^3_8$ appeared first in [25]. The expression for $H^{(6+3)}_{2B,1}$ in §B.3.6, and the expressions for $H^{(12+4)}_{4A,r}$ and $H^{(12+4)}_{8AB,r}$ in §B.3.13, appear here for the first time.

The labels for conjugacy classes in G^X are as in §A.2.

B.3.1 $\ell = 2$, $X = A^{24}_1$

We have $G^{(2)} = G^X \simeq M_{24}$ and $m^X = 2$. So for $g \in M_{24}$, the associated umbral McKay-Thompson series $H^{(2)}_g = (H^{(2)}_{g,r})$ is a 4-vector-valued function, with components indexed by $r \in \mathbb{Z}/4\mathbb{Z}$, satisfying $H^{(2)}_{g,r} = -H^{(2)}_{g,-r}$, and in particular, $H^{(2)}_{g,r} = 0$ for $r = 0 \mod 2$. So it suffices to specify the $H^{(2)}_{g,1}$ explicitly.

Define $H^{(2)}_g = (H^{(2)}_{g,r})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\varphi^{(2)}_1(\tau, z) = -24\mu_{2,0}(\tau, z) + \sum_{r \mod 4} H^{(2)}_{e,r}(\tau)\theta_{2,r}(\tau, z), \tag{B.22}$$

where

$$\varphi^{(2)}_1(\tau, z) := 4\left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right). \tag{B.23}$$

More generally, for $g \in G^{(2)}$ define

$$H^{(2)}_{g,1}(\tau) := \frac{\bar{\chi}^{(2)}_g}{24} H^{(2)}_{e,1}(\tau) - F^{(2)}_g(\tau)\frac{1}{S_{2,1}(\tau)}, \tag{B.24}$$

where $\bar{\chi}^{(2)}_g$ and $F^{(2)}_g$ are as specified in Table 36. Note that $\bar{\chi}^{(2)}_g = \bar{\chi}^{X_A}_g$, the latter appearing in Table 15. Also, $S_{2,1}(\tau) = \eta(\tau)^3$.

The functions $f_{23,a}$ and $f_{23,b}$ in Table 36 are cusp forms of weight two for $\Gamma_0(23)$, defined by

$$\begin{aligned} f_{23,a}(\tau) &:= \frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 3\eta(\tau)^2\eta(23\tau)^2 + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta(2\tau)^2\eta(46\tau)^2, \\ f_{23,b}(\tau) &:= \eta(\tau)^2\eta(23\tau)^2. \end{aligned} \tag{B.25}$$

Note that the definition of $F^{(2)}_g$ appearing here for $g \in 23A \cup 23B$ corrects errors in [11,12].

Table 36: Character Values and Weight Two Forms for $\ell = 2$, $X = A_1^{24}$

$[g]$	$\bar{\chi}_g^{(2)}$	$F_g^{(2)}(\tau)$
$1A$	24	0
$2A$	8	$16\Lambda_2(\tau)$
$2B$	0	$2\eta(\tau)^8\eta(2\tau)^{-4}$
$3A$	6	$6\Lambda_3(\tau)$
$3B$	0	$2\eta(\tau)^6\eta(3\tau)^{-2}$
$4A$	0	$2\eta(2\tau)^8\eta(4\tau)^{-4}$
$4B$	4	$4(-\Lambda_2(\tau)+\Lambda_4(\tau))$
$4C$	0	$2\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^{-2}$
$5A$	4	$2\Lambda_5(\tau)$
$6A$	2	$2(-\Lambda_2(\tau)-\Lambda_3(\tau)+\Lambda_6(\tau))$
$6B$	0	$2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^{-2}$
$7AB$	3	$\Lambda_7(\tau)$
$8A$	2	$-\Lambda_4(\tau)+\Lambda_8(\tau)$
$10A$	0	$2\eta(\tau)^3\eta(2\tau)\eta(5\tau)\eta(10\tau)^{-1}$
$11A$	2	$2(\Lambda_{11}(\tau)-11\eta(\tau)^2\eta(11\tau)^2)/5$
$12A$	0	$2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3\eta(2\tau)^{-1}\eta(3\tau)^{-1}\eta(12\tau)^{-2}$
$12B$	0	$2\eta(\tau)^4\eta(4\tau)\eta(6\tau)\eta(2\tau)^{-1}\eta(12\tau)^{-1}$
$14AB$	1	$(-\Lambda_2(\tau)-\Lambda_7(\tau)+\Lambda_{14}(\tau)-14\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau))/3$
$15AB$	1	$(-\Lambda_3(\tau)-\Lambda_5(\tau)+\Lambda_{15}(\tau)-15\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau))/4$
$21AB$	0	$(7\eta(\tau)^3\eta(7\tau)^3\eta(3\tau)^{-1}\eta(21\tau)^{-1}-\eta(\tau)^6\eta(3\tau)^{-2})/3$
$23AB$	1	$(\Lambda_{23}(\tau)-23f_{23,a}(\tau)-69f_{23,b}(\tau))/11$

B.3.2 $\ell = 3$, $X = A_2^{12}$

We have $G^{(3)} = G^X \simeq 2.M_{12}$ and $m^X = 3$. So for $g \in 2.M_{12}$, the associated umbral McKay-Thompson series $H_g^{(3)} = (H_{g,r}^{(3)})$ is a 6-vector-valued function, with components indexed by $r \in \mathbb{Z}/6\mathbb{Z}$, satisfying $H_{g,r}^{(3)} = -H_{g,-r}^{(3)}$, and in particular, $H_{g,r}^{(3)} = 0$ for $r = 0 \mod 3$. So it suffices to specify the $H_{g,1}^{(3)}$ and $H_{g,2}^{(3)}$ explicitly.

Define $H_g^{(3)} = (H_{g,r}^{(3)})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi_1^{(3)}(\tau,z) = -12\mu_{3,0}(\tau,z) + \sum_{r \mod 6} H_{e,r}^{(3)}(\tau)\theta_{3,r}(\tau,z), \tag{B.26}$$

where

$$\varphi_1^{(3)}(\tau,z) := 2\left(\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}\right). \tag{B.27}$$

More generally, for $g \in G^{(3)}$ define

$$H^{(3)}_{g,1}(\tau) := \frac{\bar{\chi}^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g + F^{(3)}_{zg}\right)\frac{1}{S_{3,1}(\tau)}, \tag{B.28}$$

$$H^{(3)}_{g,2}(\tau) := \frac{\chi^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g - F^{(3)}_{zg}\right)\frac{1}{S_{3,2}(\tau)}, \tag{B.29}$$

where $\chi^{(3)}_g$ and $F^{(3)}_g$ are as specified in Table 37, and z is the non-trivial central element of $G^{(3)}$. The action of $g \mapsto zg$ on conjugacy classes can be read off Table 37, for the horizontal lines indicate the sets $[g] \cup [zg]$.

Note the eta product identities, $S_{3,1}(\tau) = \eta(2\tau)^5/\eta(4\tau)^2$, and $S_{3,2}(\tau) = 2\eta(\tau)^2\eta(4\tau)^2/\eta(2\tau)$. Note also that $\bar{\chi}^{(3)}_g = \bar{\chi}^{X_A}_g$ and $\chi^{(3)}_g = \chi^{X_A}_g$, the latter appearing in Table 16.

Table 37: Character Values and Weight Two Forms for $\ell = 3$, $X = A_2^{12}$

$[g]$	$\bar{\chi}^{(3)}_g$	$\chi^{(3)}_g$	$F^{(3)}_g(\tau)$
1A	12	12	0
2A	12	−12	0
4A	0	0	$-2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
2B	4	4	$-16\Lambda_2(\tau)$
2C	4	−4	$16\Lambda_2(\tau) - \frac{16}{3}\Lambda_4(\tau)$
3A	3	3	$-6\Lambda_3(\tau)$
6A	3	−3	$-9\Lambda_2(\tau) - 2\Lambda_3(\tau) + 3\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau)$
3B	0	0	$8\Lambda_3(\tau) - 2\Lambda_9(\tau) + 2\,\eta^6(\tau)/\eta^2(3\tau)$
6B	0	0	$-2\eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)$
4B	0	0	$-2\eta(2\tau)^8/\eta(4\tau)^4$
4C	4	0	$-8\Lambda_4(\tau)/3$
5A	2	2	$-2\Lambda_5(\tau)$
10A	2	−2	$\sum_{d\mid 20} c_{10A}(d)\Lambda_d(\tau) + \frac{20}{3}\eta(2\tau)^2\eta(10\tau)^2$
12A	0	0	$-2\eta(\tau)\eta(2\tau)^5\eta(3\tau)/\eta(4\tau)^2\eta(6\tau)$
6C	1	1	$2(\Lambda_2(\tau) + \Lambda_3(\tau) - \Lambda_6(\tau))$
6D	1	−1	$-5\Lambda_2(\tau) - 2\Lambda_3(\tau) + \frac{5}{3}\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau)$
8AB	0	0	$-2\eta(2\tau)^4\eta(4\tau)^2/\eta(8\tau)^2$
8CD	2	0	$-2\Lambda_2(\tau) + \frac{5}{3}\Lambda_4(\tau) - \Lambda_8(\tau)$
20AB	0	0	$-2\eta(2\tau)^7\eta(5\tau)/\eta(\tau)\eta(4\tau)^2\eta(10\tau)$
11AB	1	1	$-\frac{2}{5}\Lambda_{11}(\tau) - \frac{33}{5}\eta(\tau)^2\eta(11\tau)^2$
22AB	1	−1	$\sum_{d\mid 44} c_g(d)\Lambda_d(\tau) - \frac{11}{5}\sum_{d\mid 4} c'_g(d)\eta(d\tau)^2\eta(11d\tau)^2 + \frac{22}{3}f_{44}(\tau)$

The function f_{44} is the unique new cusp form of weight 2 for $\Gamma_0(44)$, normalized so that $f_{44}(\tau) =$

$q + O(q^3)$ as $\Im(\tau) \to \infty$. The coefficients $c_g(d)$ and $c'_g(d)$ for $g \in 10A \cup 22A \cup 22B$ are given by

$$c_{10A}(2) = -5,\ c_{10A}(4) = -\frac{5}{3},\ c_{10A}(5) = -\frac{2}{3},\ c_{10A}(10) = 1,\ c_{10A}(20) = -\frac{1}{3}, \tag{B.30}$$

$$c_{22AB}(2) = -\frac{11}{5},\ c_{22AB}(4) = \frac{11}{5},\ c_{22AB}(11) = -\frac{2}{15},\ c_{22AB}(22) = \frac{1}{5},\ c_{22AB}(44) = -\frac{1}{15}, \tag{B.31}$$

$$c'_{22AB}(1) = 1,\ c'_{22AB}(2) = 4,\ c'_{22AB}(4) = 8. \tag{B.32}$$

B.3.3 $\ell = 4$, $X = A_3^8$

We have $m^X = 4$, so the umbral McKay-Thompson series $H_g^{(4)} = (H_{g,r}^{(4)})$ associated to $g \in G^{(4)}$ is an 8-vector-valued function, with components indexed by $r \in \mathbb{Z}/8\mathbb{Z}$.

Define $H_g^{(4)} = (H_{g,r}^{(4)})$ for $g \in G^{(4)}$, $g \notin 4C$, by requiring that

$$\psi_g^{(4)}(\tau, z) = -\chi_g^{(4)} \mu_{4,0}^0(\tau, z) - \bar{\chi}_g^{(4)} \mu_{4,0}^1(\tau, z) + \sum_{r \mod 8} H_{g,r}^{(4)}(\tau) \theta_{4,r}(\tau, z), \tag{B.33}$$

where $\chi_g^{(4)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(4)} := \bar{\chi}_g^{X_A}$ (cf. Table 17), and the $\psi_g^{(4)}$ are meromorphic Jacobi forms of weight 1 and index 4 given explicitly in Table 38.

Table 38: Character Values and Meromorphic Jacobi Forms for $\ell = 4$, $X = A_3^8$

$[g]$	$\chi_g^{(4)}$	$\bar{\chi}_g^{(4)}$	$\psi_g^{(4)}(\tau, z)$
1A	8	8	$2i\theta_1(\tau, 2z)^3 \theta_1(\tau, z)^{-4} \eta(\tau)^3$
2A	-8	8	$2i\theta_1(\tau, 2z)^3 \theta_2(\tau, z)^{-4} \eta(\tau)^3$
2B	0	0	$-2i\theta_1(\tau, 2z)^3 \theta_1(\tau, z)^{-2} \theta_2(\tau, z)^{-2} \eta(\tau)^3$
4A	0	0	$-2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)^2 \theta_2(2\tau, 2z)^{-2} \eta(2\tau)^2 \eta(\tau)^{-1}$
4B	0	0	$-2i\theta_1(2\tau, 2z)\theta_3(2\tau, 2z)^2 \theta_4(2\tau, 2z)\eta(2\tau)^2 \eta(\tau)^{-2} \eta(4\tau)^{-2}$
2C	0	4	$2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)^2 \theta_1(\tau, z)^{-2} \theta_2(\tau, z)^{-2} \eta(\tau)^3$
3A	2	2	$2i\theta_1(3\tau, 6z)\theta_1(\tau, z)^{-1} \theta_1(3\tau, 3z)^{-1} \eta(\tau)^3$
6A	-2	2	$-2i\theta_1(3\tau, 6z)\theta_2(\tau, z)^{-1} \theta_2(3\tau, 3z)^{-1} \eta(\tau)^3$
6BC	0	0	cf. (B.34)
8A	0	0	$-2i\theta_1(\tau, 2z)\theta_2(2\tau, 4z)\theta_2(4\tau, 4z)^{-1} \eta(\tau)\eta(4\tau)\eta(2\tau)^{-1}$
4C	0	2	$2i\theta_1(\tau, 2z)\theta_2(2\tau, 4z)\theta_1(2\tau, 2z)^{-2} \eta(2\tau)^7 \eta(\tau)^{-3} \eta(4\tau)^{-2}$
7AB	1	1	cf. (B.34)
14AB	-1	1	cf. (B.34)

$$\begin{aligned}
\psi^{(4)}_{6BC} &:= \left(\theta_1(\tau,z+\tfrac{1}{3})\theta_1(\tau,z+\tfrac{1}{6})-\theta_1(\tau,z-\tfrac{1}{3})\theta_1(\tau,z-\tfrac{1}{6})\right)\frac{-i\theta_1(3\tau,6z)}{\theta_1(3\tau,3z)\theta_2(3\tau,3z)}\eta(3\tau)\\
\psi^{(4)}_{7AB} &:= \left(\prod_{j=1}^{3}\theta_1(\tau,2z+\tfrac{j^2}{7})\theta_1(\tau,z-\tfrac{j^2}{7})+\prod_{j=1}^{3}\theta_1(\tau,2z-\tfrac{j^2}{7})\theta_1(\tau,z+\tfrac{j^2}{7})\right)\frac{-i}{\theta_1(7\tau,7z)}\frac{\eta(7\tau)}{\eta(\tau)^4}\\
\psi^{(4)}_{14AB} &:= \left(\prod_{j=1}^{3}\theta_1(\tau,2z+\tfrac{j^2}{7})\theta_2(\tau,z-\tfrac{j^2}{7})+\prod_{j=1}^{3}\theta_1(\tau,2z-\tfrac{j^2}{7})\theta_2(\tau,z+\tfrac{j^2}{7})\right)\frac{i}{\theta_2(7\tau,7z)}\frac{\eta(7\tau)}{\eta(\tau)^4}
\end{aligned} \tag{B.34}$$

For use later on, note that $\psi^{(4)}_{1A} = -2\Psi_{1,1}\varphi^{(4)}_1$, where

$$\varphi^{(4)}_1(\tau,z) := \frac{\theta_1(\tau,2z)^2}{\theta_1(\tau,z)^2}. \tag{B.35}$$

B.3.4 $\ell = 5$, $X = A_4^6$

We have $m^X = 5$, so the umbral McKay-Thompson series $H^{(5)}_g = (H^{(5)}_{g,r})$ associated to $g \in G^{(5)}$ is a 10-vector-valued function, with components indexed by $r \in \mathbb{Z}/10\mathbb{Z}$.

Define $H^{(5)}_g = (H^{(5)}_{g,r})$ for $g \in G^{(5)}$, $g \notin 5A \cup 10A$, by requiring that

$$\psi^{(5)}_g(\tau,z) = -\chi^{(5)}_g\mu^0_{5,0}(\tau,z) - \bar{\chi}^{(5)}_g\mu^1_{5,0}(\tau,z) + \sum_{r \mod 10} H^{(5)}_{g,r}(\tau)\theta_{5,r}(\tau,z), \tag{B.36}$$

where $\chi^{(5)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(5)}_g := \bar{\chi}^{X_A}_g$ (cf. Table 18), and the $\psi^{(5)}_g$ are meromorphic Jacobi forms of weight 1 and index 5 given explicitly in Table 39.

Table 39: Character Values and Meromorphic Jacobi Forms for $\ell = 5$, $X = A_4^6$

$[g]$	$\chi^{(5)}_g$	$\bar{\chi}^{(5)}_g$	$\psi^{(5)}_g(\tau,z)$
1A	6	6	$2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(\tau,z)^{-3}\eta(\tau)^3$
2A	-6	6	$-2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_2(\tau,z)^{-3}\eta(\tau)^3$
2B	-2	2	$-2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(\tau,z)^{-1}\theta_2(\tau,z)^{-2}\eta(\tau)^3$
2C	2	2	$2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_1(\tau,z)^{-2}\theta_2(\tau,z)^{-1}\eta(\tau)^3$
3A	0	0	$-2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(3\tau,3z)^{-1}\eta(3\tau)$
6A	0	0	$-2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_2(3\tau,3z)^{-1}\eta(3\tau)$
4AB	0	0	cf. (B.37)
4CD	0	2	cf. (B.37)
12AB	0	0	cf. (B.37)

$$\begin{aligned}
\psi^{(5)}_{4AB}(\tau,z) &:= -i\theta_2(\tau,2z)\frac{\theta_1(\tau,z+\frac{1}{4})\theta_1(\tau,3z+\frac{1}{4})-\theta_1(\tau,z-\frac{1}{4})\theta_1(\tau,3z-\frac{1}{4})}{\theta_2(2\tau,2z)^2}\frac{\eta(2\tau)^2}{\eta(\tau)}\\
\psi^{(5)}_{4CD}(\tau,z) &:= -i\theta_2(\tau,2z)\frac{\theta_1(\tau,z+\frac{1}{4})\theta_1(\tau,3z-\frac{1}{4})+\theta_1(\tau,z-\frac{1}{4})\theta_1(\tau,3z+\frac{1}{4})}{\theta_1(2\tau,2z)\theta_2(2\tau,2z)}\frac{\eta(2\tau)^2}{\eta(\tau)}\\
\psi^{(5)}_{12AB}(\tau,z) &:= i\frac{\theta_2(\tau,2z)}{\theta_2(6\tau,6z)}\left(\theta_1(\tau,z+\tfrac{1}{12})\theta_1(\tau,z+\tfrac{1}{4})\theta_1(\tau,z+\tfrac{5}{12})\theta_1(\tau,3z-\tfrac{1}{4})\right.\\
&\qquad\left.-\,\theta_1(\tau,z-\tfrac{1}{12})\theta_1(\tau,z-\tfrac{1}{4})\theta_1(\tau,z-\tfrac{5}{12})\theta_1(\tau,3z+\tfrac{1}{4})\right)\frac{\eta(6\tau)}{\eta(\tau)^3}
\end{aligned} \tag{B.37}$$

For $g\in 5A$ use the formulas of §B.3.20 to define

$$H^{(5)}_{5A,r}(\tau) := H^{(25)}_{1A,r}(\tau/5) - H^{(25)}_{1A,10-r}(\tau/5) + H^{(25)}_{1A,10+r}(\tau/5) - H^{(25)}_{1A,20-r}(\tau/5) + H^{(25)}_{1A,20+r}(\tau/5). \tag{B.38}$$

For $g\in 10A$ set $H^{(5)}_{10A,r}(\tau) := -(-1)^r H^{(5)}_{5A,r}(\tau)$.

For use later on we note that $\psi^{(5)}_{1A} = -2\Psi_{1,1}\varphi^{(5)}_1$, where

$$\varphi^{(5)}_1(\tau,z) := \frac{\theta_1(\tau,3z)}{\theta_1(\tau,z)}. \tag{B.39}$$

B.3.5 $\ell = 6$, $X = A_5^4 D_4$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6)}_g = (H^{(6)}_{g,r})$ associated to $g\in G^{(6)}$ is a 12-vector-valued function with components indexed by $r\in\mathbb{Z}/12\mathbb{Z}$. We have $H^{(6)}_{g,r} = -H^{(6)}_{g,-r}$, so it suffices to specify the $H^{(6)}_{g,r}$ for $r\in\{1,2,3,4,5\}$.

To define $H^{(6)}_g = (H^{(6)}_{g,r})$ for $g=e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi^{(6)}_1(\tau,z) = -24\mu_{6,0}(\tau,z) + \sum_{r \mod 12} h_r(\tau)\theta_{6,r}(\tau,z), \tag{B.40}$$

where

$$\varphi^{(6)}_1(\tau,z) := \varphi^{(2)}_1(\tau,z)\varphi^{(5)}_1(\tau,z) - \varphi^{(3)}_1(\tau,z)\varphi^{(4)}_1(\tau,z). \tag{B.41}$$

(Cf. (B.23), (B.27), (B.35), (B.39).) Now define the $H^{(6)}_{1A,r}$ by setting

$$\begin{aligned}
H^{(6)}_{1A,1}(\tau) &:= \frac{1}{24}\left(5h_1(\tau)+h_5(\tau)\right),\\
H^{(6)}_{1A,2}(\tau) &:= \frac{1}{6}h_2(\tau),\\
H^{(6)}_{1A,3}(\tau) &:= \frac{1}{4}h_3(\tau),\\
H^{(6)}_{1A,4}(\tau) &:= \frac{1}{6}h_4(\tau),\\
H^{(6)}_{1A,5}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+5h_5(\tau)\right).
\end{aligned} \tag{B.42}$$

Define $H^{(6)}_{2A,r}$ by requiring

$$H^{(6)}_{2A,r}(\tau) := -(-1)^r H^{(6)}_{1A,r}(\tau). \tag{B.43}$$

For the remaining g, recall (B.13). The $H^{(6)}_{g,r}$ for $g \notin 1A \cup 2A$ are defined as follows for $r = 2$ and $r = 4$, noting that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$.

$$\begin{aligned} H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4C,r}(\tau/2) \\ H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4B,r}(\tau/2) \\ H^{(6)}_{3A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6C,r}(\tau/2) \\ H^{(6)}_{6A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6D,r}(\tau/2) \\ H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{8CD,r}(\tau/2) \end{aligned} \tag{B.44}$$

For the $H^{(6)}_{g,3}$ we define

$$\begin{aligned} H^{(6)}_{2B,3}(\tau), H^{(6)}_{4A,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{6A,1}(\tau/3), \\ H^{(6)}_{3A,3}(\tau), H^{(6)}_{6A,3}(\tau) &:= 0, \\ H^{(6)}_{8AB,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{12A,1}(\tau/3). \end{aligned} \tag{B.45}$$

Noting that $H^{(2)}_{g,5} = H^{(2)}_{g,1}$ and $H^{(3)}_{g,5} = -H^{(3)}_{g,1}$, the $H^{(6)}_{g,1}$ and $H^{(6)}_{g,5}$ are defined for $o(g) \neq 0 \mod 3$ by setting

$$\begin{aligned} H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4C,r}(\tau/2)\right) \\ H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4B,r}(\tau/2)\right) \\ H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{12A,r}(\tau/3) + H^{(3)}_{8CD,r}(\tau/2)\right) \end{aligned} \tag{B.46}$$

It remains to specify the $H^{(6)}_{g,r}$ when $g \in 3A \cup 6A$ and r is 1 or 5. These cases are determined by using the formulas of §B.3.17 to set

$$\begin{aligned} H^{(6)}_{3A,1}(\tau), H^{(6)}_{6A,1}(\tau) &:= H^{(18)}_{1A,1}(3\tau) - H^{(18)}_{1A,11}(3\tau) + H^{(18)}_{1A,13}(3\tau), \\ H^{(6)}_{3A,5}(\tau), H^{(6)}_{6A,5}(\tau) &:= H^{(18)}_{1A,5}(3\tau) - H^{(18)}_{1A,7}(3\tau) + H^{(18)}_{1A,17}(3\tau). \end{aligned} \tag{B.47}$$

B.3.6 $\ell = 6 + 3$, $X = D_4^6$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6+3)}_g = (H^{(6+3)}_{g,r})$ associated to $g \in G^{(6+3)}$ is a 12-vector-valued function with components indexed by $r \in \mathbb{Z}/12\mathbb{Z}$. In addition to the identity $H^{(6+3)}_{g,r} = -H^{(6+3)}_{g,-r}$, we have $H^{(6+3)}_{g,r} = 0$ for $r = 0 \mod 2$. Thus it suffices to specify the $H^{(6+3)}_{g,r}$ for $r \in \{1, 3, 5\}$.

Recall (B.13). For $r = 1$, define

$$\begin{aligned}
H^{(6+3)}_{1A,1}(\tau), H^{(6+3)}_{3A,1}(\tau) &:= H^{(6)}_{1A,1}(\tau) + H^{(6)}_{1A,5}(\tau),\\
H^{(6+3)}_{2A,1}(\tau), H^{(6+3)}_{6A,1}(\tau) &:= H^{(6)}_{2B,1}(\tau) + H^{(6)}_{2B,5}(\tau),\\
H^{(6+3)}_{3B,1}(\tau) &:= H^{(6)}_{3A,1}(\tau) + H^{(6)}_{3A,5}(\tau),\\
H^{(6+3)}_{3C,1}(\tau) &:= -2\frac{\eta(\tau)^2}{\eta(3\tau)},\\
H^{(6+3)}_{4A,1}(\tau), H^{(6+3)}_{12A,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) + H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{5A,1}(\tau), H^{(6+3)}_{15A,1}(\tau) &:= [-\tfrac{1}{24}]H^{(2)}_{15AB,1}(\tau/3),\\
H^{(6+3)}_{2C,1}(\tau) &:= H^{(6)}_{4A,1}(\tau) - H^{(6)}_{4A,5}(\tau),\\
H^{(6+3)}_{4B,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) - H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{6B,1}(\tau) &:= H^{(6)}_{6A,1}(\tau) - H^{(6)}_{6A,5}(\tau),\\
H^{(6+3)}_{6C,1}(\tau) &:= -2\frac{\eta(2\tau)\,\eta(3\tau)}{\eta(6\tau)}.
\end{aligned} \tag{B.48}$$

Then define $H^{(6+3)}_{2B,1}$ by setting

$$H^{(6+3)}_{2B,1}(\tau) := 2H^{(6+3)}_{4B,1}(\tau) + 2\frac{\eta(\tau)^3}{\eta(2\tau)^2}. \tag{B.49}$$

For $r = 3$ set

$$\begin{aligned}
H^{(6+3)}_{1A,3}(\tau) &:= 2H^{(6)}_{1A,3}(\tau),\\
H^{(6+3)}_{3A,3}(\tau) &:= -H^{(6)}_{1A,3}(\tau),\\
H^{(6+3)}_{2A,3}(\tau) &:= 2H^{(6)}_{2B,3}(\tau),\\
H^{(6+3)}_{6A,3}(\tau) &:= -H^{(6)}_{2B,3}(\tau),\\
H^{(6+3)}_{4A,3}(\tau) &:= 2H^{(6)}_{8AB,3}(\tau),\\
H^{(6+3)}_{12A,3}(\tau) &:= -H^{(6)}_{8AB,3}(\tau),\\
H^{(6+3)}_{5A,3}(\tau) &:= -2[-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau),\\
H^{(6+3)}_{15A,3}(\tau) &:= [-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau),
\end{aligned} \tag{B.50}$$

and

$$H^{(6+3)}_{3B,3}(\tau), H^{(6+3)}_{3C,3}(\tau), H^{(6+3)}_{2B,3}(\tau), H^{(6+3)}_{2C,3}(\tau), H^{(6+3)}_{4B,3}(\tau), H^{(6+3)}_{6B,3}(\tau), H^{(6+3)}_{6C,3}(\tau) := 0. \tag{B.51}$$

For $r = 5$ define $H^{(6+3)}_{g,5}(\tau) := H^{(6+3)}_{g,1}(\tau)$ for $[g] \in \{1A, 3A, 2A, 6A, 3B, 3C, 4A, 12A, 5A, 15AB\}$, and set $H^{(6+3)}_{g,5}(\tau) := -H^{(6+3)}_{g,1}(\tau)$ for the remaining cases, $[g] \in \{2B, 2C, 4B, 6B, 6C\}$.

B.3.7 $\ell = 7$, $X = A_6^4$

We have $m^X = 7$, so the umbral McKay-Thompson series $H_g^{(7)} = (H_{g,r}^{(7)})$ associated to $g \in G^{(7)} = G^X \simeq SL_2(3)$ is a 14-vector-valued function, with components indexed by $r \in \mathbb{Z}/14\mathbb{Z}$.

Define $H_g^{(7)} = (H_{g,r}^{(7)})$ for $g \in G^{(7)}$ by requiring that

$$\psi_g^{(7)}(\tau, z) = -\chi_g^{(7)}\mu_{7,0}^0(\tau, z) - \bar{\chi}_g^{(7)}\mu_{7,0}^1(\tau, z) + \sum_{r \mod 14} H_{g,r}^{(7)}(\tau)\theta_{7,r}(\tau, z), \tag{B.52}$$

where $\chi_g^{(7)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(7)} := \bar{\chi}_g^{X_A}$ (cf. Table 21), and the $\psi_g^{(7)}$ are meromorphic Jacobi forms of weight 1 and index 7 given explicitly in Table 40.

Table 40: Character Values and Meromorphic Jacobi Forms for $\ell = 7$, $X = A_6^4$

$[g]$	$\chi_g^{(7)}$	$\bar{\chi}_g^{(7)}$	$\psi_g^{(7)}(\tau, z)$
1A	4	4	$2i\theta_1(\tau, 4z)\theta_1(\tau, z)^{-2}\eta(\tau)^3$
2A	-4	4	$-2i\theta_1(\tau, 4z)\theta_2(\tau, z)^{-2}\eta(\tau)^3$
4A	0	0	$-2i\theta_1(\tau, 4z)\theta_2(2\tau, 2z)^{-1}\eta(2\tau)\eta(\tau)$
3A	1	1	cf. (B.53)
6A	-1	1	cf. (B.53)

$$\begin{aligned}
\psi_{3A}^{(7)}(\tau, z) &:= -i\frac{\theta_1(\tau, 4z + \frac{1}{3})\theta_1(\tau, z - \frac{1}{3}) + \theta_1(\tau, 4z - \frac{1}{3})\theta_1(\tau, z + \frac{1}{3})}{\theta_1(3\tau, 3z)}\eta(3\tau) \\
\psi_{6A}^{(7)}(\tau, z) &:= -i\frac{\theta_1(\tau, 4z + \frac{1}{3})\theta_1(\tau, z - \frac{1}{6}) - \theta_1(\tau, 4z - \frac{1}{3})\theta_1(\tau, z + \frac{1}{6})}{\theta_2(3\tau, 3z)}\eta(3\tau)
\end{aligned} \tag{B.53}$$

For use later on we note that $\psi_{1A}^{(7)} = -2\Psi_{1,1}\varphi_1^{(7)}$, where

$$\varphi_1^{(7)}(\tau, z) := \frac{\theta_1(\tau, 4z)}{\theta_1(\tau, 2z)}. \tag{B.54}$$

B.3.8 $\ell = 8$, $X = A_7^2 D_5^2$

We have $m^X = 8$, so the umbral McKay-Thompson series $H_g^{(8)} = (H_{g,r}^{(8)})$ associated to $g \in G^{(8)}$ is a 16-vector-valued function with components indexed by $r \in \mathbb{Z}/16\mathbb{Z}$. We have $H_{g,r}^{(8)} = -H_{g,-r}^{(8)}$, so it suffices to specify the $H_{g,r}^{(8)}$ for $r \in \{1, 2, 3, 4, 5, 6, 7\}$.

To define $H_g^{(8)} = (H_{g,r}^{(8)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\left(\varphi_1^{(8)}(\tau, z) + \frac{1}{2}\varphi_2^{(8)}(\tau, z)\right) = -24\mu_{8,0}(\tau, z) + \sum_{r \mod 16} h_r(\tau)\theta_{8,r}(\tau, z), \tag{B.55}$$

where

$$\begin{aligned}
\varphi_1^{(8)}(\tau, z) &:= \varphi_1^{(3)}(\tau, z)\varphi_1^{(6)}(\tau, z) - 5\varphi_1^{(4)}(\tau, z)\varphi_1^{(5)}(\tau, z), \\
\varphi_2^{(8)}(\tau, z) &:= \varphi_1^{(4)}(\tau, z)\varphi_1^{(5)}(\tau, z) - \varphi_1^{(8)}(\tau, z).
\end{aligned} \tag{B.56}$$

(Cf. (B.27), (B.35), (B.39), (B.41).) Now define the $H^{(8)}_{1A,r}$ by setting

$$H^{(8)}_{1A,r}(\tau) := \frac{1}{6}h_r(\tau), \tag{B.57}$$

for $r \in \{1,3,4,5,7\}$, and

$$H^{(8)}_{1A,2}(\tau), H^{(8)}_{1A,6}(\tau) := \frac{1}{12}\left(h_2(\tau) + h_6(\tau)\right). \tag{B.58}$$

Define $H^{(8)}_{2A,r}$ for $1 \leq r \leq 7$ by requiring

$$H^{(8)}_{2A,r}(\tau) := -(-1)^r H^{(8)}_{1A,r}(\tau). \tag{B.59}$$

For the remaining g, recall (B.13). The $H^{(8)}_{g,r}$ for $g \in 2B \cup 2C \cup 4A$ are defined as follows for $r \in \{1,3,5,7\}$, noting that $H^{(4)}_{g,7} = H^{(4)}_{g,-1} = -H^{(4)}_{g,1}$, &c.

$$\begin{aligned} H^{(8)}_{2BC,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4C,r}(\tau/2) \\ H^{(8)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4B,r}(\tau/2) \end{aligned} \tag{B.60}$$

The $H^{(8)}_{2BC,r}$ and $H^{(8)}_{4A,r}$ vanish for $r = 0 \mod 2$.

B.3.9 $\ell = 9$, $X = A_8^3$

We have $m^X = 9$, so for $g \in G^{(9)}$ the associated umbral McKay-Thompson series $H^{(9)}_g = (H^{(9)}_{g,r})$ is a 18-vector-valued function, with components indexed by $r \in \mathbb{Z}/18\mathbb{Z}$, satisfying $H^{(9)}_{g,r} = -H^{(9)}_{g,-r}$, and in particular, $H^{(9)}_{g,r} = 0$ for $r = 0 \mod 9$. So it suffices to specify the $H^{(9)}_{g,r}$ for $r \in \{1,2,3,4,5,6,7,8\}$.

Define $H^{(9)}_g = (H^{(9)}_{g,r})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau,z)\varphi^{(9)}_1(\tau,z) = -3\mu_{9,0}(\tau,z) + \sum_{r \mod 18} H^{(9)}_{e,r}(\tau)\theta_{9,r}(\tau,z), \tag{B.61}$$

where

$$\varphi^{(9)}_1(\tau,z) := \varphi^{(3)}_1(\tau,z)\varphi^{(7)}_1(\tau,z) - \varphi^{(5)}_1(\tau,z)^2. \tag{B.62}$$

(Cf. (B.27), (B.39), (B.54).)

Recall (B.13). The $H^{(9)}_{2B,r}$ are defined for $r \in \{1,2,4,5,7,8\}$ by setting

$$H^{(9)}_{2B,r}(\tau) := [-\tfrac{r^2}{36}]H^{(3)}_{6C,r}(\tau/3), \tag{B.63}$$

where we note that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$, &c. We determine $H^{(9)}_{2B,3}$ and $H^{(9)}_{2B,6}$ by using §B.3.17 to set

$$H^{(9)}_{2B,r}(\tau) := H^{(18)}_{1A,r}(2\tau) - H^{(18)}_{1A,18-r}(2\tau) \tag{B.64}$$

for $r \in \{3,6\}$.

The $H^{(9)}_{3A,r}$ are defined by the explicit formulas

$$\begin{aligned}
H^{(9)}_{3A,1}(\tau) &:= [-\tfrac{1}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,2}(\tau) &:= [-\tfrac{4}{36}]f^{(9)}_2(\tau/3),\\
H^{(9)}_{3A,3}(\tau) &:= -\theta_{3,3}(\tau,0),\\
H^{(9)}_{3A,4}(\tau) &:= -[-\tfrac{16}{36}]f^{(9)}_2(\tau/3),\\
H^{(9)}_{3A,5}(\tau) &:= -[-\tfrac{25}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,6}(\tau) &:= \theta_{3,0}(\tau,0),\\
H^{(9)}_{3A,7}(\tau) &:= [-\tfrac{13}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,8}(\tau) &:= [-\tfrac{28}{36}]f^{(9)}_2(\tau/3),
\end{aligned} \tag{B.65}$$

where

$$\begin{aligned}
f^{(9)}_1(\tau) &:= -2\frac{\eta(\tau)\eta(12\tau)\eta(18\tau)^2}{\eta(6\tau)\eta(9\tau)\eta(36\tau)},\\
f^{(9)}_2(\tau) &:= \frac{\eta(2\tau)^6\eta(12\tau)\eta(18\tau)^2}{\eta(\tau)\eta(4\tau)^4\eta(6\tau)\eta(9\tau)\eta(36\tau)} - \frac{\eta(\tau)\eta(2\tau)\eta(3\tau)^2}{\eta(4\tau)^2\eta(9\tau)}.
\end{aligned} \tag{B.66}$$

Finally, the $H^{(9)}_{g,r}$ are determined for $g \in 2A \cup 2C \cup 6A$ by setting

$$\begin{aligned}
H^{(9)}_{2A,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{1A,r}(\tau),\\
H^{(9)}_{2C,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{2B,r}(\tau),\\
H^{(9)}_{6A,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{3A,r}(\tau).
\end{aligned} \tag{B.67}$$

B.3.10 $\ell = 10$, $X = A_9^2D_6$

We have $m^X = 10$, so the umbral McKay-Thompson series $H^{(10)}_g = (H^{(10)}_{g,r})$ associated to $g \in G^{(10)}$ is a 20-vector-valued function with components indexed by $r \in \mathbb{Z}/20\mathbb{Z}$. We have $H^{(10)}_{g,r} = -H^{(10)}_{g,-r}$, so it suffices to specify the $H^{(10)}_{g,r}$ for $1 \le r \le 9$.

To define $H^{(10)}_g = (H^{(10)}_{g,r})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau,z)\varphi^{(10)}_1(\tau,z) = -24\mu_{10,0}(\tau,z) + \sum_{r \mod 20} h_r(\tau)\theta_{10,r}(\tau,z), \tag{B.68}$$

where

$$\varphi^{(10)}_1(\tau,z) := 5\varphi^{(4)}_1(\tau,z)\varphi^{(7)}_1(\tau,z) - \varphi^{(5)}_1(\tau,z)\varphi^{(6)}_1(\tau,z). \tag{B.69}$$

(Cf. (B.35), (B.39), (B.41), (B.54).) Now define the $H^{(10)}_{1A,r}$ for r odd by setting

$$\begin{aligned} H^{(10)}_{1A,1}(\tau) &:= \frac{1}{24}\left(3h_1(\tau)+h_9(\tau)\right), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(3h_3(\tau)+h_7(\tau)\right), \\ H^{(10)}_{1A,5}(\tau) &:= \frac{1}{6}h_5(\tau), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(h_3(\tau)+3h_7(\tau)\right), \\ H^{(10)}_{1A,9}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+3h_9(\tau)\right). \end{aligned} \tag{B.70}$$

For $r = 0 \mod 2$ set

$$H^{(10)}_{1A,r}(\tau) := \frac{1}{12}h_r(\tau), \tag{B.71}$$

and define $H^{(10)}_{2A,r}$ for $1 \le r \le 9$ by requiring

$$H^{(10)}_{2A,r}(\tau) := -(-1)^r H^{(10)}_{1A,r}(\tau). \tag{B.72}$$

It remains to specify $H^{(10)}_{g,r}$ for $g \in 4A \cup 4B$. For $r = 0 \mod 2$ set

$$H^{(10)}_{4AB,r}(\tau) := 0. \tag{B.73}$$

For r odd, recall (B.13), and define

$$H^{(10)}_{4A,r}(\tau) := [-\tfrac{r^2}{40}]\frac{1}{2}\left(H^{(2)}_{10A,r}(\tau/5) + H^{(5)}_{4CD,r}(\tau/2)\right). \tag{B.74}$$

B.3.11 $\ell = 10+5$, $X = D_6^4$

We have $m^X = 10$, so the umbral McKay-Thompson series $H^{(10+5)}_g = (H^{(10+5)}_{g,r})$ associated to $g \in G^{(10+5)}$ is a 20-vector-valued function with components indexed by $r \in \mathbb{Z}/20\mathbb{Z}$. We have $H^{(10+5)}_{g,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(10+5)}_{g,r}$ for r odd. Observing that $H^{(10+5)}_{g,r} = -H^{(10+5)}_{g,-r}$ we may determine $H^{(10+5)}_g$ by requiring that

$$\psi^{(5/2)}_g(\tau,z) = -2\chi^{(5/2)}_g i\mu_{5/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2\\ r \mod 5}} e(-r/2)H^{(10+5)}_{g,2r}(\tau)\theta_{5/2,r}(\tau,z), \tag{B.75}$$

where $\chi^{(5/2)}_g := \bar{\chi}^{X_D}_g$ as in Table 25, and the $\psi^{(5/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 5/2 defined as follows.

Table 41: Character Values and Meromorphic Jacobi Forms for $\ell = 10 + 5$, $X = D_6^4$

$[g]$	$\bar{\chi}_g^{(5/2)}$	$\psi_g^{(5/2)}(\tau, z)$
1A	4	$2i\theta_1(\tau, 2z)^2\theta_1(\tau, z)^{-3}\eta(\tau)^3$
2A	0	$-2i\theta_1(\tau, 2z)^2\theta_1(\tau, z)^{-1}\theta_2(\tau, z)^{-2}\eta(\tau)^3$
3A	1	$2i\theta_1(3\tau, 6z)\theta_1(\tau, 2z)^{-1}\theta_1(3\tau, 3z)^{-1}\eta(\tau)^3$
2B	2	$2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)\theta_1(\tau, z)^{-2}\theta_2(\tau, z)^{-1}\eta(\tau)^3$
4A	0	$-2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)\theta_2(2\tau, 2z)^{-1}\eta(\tau)\eta(2\tau)$

B.3.12 $\ell = 12$, $X = A_{11}D_7E_6$

We have $m^X = 12$, so the umbral McKay-Thompson series $H_g^{(12)} = (H_{g,r}^{(12)})$ associated to $g \in G^{(12)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. We have $H_{g,r}^{(12)} = -H_{g,-r}^{(12)}$, so it suffices to specify the $H_{g,r}^{(12)}$ for $1 \leq r \leq 11$.

To define $H_e^{(12)} = (H_{e,r}^{(12)})$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\left(\varphi_1^{(12)}(\tau, z) + \varphi_2^{(12)}(\tau, z)\right) = -24\mu_{12,0}(\tau, z) + \sum_{r \mod 24} h_r(\tau)\theta_{12,r}(\tau, z), \tag{B.76}$$

where

$$\begin{aligned} \varphi_1^{(12)}(\tau, z) &:= 3\varphi_1^{(3)}(\tau, z)\varphi_1^{(10)}(\tau, z) - 8\varphi_1^{(4)}(\tau, z)\varphi_1^{(9)}(\tau, z) + \varphi_1^{(5)}(\tau, z)\varphi_1^{(8)}(\tau, z), \\ \varphi_2^{(12)}(\tau, z) &:= 4\varphi_1^{(4)}(\tau, z)\varphi_1^{(9)}(\tau, z) - \varphi_1^{(5)}(\tau, z)\varphi_1^{(8)}(\tau, z) - \varphi_1^{(12)}(\tau, z). \end{aligned} \tag{B.77}$$

(Cf. (B.27), (B.35), (B.39), (B.54), (B.56), (B.62), (B.69).) Now define the $H_{1A,r}^{(12)}$ for $r \neq 0 \mod 3$ by setting

$$\begin{aligned} H_{1A,1}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_1(\tau) + h_7(\tau)\right), \\ H_{1A,2}^{(12)}(\tau), H_{1A,10}^{(12)}(\tau) &:= \frac{1}{24}\left(h_2(\tau) + h_{10}(\tau)\right), \\ H_{1A,4}^{(12)}(\tau), H_{1A,8}^{(12)}(\tau) &:= \frac{1}{12}\left(h_4(\tau) + h_8(\tau)\right), \\ H_{1A,5}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_5(\tau) + h_{11}(\tau)\right), \\ H_{1A,7}^{(12)}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 3h_7(\tau)\right), \\ H_{1A,11}^{(12)}(\tau) &:= \frac{1}{24}\left(h_5(\tau) + 3h_{11}(\tau)\right). \end{aligned} \tag{B.78}$$

For $r = 0 \mod 3$ set

$$H_{1A,r}^{(12)}(\tau) := \frac{1}{12}h_r(\tau), \tag{B.79}$$

and define $H_{2A,r}^{(12)}$ by requiring

$$H_{2A,r}^{(12)}(\tau) := -(-1)^r H_{1A,r}^{(12)}(\tau). \tag{B.80}$$

B.3.13 $\ell = 12+4$, $X = E_6^4$

We have $m^X = 12$, so the umbral McKay-Thompson series $H_g^{(12+4)} = (H_{g,r}^{(12+4)})$ associated to $g \in G^{(12+4)}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. In addition to the identity $H_{g,r}^{(12+4)} = -H_{g,-r}^{(12+4)}$, we have $H_{g,r}^{(12+4)} = 0$ for $r \in \{2,3,6,9,10\}$, $H_{g,1}^{(12+4)} = H_{g,7}^{(12+4)}$, $H_{g,4}^{(12+4)} = H_{g,8}^{(12+4)}$, and $H_{g,5}^{(12+4)} = H_{g,11}^{(12+4)}$. Thus it suffices to specify the $H_{g,1}^{(12+4)}$, $H_{g,4}^{(12+4)}$ and $H_{g,5}^{(12+4)}$.

Recall (B.13). Also, set $S_1^{E_6}(\tau) := S_{12,1}(\tau) + S_{12,7}(\tau)$, and $S_5^{E_6}(\tau) := S_{12,5}(\tau) + S_{12,11}(\tau)$. For $r = 1$ define

$$\begin{aligned}
H_{1A,1}^{(12+4)}(\tau) &:= H_{1A,1}^{(12)}(\tau) + H_{1A,7}^{(12)}(\tau),\\
H_{2B,1}^{(12+4)}(\tau) &:= [-\tfrac{1}{48}]\left(H_{8AB,1}^{(6)}(\tau/2) - H_{8AB,5}^{(6)}(\tau/2)\right),\\
H_{4A,1}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(-2\frac{\eta(2\tau)^8}{\eta(\tau)^4}S_1^{E_6}(\tau) + 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4}S_5^{E_6}(\tau)\right),\\
H_{3A,1}^{(12+4)}(\tau) &:= [-\tfrac{1}{48}]\left(H_{3A,1}^{(6)}(\tau/2) - H_{3A,5}^{(6)}(\tau/2)\right),\\
H_{8AB,1}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(-2F_{8AB,1}^{(12+4)}(\tau)S_1^{E_6}(\tau) + 12F_{8AB,5}^{(12+4)}(\tau/2)S_5^{E_6}(\tau)\right).
\end{aligned} \tag{B.81}$$

In the expression for $g \in 8AB$, we write $F_{8AB,1}^{(12+4)}$ for the unique modular form of weight 2 for $\Gamma_0(32)$ such that

$$F_{8AB,1}^{(12+4)}(\tau) = 1 + 12q + 4q^2 - 24q^5 - 16q^6 - 8q^8 + O(q^9), \tag{B.82}$$

and we write $F_{8AB,5}^{(12+4)}$ for the unique modular form of weight 2 for $\Gamma_0(64)$ such that

$$F_{8AB,5}^{(12+4)}(\tau) = 3q + 4q^3 + 6q^5 - 8q^7 - 9q^9 + 12q^{11} - 18q^{13} - 24q^{15} + O(q^{17}). \tag{B.83}$$

For $r = 4$ define

$$\begin{aligned}
H_{1A,4}^{(12+4)}(\tau) &:= H_{1A,4}^{(12)}(\tau) + H_{1A,8}^{(12)}(\tau),\\
H_{3A,4}^{(12+4)}(\tau) &:= H_{3A,2}^{(6)}(\tau/2) + H_{3A,4}^{(6)}(\tau/2),
\end{aligned} \tag{B.84}$$

and set $H_{g,4}^{(12+4)}(\tau) := 0$ for $g \in 2B \cup 4A \cup 8AB$.

For $r = 5$ define

$$\begin{aligned}
H_{1A,5}^{(12+4)}(\tau) &:= H_{1A,5}^{(12)}(\tau) + H_{1A,11}^{(12)}(\tau),\\
H_{2B,5}^{(12+4)}(\tau) &:= [-\tfrac{25}{48}]\left(H_{8AB,5}^{(6)}(\tau/2) - H_{8AB,1}^{(6)}(\tau/2)\right),\\
H_{4A,5}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(2\frac{\eta(2\tau)^8}{\eta(\tau)^4}S_5^{E_6}(\tau) - 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4}S_1^{E_6}(\tau)\right),\\
H_{3A,5}^{(12+4)}(\tau) &:= [-\tfrac{25}{48}]\left(H_{3A,5}^{(6)}(\tau/2) - H_{3A,1}^{(6)}(\tau/2)\right),\\
H_{8AB,5}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(2F_{8AB,1}^{(12+4)}(\tau)S_5^{E_6}(\tau) - 12F_{8AB,5}^{(12+4)}(\tau/2)S_1^{E_6}(\tau)\right).
\end{aligned} \tag{B.85}$$

Finally, define $H^{(12+4)}_{g,r}$ for $g \in 2A \cup 6A$ by setting

$$\begin{aligned} H^{(12+4)}_{2A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{1A,r}(\tau), \\ H^{(12+4)}_{6A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{3A,r}(\tau). \end{aligned} \tag{B.86}$$

B.3.14 $\ell = 13$, $X = A_{12}^2$

We have $m^X = 13$, so the umbral McKay-Thompson series $H^{(13)}_g = (H^{(13)}_{g,r})$ associated to $g \in G^{(13)} = G^X \simeq \mathbb{Z}/4\mathbb{Z}$ is a 26-vector-valued function, with components indexed by $r \in \mathbb{Z}/26\mathbb{Z}$.

Define $H^{(13)}_g = (H^{(13)}_{g,r})$ for $g \in G^{(13)}$ by requiring that

$$\psi^{(13)}_g(\tau, z) = -\chi^{(13)}_g \mu^0_{13,0}(\tau, z) - \bar{\chi}^{(13)}_g \mu^1_{13,0}(\tau, z) + \sum_{r \mod 26} H^{(13)}_{g,r}(\tau)\theta_{13,r}(\tau, z), \tag{B.87}$$

where $\chi^{(13)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(13)}_g := \bar{\chi}^{X_A}_g$ (cf. Table 28), and the $\psi^{(13)}_g$ are meromorphic Jacobi forms of weight 1 and index 13 given explicitly in Table 42.

Table 42: Character Values and Meromorphic Jacobi Forms for $\ell = 13$, $X = A_{12}^2$

$[g]$	$\chi^{(13)}_g$	$\bar{\chi}^{(13)}_g$	$\psi^{(13)}_g(\tau, z)$
1A	2	2	$2i\theta_1(\tau, 6z)\theta_1(\tau, z)^{-1}\theta_1(\tau, 3z)^{-1}\eta(\tau)^3$
2A	-2	2	$-2i\theta_1(\tau, 6z)\theta_2(\tau, z)^{-1}\theta_2(\tau, 3z)^{-1}\eta(\tau)^3$
4A	0	0	cf. (B.88)

$$\psi^{(13)}_{4AB}(\tau, z) := -i\theta_2(\tau, 6z)\frac{\theta_1(\tau, z + \frac{1}{4})\theta_1(\tau, 3z + \frac{1}{4}) - \theta_1(\tau, z - \frac{1}{4})\theta_1(\tau, 3z - \frac{1}{4})}{\theta_2(2\tau, 2z)\theta_2(2\tau, 6z)}\frac{\eta(2\tau)^2}{\eta(\tau)} \tag{B.88}$$

For use later on we note that $\psi^{(13)}_{1A} = -2\Psi_{1,1}\varphi^{(13)}_1$, where

$$\varphi^{(13)}_1(\tau, z) := \frac{\theta_1(\tau, z)\theta_1(\tau, 6z)}{\theta_1(\tau, 2z)\theta_1(\tau, 3z)}. \tag{B.89}$$

B.3.15 $\ell = 14 + 7$, $X = D_8^3$

We have $m^X = 14$, so the umbral McKay-Thompson series $H^{(14+7)}_g = (H^{(14+7)}_{g,r})$ associated to $g \in G^{(14+7)}$ is a 28-vector-valued function with components indexed by $r \in \mathbb{Z}/28\mathbb{Z}$. We have $H^{(14+7)}_{g,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(14+7)}_{g,r}$ for r odd. Observing that $H^{(14+7)}_{g,r} = -H^{(14+7)}_{g,-r}$ we may determine $H^{(14+7)}_g$ by requiring that

$$\psi^{(7/2)}_g(\tau, z) = -2\bar{\chi}^{(7/2)}_g i\mu_{7/2,0}(\tau, z) + \sum_{\substack{r \in \mathbb{Z}+1/2 \\ r \mod 7}} e(-r/2)H^{(14+7)}_{g,2r}(\tau)\theta_{7/2,r}(\tau, z), \tag{B.90}$$

where $\bar{\chi}^{(7/2)}_g := \bar{\chi}^{X_D}_g$ is the number of fixed points of $g \in G^{(14+7)} \simeq S_3$ in the defining permutation representation on 3 points. The $\psi^{(7/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 7/2 defined in Table 43.

Table 43: Character Values and Meromorphic Jacobi Forms for $\ell = 14 + 7$, $X = D_8^3$

$[g]$	$\bar{\chi}_g^{(7/2)}$	$\psi_g^{(7/2)}(\tau, z)$
1A	3	$2i\theta_1(\tau, 3z)\theta_1(\tau, z)^{-2}\eta(\tau)^3$
2A	1	$2i\theta_2(\tau, 3z)\theta_1(\tau, z)^{-1}\theta_2(\tau, z)^{-1}\eta(\tau)^3$
3A	0	$-2i\theta_1(\tau, z)\theta_1(\tau, 3z)\theta_1(3\tau, 3z)^{-1}\eta(3\tau)$

B.3.16 $\ell = 16$, $X = A_{15}D_9$

We have $m^X = 16$, so the umbral McKay-Thompson series $H_g^{(16)} = (H_{g,r}^{(16)})$ associated to $g \in G^{(16)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 32-vector-valued function with components indexed by $r \in \mathbb{Z}/32\mathbb{Z}$. We have $H_{g,r}^{(16)} = -H_{g,-r}^{(16)}$, so it suffices to specify the $H_{g,r}^{(16)}$ for $1 \leq r \leq 15$.

To define $H_g^{(16)} = (H_{g,r}^{(16)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau, z)\left(\varphi_1^{(16)}(\tau, z) + \frac{1}{2}\varphi_2^{(16)}(\tau, z)\right) = -24\mu_{16,0}(\tau, z) + \sum_{r \mod 32} h_r(\tau)\theta_{16,r}(\tau, z), \tag{B.91}$$

where

$$\begin{aligned} \varphi_1^{(16)}(\tau, z) &:= 8\varphi_1^{(4)}(\tau, z)\varphi_1^{(13)}(\tau, z) - \varphi_1^{(5)}(\tau, z)\varphi_1^{(12)}(\tau, z) + \varphi_1^{(7)}(\tau, z)\varphi_1^{(10)}(\tau, z), \\ \varphi_2^{(16)}(\tau, z) &:= 12\varphi_1^{(4)}(\tau, z)\varphi_1^{(13)}(\tau, z) - \varphi_1^{(5)}(\tau, z)\varphi_1^{(12)}(\tau, z) - 3\varphi_1^{(16)}(\tau, z). \end{aligned} \tag{B.92}$$

(Cf. (B.35), (B.39), (B.54), (B.69), (B.77), (B.89).) Now define the $H_{1A,r}^{(16)}$ by setting

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{12}h_r(\tau) \tag{B.93}$$

for r odd. For r even, $2 \leq r \leq 14$, use

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{24}\left(h_r(\tau) + h_{16-r}(\tau)\right). \tag{B.94}$$

Define $H_{2A,r}^{(16)}$ by requiring

$$H_{2A,r}^{(16)}(\tau) := -(-1)^r H_{1A,r}^{(16)}(\tau). \tag{B.95}$$

B.3.17 $\ell = 18$, $X = A_{17}E_7$

We have $m^X = 18$, so the umbral McKay-Thompson series $H_g^{(18)} = (H_{g,r}^{(18)})$ associated to $g \in G^{(18)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H_{g,r}^{(18)} = -H_{g,-r}^{(18)}$, so it suffices to specify the $H_{g,r}^{(18)}$ for $1 \leq r \leq 17$.

To define $H_g^{(18)} = (H_{g,r}^{(18)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-24\Psi_{1,1}(\tau, z)\phi^{(18)}(\tau, z) = -24\mu_{18,0}(\tau, z) + \sum_{r \mod 36} h_r(\tau)\theta_{18,r}(\tau, z), \tag{B.96}$$

where

$$\phi^{(18)} := \frac{1}{12}\left(\varphi_1^{(18)} + \frac{1}{3}\varphi_3^{(18)} + 4\frac{\theta_1^{12}}{\eta^{12}}\left(\varphi_1^{(12)} + 2\varphi_2^{(12)} + \frac{1}{3}\varphi_3^{(12)}\right)\right). \tag{B.97}$$

For the definition of $\phi^{(18)}$ we require

$$\begin{aligned}
\varphi_2^{(9)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(6)}(\tau,z) - 4\varphi_1^{(5)}(\tau,z)^2 - 4\varphi_1^{(9)}(\tau,z),\\
\varphi_1^{(11)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(7)}(\tau,z) + 2\varphi_1^{(3)}(\tau,z)\varphi_1^{(9)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(8)}(\tau,z),\\
\varphi_3^{(12)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(9)}(\tau,z),\\
\varphi_1^{(14)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(10)}(\tau,z) + \varphi_1^{(3)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(11)}(\tau,z),\\
\varphi_1^{(15)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) + 6\varphi_1^{(3)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z),\\
\varphi_2^{(15)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 2\varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) - 2\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(18)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(14)}(\tau,z) + 3\varphi_1^{(3)}(\tau,z)\varphi_1^{(16)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_3^{(18)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(15)}(\tau,z),
\end{aligned} \tag{B.98}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Now define the $H_{1A,r}^{(18)}$ by setting

$$H_{1A,r}^{(18)}(\tau) := \frac{1}{24}h_r(\tau) \tag{B.99}$$

for r even. For r odd, use

$$\begin{aligned}
H_{1A,1}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_1(\tau) + h_{17}(\tau)\right),\\
H_{1A,3}^{(18)}(\tau) &:= \frac{1}{24}\left(h_3(\tau) + h_9(\tau)\right),\\
H_{1A,5}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_5(\tau) + h_{13}(\tau)\right),\\
H_{1A,7}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_7(\tau) + h_{11}(\tau)\right),\\
H_{1A,9}^{(18)}(\tau) &:= \frac{1}{24}\left(h_3(\tau) + 2h_9(\tau) + h_{15}(\tau)\right),\\
H_{1A,11}^{(18)}(\tau) &:= \frac{1}{24}\left(h_7(\tau) + 2h_{11}(\tau)\right),\\
H_{1A,13}^{(18)}(\tau) &:= \frac{1}{24}\left(h_5(\tau) + 2h_{13}(\tau)\right),\\
H_{1A,15}^{(18)}(\tau) &:= \frac{1}{24}\left(h_{15}(\tau) + h_9(\tau)\right),\\
H_{1A,17}^{(18)}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 2h_{17}(\tau)\right).
\end{aligned} \tag{B.100}$$

Define $H_{2A,r}^{(18)}$ in the usual way for root systems with a type A component, by requiring

$$H_{2A,r}^{(18)}(\tau) := -(-1)^r H_{1A,r}^{(18)}(\tau). \tag{B.101}$$

B.3.18 $\ell = 18 + 9$, $X = D_{10}E_7^2$

We have $m^X = 18$, so the umbral McKay-Thompson series $H_g^{(18+9)} = (H_{g,r}^{(18+9)})$ associated to $g \in G^{(18+9)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H_{g,r}^{(18+9)} = -H_{g,-r}^{(18+9)}$, $H_{g,r}^{(18+9)} = H_{g,18-r}^{(18+9)}$ for $1 \leq r \leq 17$, and $H_{g,r}^{(18+9)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{g,r}^{(18+9)}$ for $r \in \{1, 3, 5, 7, 9\}$.

Define

$$\begin{aligned} H_{1A,r}^{(18+9)}(\tau) &:= H_{1A,r}^{(18)}(\tau) + H_{1A,18-r}^{(18)}(\tau), \\ H_{2A,r}^{(18+9)}(\tau) &:= H_{1A,r}^{(18)}(\tau) - H_{1A,18-r}^{(18)}(\tau), \end{aligned} \tag{B.102}$$

for $r \in \{1, 3, 5, 7, 9\}$.

B.3.19 $\ell = 22 + 11$, $X = D_{12}^2$

We have $m^X = 22$, so the umbral McKay-Thompson series $H_g^{(22+11)} = (H_{g,r}^{(22+11)})$ associated to $g \in G^{(22+11)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 44-vector-valued function with components indexed by $r \in \mathbb{Z}/44\mathbb{Z}$. We have $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ and $H_{g,r}^{(22+11)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{g,r}^{(22+11)}$ for r odd. Observing that $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ we may determine $H_g^{(22+11)}$ by requiring that

$$\psi_g^{(11/2)}(\tau, z) = -2\bar{\chi}_g^{(11/2)} i\mu_{11/2,0}(\tau, z) + \sum_{\substack{r \in \mathbb{Z}+1/2 \\ r \mod 11}} e(-r/2) H_{g,2r}^{(22+11)}(\tau)\theta_{11/2,r}(\tau, z), \tag{B.103}$$

where $\bar{\chi}_{1A}^{(11/2)} := 2$, $\bar{\chi}_{2A}^{(11/2)} := 0$, and the $\psi_g^{(11/2)}$ are the meromorphic Jacobi forms of weight 1 and index 11/2 defined as follows.

$$\begin{aligned} \psi_{1A}^{(11/2)}(\tau, z) &:= 2i \frac{\theta_1(\tau, 4z)}{\theta_1(\tau, z)\theta_1(\tau, 2z)} \eta(\tau)^3 \\ \psi_{2A}^{(11/2)}(\tau, z) &:= -2i \frac{\theta_1(\tau, 4z)}{\theta_2(\tau, z)\theta_2(\tau, 2z)} \eta(\tau)^3 \end{aligned} \tag{B.104}$$

B.3.20 $\ell = 25$, $X = A_{24}$

We have $m^X = 25$, so for $g \in G^{(25)} \simeq \mathbb{Z}/2\mathbb{Z}$, the associated umbral McKay-Thompson series $H_g^{(25)} = (H_{g,r}^{(25)})$ is a 50-vector-valued function, with components indexed by $r \in \mathbb{Z}/50\mathbb{Z}$, satisfying $H_{g,r}^{(25)} = -H_{g,-r}^{(25)}$, and in particular, $H_{g,r}^{(25)} = 0$ for $r = 0 \mod 25$. So it suffices to specify the $H_{g,r}^{(25)}$ for $1 \leq r \leq 24$.

Define $H_g^{(25)} = (H_{g,r}^{(25)})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau, z)\varphi_1^{(25)}(\tau, z) = -\mu_{25,0}(\tau, z) + \sum_{r \mod 50} H_{e,r}^{(25)}(\tau)\theta_{25,r}(\tau, z), \tag{B.105}$$

where

$$\varphi_1^{(25)}(\tau, z) := \frac{1}{2}\varphi_1^{(5)}(\tau, z)\varphi_1^{(21)}(\tau, z) - \varphi_1^{(7)}(\tau, z)\varphi_1^{(19)}(\tau, z) + \frac{1}{2}\varphi_1^{(13)}(\tau, z)^2. \tag{B.106}$$

For the definition of $\varphi_1^{(25)}$ we require

$$\begin{aligned}
\varphi_1^{(17)}(\tau,z) &:= 4\varphi_1^{(5)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(9)}(\tau,z)^2,\\
\varphi_1^{(19)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(16)}(\tau,z) + 2\varphi_1^{(7)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(21)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(17)}(\tau,z) - 2\varphi_1^{(9)}(\tau,z)\varphi_1^{(13)}(\tau,z),
\end{aligned} \tag{B.107}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Define $H_{2A,r}^{(25)}$ in the usual way for root systems with a type A component, by requiring

$$H_{2A,r}^{(18)}(\tau) := -(-1)^r H_{1A,r}^{(18)}(\tau). \tag{B.108}$$

B.3.21 $\ell = 30+15$, $X = D_{16}E_8$

We have $m^X = 30$, so the umbral McKay-Thompson series $H_g^{(30+15)} = (H_{g,r}^{(30+15)})$ associated to $g \in G^{(30+15)} = \{e\}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have $H_{e,r}^{(30+15)} = -H_{e,-r}^{(30+15)}$, $H_{e,r}^{(30+15)} = H_{e,30-r}^{(30+15)}$ for $1 \le r \le 29$, and $H_{e,r}^{(30+15)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{e,r}^{(30+15)}$ for $r \in \{1,3,5,7,9,11,13,15\}$.

Define

$$\begin{aligned}
H_{1A,1}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} + [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,3}^{(30+15)}(\tau) &:= [-\tfrac{9}{120}]H_{3A,3}^{(10+5)}(\tau/3),\\
H_{1A,5}^{(30+15)}(\tau) &:= [-\tfrac{25}{120}]H_{3A,5}^{(10+5)}(\tau/3),\\
H_{1A,7}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} + [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,11}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} - [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,13}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} - [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,15}^{(30+15)}(\tau) &:= -[-\tfrac{105}{120}]H_{3A,5}^{(10+5)}(\tau/3).
\end{aligned} \tag{B.109}$$

B.3.22 $\ell = 30+6,10,15$, $X = E_8^3$

We have $m^X = 30$, and $G^{(30+6,10,15)} = G^X \simeq S_3$. The umbral McKay-Thompson series $H^{(30+6,10,15)}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have

$$H_{g,r}^{(30+6,10,15)}(\tau) = \begin{cases} \pm H_{g,1}^{(30+6,10,15)} & \text{if } r = \pm1,\pm11,\pm19,\pm29 \mod 60,\\ \pm H_{g,7}^{(30+6,10,15)} & \text{if } r = \pm7,\pm13,\pm17,\pm27 \mod 60,\\ 0 & \text{else,} \end{cases} \tag{B.110}$$

so it suffices to specify the $H_{g,r}^{(30+6,10,15)}$ for $r = 1$ and $r = 7$. These functions may be defined as follows.

$$\begin{aligned}
H_{1A,1}^{(30+6,10,15)} &:= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+(k+l+m)/2+3/40}\\
H_{2A,1}^{(30+6,10,15)} &:= -2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0} - \sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+(2k+m)/2+3/40}\\
H_{3A,1}^{(30+6,10,15)} &:= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+3k/2+3/40}\\
H_{1A,7}^{(30+6,10,15)} &= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+3(k+l+m)/2+27/40}\\
H_{2A,7}^{(30+6,10,15)} &= 2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0} - \sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+3(2k+m)/2+27/40}\\
H_{3A,7}^{(30+6,10,15)} &= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+9k/2+27/40}
\end{aligned} \tag{B.111}$$

B.3.23 $\ell = 46 + 23$, $X = D_{24}$

We have $m^X = 22$, and $G^{(46+23)} = \{e\}$. The umbral McKay-Thompson series $H_e^{(46+23)} = (H_{e,r}^{(46+23)})$ is a 92-vector-valued function with components indexed by $r \in \mathbb{Z}/92\mathbb{Z}$. We have $H_{e,r}^{(46+23)} = -H_{e,-r}^{(46+23)}$ and $H_{e,r}^{(46+23)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{e,r}^{(46+23)}$ for r odd. Observing that $H_{e,r}^{(46+23)} = -H_{e,-r}^{(46+23)}$ we may determine $H_e^{(46+23)}$ by requiring that

$$\psi_e^{(23/2)}(\tau,z) = -2i\mu_{23/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2\\ r \mod 23}} e(-r/2)H_{g,2r}^{(46+23)}(\tau)\theta_{23/2,r}(\tau,z), \tag{B.112}$$

where $\psi_e^{(23/2)}$ is the meromorphic Jacobi forms of weight 1 and index 23/2 defined by setting

$$\psi_e^{(23/2)}(\tau,z) := 2i\frac{\theta_1(\tau,6z)}{\theta_1(\tau,2z)\theta_1(\tau,3z)}\eta(\tau)^3. \tag{B.113}$$

B.4 Rademacher Sums

Let Γ_∞ denote the subgroup of upper-triangular matrices in $SL_2(\mathbb{Z})$. Given $\alpha \in \mathbb{R}$ and $\gamma \in SL_2(\mathbb{Z})$, define $\mathrm{r}_{1/2}^{[\alpha]}(\gamma,\tau) := 1$ if $\gamma \in \Gamma_\infty$. Otherwise, set

$$\mathrm{r}_{1/2}^{[\alpha]}(\gamma,\tau) := e(-\alpha(\gamma\tau-\gamma\infty))\sum_{k\geq 0}\frac{(2\pi i\alpha(\gamma\tau-\gamma\infty))^{n+1/2}}{\Gamma(n+3/2)}, \tag{B.114}$$

where $e(x) := e^{2\pi i x}$. Let n be a positive integer, and suppose that ν is a multiplier system for vector-valued modular forms of weight $1/2$ on $\Gamma = \Gamma_0(n)$. Assume that $\nu = (\nu_{ij})$ satisfies $\nu_{11}(T) = e^{\pi i/2m}$, for some basis $\{\mathfrak{e}_i\}$, for some positive integer m, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. To this data, attach the Rademacher sum

$$R_{\Gamma,\nu}(\tau) := \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} \nu(\gamma) e\left(-\frac{\gamma\tau}{4m}\right) \mathfrak{e}_1 \, \mathrm{j}(\gamma,\tau)^{1/2} \, \mathrm{r}_{1/2}^{[-1/4m]}(\gamma,\tau), \tag{B.115}$$

where $\Gamma_{K,K^2} := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid 0 \le c < K, \, |d| < K^2 \right\}$, and $\mathrm{j}(\gamma,\tau) := (c\tau + d)^{-1}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If the expression (B.115) converges then it defines a mock modular form of weight $1/2$ for Γ whose shadow is given by an explicitly identifiable Poincaré series. We refer to [13] for a review of this, and to [53] for a more general and detailed discussion.

Convergence of (B.115) can be shown by rewriting the Fourier expansion as in [53, Theorem 2] in terms of a sum of Kloostermann sums weighted by Bessel functions. This expression converges at $w = 1/2$ by the analysis discussed at the end of §3, following the method of Hooley as adapted by Gannon. That analysis requires not only establishing that the expressions converge, but also explicitly bounding the rates of convergence.

For the special case that $X = A_8^3$ we require 8-vector-valued functions $\check{t}_g^{(9)} = (\check{t}_{g,r}^{(9)})$ for $g \in G^X$ with order 3 or 6. For such g, define $\check{t}_{g,r}^{(9)}$, for $0 < r < 9$, by setting

$$\check{t}_{3A,r}^{(9)}(\tau) := \begin{cases} 0, & \text{if } r \neq 0 \mod 3, \\ -\theta_{3,3}(\tau,0), & \text{if } r = 3, \\ \theta_{3,0}(\tau,0), & \text{if } r = 6, \end{cases} \tag{B.116}$$

in the case that g has order 3, and

$$\check{t}_{6A,r}^{(9)}(\tau) := \begin{cases} 0, & \text{if } r \neq 0 \mod 3, \\ -\theta_{3,3}(\tau,0), & \text{if } r = 3, \\ -\theta_{3,0}(\tau,0), & \text{if } r = 6, \end{cases} \tag{B.117}$$

when $o(g) = 6$. Here $\theta_{m,r}(\tau,z)$ is as defined in (B.5).

The following result is proved in [17], using an analysis of representations of the metaplectic double cover of $SL_2(\mathbb{Z})$.

Theorem B.1 ([17])**.** Let X be a Niemeier root system and let $g \in G^X$. Assume that the Rademacher sum $R^X_{\Gamma_0(n_g),\check{\nu}_g^X}$ converges. If $X \neq A_8^3$, or if $X = A_8^3$ and $g \in G^X$ does not satisfy $o(g) = 0 \mod 3$, then we have

$$\check{H}_g^X(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}_g^X}. \tag{B.118}$$

If $X = A_8^3$ and $g \in G^X$ satisfies $o(g) = 0 \mod 3$ then

$$\check{H}_{g,r}^X(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}_g^X}(\tau) + \check{t}_g^{(9)}(\tau). \tag{B.119}$$

The $X = A_1^{24}$ case of Theorem B.1 was proven first in [12], via different methods.

References

[1] D. Alexander, C. Cummins, J. McKay, and C. Simons, *Completely replicable functions*, Groups, combinatorics & geometry (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 165, Cambridge Univ. Press, Cambridge, 1992, 87–98. MR 1200252 (94g:11029)

[2] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proceedings of the National Academy of Sciences, U.S.A. **83** (1986), no. 10, 3068–3071.

[3] ———, *Generalized Kac-Moody algebras*, J. Algebra **115** (1988), no. 2, 501–512. MR 943273 (89g:17004)

[4] ———, *Central extensions of generalized Kac-Moody algebras*, J. Algebra **140** (1991), no. 2, 330–335. MR 1120425 (92g:17031)

[5] ———, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), no. 2, 405–444.

[6] ———, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562. MR 1625724 (99c:11049)

[7] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), no. 2, 243–266. MR 2231957 (2007e:11127)

[8] ———, *Coefficients of harmonic maass forms*, Partitions, q-Series, and Modular Forms (Krishnaswami Alladi and Frank Garvan, eds.), Developments in Mathematics, vol. 23, Springer New York, 2012, pp. 23–38.

[9] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90. MR 2097357 (2005m:11089)

[10] M. C. Cheng, *$K3$ surfaces, $N = 4$ dyons and the Mathieu group M_{24}*, Commun. Number Theory Phys. **4** (2010), no. 4, 623–657. MR 2793423 (2012e:11076)

[11] M. Cheng and J. Duncan, *The largest Mathieu group and (mock) automorphic forms*, String-Math 2011, Proc. Sympos. Pure Math., vol. 85, Amer. Math. Soc., Providence, RI, 2012, pp. 53–82. MR 2985326

[12] ———, *On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine*, Commun. Number Theory Phys. **6** (2012), no. 3.

[13] ———, *Rademacher Sums and Rademacher Series*, Conformal Field Theory, Automorphic Forms and Related Topics, 2014, pp.143-182, Contributions in Mathematical and Computational Sciences.

[14] ———, *Meromorphic Jacobi Forms of Half-Integral Index and Umbral Moonshine Modules*, in preparation.

[15] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, *Umbral Moonshine*, Commun. Number Theory Phys. **8** (2014), no. 2.

[16] ———, *Umbral Moonshine and the Niemeier Lattices*, Research in the Mathematical Sciences **1** (2014), no. 3.

[17] ———, *Weight One Jacobi Forms and Umbral Moonshine*, in preparation.

[18] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), no. 3, 308–339. MR 554399 (81j:20028)

[19] J. H. Conway and N. J. A. Sloane, *On the enumeration of lattices of determinant one*, J. Number Theory **15** (1982), no. 1, 83–94. MR 666350 (84b:10047)

[20] ———, *Sphere packings, lattices and groups*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1999, With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. MR 1662447 (2000b:11077)

[21] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.*, Oxford: Clarendon Press, 1985.

[22] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms*, arXiv:1208.4074.

[23] L. Dixon, P. Ginsparg and J. Harvey, *Beauty and the beast: superconformal symmetry in a Monster module*, Comm. Math. Phys. **119** (1998), no. 2, 221–241. MR 968697 (90b:81119)

[24] J. Duncan, M. Griffin, and K. Ono, *Moonshine*, Research in the Mathematical Sciences (2015) 2:11.

[25] J. Duncan and J. Harvey, *The Umbral Mooonshine Module for the Unique Unimodular Niemeier Root System*, arXiv:1412.8191.

[26] J. Duncan and A. O'Desky, *Super Vertex Algebras, Meromorphic Jacobi Forms, and Umbral Moonshine*, in preparation.

[27] T. Eguchi and K. Hikami, *Note on Twisted Elliptic Genus of K3 Surface*, Phys.Lett. **B694** (2011), 446–455.

[28] T. Eguchi, H. Ooguri, and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group* M_{24}, Exper.Math. **20** (2011), 91–96.

[29] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Birkhäuser, 1985.

[30] I. Frenkel, J. Lepowsky, and A. Meurman, *A natural representation of the Fischer-Griess Monster with the modular function J as character*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), no. 10, Phys. Sci., 3256–3260. MR MR747596 (85e:20018)

[31] ———, *A moonshine module for the Monster*, Vertex operators in mathematics and physics (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., vol. 3, Springer, New York, 1985, pp. 231–273. MR 86m:20024

[32] ———, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press Inc., Boston, MA, 1988. MR 90h:17026

[33] M. Gaberdiel, S. Hohenegger, and R. Volpato, *Mathieu Moonshine in the elliptic genus of K3*, JHEP **1010** (2010), 062.

[34] ———, *Mathieu twining characters for K3*, JHEP **1009** (2010), 058, 19 pages.

[35] T. Gannon, *Monstrous moonshine: the first twenty-five years*, Bull. London Math. Soc. **38** (2006), no. 1, 1–33. MR 2201600 (2006k:11071)

[36] ———, *Moonshine beyond the Monster*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2006, The bridge connecting algebra, modular forms and physics. MR 2257727 (2008a:17032)

[37] ———, *Much ado about Mathieu*, arXiv:1211.5531.

[38] C. Hooley, *On the number of divisors of a quadratic polynomial*, Acta Math. **110** (1963), 97-114. MR 0153648

[39] J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972, Graduate Texts in Mathematics, Vol. 9. MR 0323842 (48 #2197)

[40] W. Kohnen, *Fourier coefficients of half-integral weight*, Math. Ann. **271** (1985), 237–268. MR 0783554

[41] Ö. Imamoğlu, M. Raum, and O. Richer, *Holomorphic projections and Ramanujan's mock theta functions*, Proc. Natl. Acad. Sci., USA, **111** (2014), 3961–3967.

[42] M. Mertens, *Eichler-Selberg type identities for mixed mock modular forms*, arXiv:1404.5491.

[43] H. V. Niemeier, *Definite quadratische Formen der Dimension* 24 *und Diskriminante* 1, J. Number Theory **5** (1973), 142–178. MR 0316384 (47 #4931)

[44] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conf. Ser. No. 102, Amer. Math. Soc., 2004, Providence MR 2020489 (2005c:11053)

[45] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 347–454. MR 2555930 (2010m:11060)

[46] S. Ramanujan, *The lost notebook and other unpublished papers*, Springer-Verlag, Berlin, 1988, With an introduction by George E. Andrews. MR 947735 (89j:01078)

[47] W. A. Stein et al., *Sage Mathematics Software (Version 6.5)*, The Sage Development Team, 2015, `http://www.sagemath.org`.

[48] J.-P. Serre and H. M. Stark, *Modular forms of weight* 1/2, in Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 27–67. Lecture Notes in Math., Vol. 627. Springer, Berlin, 1977.

[49] J. Sturm, *On the congruence of modular forms*, Springer Lect. Notes Math. **1240** (1984), 275-280. MR 0894516 (88h:11031)

[50] J. G. Thompson, *Finite groups and modular functions*, Bull. London Math. Soc. **11** (1979), no. 3, 347–351. MR 554401 (81j:20029)

[51] ———, *Some numerology between the Fischer-Griess Monster and the elliptic modular function*, Bull. London Math. Soc. **11** (1979), no. 3, 352–353. MR MR554402 (81j:20030)

[52] B. B. Venkov, *On the classification of integral even unimodular* 24*-dimensional quadratic forms*, Trudy Mat. Inst. Steklov. **148** (1978), 65–76, 273, Algebra, number theory and their applications. MR 558941 (81d:10024)

[53] D. Whalen, *Vector-Valued Rademacher Sums and Automorphic Integrals*, arXiv:1406.0571.

[54] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)*, Astérisque (2009), no. 326, Exp. No. 986, vii–viii, 143–164 (2010), Séminaire Bourbaki. Vol. 2007/2008. MR 2605321 (2011h:11049)

[55] S. Zwegers, *Mock Theta Functions*, Ph.D. thesis, Utrecht University, 2002. arXiv:0807.4834.

J. F. R. Duncan, Dept. of Mathematics & Computer Science, Emory University, Atlanta, GA 30322
E-mail address, J. F. R. Duncan: `john.duncan@emory.edu`

M. J. Griffin, Dept. of Mathematics, Princeton University, Princeton, NJ 08544
E-mail address, M. J. Griffin: `mjg4@princeton.edu`

K. Ono, Dept. of Mathematics & Computer Science, Emory University, Atlanta, GA 30322
E-mail address, K. Ono: `ono@mathcs.emory.edu`

Title: Umbral Moonshine and K3 Surfaces

Date: Apr 17, 2015 11:00 AM

URL: http://pirsa.org/15040135

Abstract: Recently, 23 cases of umbral moonshine, relating mock modular forms and finite groups, have been discovered in the context of the 23 even unimodular Niemeier lattices. One of the 23 cases in fact coincides with the so-called Mathieu moonshine, discovered in the context of K3 non-linear sigma models. Here we establish a uniform relation between all 23 cases of umbral moonshine and K3 sigma models, and thereby take a first step in placing umbral moonshine into a geometric and physical context. This is achieved by relating the ADE root systems of the Niemeier lattices to the ADE du Val singularities that a K3 surface can develop, and the configuration of smooth rational curves in their resolutions. A geometric interpretation of our results is given in terms of the marking of K3 surfaces by Niemeier lattices.

- We are all here for some reason or another because of the observation of EOT relating the group M_{24} and K3 elliptic genus
- This has led to a variety of fascinating progress in the area of moonshine, physics, and mathematics, including the still-mysterious umbral moonshine, of which M_{24} moonshine is only the first case
- What about the other cases of umbral moonshine? Do they have some relation to K3? Can this help us find a unifying structure?

- We are all here for some reason or another because of the observation of EOT relating the group M_{24} and K3 elliptic genus
- This has led to a variety of fascinating progress in the area of moonshine, physics, and mathematics, including the still-mysterious umbral moonshine, of which M_{24} moonshine is only the first case
- What about the other cases of umbral moonshine? Do they have some relation to K3? Can this help us find a unifying structure?

Elliptic genus of singularities

Umbral moonshine and the Niemeier lattices

Umbral moonshine and the K3 elliptic genus

Comments and open questions

Elliptic genus of singularities

- Consider singularities of the form $\mathbb{C}^2/G$ with G a subgroup of $SU_2(\mathbb{C})$
- Local description of singularities of K3 surfaces
- ADE classification: Intersection matrix of curves in resolution yields an ADE Dynkin diagram
- In terms of hypersurfaces, it is given by $W^0_\Phi = 0$ with

$$\begin{aligned} W^0_{A_{m-1}} &= x_1^2 + x_2^2 + x_3^m \\ W^0_{D_{m/2+1}} &= x_1^2 + x_2^2 x_3 + x_3^{m/2} \\ W^0_{E_6} &= x_1^2 + x_2^3 + x_3^4 \\ W^0_{E_7} &= x_1^2 + x_2^3 + x_2 x_3^3 \\ W^0_{E_8} &= x_1^2 + x_2^3 + x_3^5. \end{aligned}$$

Elliptic genus of singularities

The 2d conformal field theory description of these (isolated) singularities was proposed by Ooguri and Vafa to be:

$$\text{Minkowski } \mathbb{R}^{5,1} \otimes \left(\mathcal{N} = 2 \text{ minimal } \otimes \mathcal{N} = 2 \ \frac{SL(2,\mathbb{R})}{U(1)} \right) / (\mathbb{Z}/m\mathbb{Z}). \tag{1.1}$$

where m is the coxeter number of the corresponding simply laced root system. For A_{m-1}, this describes near-horizon geometry of m NS5-branes. Interesting work by Harvey and Murthy where one replaces $\mathbb{R}^{5,1}$ with $K3 \times \mathbb{R}^{1,1}$.

Elliptic genus of singularities

The 2d conformal field theory description of these (isolated) singularities was proposed by Ooguri and Vafa to be:

$$\text{Minkowski } \mathbb{R}^{5,1} \otimes \left(\mathcal{N} = 2 \text{ minimal } \otimes \mathcal{N} = 2 \ \frac{SL(2,\mathbb{R})}{U(1)} \right) / (\mathbb{Z}/m\mathbb{Z}). \tag{1.1}$$

where m is the coxeter number of the corresponding simply laced root system. For A_{m-1}, this describes near-horizon geometry of m NS5-branes. Interesting work by Harvey and Murthy where one replaces $\mathbb{R}^{5,1}$ with $K3 \times \mathbb{R}^{1,1}$.

Elliptic genus of singularities

One can resolve the singularity by considering $W_\Phi^0 = \mu$.
It was proposed that the sigma model with the non-compact target space $W_\Phi^0 = \mu$ has an alternative description as the Landau–Ginsburg model with superpotential

$$\tilde{W}_\Phi = -\mu x_0^{-m} + W_\Phi^0,$$

where x_0 is an additional chiral superfield and m is again given by the Coxeter number of Φ.
In order to compute the elliptic genus, we need to understand both the contribution from the $\mathcal{N} = 2$ minimal model and the noncompact coset model.

Elliptic genus of singularities

One can resolve the singularity by considering $W_\Phi^0 = \mu$.
It was proposed that the sigma model with the non-compact target space $W_\Phi^0 = \mu$ has an alternative description as the Landau–Ginsburg model with superpotential

$$\tilde{W}_\Phi = -\mu x_0^{-m} + W_\Phi^0,$$

where x_0 is an additional chiral superfield and m is again given by the Coxeter number of Φ.
In order to compute the elliptic genus, we need to understand both the contribution from the $\mathcal{N} = 2$ minimal model and the noncompact coset model.

The $\mathcal{N} = 2$ minimal models

The $\mathcal{N} = 2$ minimal models are known to have an ADE classification based on an ADE classification of the modular invariant combinations of chiral (holomorphic) and anti-chiral (anti-holomorphic) characters of the $A_1^{(1)}$ Kac–Moody algebra. These correspond to a $2m \times 2m$ matrix Ω^{Φ} for each ADE root system Φ.

The central charge is given by the coxeter number of the root system:

$$\hat{c} = c/3 = 1 - \frac{2}{m}. \tag{1.2}$$

The $\mathcal{N} = 2$ minimal models

The $\mathcal{N} = 2$ minimal models are known to have an ADE classification based on an ADE classification of the modular invariant combinations of chiral (holomorphic) and anti-chiral (anti-holomorphic) characters of the $A_1^{(1)}$ Kac–Moody algebra. These correspond to a $2m \times 2m$ matrix Ω^{Φ} for each ADE root system Φ.

The central charge is given by the coxeter number of the root system:

$$\hat{c} = c/3 = 1 - \frac{2}{m}. \tag{1.2}$$

The $\mathcal{N} = 2$ minimal models

The $\mathcal{N} = 2$ minimal models are known to have an ADE classification based on an ADE classification of the modular invariant combinations of chiral (holomorphic) and anti-chiral (anti-holomorphic) characters of the $A_1^{(1)}$ Kac–Moody algebra. These correspond to a $2m \times 2m$ matrix Ω^{Φ} for each ADE root system Φ.

The central charge is given by the coxeter number of the root system:

$$\hat{c} = c/3 = 1 - \frac{2}{m}. \tag{1.2}$$

Elliptic genus of singularities

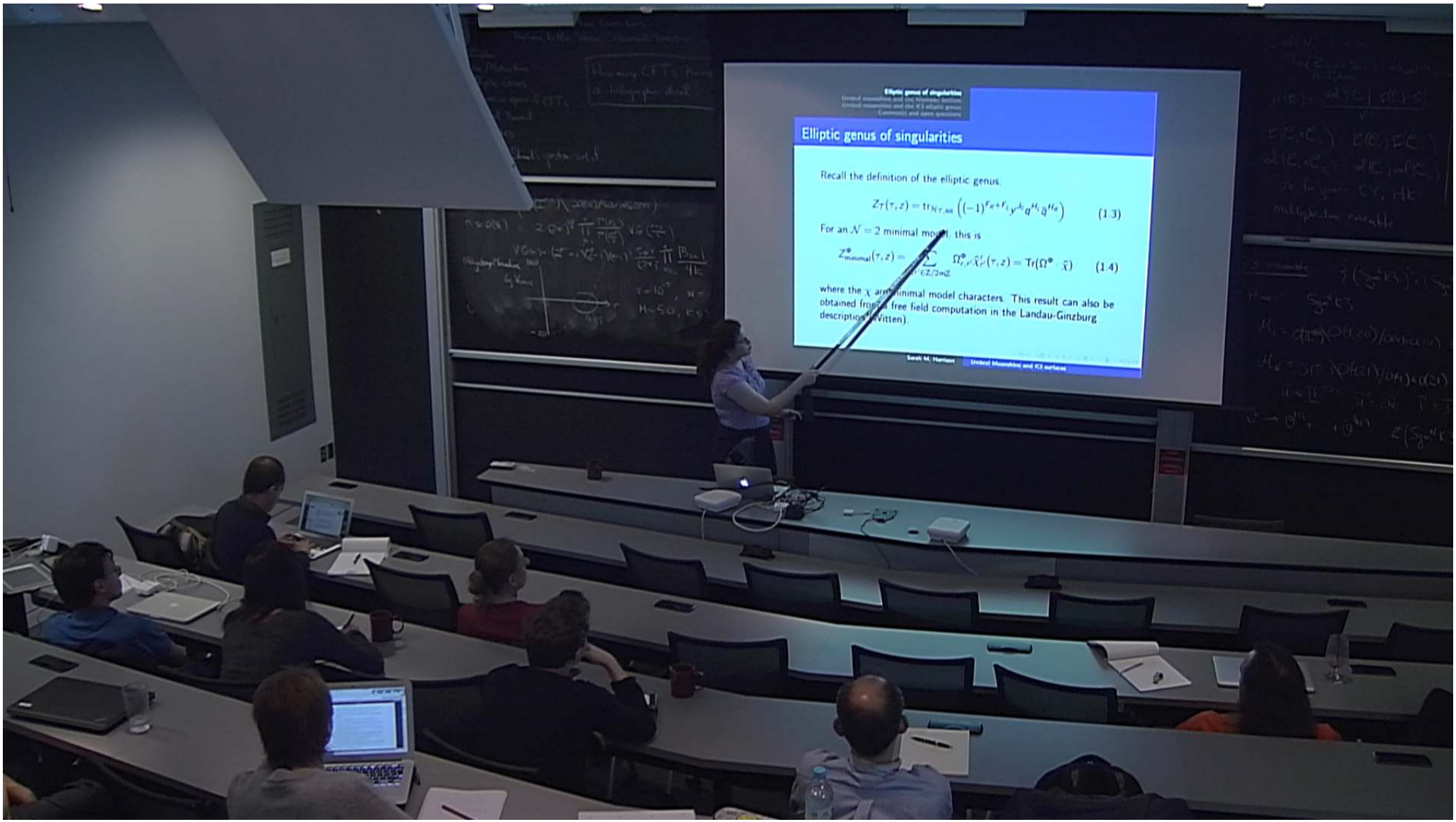
Recall the definition of the elliptic genus:

$$Z_{\mathcal{T}}(\tau, z) = \mathrm{tr}_{\mathcal{H}_{\mathcal{T},\mathrm{RR}}}\left((-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R}\right) \tag{1.3}$$

For an $\mathcal{N} = 2$ minimal model, this is

$$Z^{\Phi}_{\mathrm{minimal}}(\tau, z) = \sum_{r,r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega^{\Phi}_{r,r'} \tilde{\chi}^{r}_{r'}(\tau, z) = \mathrm{Tr}(\Omega^{\Phi} \cdot \tilde{\chi}). \tag{1.4}$$

where the χ are minimal model characters. This result can also be obtained from a free field computation in the Landau-Ginzburg description (Witten).


Elliptic genus of singularities
Recall the definition of the elliptic genus.
$Z_T(\tau,z) = \mathrm{tr}_{\mathcal{H}_{T,\mathrm{RR}}}\left((-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R}\right)$ (1.3)
For an $\mathcal{N}=2$ minimal model, this is
$Z^*_{\mathrm{minimal}}(\tau,z) = \sum_{\Gamma\in\mathbb{Z}/2m\mathbb{Z}} \Omega^*_{\Gamma,r}\bar{\chi}^r_\Gamma(\tau,z) = \mathrm{Tr}(\Omega^* \cdot \bar{\chi})$ (1.4)
where the χ are minimal model characters. This result can also be obtained from a free field computation in the Landau-Ginzburg description (Witten).
Sarah M. Harrison
Umbral Moonshine and K3 surfaces

Elliptic genus of singularities

Recall the definition of the elliptic genus:

$$Z_{\mathcal{T}}(\tau, z) = \mathrm{tr}_{\mathcal{H}_{\mathcal{T},\mathrm{RR}}}\left((-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R}\right) \tag{1.3}$$

For an $\mathcal{N} = 2$ minimal model, this is

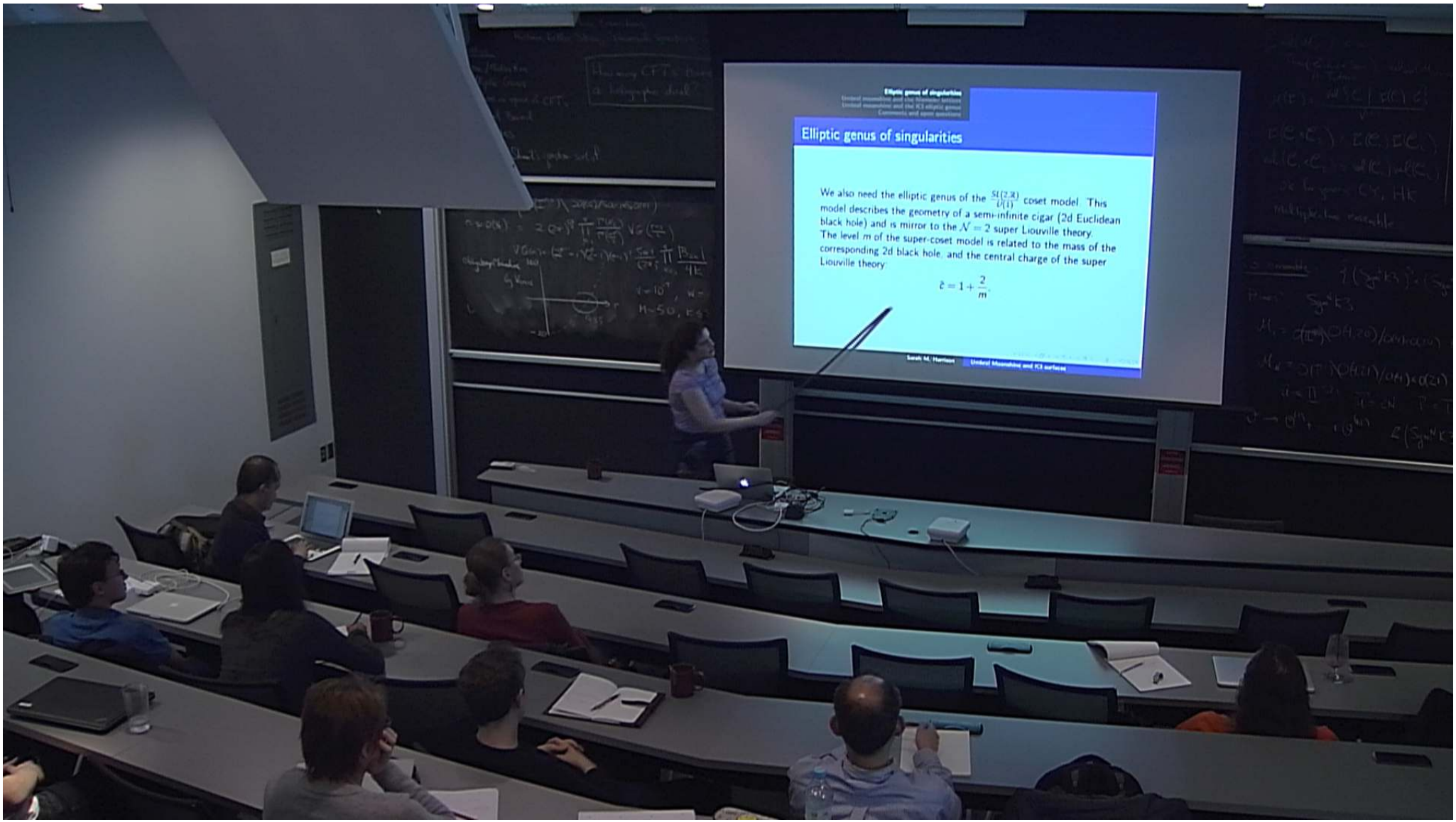
$$Z^{\Phi}_{\mathrm{minimal}}(\tau, z) = \sum_{r,r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega^{\Phi}_{r,r'} \tilde{\chi}^{r}_{r'}(\tau, z) = \mathrm{Tr}(\Omega^{\Phi} \cdot \tilde{\chi}). \tag{1.4}$$

where the χ are minimal model characters. This result can also be obtained from a free field computation in the Landau-Ginzburg description (Witten).

Elliptic genus of singularities

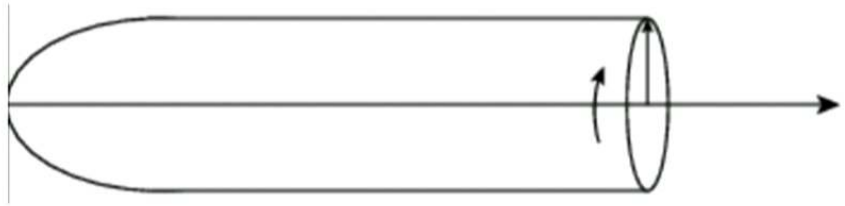
We also need the elliptic genus of the $\frac{SL(2,\mathbb{R})}{U(1)}$ coset model. This model describes the geometry of a semi-infinite cigar (2d Euclidean black hole) and is mirror to the $\mathcal{N}=2$ super Liouville theory.
The level m of the super-coset model is related to the mass of the corresponding 2d black hole, and the central charge of the super Liouville theory:

$$\hat{c} = 1 + \frac{2}{m}.$$


Elliptic genus of singularities
We also need the elliptic genus of the $\frac{SL(2,\mathbb{R})}{U(1)}$ coset model. This model describes the geometry of a semi-infinite cigar (2d Euclidean black hole) and is mirror to the $\mathcal{N}=2$ super Liouville theory.
The level m of the super-coset model is related to the mass of the corresponding 2d black hole, and the central charge of the super Liouville theory:
$\hat{c} = 1 + \frac{2}{m}$.

Elliptic genus of singularities

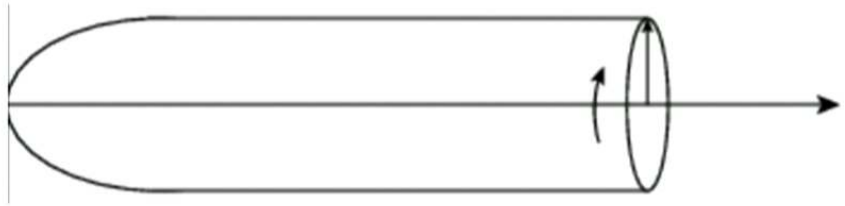
This model is noncompact $\rightarrow$ spectrum contains both discrete and continuous states. (DVV) Geometrically, the discrete states are localized at the tip of the cigar, and the continuous ones are those states whose wave-functions spread into the infinitely long half-cylinder and are only present above a "mass gap" $\frac{1}{4m}$ on the conformal weight.



The continuous states correspond to massive (or long) $\mathcal{N} = 2$ highest weight representations while the discrete states correspond to massless (or short) ones.

Elliptic genus of singularities

This model is noncompact $\rightarrow$ spectrum contains both discrete and continuous states. (DVV) Geometrically, the discrete states are localized at the tip of the cigar, and the continuous ones are those states whose wave-functions spread into the infinitely long half-cylinder and are only present above a "mass gap" $\frac{1}{4m}$ on the conformal weight.



The continuous states correspond to massive (or long) $\mathcal{N} = 2$ highest weight representations while the discrete states correspond to massless (or short) ones.

Elliptic genus of singularities

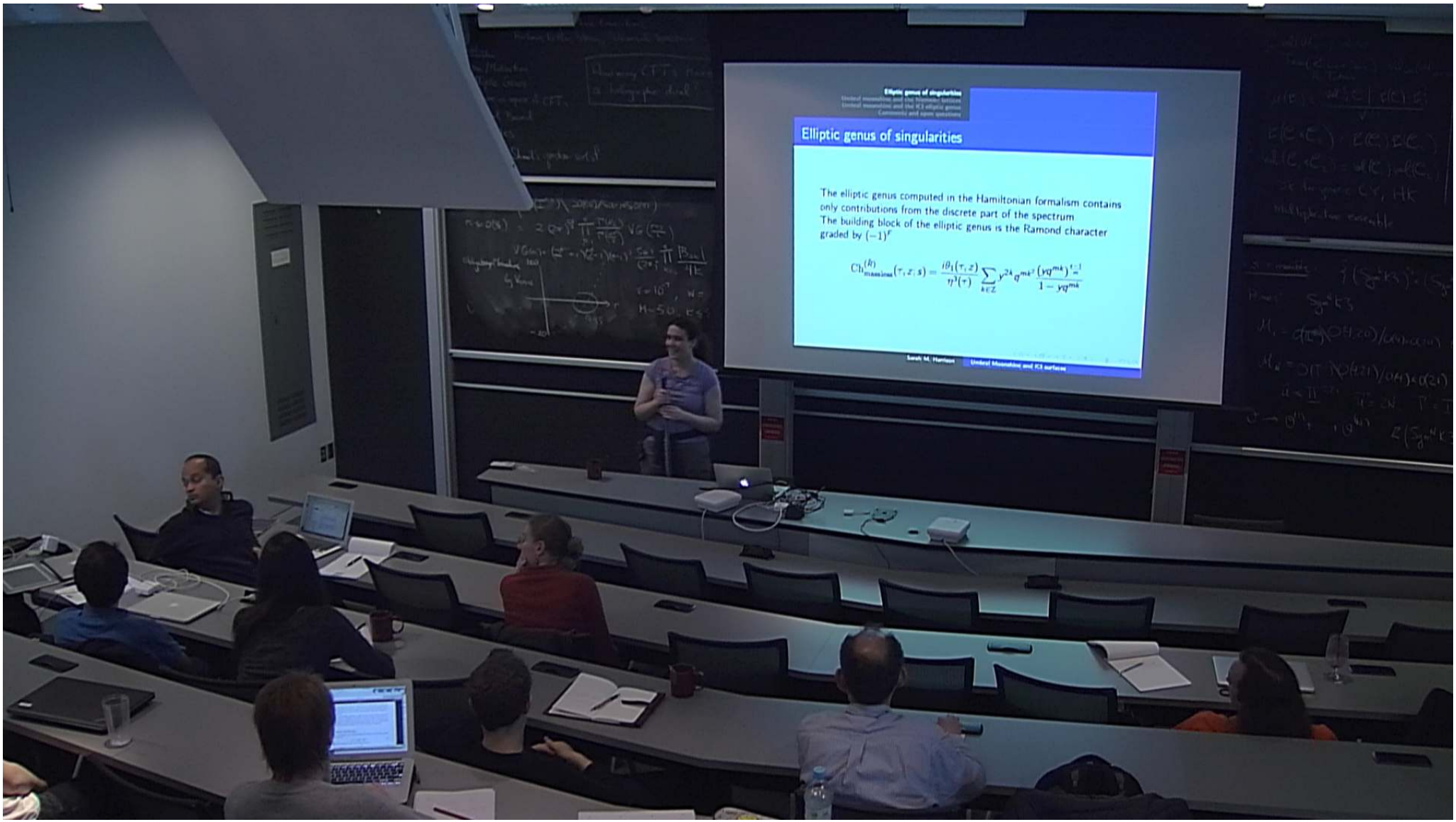
The elliptic genus computed in the Hamiltonian formalism contains only contributions from the discrete part of the spectrum.
The building block of the elliptic genus is the Ramond character graded by $(-1)^F$

$$\mathrm{Ch}^{(\tilde{R})}_{\mathrm{massless}}(\tau, z; s) = \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \sum_{k\in\mathbb{Z}} y^{2k} q^{mk^2} \frac{(yq^{mk})^{\frac{s-1}{m}}}{1 - yq^{mk}}$$

Elliptic genus of singularities

The elliptic genus computed in the Hamiltonian formalism contains only contributions from the discrete part of the spectrum.
The building block of the elliptic genus is the Ramond character graded by $(-1)^F$

$$\mathrm{Ch}^{(\tilde{R})}_{\mathrm{massless}}(\tau, z; s) = \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \sum_{k\in\mathbb{Z}} y^{2k} q^{mk^2} \frac{(yq^{mk})^{\frac{s-1}{m}}}{1 - yq^{mk}}$$


Elliptic genus of singularities
The elliptic genus computed in the Hamiltonian formalism contains only contributions from the discrete part of the spectrum
The building block of the elliptic genus is the Ramond character graded by $(-1)^F$
$$\mathrm{Ch}^{(R)}_{\mathrm{massless}}(\tau,z;s) = \frac{i\theta_1(\tau,z)}{\eta^3(\tau)} \sum_{k\in\mathbb{Z}} y^{2k} q^{mk^2} \frac{(yq^{mk})^{s-1}}{1-yq^{mk}}$$

Elliptic genus of singularities

Putting them together, from the spectrum of the super-coset model it is straightforward to work out the elliptic genus of the theory

$$Z_{L_m}(\tau,z) = \frac{1}{2}\sum_{s=1}^{m} \mathrm{Ch}^{(\tilde{R})}_{\text{massless}}(\tau,z;m+2-s) + \mathrm{Ch}^{(\tilde{R})}_{\text{massless}}(\tau,z;s)$$

$$= \frac{1}{2}\mu_{m,0}\big(\tau,\frac{z}{m}\big)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3},$$

where we have used the (specialised) Appell–Lerch sum

$$\mu_{m,0}(\tau,z) = -\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2km} \frac{1+yq^k}{1-yq^k}.$$

[Troost, Ashok-Troost, Eguchi-Sugawara]

Elliptic genus of singularities

One could also consider a path integral approach to the computation of the elliptic genus, which should naturally yield a modular object:

$$\hat{Z}^{\Phi,S}(\tau,z) = \frac{1}{2m}\frac{i\theta_1(\tau,z)}{\eta^3(\tau)} \sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} (-1)^{a+b} q^{a^2/2} y^a \, Z^{\Phi}_{\mathrm{minimal}}(\tau, z+a\tau+b)\, \hat{\mu}_{m,0}(\tau, \frac{z+a\tau+b}{m})$$

The Niemeier lattices

Here we review the main components of umbral moonshine.

- Even, unimodular, positive-definite lattices of rank 24
- More broadly relevant because their theta functions are modular invariant
- 24 such lattices, classified by Niemeier: Leech lattice + 23 others which have ADE classification
- Uniquely determined by their root systems $\Delta(L)$, that are all unions of the simply-laced root systems

The Niemeier lattices

Two conditions: all of the irreducible components have the same Coxeter numbers; total rank is 24:

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4 D_4$	A_6^4	$A_7^2 D_5^2$
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2 D_6$	$A_{11} D_7 E_6$	A_{12}^2	$A_{15} D_9$	$A_{17} E_7$	A_{24}
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10} E_7^2$	D_{12}^2	$D_{16} E_8$	D_{24}
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

The Niemeier lattices

Umbral groups come from the automorphism group of the lattice mod the Weyl group generated by reflections of the roots:

$$G^X = \mathrm{Aut}(L^X)/\mathrm{Weyl}(X).$$

The mock modular forms

We will use these to define special mock modular forms. Mock modular property: require H^X to be a weight $1/2$ vector-valued mock modular form whose shadow is given by S^X:

$$\hat{H}_r^X(\tau) = H_r^X(\tau) + \mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}}\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-\frac{1}{2}}\,\overline{S_r^X(-\bar{\tau}')}\,\mathrm{d}\tau',$$

then

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \hat{H}_r^X(\tau)\,\theta_{m,r}(\tau,z)$$

transforms as a Jacobi form of weight 1 and index m under the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

The mock modular forms

We will use these to define special mock modular forms. Mock modular property: require H^X to be a weight $1/2$ vector-valued mock modular form whose shadow is given by S^X:

$$\hat{H}_r^X(\tau) = H_r^X(\tau) + \mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}}\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-\frac{1}{2}}\,\overline{S_r^X(-\bar{\tau}')}\,\mathrm{d}\tau',$$

then

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \hat{H}_r^X(\tau)\,\theta_{m,r}(\tau,z)$$

transforms as a Jacobi form of weight 1 and index m under the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

The mock modular forms

Analyticity condition: require its growth near the cusp to be

$$q^{1/4m} H_r^X(\tau) = O(1) \quad \text{as } \tau \to i\infty \tag{2.1}$$

for every element $r \in \mathbb{Z}/2m\mathbb{Z}$.
The above two conditions turn out to be sufficient to determine H^X uniquely (up to a rescaling), as shown. We also fix the scaling by requiring $q^{1/4m} H_1^X(\tau) = -2 + O(q)$.

The mock modular forms

Modular properties?

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} H_r^X \, \theta_{m,r}$$

is a mock Jacobi form which is the finite part of a meromorphic Jacobi form with simple poles at m-torsion points.

The mock modular forms

To fix modular properties of the twining functions H_g^X we need

- The shadow $S_g^X = \Omega_g^X S_m = S_{g,r}^X$
- The multiplier system

Given this info, we can define:

$$\hat{H}_{g,r}^X(\tau) = H_{g,r}^X(\tau) + \mathrm{e}(-\tfrac{1}{8}) \frac{1}{\sqrt{2m}} \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-1/2} \overline{S_{g,r}^X(-\bar{\tau}')} \, \mathrm{d}\tau',$$

which appears in the theta decomposition of a weight 1 index m Jacobi form under a modular subgroup:

$$\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \hat{H}_{g,r}^X(\tau) \, \theta_{m,r}(\tau, z).$$

The mock modular forms

$$\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H^X_{g,r}\, \theta_{m,r} \tag{2.2}$$

is a mock Jacobi form of weight 1 and index m under a subgroup The functions H^X_g also satisfy a similar growth condition as in the case of the identity. These are defined for every $[g] \subset G^X$ and for all 23 Niemeier lattices L^X.

Umbral moonshine and the K3 elliptic genus

Now let's see how we can relate all of these mock modular forms to the $K3$ elliptic genus.
Recall the $\mathcal{N} = 4$ decomposition of the K3 elliptic genus

$$\begin{aligned}\mathbf{EG}(\tau, z; K3) &= 20\,\mathrm{ch}_{2;\frac{1}{4},0} - 2\,\mathrm{ch}_{2;\frac{1}{4},\frac{1}{2}} + \big(90\,\mathrm{ch}_{2;\frac{5}{4},\frac{1}{2}} + 462\,\mathrm{ch}_{2;\frac{9}{4},\frac{1}{2}} \\ &\quad + 1540\,\mathrm{ch}_{2;\frac{13}{4},\frac{1}{2}} + \dots\big) \\ &= \frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\Big\{24\,\mu_{2,0}(\tau,z) + (\theta_{2,-1}(\tau,z) - \theta_{2,1}(\tau,z)) \\ &\quad \times (-2q^{-1/8} + 90q^{7/8} + 462q^{15/8} + 1540q^{23/8} + \dots)\Big\}\end{aligned}$$

Umbral moonshine and the K3 elliptic genus

We can view the two contributions to $\mathbf{EG}(\tau, z; K3)$, given by

$$24\,\mu_{2,0}(\tau, z)$$

and

$$-\sum_{r\in\mathbb{Z}/4\mathbb{Z}} H_r^{X=A_1^{24}}(\tau)\theta_{2,r}(\tau, z),$$

as contributions from the BPS and non-BPS $\mathcal{N}=4$ multiplets respectively (up to the polar term in $H_r^{X=A_1^{24}}(\tau)$).

Umbral moonshine and the K3 elliptic genus

There is an alternative interpretation due to the identity between the short $\mathcal{N} = 4$ characters and the elliptic genus of an $\Phi = A_1$ singularity:

$$Z^{A_1,S}(\tau, z) = \mathrm{ch}_{2;\frac{1}{4},0}(\tau, z),$$

In other words, we can re-express the elliptic genus of $K3$ as

$$\mathbf{EG}(\tau, z; K3) = 24 Z^{A_1,S}(\tau, z) - \frac{i\,\theta_1(\tau, z)^2}{\eta^3(\tau)\theta_1(\tau, 2z)} \sum_{r \in \mathbb{Z}/4\mathbb{Z}} H_r^{A_1^{24}} \theta_{2,r}(\tau, z).$$

This provides a geometric interpretation of this decomposition.

Umbral moonshine and the K3 elliptic genus

We can rewrite the former expression as

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \, \phi^X(\tau, \frac{z + a\tau + b}{m})$$

for $X = A_1^{24}$, where

$$\phi^X = \frac{i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)} \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X(\tau) \, \theta_{m,r}(\tau, z)$$

encodes the umbral moonshine mock modular form.

Umbral moonshine and the K3 elliptic genus

Where, for

$$X = A_{m-1}^{d_A} D_{m/2+1}^{d_D} (E^{(m)})^{d_E},$$

we write

$$Z^{X,S} = d_A Z^{A_{m-1}} + d_D Z^{D_{m/2+1}} + d_E Z^{E^{(m)}}.$$

corresponding to a collection of no n-interacting ADE theories with the total Hilbert space given by the direct sum of the Hilbert spaces of the component theories.
So we can offer an alternate interpretation of this decomposition as a contribution from 24 copies of A_1-type surface singularities and an "umbral moonshine" contribution given by the umbral moonshine mock modular forms H^X with $X = A_1^{24}$

Umbral moonshine and the K3 elliptic genus

In fact, this type of decomposition holds for all 23 cases of umbral moonshine!

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \, \phi^X(\tau, \frac{z + a\tau + b}{m})$$

In other words: for the 23 Niemeier lattices L^X we have 23 different ways of separating $\mathbf{EG}(K3)$ into two parts.

- Replace the Niemeier root system X with the corresponding configuration of singularities to obtain a contribution to the $K3$ elliptic genus by the singularities.
- Use the umbral moonshine construction for the mock Jacobi form ϕ^X associated to each L^X to get the rest of $\mathbf{EG}(K3)$ after a summation procedure reminiscent of the "orbifoldization" formula for the elliptic genus of orbifold SCFTs.

Umbral moonshine and the K3 elliptic genus

We can also define a twisted version of this relation:

$$Z_g^X(\tau, z) = Z_g^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \, \phi_g^X(\tau, \frac{z + a\tau + b}{m}).$$

Umbral moonshine and the K3 elliptic genus

Consequences of this relation:
Consider [g] which obey a geometric condition, $g \in G^X$ has at least 5 orbits and one fixed point on the 24-dimensional representation. For these elements,

- Whenever $g_1 \in G^{X_1}$ and $g_2 \in G^{X_2}$ both satisfy the geometric condition and have the same 24-dimensional cycle shape $\Pi_{g_1}^{X_1} = \Pi_{g_2}^{X_2}$, we obtain
$$Z_{g_1}^{X_1} = Z_{g_2}^{X_2} \tag{3.1}$$
- The result coincides with the geometrically twined elliptic genus for a $K3$ admitting $\langle g \rangle$-symmetry
$$Z_g^X = \mathbf{EG}_g(K3) \tag{3.2}$$
whose induced action on 24-dimensional representation is isomorphic to that of $g \in G^X$

Umbral moonshine and the K3 elliptic genus

Consequences of this relation:
Consider [g] which obey a geometric condition, $g \in G^X$ has at least 5 orbits and one fixed point on the 24-dimensional representation. For these elements,

- Whenever $g_1 \in G^{X_1}$ and $g_2 \in G^{X_2}$ both satisfy the geometric condition and have the same 24-dimensional cycle shape $\Pi_{g_1}^{X_1} = \Pi_{g_2}^{X_2}$, we obtain
$$Z_{g_1}^{X_1} = Z_{g_2}^{X_2} \tag{3.1}$$
- The result coincides with the geometrically twined elliptic genus for a $K3$ admitting $\langle g \rangle$-symmetry
$$Z_g^X = \mathbf{EG}_g(K3) \tag{3.2}$$
whose induced action on 24-dimensional representation is isomorphic to that of $g \in G^X$

Umbral moonshine and the K3 elliptic genus

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$4A$	4^6	H_{4C}
$2B$	$1^8 2^8$	H_{2A}
$2C$	2^{12}	H_{2B}
$3A$	$1^6 3^6$	H_{3A}
$6A$	$2^3 6^3$	$\tilde{T}_{6A}^X = 3\Lambda_2 + 2\Lambda_3 - \Lambda_4 - 3\Lambda_6 + \Lambda_{12}$
$3B$	3^8	$\tilde{T}_{3B}^X = 2(-4\Lambda_3 + \Lambda_9 - (1)^6/(3)^2)$
$6B$	6^4	$\tilde{T}_{6B}^X = 2\frac{(1)^5(3)}{(2)(6)}$
$4B$	$2^4 4^4$	H_{4A}
$4C$	$1^4 2^2 4^4$	H_{4B}
$5A$	$1^4 5^4$	H_{5A}
$10A$	$2^2 10^2$	H_{10A}
$12A$	12^2	$\tilde{T}_{12A}^X = 2\frac{(1)(2)^5(3)}{(4)^2(6)}$
$6C$	$1^2 2^2 3^2 6^2$	H_{6A}
...	...	...

Geometric interpretation

Main results of Nikulin:

- Every $K3$ surface admits a marking by (at least) one of the 23 Niemeier lattices
- For every L^X with the exception of $X = A_{24}$ and $X = A_{12}^2$, there exists a $K3$ surface that can only be marked using L^X and not by any other Niemeier lattice (the exceptions are a conjecture)
- For any L^X, any primitive sublattice of L^X which can be primitively embedded into $\Gamma_{3,19}(-1)$ arises from the Picard lattice $\mathrm{Pic}(M)$ of a certain $K3$ surface M

Geometric interpretation

Main results of Nikulin:

- Every $K3$ surface admits a marking by (at least) one of the 23 Niemeier lattices
- For every L^X with the exception of $X = A_{24}$ and $X = A_{12}^2$, there exists a $K3$ surface that can only be marked using L^X and not by any other Niemeier lattice (the exceptions are a conjecture)
- For any L^X, any primitive sublattice of L^X which can be primitively embedded into $\Gamma_{3,19}(-1)$ arises from the Picard lattice $\mathrm{Pic}(M)$ of a certain $K3$ surface M

Geometric interpretation

Applications:

- For the generic cases, a $K3$ surface M that can be marked by L^X has the configuration of all smooth rational curves given by $X \cap S_M$. In particular, if one thinks of the rational curves as arising from the minimal resolutions of the $\mathbb{C}^2/G$ singularities, then the singularities have to be given by a sub-diagram of the Dynkin diagram corresponding to X.
- If M is a generic $K3$ surface admits a marking by L^X, then the finite symplectic automorphism group G_M of M is a subgroup of G^X.

Comments

We find some evidence for the relation between groups for index m umbral moonshine and $\mathbb{Z}_m$ orbifold K3 sigma models. [WIP with Francesca Ferrari and Natalie Paquette] In particular, we find in a $\mathbb{Z}_3$ orbifold model an order 11 symmetry which comes from the $m = 3$ decomposition of $\mathbf{EG}(\tau, z; K3)$, and is a different function from the order 11 twining of $\mathbf{EG}(\tau, z; K3)$ coming from $M_{24}(m = 2)$.

Comments

We find some evidence for the relation between groups for index m umbral moonshine and $\mathbb{Z}_m$ orbifold K3 sigma models. [WIP with Francesca Ferrari and Natalie Paquette] In particular, we find in a $\mathbb{Z}_3$ orbifold model an order 11 symmetry which comes from the $m = 3$ decomposition of $\mathbf{EG}(\tau, z; K3)$, and is a different function from the order 11 twining of $\mathbf{EG}(\tau, z; K3)$ coming from $M_{24}(m = 2)$.

Comments

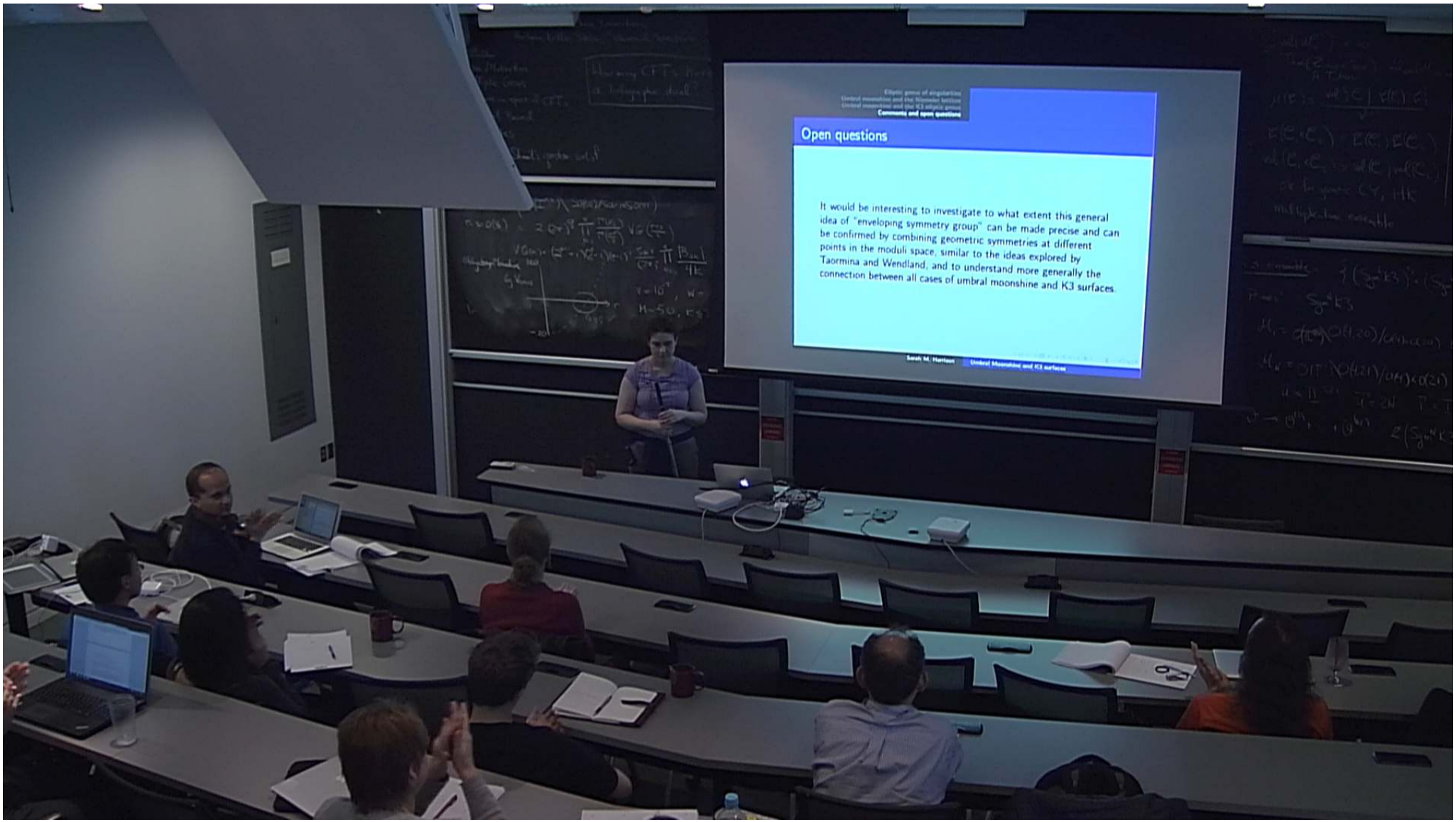
There is another rewriting of the formula

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \, \phi^X(\tau, \frac{z + a\tau + b}{m})$$

which is mathematically equivalent:

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \phi_{-2,1}(\tau, z) \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X(\tau) \, S_{m,r}(\tau)$$

Notably, the piece coming from the umbral mock modular forms, appears in other contexts, e.g. the spacetime BPS index χ_2 computed by Harvey and Murthy coming from NS5-branes wrapped on K3.


Open questions
It would be interesting to investigate to what extent this general idea of "enveloping symmetry group" can be made precise and can be confirmed by combining geometric symmetries at different points in the moduli space, similar to the ideas explored by Taormina and Wendland, and to understand more generally the connection between all cases of umbral moonshine and K3 surfaces.

UMBRAL MOONSHINE

MICHAEL GRIFIN

Monstrous Moonshine asserts the existence of a natural, infinite-dimensional, graded representation of the Monster group, whose traces at the various conjugacy classes give rise to the Fourier expansions of important modular functions. Building on Borcherds' proof of the conjecture, Witten conjectured deep connections between Monstrous Moonshine and quantum gravity in three dimensions. In 2013 Gannon proved a new Moonshine for the Mathieu group M_{24}. Cheng, Duncan, and Harvey placed this phenomenon within a larger conjectural framework of Moonshines connecting the automorphism groups of each of the 24 Niemeier lattices to the Fourier coefficients of mock modular forms explicitly constructed using indefinite theta functions. This "Umbral Moonshine" conjecture was verified in joint work with John Duncan and Ken Ono.

arXiv:1208.6254v3 [math.RT] 20 Aug 2016

GENERALIZED MOONSHINE IV: MONSTROUS LIE ALGEBRAS

SCOTT HUAI-LEI CARNAHAN

Abstract. For each element of the Fischer-Griess Monster sporadic simple group, we construct an infinite dimensional Lie algebra equipped with a projective action of the centralizer of that element. Our construction is given by a string-theoretic "add a spacetime torus and quantize" functor applied to an abelian intertwining algebra that is formed from a family of twisted modules of the Monster vertex operator algebra. We prove that for all Fricke elements in the Monster, the characters of centralizers acting on the corresponding irreducible twisted modules are Hauptmoduln. From these results, we resolve Norton's Generalized Moonshine Conjecture.

Contents

1. Introduction 1
1.1. Generalized Moonshine 1
1.2. Overview 6
1.3. Acknowledgments 7
2. Vertex algebraic input 7
2.1. Vertex algebras and twisted modules 7
2.2. Abelian intertwining algebras 8
2.3. Unrolling 12
2.4. Adding a spacetime torus 16
2.5. Projective action of centralizer 17
3. Lie algebras 19
3.1. Virasoro preliminaries 19
3.2. Old covariant quantization 22
3.3. BRST cohomology 24
3.4. Borcherds-Kac-Moody property 27
4. Comparison theorems 30
4.1. Lie algebra comparison 30
4.2. Main theorem 31
References 32

1. Introduction

For each element g of the Monster, we construct an infinite dimensional Lie algebra equipped with a canonical projective action of the centralizer of g. For those g satisfying the Fricke property, we show that the Lie algebras are Borcherds-Kac-Moody with a real simple root, and we use the resulting twisted denominator identities to prove Norton's Generalized Moonshine Conjecture for the class of irreducible twisted $V^\natural$-modules.

1.1. **Generalized Moonshine.** We give a brief reminder of monstrous moonshine before describing its generalization. In the late 1970s, McKay observed that the q^1-term in the q-expansion of the normalized modular function $J = q^{-1} + 196884q + 21493760q^2 + \cdots$ is one more than the dimension of the smallest nontrivial complex representation of the Fischer-Griess Monster simple

group $\mathbb{M}$, which is a sporadic simple group of order about 8×10^{53}. This observation suggested a relationship between the J-function and representations of the Monster that was further reinforced by additional computations by McKay and Thompson. Conway and Norton then did a massive computation with characters, producing evidence that there is an infinite dimensional graded representation $V = \bigoplus_{n=0}^{\infty} V_n$ of the Monster, such that the graded trace $T_g(\tau) = \sum_{n=0}^{\infty} \mathrm{Tr}(g|V_n)q^{n-1}$ of any element is the Fourier expansion (under $q = e^{2\pi i\tau}$) of a holomorphic function on the complex upper half-plane $\mathfrak{H}$ that is moreover a Hauptmodul [Conway-Norton-1979]. That is, for each $g \in \mathbb{M}$, there is a group $\Gamma_g \subset SL_2(\mathbb{R})$ under which $T_g(\tau)$ is invariant, such that the quotient curve $X_g = \mathfrak{H}/\Gamma_g$ is genus zero, and T_g generates the field of meromorphic functions on X_g. A candidate representation $V^\natural$ was constructed by Frenkel, Lepowsky, and Meurman, who showed that it is a vertex operator algebra whose automorphism group is precisely $\mathbb{M}$, and its graded dimension T_1 is J [Frenkel, et al, 1988]. Borcherds then proved Conway and Norton's conjecture for $V^\natural$ [Borcherds-1992].

In the Conway-Norton paper, the authors suggest that there may be moonshine-like phenomena for groups other than $\mathbb{M}$, and shortly afterward, Queen did some computations that yielded concrete evidence that one can obtain Hauptmoduln from characters of a variety of large finite groups [Queen-1981]. In particular, the low-order coefficients of many such modular functions can be assembled from representations of sporadic simple groups other than the Monster.

For illumination, we consider a level 2 example. The irreducible representations of the Baby Monster sporadic simple group $\mathbb{B}$ have dimension $1, 4371, 96255, \ldots$, while the power series $q^{-1} + 4372q + 96256q^2 + \cdots$ is the q-expansion of $\frac{\Delta(\tau)}{\Delta(2\tau)} + \frac{\Delta(2\tau)}{\Delta(\tau)} + 24$, which is a Hauptmodul for $\Gamma_0(2)^+$. It is difficult to resist the temptation to decompose the coefficients as small combinations of dimensions of irreducible representations: $4372 = 4371 + 1$ and $96256 = 96255 + 1$. Furthermore, we encounter promising evidence in the form of the character of an element in conjugacy class 2A: the first term yields $q^{-1} - 492q + O(q^2)$, and there is a well-known modular function $\sqrt{j(2\tau) - 1728} = q^{-1} - 492q - 22590q^3 - \cdots$ that is a Hauptmodul for $\Gamma_0(2|2)$. However, a complication arises: the q^2 term in our character does not vanish, since the trace of 2A on $\underline{96255}$ is 4863, not -1. Despite the initial promising evidence, we find that the Baby Monster group does not yield moonshine-like behavior. However, there is a central extension $2.\mathbb{B}$, with an irreducible representation of dimension 96256, where the trace of 2A is zero. By replacing the Baby Monster with its central extension, we eliminate this obstruction to assembling the characters into modular functions. Similar behavior appears when considering the sporadic group Fi_{24}, where one needs to pass to a triple cover to get a representation of dimension 783, yielding a level 3 modular function $q^{-1} + 783q + \cdots$. What these examples have in common is that both $2.\mathbb{B}$ and $3.Fi_{24}$ are centralizers of elements of the Monster.

Norton employed these computations and many of his own to formulate the following conjecture:

Conjecture. *(Generalized Moonshine [Norton-1987], revised in [Norton-2001]) There exists a rule that assigns to each element g of the Monster simple group $\mathbb{M}$ a graded projective representation $V(g) = \bigoplus_{n\in\mathbb{Q}} V(g)_n$ of the centralizer $C_{\mathbb{M}}(g)$, and to each pair (g, h) of commuting elements of $\mathbb{M}$ a holomorphic function $Z(g, h, \tau)$ on the complex upper half-plane $\mathfrak{H}$, satisfying the following conditions:*

(1) *There is some lift $\tilde{h}$ of h to a linear transformation on $V(g)$ such that*

$$Z(g, h, \tau) = \sum_{n\in\mathbb{Q}} Tr(\tilde{h}|V(g)_n)q^{n-1}.$$

(2) *$Z(g, h, \tau)$ is invariant up to constant multiplication under simultaneous conjugation of the pair (g, h) in $\mathbb{M}$.*
(3) *$Z(g, h, \tau)$ is either constant, or a Hauptmodul for some genus zero congruence group.*
(4) *For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $Z(g^a h^c, g^b h^d, \tau)$ is proportional to $Z(g, h, \frac{a\tau+b}{c\tau+d})$.*

(5) $Z(g,h,\tau) = J(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = q^{-1} + 196884q + \cdots$ *if and only if* $g = h = 1 \in \mathbb{M}$.

This is a generalization of the Monstrous Moonshine Conjecture, in the sense that fixing $g = 1$ yields an assertion that one has a graded representation of $\mathbb{M}$ whose characters form Hauptmoduln. It also subsumes the example observation above, in the sense that if $q^{-1}+4372q+96256q^2+\cdots$ is the Moonshine character of an element g in class 2A of the Monster, then $Z(g,1,\tau)$ is proportional to $q^{-1/2}+4372q^{1/2}+96256q+\cdots$, and these coefficients decompose into well-behaved representations of $2.\mathbb{B}$, the centralizer of an element in class 2A of the Monster.

In contrast to the situation with the original Monstrous Moonshine conjecture, we do not have a list of conjugacy classes of commuting pairs of elements in $\mathbb{M}$, or a list of character tables of central extensions of centralizers of elements at our disposal. Conversations with group theorists suggest that these explicit data are still beyond current technology, so at the moment, we cannot solve the conjecture by computational brute force. Instead, one needs to find an interpretation of the spaces $V(g)$ and the functions Z, and one must seek general methods for proving that traces like Z yield Hauptmoduln, and for showing that traces are $SL_2(\mathbb{Z})$-compatible.

Shortly after Norton published this conjecture, Dixon, Ginsparg, and Harvey found a physical interpretation for claims 1,2,4, and 5: For each $g \in \mathbb{M}$, the space $V(g)$ is the "Hilbert space twisted by g" in a conformal field theory with $\mathbb{M}$ symmetry, and $Z(g,h)$ is a genus one partition function for a torus twisted in "space" by g and in "time" by h [Dixon-Ginsparg-Harvey-1988]. In algebraic language, we would say that for each $g \in \mathbb{M}$, we have a distinguished irreducible g-twisted module $V^\natural(g)$ of the Monster vertex operator algebra, and one may hope that such objects exist and are unique up to isomorphism.

The first major mathematical breakthrough toward a solution of this conjecture then came in [Dong-Li-Mason-2000], with the following results:

(1) Theorem 10.3 asserts that for any $g \in \mathbb{M}$, there exists an irreducible g-twisted $V^\natural$-module, and it is unique up to isomorphism. By Schur's Lemma, such a module $V^\natural(g)$ admits a canonical projective action of $C_{\mathbb{M}}(g)$. By general nonsense about conjugation, this resolves Claim 2, at least on the level of formal power series.
(2) Theorem 8.1 implies the series $\sum_{n\in\mathbb{Q}} Tr(\tilde{h}|V(g)_n)q^{n-1}$ converges absolutely uniformly on compact subsets of the complex upper half-plane to a holomorphic function, for any lift $\tilde{h}$ of an element $h \in C_{\mathbb{M}}(g)$. We can therefore define $Z(g,h,\tau)$ to be this function for some choice of $\tilde{h}$. This resolves Claim 1, and translates the power-series version of Claim 2 to a holomorphic function version.
(3) Theorem 10.3 also contains an assertion that implies $Z(g,1,\tau)$ is a constant multiple of $Z(1,g,-1/\tau)$. This resolves Claim 5, essentially by the fact that $Z(1,g,\tau)$ has level greater than 1 for any $g \neq 1$.
(4) Theorem 10.1 reduces Claim 4 to an conjecture about g-rationality of $V^\natural$, i.e., that any g-twisted $V^\natural$-module is isomorphic to a direct sum of copies of $V^\natural(g)$.

Claim 4 was recently resolved: By Corollary 5.26 of [Carnahan-Miyamoto-2016], g-rationality holds for any simple regular non-negatively graded vertex operator algebra with an invariant bilinear form, and $V^\natural$ satisfies these properties. The final remaining claim is Claim 3, the Hauptmodul-or-constant property, and this is the subject of this paper.

Under the current interpretation, Claims 1,2,4, and 5 are in fact general claims about non-negatively graded holomorphic C_2-cofinite vertex operator algebras and their automorphisms. That is, these claims hold true for an infinite class of examples, and don't have much to do with the Monster, except for the fact that $V^\natural$ is a particularly well-behaved vertex operator algebra. Claim 3 has a more exceptional, moonshine-like quality, since there are only finitely many Hauptmoduln, and the restriction on level gives an indication of smallness. For this reason, a solution cannot be expected to appear directly from general facts about vertex operator algebras or conformal field

theory. Perhaps unsurprisingly, this is the only claim that was not given a physical interpretation by Dixon-Ginsparg-Harvey. Proving such an exceptional claim seems to require a specialized strategy, and this is where we come to our second major breakthrough.

In an unpublished preprint [Höhn-2003b], Höhn resolved the Hauptmodul claim for the case where g is an element in the conjugacy class 2A. The resolution of this case is by itself a substantial advance, but more importantly, Höhn's preprint contained a general strategy for proving that Generalized Moonshine trace functions are Hauptmoduln. Alternatively, one may say that Höhn worked out precisely the ingredients needed to extend Borcherds's proof of the Monstrous Moonshine Conjecture to the general case in question. We shall call this the Borcherds-Höhn program:

(1) Build an abelian intertwining algebra structure on the direct sum $\bigoplus_{i=0}^{|g|-1} V^\natural(g^i)$.
(2) Add a spacetime torus: Make a conformal vertex algebra of central charge 26 by taking a graded tensor product with the abelian intertwining algebra of a suitable hyperbolic lattice.
(3) Quantize: Apply a bosonic string quantization functor to the conformal vertex algebra. This yields a Lie algebra $\mathfrak{m}_g$ equipped with a canonical projective action of $C_{\mathbb{M}}(g)$ by automorphisms.
(4) Generate a Lie algebra L_g whose denominator identity is some automorphic infinite product. This Lie algebra has a "nice shape", in the sense that its simple roots and homology are well-controlled.
(5) Comparison: Show that the Lie algebra $\mathfrak{m}_g$ is isomorphic to L_g. By *transport de structure*, we obtain a Lie algebra with both a group action and nice shape.
(6) Hauptmodul conclusion: Use the twisted denominator identity to produce recursion relations on the characters that are strong enough to conclude that the characters are Hauptmoduln.

When $g = 1$, this program gives an outline of Borcherds's proof [Borcherds-1992], with the first step given by Frenkel-Lepowsky-Meurman's construction of $V^\natural$. When g is an element of type 2A, this program gives an outline of Höhn's proof of Generalized Moonshine for $2.\mathbb{B}$ [Höhn-2003b]. In both of those cases, the last step was settled by combining recursion relations with an explicit computation of the first few terms, to match the proposed functions with the actual characters. In general, we do not have character tables for central extensions of centralizers in the Monster, so such explicit computations are out of the question. Instead, we found a general Hauptmodul criterion involving equivariant Hecke operators in [Carnahan-2010], generalizing the results of [Cummins-Gannon-1997]. While Cummins-Gannon could be applied to remove the computations from Borcherds's proof of the Monstrous Moonshine conjecture [Borcherds-1992], our methods could be used to eliminate explicit computations for all cases we expected to encounter, including Höhn's proof. This settles the last step, conditional on the constructions in the previous steps.

The step with L_g was resolved in [Carnahan-2012], i.e., we constructed automorphic infinite products and candidate Lie algebras for each element of the Monster. We also defined a precise condition, called "Fricke-compatibility", on a projective action of $C_{\mathbb{M}}(g)$ on L_g that suffices for showing that the Hauptmodul condition holds for g of Fricke type. The question of producing such an action is essentially equivalent to the question of constructing $\mathfrak{m}_g$ and showing that $\mathfrak{m}_g \cong L_g$, and this is precisely what we resolve in this paper.

The first step is the most technically challenging. For example, in the case $g = 1$, this is the construction of the moonshine module $V^\natural$, which takes up the whole book [Frenkel, et al, 1988]. Before my recent project with Miyamoto culminating in [Carnahan-Miyamoto-2016], I had formed an elaborate plan to construct suitable abelian intertwining algebras using the theory of conformal blocks on twisted nodal curves together with some work eliminating homological obstructions. Some of this theory was sketched in my dissertation [Carnahan-2007], and the third paper in this series

[Carnahan>2016] is intended to present a full development. I believe the methods are still useful in a general mathematical sense, and the first installment of results should appear in the near future.

We therefore come to the third major breakthrough, which rendered my plan superfluous for the purpose of proving Generalized Moonshine. In [van Ekeren, et al, 2015], the authors showed that there is an abelian intertwining algebra structure on a sum of twisted modules of a well-behaved vertex operator algebra, by an ingenious application of the Verlinde formula [Huang-2008] together with the fixed-point regularity results of [Carnahan-Miyamoto-2016]. Furthermore, they determined the precise characters of modules of the fixed point vertex algebra $(V^\natural)^{\langle g\rangle}$. Thus, their results essentially establish the first step, and go a long way toward the comparison step.

We now describe how the results of this paper fit in. In order to construct $\mathfrak{m}_g$, we need a functorial way to pass from abelian intertwining algebras to Lie algebras. We will employ an "add a spacetime torus and quantize" functor that has existed in some form for over 40 years, but we supply some necessary details that have not appeared in the literature. Adding a torus is just a degree-wise tensor product with the abelian intertwining algebra of a well-chosen hyperbolic rational lattice. The quantization functor takes conformal vertex algebras of central charge 26 to Lie algebras. We exploit the "cancellation of oscillators" phenomenon to describe the homogeneous spaces of the Lie algebras explicitly in terms of the input object. This theory was one of the main engines behind Frenkel's bound on root multiplicities of the Lie algebra whose simple roots form the Leech Lattice as its Dynkin diagram [Frenkel-1985], Borcherds's proof of the Monstrous Moonshine conjecture, and Hoehn's proof of Generalized Moonshine in the 2A case.

Once $\mathfrak{m}_g$ is constructed, we show that it is a Borcherds-Kac-Moody algebra by introducing a new criterion that is particularly well-suited to considering the output of the quantization functor. Then, the homogeneous subspaces become root spaces, and they are canonically identified with certain subspaces of twisted modules as representations of some central extension of $C_{\mathbb{M}}(g)$. The comparison isomorphism $\mathfrak{m}_g \cong L_g$ then follows from a comparison of root multiplicities, and this is made possible by the precise character work in [van Ekeren, et al, 2015].

The reader may be interested to know just how much information about $V^\natural$ and the Monster was necessary for our proof to work. In fact, we need almost no fine structure at all. By examining the proof in this paper, together with the work on which it depends, we see that in fact we have the following result:

Theorem 1.1.1. *Let V be a holomorphic C_2-cofinite vertex operator algebra of central charge 24 whose character is $J(\tau)$, and let G be a finite order group of automorphisms of V. Then any rule assigning to each $g \in G$ the irreducible g-twisted module $V(g)$, and to each commuting pair (g,h) the function $Z(g,h;\tau) = \mathrm{Tr}(\tilde{h}q^{L_0-1}|V(g))$ for some lift $\tilde{h} \in \widetilde{C_G(g)}$ of h to a linear transformation on $V(g)$, satisfies the conditions of Norton's conjecture, with G in place of $\mathbb{M}$.*

This is quite similar to the situation with Borcherds's proof of the Monstrous moonshine conjectures - one only needs central charge 24, an invariant bilinear form, and the character J to get completely replicable functions (which were later shown to be either highly degenerate or Hauptmoduln in [Cummins-Gannon-1997]). However, without the construction of $V^\natural$ in [Frenkel, et al, 1988], this result loses most of its value, since the connection to the Monster is lost. As it happens, Frenkel-Lepowsky-Meurman conjectured in section 3 of the introduction of *loc. cit.* that any V satisfying (more or less) the hypotheses in this theorem is isomorphic to $V^\natural$, so if we assume a suitable variant of the FLM Uniqueness Conjecture, then the statement given here becomes essentially the same as our Main Theorem (Theorem 4.2.2).

We note that there is previous work on Generalized Moonshine that establishes some Hautpmodul results for twisted modules - perhaps the most advanced was [Ivanov-Tuite-2002]. However, these results were conditional on both the Uniqueness Conjecture and the construction of holomorphic orbifolds established in [van Ekeren, et al, 2015].

We do not consider this the end of the Generalized Moonshine story. There are conjectural refinements of our main theorem, and intriguing conjectural relations between Generalized Moonshine and other fields of mathematics and physics. We focus on three further questions:

(1) Norton has suggested stronger versions of this conjecture based on computational evidence [Norton-2001]. In particular, he asserts that the constant ambiguity can be reduced to a root of unity (in particular a 24th root), and that for any (g, h), there is some rational k such that each coefficient of $Z(g, h, \tau + k)$ generates a field extension of $\mathbb{Q}$ with Galois group of exponent 2. Terry Gannon has proposed a further refinement of the root-of-unity restriction, asserting that the proportionality constants are completely controlled by a 3-cocycle in the cohomology of $\mathbb{M}$, with coefficients in $U(1)$. The precise conjectural relationship between a suitable 3-cocycle and the proportionality constants is detailed in section 2 of [Gaberdiel-Persson-Volpato-2013]. Such a cocycle is expected from orbifold conformal field theory considerations [Dijkgraaf-Pasquier-Roche-1990], and an analogue of this refinement for M_{24} moonshine was established in [Gaberdiel, et al, 2012]. We note that this refinement is a general statement about twistings of holomorphic C_2-cofinite vertex operator algebras, and does not have a substantial connection to the Hauptmodul problem we solve in this paper.

(2) Recall that almost all of the claims in the Generalized Moonshine conjecture were given physical interpretations in [Dixon-Ginsparg-Harvey-1988], and the remaining claim concerns Hauptmoduln. As it happens, there is a conjectural connection between Hauptmoduln and physics through Rademacher sums, as suggested in [Duncan-Frenkel-2011]. The general picture is that AdS/CFT correspondence seems to relate 2+1-dimensional gravity in the extreme quantum regime with conformal field theories satisfying smallness conditions. In particular, Duncan and Frenkel propose the existence of a class of second-quantized twisted chiral quantum gravity theories that can be manipulated to produce the monstrous Lie algebras $\mathfrak{m}_g$ constructed in this paper.

(3) There are intriguing conjectural connections between moonshine and equivariant elliptic cohomology. The folklore idea is that elliptic cohomology is somehow controlled by, or is a shadow of, conformal field theory. Concretely, an element of the elliptic cohomology of the classifying space of a finite group G is a datum that includes a family of modular forms parametrized by a pair of commuting elements of G, satisfying an $SL_2(\mathbb{Z})$-compatibility condition. It was noted quite early (and I am not sure who was first - perhaps Mason) that the weight zero version of this condition is precisely given as claim 4 in the Generalized Moonshine conjecture. While elliptic cohomology does not play a direct role in our proof of the Generalized Moonshine conjecture, it was essential in an inspirational sense. Indeed, equivariant Hecke operators, seen as power operations in equivariant elliptic cohomology, were the key to the formulation of a good Hauptmodul criterion in [Carnahan-2010]. I believe the precise formula was first written in [Ganter-2009], where one may also find a topological interpretation of replicability. This piece of history, together with hints relating the formation of $\mathfrak{m}_g$ to exponential operations, suggests that the Hauptmodul claim may be amenable to a topological interpretation.

We would be interested in seeing any developments toward the refined versions of this conjecture, and the establishment of stronger connections to other mathematical disciplines.

1.2. **Overview.** In section 2, we introduce some vertex algebraic constructions, such as abelian intertwining algebras and the "add a torus" functors. The main purpose of this section is to take the results of [van Ekeren, et al, 2015], and transmute their construction into a form that is amenable to quantization.

In section 3, we construct Lie algebras, using a generalization of Borcherds's method for constructing the Monster Lie algebra. Due to the functorial nature of the construction, we obtain

canonical group actions by homogeneous automorphisms. We show that for Fricke elements, the resulting Lie algebras are necessarily Borcherds-Kac-Moody, and we show that the root multiplicities match the dimensions of certain eigenspaces in twisted modules.

In section 4, we compare root multiplicities. For Fricke elements, this then yields isomorphisms with the Lie algebras constructed in [Carnahan-2012], and we find that characters of centralizers are Hauptmoduln. This then implies the full conjecture.

1.3. **Acknowledgments.** The author would like to thank Richard Borcherds, Jethro van Ekeren, Jeff Harvey, Yoshitake Hashimoto, Gerald Höhn, Satoshi Kondo, Gregory Moore, Luboš Motl, Yu Nakayama, Simon Norton, and Nils Scheithauer for helpful conversations and advice. Kazuya Kawasetsu, Sven Möller, and Hiromichi Yamada offered suggestions and corrections to an earlier version of this article.

This research was partly supported by NSF grant DMS-0354321, JSPS Kakenhi Grant-in-Aid for Young Scientists (B) 24740005, the Program to Disseminate Tenure Tracking System, MEXT, Japan, and the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

2. Vertex algebraic input

Algebraic structures related to vertex algebras are essential to moonshine, but we will not need to spend much time on fine details, since most of the technical aspects have been worked out. We will review some basic facts, then pursue some consequences of the work of [van Ekeren, et al, 2015].

We will use the notation $e(s)$ to denote the normalized exponential $e^{2\pi i s}$.

2.1. Vertex algebras and twisted modules.

Definition 2.1.1. A vertex algebra over $\mathcal{C}$ is a complex vector space V equipped with a distinguished vector $\mathbf{1} \in V$, a distinguished linear transformation $T : V \to V$, and a left multiplication map $Y : V \to (\operatorname{End} V)[[z, z^{-1}]]$, written $Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ satisfying the following conditions:

(1) $Y(\mathbf{1}, z) = id_V = id_V z^0$, and $Y(a, z)\mathbf{1} \in a + zV[[z]]$.
(2) For any $a, b \in V$, $Y(a, z)b \in V((z))$, i.e., $a_n b = 0$ for n sufficiently large.
(3) $[T, Y(a, z)] = \frac{d}{dz}Y(a, z)$
(4) The Jacobi identity: for any $a, b \in V$,

$$x^{-1}\delta\left(\frac{y-z}{x}\right)Y(a,y)Y(b,z) - x^{-1}\delta\left(\frac{z-y}{-x}\right)Y(b,z)Y(a,y) = z^{-1}\delta\left(\frac{y-x}{z}\right)Y(Y(a,x)b,z)$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ and $\delta(\frac{y-z}{x})$ is expanded as a formal power series in z.

An automorphism of a vertex algebra V is a linear transformation $g : V \to V$ that fixes $\mathbf{1}$, and satisfies $Y(ga, z)gb = gY(a, z)b$ for all $a, b \in V$.

Definition 2.1.2. A conformal vertex algebra of central charge $c \in \mathbb{C}$ is a vertex algebra equipped with a distinguished nonzero vector ω, satisfying the following conditions:

(1) If we write $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, then the coefficients $L_n \in \operatorname{End} V$ satisfy the Virasoro relations:
$$[L_m, L_n] = (m-n)L_{m+n} + c\frac{m^3-m}{12}\delta_{m+n,0}id_V$$
(2) L_0 acts semisimply on V, with integer eigenvalues.
(3) $L_{-1} = T$.

An automorphism of a conformal vertex algebra is an automorphism of the underlying vertex algebra that fixes ω. A vertex operator algebra is a conformal vertex algebra for which the eigenvalues of L_0 are bounded below and have finite multiplicity.

We will only consider one vertex operator algebra in this paper, namely the Monster vertex operator algebra $V^\natural$ that was constructed in [Frenkel, et al, 1988].

Vertex algebras have a more complicated representation theory than ordinary algebras, because modules may be twisted by automorphisms of the vertex algebra.

Definition 2.1.3. Let V be a vertex algebra over $\mathbb{C}$, and let g be an automorphism of order n. Then a g-twisted V-module is a complex vector space M equipped with an action map $Y^M : V \to (\operatorname{End} M)[[z^{1/n}, z^{-1/n}]]$, written $Y^M(a,z) = \sum_{k\in\frac{1}{n}\mathbb{Z}} a_k z^{-k-1}$ satisfying the following conditions:

(1) $Y^M(\mathbf{1}, z) = id_M$
(2) For any $a \in V$, $u \in M$, $Y^M(a,z)u \in M((z^{1/n}))$, i.e., $a_k u = 0$ if k is sufficiently large.
(3) If $a \in V$ satisfies $ga = e(k/n)a$ for some $k \in \mathbb{Z}$, then $Y^M(a,z) \in z^{k/n}(\operatorname{End} M)[[z, z^{-1}]]$.
(4) The twisted Jacobi identity holds: if $ga = e(k/n)a$, then for any $b \in V$,

$$x^{-1}\delta\left(\frac{y-z}{x}\right)Y^M(a,y)Y^M(b,z) - x^{-1}\delta\left(\frac{z-y}{-x}\right)Y^M(b,z)Y^M(a,y) =$$
$$= z^{-1}\left(\frac{y-x}{z}\right)^{-k/n}\delta\left(\frac{y-x}{z}\right)Y^M(Y(a,x)b,z)$$

If V is a vertex operator algebra, these conditions define the notion of "weak g-twisted V-module". An "admissible g-twisted V-module" is a weak g-twisted V-module that admits a $\frac{1}{n}\mathbb{Z}_{\geq 0}$-grading that is compatible with the L_0-grading on V. An "ordinary g-twisted V-module" is an admissible g-twisted V-module for which L_0 acts semisimply, with finite dimensional eigenspaces, and eigenvalues that are bounded below in each coset of $\mathbb{Z}$ in $\mathbb{C}$. When $g = 1$, we replace "g-twisted V-module" with "V-module".

Definition 2.1.4. Here are a few more technical conditions about vertex algebras. We won't use them in any specific way, but they appear as conditions in the theorems we need.

(1) A vertex algebra V is C_2-cofinite if the subspace $C_2(V)$ spanned by $\{a_2 b | a, b \in V\}$ satisfies the property that $V/C_2(V)$ is finite dimensional.
(2) A vertex operator algebra V is g-rational if all admissible g-twisted V-modules are direct sums of irreducible ordinary g-twisted V-modules.
(3) A vertex operator algebra V is holomorphic if all admissible V-modules are isomorphic to direct sums of V.
(4) A vertex operator algebra V is of CFT type if L_0 has only non-negative eigenvalues, and the kernel of L_0 is spanned by $\mathbf{1}$.

Remark 2.1.5. Theorem 10.3 in [Dong-Li-Mason-2000] asserts that if a vertex operator algebra V is C_2-cofinite and holomorphic, then there exists a unique irreducible g-twisted V-module, which we will write as $V(g)$. For any automorphism h of V, one may change the action of V on the underlying vector space of $V(g)$ to produce a module isomorphic to $V(h^{-1}gh)$. By Schur's Lemma, we obtain a canonical projective action of $C_{\operatorname{Aut} V}(g)$ on the underlying vector space of $V(g)$, which is compatible with the action on V (but it is not an action by twisted module automorphisms, since those are just scalars).

Definition 2.1.6. We write $\operatorname{Aut} V(g)$ to denote the group of vector space automorphisms of $V(g)$ that are compatible with automorphisms of V - it is a central extension of $C_{\operatorname{Aut} V}(g)$ by $\mathbb{C}^\times$.

2.2. **Abelian intertwining algebras.** Dong and Lepowsky introduced abelian intertwining algebras essentially as vertex algebras with some monodromy obstructing commutativity. We describe them briefly, and we describe the theorem we need from [van Ekeren, et al, 2015].

Definition 2.2.1. ([MacLane-1952]) Let A be an abelian group. The Eilenberg-Mac Lane cochain complex with coefficients in $\mathbb{C}^\times$ has the following description in low degree:

$$K^1 \xrightarrow{d^1} K^2 = \{f : A \times A \to \mathbb{C}^\times\} \xrightarrow{d^2} K^3 = \{(F, \Omega) : A^{\oplus 3} \times A^{\oplus 2} \to \mathbb{C}^\times\} \xrightarrow{d^3} \cdots$$

where:

(1) $K^1 = \{\phi : A \to \mathbb{C}^\times\}$ and $K^2 = \{f : A \times A \to \mathbb{C}^\times\}$ are groups of set-theoretic maps.
(2) $K^3 = \{(F, \Omega) : A^{\oplus 3} \times A^{\oplus 2} \to \mathbb{C}^\times\}$ is a group of pairs of set-theoretic maps.
(3) The map $d^1 : K^1 \to K^2$ is given by the usual group cohomology coboundary $\phi \mapsto d^1\phi$, defined by $d^1\phi(i, j) = \phi(j) - \phi(i + j) + \phi(i)$.
(4) The map $d^2 : K^2 \to K^3$ is given by the group cohomology coboundary $d^2 f(i, j, k) = f(j, k) - f(i + j, k) + f(i, j + k) - f(i, j)$ together with the antisymmetrizer: $(i, j) \mapsto \frac{f(i,j)}{f(j,i)}$.
(5) The map $d^3 : K^3 \to K^4$ vanishes if and only if the following conditions are satisfied:
   (a) $F(i, j, k)F(i, j + k, \ell)F(j, k, \ell) = F(i + j, k, \ell)F(i, j, k + \ell)$ for all $i, j, k, \ell \in A$
   (b) $F(i, j, k)^{-1}\Omega(i, j + k)F(j, k, i)^{-1} = \Omega(i, j)F(j, i, k)^{-1}\Omega(i, k)$
   (c) $F(i, j, k)\Omega(i + j, k)F(k, i, j) = \Omega(j, k)F(i, k, j)\Omega(i, k)$

The kernel of d^n is written $Z^n_{ab}(A, \mathbb{C}^\times)$, and elements are called abelian n-cocycles. The cohomology of the Eilenberg-MacLane complex is called abelian cohomology, and written $H^n_{ab}(A, \mathbb{C}^\times)$. An abelian 3-cocycle is normalized if $F(i, j, 0) = F(i, 0, k) = F(0, j, k) = 1$ and $\Omega(i, 0) = \Omega(0, j) = 1$ for all $i, j, k \in A$.

We note that any cohomology class has a normalized representative.

Lemma 2.2.2. *([MacLane-1952] Theorem 3) If (F, Ω) is an abelian 3-cocycle, then the map $Q : A \to \mathbb{C}^\times$ defined by $i \mapsto \Omega(i, i)$ is a quadratic function, i.e., $Q(i) = Q(-i)$ for all $i \in A$, and $\frac{Q(i+j+k)Q(i)Q(j)Q(k)}{Q(i+j)Q(i+k)Q(j+k)} = 1$ for all $i, j, k \in A$. Furthermore, the trace map $(F, \Omega) \mapsto (i \mapsto \Omega(i, i))$ induces a bijection from $H^3_{ab}(A, \mathbb{C}^\times)$ to the set of $\mathbb{C}^\times$-valued quadratic functions on A.*

Proof. This is proved as [Eilenberg-Mac Lane-1954], theorem 26.1. □

Definition 2.2.3. Let A be an abelian group, let (F, Ω) be an abelian 3-cocycle on A. We write $q_\Omega : A \to \mathbb{C}/\mathbb{Z}$ for the unique quadratic map such that $e(q_\Omega(a)) = \Omega(a, a)$ for all $a \in A$, and we write $b_\Omega : A \times A \to \mathbb{C}/\mathbb{Z}$ for the induced bilinear form.

We take the following definition from [Dong-Lepowsky-1993]

Definition 2.2.4. Let A be an abelian group, let (F, Ω) be a normalized abelian 3-cocycle on A. Then an abelian intertwining algebra of level $N \in \mathbb{Z}_{\geq 1}$ and central charge $c \in \mathbb{Q}$ associated to the datum (A, F, Ω) is a complex vector space V equipped with

(1) a $\frac{1}{N}\mathbb{Z} \times A$-grading $V = \bigoplus_{n \in \frac{1}{N}\mathbb{Z}} V_n = \bigoplus_{i \in A} V^i = \bigoplus_{n \in \frac{1}{N}\mathbb{Z}, i \in A} V^i_n$
(2) a left-multiplication $Y : V \to (\operatorname{End} V)[[z^{1/N}, z^{-1/N}]]$ written $Y(a, z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} a_n z^{-n-1}$, and
(3) distinguished vectors $\mathbf{1} \in V^0_0$ and $\omega \in V^0_2$

satisfying the following conditions for any $i, j, k \in A$, $a \in V^i$, $b \in V^j$, $u \in V^k$, and $n \in \frac{1}{N}\mathbb{Z}$:

(1) $a_n b \in V^{i+j}$.
(2) $a_n b = 0$ for n sufficiently large.
(3) $Y(\mathbf{1}, z)a = a$
(4) $Y(a, z)\mathbf{1} \in a + zV[[z]]$.
(5) $Y(a, z)b = \sum_{k \in b_\Omega(i,j)+\mathbb{Z}} a_k b z^{-k-1}$.

(6) The Jacobi identity:

$$x^{-1}\left(\frac{y-z}{x}\right)^{b_\Omega(i,j)}\delta\left(\frac{y-z}{x}\right)Y(a,y)Y(b,z)u$$
$$-B(i,j,k)x^{-1}\left(\frac{z-y}{e^{i\pi}x}\right)^{b_\Omega(i,j)}\delta\left(\frac{z-y}{-x}\right)Y(b,z)Y(a,y)u=$$
$$=F(i,j,k)z^{-1}\delta\left(\frac{y-x}{z}\right)Y(Y(a,x)b,z)\left(\frac{y-x}{z}\right)^{-b_\Omega(i,k)}u$$

where $B(i,j,k)=\frac{\Omega(i,j)F(i,j,k)}{F(j,i,k)}$.

(7) The coefficients of $Y(\omega,z)=\sum_{k\in\mathbb{Z}}L_kz^{-k-2}$ satisfy the Virasoro relations at central charge c.

(8) $L_0a=na$ if $a\in V_n$.

(9) $\frac{d}{dz}Y(a,z)=Y(L_{-1}a,z)$.

Given a suitably nondegenerate abelian intertwining algebra (namely one where left multiplication by elements of V^i is not the zero map), one has an additional $\mathbb{C}/\mathbb{Z}$-valued quadratic form q_Δ on A, given by setting $q_\Delta(i)$ to be the conformal weight of V^i. It is not hard to show that the associated bilinear forms satisfy $b_\Omega=-b_\Delta$ (see, e.g., the discussion in [van Ekeren, et al, 2015] after Theorem 4.2). Furthermore, it is often the case that we have $q_\Omega=-q_\Delta$ - when this holds, we say that the abelian intertwining algebra is "even".

Dong and Lepowsky give the following generalization of the lattice vertex algebra construction: for any finite rank free abelian group L with a $\frac{1}{N}\mathbb{Z}$-valued quadratic form, we may produce the abelian intertwining algebra $V=V_L$ attached to L. It is L-graded, and the L-homogeneous pieces are the usual modules $\pi_\lambda^{L\otimes\mathbb{C}}$ of the Heisenberg vertex algebra $\pi_0^{L\otimes\mathbb{C}}$ for the quadratic vector space $L\otimes\mathbb{C}$ attached to vectors $\lambda\in L$. This abelian intertwining algebra is even, i.e., it satisfies $q_\Delta=-q_\Omega$. One needs to choose a cocycle (F,Ω) in the cohomology class of q_Ω to obtain an abelian intertwining algebra, but we have the following way to pass between different choices.

Lemma 2.2.5. *Given an abelian intertwining algebra V with cocycle (F,Ω), and any function $f\in K^2$ satisfying $f(i,0)=f(0,j)=1$, the tuple $(V,1,\omega,f\cdot Y)$ is an abelian intertwining algebra with cocycle $d_2f\cdot(F,\Omega)$. Here, $f\cdot Y$ is the map $V\to(\operatorname{End}V)[[z^{1/N},z^{-1/N}]]$ that takes a pair $a\in V^i$, $b\in V^j$ to $f(i,j)\cdot Y(a,z)b$.*

Proof. This is Remark 12.23 in [Dong-Lepowsky-1993]. □

Definition 2.2.6. Let q be a quadratic form on an abelian group A that takes values in $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$. We define $AIA^c_{(A,q)}$ to be the following groupoid: objects are abelian intertwining algebras of level N with central charge c, associated to the datum (A,F,Ω) for some abelian 3-cocycle (F,Ω) such that $q_\Omega=e(q)$. Morphisms are homogeneous isomorphisms of abelian intertwining algebras.

By Lemma 2.2.5, this groupoid is equipped with an action of the abelian 3-coboundary group by autofunctors.

The last structure we introduce is a graded tensor product.

Definition 2.2.7. Let V be an object in $AIA^c_{(A,q)}$ with cocycle (F,Ω), and let W be an object in $AIA^{c'}_{(A,q')}$ with cocycle (F',Ω'). Then we define $V\otimes^A W$ to be the following object in $AIA^{c+c'}_{(A,qq')}$ with cocycle $(FF',\Omega\Omega')$:

(1) The underlying graded vector space is $\bigoplus_{i\in A}V^i\otimes W^i$

(2) Multiplication is induced by tensor product maps. That is, $Y(a\otimes a',z)(b\otimes b')=Y(a,z)b\otimes Y(a',z)b'$ for $a,b\in V$, and $a',b'\in W$.

(3) $\mathbf{1}_{V\otimes^A W}=\mathbf{1}_V\otimes\mathbf{1}_W$

(4) $\omega_{V\otimes^A W} = \omega_V \otimes \mathbf{1}_W + \mathbf{1}_V \otimes \omega_W$

The following theorem describes the results of van Ekeren, Möller, and Scheithauer that we need.

Theorem 2.2.8. *Let V be a holomorphic C_2-cofinite vertex operator algebra of CFT type, let g be an automorphism of V of order $n \in \mathbb{Z}_{\geq 1}$, and let $G = \langle g \rangle$ be the cyclic group generated by g. Suppose the L_0-spectrum of the irreducible twisted module $V(g^i)$ is strictly positive whenever $g^i \neq 1$. Then there is a distinguished element $t \in \mathbb{Z}/n\mathbb{Z}$ and a unique assignment of homomorphisms $\phi_i : G \to \operatorname{Aut} V(g^i)$ satisfying the following conditions:*

(1) *If we let $V^{(i,j)} = \{v \in V(g^i) | \phi_i(g)v = e(j/n)v\}$, then the set $\{V^{(i,j)}\}_{i,j\in\mathbb{Z}/n\mathbb{Z}}$ forms a complete list of isomorphism classes of irreducible V^G-modules. They are simple currents, whose fusion rules satisfy $V^{(i,j)} \boxtimes V^{(k,l)} \cong V^{(i+k,j+l+c_{2t}(i,k))}$, where c_{2t} is the "add with carry" 2-cocycle $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by $c_{2t}(i,k) = \begin{cases} 0 & i_n + k_n < n \\ 2t & i_n + k_n \geq n \end{cases}$. Here, i_n and k_n denote the unique elements of $\{0, \ldots, n-1\}$ to which i and k reduce modulo n, respectively, and we let D denote the central extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$ defined by the 2-cocycle c_{2t}.*

(2) *There exists an abelian intertwining algebra ${}^gV \cong \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} V(g^i) \cong \bigoplus_{i,j\in\mathbb{Z}/n\mathbb{Z}} V^{(i,j)}$, whose multiplication is compatible with the fusion rules on $V^{(i,j)}$, graded by the abelian group D, with $q_\Omega(i,j) = -\frac{ij}{n} - \frac{i_n^2 t_n}{n^2}$. Furthermore, ${}^gV$ is even, i.e., $q_\Delta(i,j) = -q_\Omega(i,j)$.*

Furthermore, if we write

$$T(v,i,j,\tau) = \operatorname{Tr}(o(v)\phi_i(g^j)q^{L_0-c/24}|V(g^i))$$
$$T_{(i,j)}(v;\tau) = \operatorname{Tr}(o(v)q^{L_0-c/24}|V^{(i,j)})$$

for all $v \in V^G$, $i,j \in \mathbb{Z}/n\mathbb{Z}$ and $q = e(\tau)$, where $o(v)$ is the zero-mode map (which is $v_{wt(v)-1}$ for homogeneous v), we obtain holomorphic functions on $\mathfrak{H}$ that satisfy the following modular transformation rules:

$$T(v,i,j,\tau+1) = e(\frac{i_n^2 t_n}{n^2} - \frac{c}{24})T(v,i,j+i,\tau)$$
$$T(v,i,j,\frac{-1}{\tau}) = \tau^{wt[v]}e(\frac{-2ti_nj_n}{n^2})T(v,j,-i,\tau)$$
$$T_{(i,j)}(v;\tau+1) = e(\frac{ij}{n} + \frac{i_n^2 t_n}{n^2} - \frac{c}{24})T_{(i,j)}(v;\tau)$$
$$T_{(i,j)}(v;\frac{-1}{\tau}) = \frac{1}{n}\tau^{wt[v]}\sum_{k,l\in\mathbb{Z}/n\mathbb{Z}} e(-\frac{kj+il}{n})e(\frac{-2ti_nk_n}{n^2})T_{(k,l)}(v;\tau)$$

where $wt[v]$ denotes the conformal weight with respect to Zhu's bracket grading [Zhu-1996].

Proof. We point out where the specific claims are given in [van Ekeren, et al, 2015]. The identification of eigenspaces with irreducible V^G-modules is Proposition 5.2, and the fact that they are simple currents is Proposition 5.6.

The fusion rules, conformal weights and S-matrix entries are given in Theorem 5.12. The modular transformation rules for $T(v,i,j,\tau)$ and $T_{(i,j)}(v;\tau)$ then follow from the identification of S-matrix entries and the conformal weights of modules.

The existence and properties of the abelian intertwining algebra are given in Theorem 4.1. □

We now show that this theorem can be applied to the object of our interest. Recall that an element g in the Monster is called Fricke if its McKay-Thompson series $T_g(\tau)$ is proportional to $T_g(-1/N\tau)$ for some N. Note that this is a definition that depends on the construction of the moonshine module $V^\natural$ in [Frenkel, et al, 1988]. By Borcherds's proof of the Monstrous Moonshine

Conjecture [Borcherds-1992], we have the following necessary and sufficient conditions for g to be Fricke: T_g has a pole at zero, or $T_g(\tau)$ is equal to $T_g(-1/N\tau)$ for some N. The Fricke property is conjugation-invariant, since the McKay-Thompson series is a trace. The conjectural enumeration of functions in [Conway-Norton-1979] (proved to be valid by Borcherds) shows that 120 of the 171 distinct McKay-Thompson series are Fricke, and 143 of the 194 conjugacy classes in the Monster are Fricke.

Theorem 2.2.9. *The Monster vertex operator algebra $V^\natural$ is holomorphic, C_2-cofinite, and of CFT type. For any nontrivial element g of the Monster, the L_0-spectrum of $V(g)$ is strictly positive. That is, the conditions of Theorem 2.2.8 are satisfied for any automorphism of $V^\natural$.*

Proof. The holomorphic property is proved in [Dong-1994], the C_2-cofinite property is proved in [Dong-Li-Mason-2000], and CFT type is clear from the construction [Frenkel, et al, 1988]. By Theorem 13.1 of [Dong-Li-Mason-2000], the character of $V^\natural(g)$ is a constant multiple of $T_g(-1/\tau)$, where $T_g(\tau) = Tr(gq^{L_0-1}|V^\natural)$ is the McKay-Thompson series. If T_g is regular at $\tau = 0$ (i.e., g is non-Fricke), then the lowest-weight space of $V(g)$ has L_0-eigenvalue at least 1. If T_g has a pole at $\tau = 0$ (i.e., g is Fricke), then the pole has order $1/N$, where N is the level of T_g, and the lowest-weight space of $V^\natural(g)$ has L_0-eigenvalue $1 - 1/N$. We conclude that as long as g is nontrivial, the lowest weight space has positive L_0-eigenvalue. □

We may then determine the precise graded dimension of twisted modules of $V^\natural$ - this eliminates the scalar ambiguity in [Dong-Li-Mason-2000].

Corollary 2.2.10. *For any $g \in \mathbb{M}$, hypothesis A_g from section 4.4 of [Carnahan-2012] is satisfied, i.e., $\sum_{n\in\mathbb{Q}} \dim V^\natural(g)_n q^{n-1} = T_g(-1/\tau)$.*

Proof. The left side of the equation is $T(\mathbf{1}, 1, 0, \tau)$, and the right side is $T(\mathbf{1}, 0, 1, -1/\tau)$, as defined in Theorem 2.2.8. By Theorem 2.2.9, we may apply Theorem 2.2.8 to $V^\natural$ and its twisted modules, so the two quantities are equal. □

Corollary 2.2.11. *If $V = V^\natural$, and $g \in \mathbb{M}$ has eigengroup $\Gamma_0(n|h) + e_1, e_2, \ldots$ (in the notation of [Conway-Norton-1979]), then the distinguished element $t \in \mathbb{Z}/n\mathbb{Z}$ given in Theorem 2.2.8 is equal to $-n/h$ if g is Fricke and n/h if g is non-Fricke.*

Proof. The value of t is uniquely determined by the transformation $\tau \mapsto \tau + n$ on the graded dimension of $V^\natural(g)$. Theorem 2.2.8 gives us $T(\mathbf{1}, 1, 0, \tau - n) = e(-\frac{t_n}{n})T(\mathbf{1}, 1, 0, \tau)$, and Section 5 of [Conway-Norton-1979] gives us the transformation rule $T_g(\tau)|_0 \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right) = e(\pm 1/h)T_g(\tau)$, where we have a plus sign if and only if g is Fricke. The latter Möbius transformation conjugates to the $\tau \mapsto \tau - n$ transformation on $T(\mathbf{1}, 1, 0, \tau)$, so we have $t_n = \mp n/h$. □

2.3. **Unrolling.** In Section 3.1 of [Carnahan-2012] we introduce vector-valued modular functions $\hat{F}$ and F, which are uniquely determined by the $SL_2(\mathbb{Z})$ action on McKay-Thompson series. In section 4.1, we will need to match the functions $T(\mathbf{1}, i, j, \tau)$ with $\hat{F}_{i,j}(\tau)$ and $T_{(i,j)}(\tau)$ with $F_{i,j}(\tau)$ in order to show that certain Lie algebras are isomorphic. However, this is only possible when $t = 0$ given our current setup, because that is the only way the modular transformation rules can match. We will fix this problem by "unrolling" the abelian intertwining algebra into a $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$-graded object in a unique way, for N any multiple of $\frac{n^2}{(n,t)}$. By passing to the unrolled object, the modular transformation rules match in all cases.

Lemma 2.3.1. *Let $V_A = \bigoplus_{a\in A} V_a$ be an abelian intertwining algebra with cocycle (F_A, Ω_A). Let (B, q) be an abelian group with $\mathbb{C}/\mathbb{Z}$-valued quadratic form, and let $\pi : (B, q) \to (A, (a \mapsto \Omega_A(a, a)))$ be the quotient by a null subgroup $I \subset B^\perp$. Then there is a unique "pullback" abelian intertwining algebra $W_B = \bigoplus_{b\in B} W_b$ satisfying the following properties:*

(1) $W_b = \begin{cases} V_a & b \in B, \pi(b) = a \\ 0 & b \notin B \end{cases}$

(2) *The multiplication map* $W_b \otimes W_{b'} \to W_{b+b'}\{z\}$ *is given by the multiplication* $V_a \otimes V_{a'} \to V_{a+a'}\{z\}$ *from* V_A *when* $b, b' \in B$, $\pi(b) = a$, *and* $\pi(b') = a'$.

The cocycle (F_B, Ω_B) *attached to* W_B *is precisely the pullback of the cocycle* (F_A, Ω_A) *attached to* V_A.

Proof. The fact that W_B satisfies the axioms of an abelian intertwining algebra follows straightforwardly from our definition of the multiplication operation - the Jacobi identity for W_B reduces immediately to the Jacobi identity for V_A. □

Lemma 2.3.2. *If* $i : (A, q_A) \to (B, q_B)$ *is a quadratic embedding (i.e., i is an injective homomorphism of abelian groups, and* $q_B \circ i = q_A$*), and* V_A *is an* (A, q_A)*-graded abelian intertwining algebra, then there is a unique "extension by zero" abelian intertwining algebra* $W_B = \bigoplus_{b \in B} W_b$ *satisfying the following properties*

(1) $W_b = \begin{cases} V_a & b = i(a) \\ 0 & b \notin i(A) \end{cases}$

(2) *The multiplication map* $W_b \otimes W_{b'} \to W_{b+b'}\{z\}$ *is given by the multiplication* $V_a \otimes V_{a'} \to V_{a+a'}\{z\}$ *from* V_A *when* $b = i(a)$, *and* $b' = i(a')$. *For all other values of* b *and* b', *at least one of the input spaces is zero, so multiplication is trivial.*

The cocycle (F_B, Ω_B) *attached to* W_B *is uniquely determined by* (F_A, Ω_A) *up to a coboundary that vanishes identically on* $i(A)$.

Proof. Omitted. □

Definition 2.3.3. Given a diagram $(A, q) \overset{\pi}{\twoheadleftarrow} (A', q') \overset{i}{\hookrightarrow} (A'', q'')$ of quadratic spaces, we define the unrolling functor $AIA^c_{(A,q)} \to AIA^c_{(A'',q'')}$ as the composite of the pullback given in Lemma 2.3.1 and the extension by zero given in Lemma 2.3.2.

The following proposition shows that we can make the modular transformation rules in Theorem 2.2.8 somewhat cleaner by unrolling.

Proposition 2.3.4. *Let* V *be a holomorphic* C_2*-cofinite vertex operator algebra of CFT type, let* g *be an automorphism of* V *of order* $n \in \mathbb{Z}_{\geq 1}$, *such that the* L_0*-spectrum of the irreducible twisted module* $V(g^i)$ *is strictly positive whenever* $g^i \neq 1$. *Let* $t \in \mathbb{Z}/n\mathbb{Z}$ *be the distinguished element given by Theorem 2.2.8, and let* $N = \frac{rn^2}{(n,t)}$ *for some* $r \in \mathbb{Z}_{\geq 1}$. *Finally, let* B *denote the group* $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ *with quadratic form* $(a, b) \mapsto e(\frac{ab}{N})$. *Then the subgroup* $B' \subset B$ *generated by* $(0, \frac{rn}{(n,t)})$ *and* $(1, \frac{t_n r}{(n,t)})$ *admits a quadratic-form-preserving quotient map* $B' \to D$ *by* $a(1, \frac{t_n r}{(n,t)}) + b(0, \frac{rn}{(n,t)}) \mapsto (a_n, b_n + 2t_n \lfloor a/n \rfloor)$, *and there exists a unique set of homomorphisms* $\{\psi_a : \mathbb{Z}/N\mathbb{Z} \to \operatorname{Aut} V(g^a)\}_{a=0}^{N-1}$ *such that the abelian intertwining algebra* ${}^g_N W \cong \bigoplus_{(a,b) \in B} W^{(a,b)} \cong \bigoplus_{a=0}^{N-1} V(g^a)$ *unrolled from* ${}^g V$ *satisfies the following conditions:*

(1) *For each* $a, b \in \mathbb{Z}/N\mathbb{Z}$, *we have* $W^{(a,b)} = \{v \in V(g^a) | \psi_a(1)v = e(\frac{b}{N})v\}$, *and when* $W^{(a,b)} \neq 0$, *it is a simple current (in particular, an irreducible module) for the fixed-point algebra* V^g.

(2) *For any* $a, b \in \mathbb{Z}/N\mathbb{Z}$, *we have the equality* $\psi_a(b) = \phi_a(g^b) e(b \frac{a_n t_n}{n^2} - b \frac{t_n \lfloor a/n \rfloor}{n})$ *of automorphisms of* $V(g^a)$.

(3) *If we define* $T(a, b, \tau) = \operatorname{Tr}(\psi_a(b) q^{L_0 - c/24} | V(g^a))$ *and* $T_{(a,b)}(\tau) = \operatorname{Tr}(q^{L_0 - c/24} | W^{(a,b)})$ *for all* $a, b \in \mathbb{Z}/N\mathbb{Z}$ *and* $q = e(\tau)$, *then we obtain holomorphic functions on* $\mathfrak{H}$ *that satisfy the*

following modular transformation rules:

$$T(a,b,\tau+1) = e(-\frac{c}{24})T(a,b+a,\tau)$$
$$T(a,b,\frac{-1}{\tau}) = T(b,-a,\tau)$$
$$T_{(a,b)}(\tau+1) = e(\frac{ab}{N}-\frac{c}{24})T_{(a,b)}(\tau)$$
$$T_{(a,b)}(\frac{-1}{\tau}) = \frac{1}{N}\sum_{p,q\in\mathbb{Z}/n\mathbb{Z}} e(-\frac{bp+aq}{N})T_{(p,q)}(\tau)$$

In particular (following [Carnahan-2012] Lemma 3.6), if c is a multiple of 24, then the collection $\{T_{(a,b)}\}_{a,b\in\mathbb{Z}/N\mathbb{Z}}$ forms a vector-valued modular function of type ρ_M for $M = \Pi_{1,1}(N)$, and the collection $\{T(a,b,\tau)\}_{a,b\in\mathbb{Z}/N\mathbb{Z}}$ describes a function on the moduli space $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ of elliptic curves with $\mathbb{Z}/N\mathbb{Z}$-torsors.

Proof. We first check that $B' \to D$ preserves the quadratic form: $q_{B'}(a(1,\frac{t_n r}{(n,t)}) + b(0,\frac{rn}{(n,t)})) = \frac{ab}{n} + \frac{a^2 t_n}{n^2} + \mathbb{Z}$, while $q_D(a_n, b_n + 2t_n\lfloor a/n\rfloor) = \frac{a_n b_n + 2a_n t_n\lfloor a/n\rfloor}{n} + \frac{a_n^2 t_n}{n^2} + \mathbb{Z}$. Because $a = n\lfloor a/n\rfloor + a_n$, we have $\frac{a^2 t_n}{n^2} + \mathbb{Z} = \frac{a_n^2 t_n}{n^2} + \frac{2a_n t_n\lfloor a/n\rfloor}{n} + \mathbb{Z}$, so the two expressions agree.

We therefore have a well-defined unrolling functor, and an abelian intertwining algebra ${}^g_N W$ whose B-homogeneous components and multiplication maps are either pulled back from ${}^g V$ or zero. The homomorphisms ψ_a are uniquely determined by the B-grading on ${}^g_N W$. The simple current property follows from the first listed claim in 2.2.8.

We now check that $\psi_a(b) = \phi_a(g^b)e(b\frac{a_n t_n}{n^2} - b\frac{t_n\lfloor a/n\rfloor}{n})$ on $V(g^a)$. For any $a(1,\frac{t_n r}{(n,t)}) + j(0,\frac{rn}{(n,t)}) \in B'$, we have $\psi_a(1)|_{W^{a(1,\frac{t_n r}{(n,t)})+j(0,\frac{rn}{(n,t)})}} = e(\frac{at_n}{n^2} + \frac{j}{n})$. The module is the pullback of $V^{(a_n, j_n + 2t_n\lfloor a/n\rfloor)}$, where $\phi_a(g)$ acts as $e(\frac{j_n + 2t_n\lfloor a/n\rfloor}{n})$. Dividing the two answers, and taking the bth power yields the answer we want.

It remains to consider the modular transformation properties. We reduce the first two equations to the last two equations by noting that $\{T(a,b,\tau)\}$ and $T_{(a,b)}(\tau)$ are related by a discrete Fourier transform. That is, the equations

$$T(a,b,\tau) = \sum_{k\in\mathbb{Z}/N\mathbb{Z}} e\left(\frac{bk}{N}\right) T_{(a,k)}(\tau)$$
$$T_{(a,b)}(\tau) = \frac{1}{N}\sum_{k\in\mathbb{Z}/N\mathbb{Z}} e\left(\frac{-bk}{N}\right) T(a,k,\tau)$$

allow us to pass from one equation to the other - this is essentially Lemma 3.3 of [Carnahan-2012] but we multiply by the additional character $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \mapsto e(-c/24)$, $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) \mapsto 1$ of $SL_2(\mathbb{Z})$.

Since $W^{(a(1,\frac{t_n r}{(n,t)})+b(0,\frac{rn}{(n,t)}))} = V^{(a_n, b_n + 2t_n\lfloor a/n\rfloor)}$, we have

$$T_{(a(1,\frac{t_n r}{(n,t)})+b(0,\frac{rn}{(n,t)}))}(\tau) = T_{(a_n, b_n+2t_n\lfloor a/n\rfloor)}(\mathbf{1},\tau)$$

for all $a,b \in \mathbb{Z}/N\mathbb{Z}$, and $T_{(a,b)}(\tau) = 0$ for $(a,b) \notin B'$. By our computation of the quadratic form in the first paragraph of this proof, we find that

$$\begin{aligned} T_{(a(1,\frac{t_n r}{(n,t)})+b(0,\frac{rn}{(n,t)}))}(\tau+1) &= T_{(a_n, b_n+2t_n\lfloor a/n\rfloor)}(\mathbf{1},\tau+1) \\ &= e\left(\frac{a_n(b_n+2t_n\lfloor a/n\rfloor)}{n} + \frac{a_n^2 t_n}{n^2} - \frac{c}{24}\right) T_{(a_n, b_n+2t_n\lfloor a/n\rfloor)}(\mathbf{1},\tau) \end{aligned}$$

Then $\frac{a_n(b_n+2t_n\lfloor a/n\rfloor)}{n}+\frac{a_n^2t_n}{n^2}=\frac{ab}{n}+\frac{a^2t_n}{n^2}=\frac{abrn+a^2t_nr}{N(n,t)}=\frac{a\left(\frac{at_nr}{(n,t)}+\frac{brn}{(n,t)}\right)}{N}$, so we obtain

$$T_{(a(1,\frac{t_nr}{(n,t)})+b(0,\frac{rn}{(n,t)}))}(\tau+1)=e\left(\frac{a\left(\frac{at_nr}{(n,t)}+\frac{brn}{(n,t)}\right)}{N}-\frac{c}{24}\right)T_{(a(1,\frac{t_nr}{(n,t)})+b(0,\frac{rn}{(n,t)}))}(\tau).$$

This proves the third equation.

For the fourth equation, we note that by Theorem 2.2.8, the S-matrix entries $S_{(i,j),(k,l)}$ for ${}^gV$ are given by $\frac{1}{n}e(b_\Omega((i,j),(k,l)))$, where b_Ω is -1 times the inner product induced by q_D. Since the map $B'\to D$ preserves the quadratic form, hence -1 times the $\mathbb{Q}/\mathbb{Z}$-valued inner product, if we write b'_Ω for the pullback of b_Ω, we have

$$\begin{aligned}
T_{(a(1,\frac{t_nr}{(n,t)})+b(0,\frac{rn}{(n,t)}))}(\frac{-1}{\tau})&=T_{(a_n,b_n+2t_n\lfloor a/n\rfloor)}(\mathbf{1},\frac{-1}{\tau})\\
&=\frac{1}{n}\sum_{x\in D}e(b_\Omega((a_n,b_n+2t_n\lfloor a/n\rfloor),x))T_x(\mathbf{1},\tau)\\
&=\frac{1}{N}\sum_{y\in B'}e(b'_\Omega((a,\frac{at_nr+brn}{(n,t)}),y))T_y(\tau)\\
&=\frac{1}{N}\sum_{p,q\in\mathbb{Z}/n\mathbb{Z}}e\left(-\frac{\frac{at_nr+brn}{(n,t)}p+aq}{N}\right)T_{(p,q)}(\tau),
\end{aligned}$$

so the fourth equation holds for $(a,b)\in B'$. For $(a,b)\notin B'$, we need a vanishing result, and it suffices to show that $\sum_{p,q\in\mathbb{Z}/n\mathbb{Z}}e(-\frac{bp+aq}{N})T_{(p,q)}(\tau)=0$. This sum reduces to a sum over B', and it suffices to show that for any $x\in D$, the sum over preimages of x in B' vanishes. The kernel of $B'\to D$ is generated by $(n,\frac{-nrt_n}{(n,t)})=n(1,\frac{t_nr}{(n,t)})-2t_n(0,\frac{rn}{(n,t)})$, so we fix $p(1,\frac{t_nr}{(n,t)})+q(0,\frac{rn}{(n,t)})\in B'$ and compute the coefficient on $T_{p(1,\frac{t_nr}{(n,t)})+q(0,\frac{rn}{(n,t)})}(\mathbf{1},\tau)$:

$$\begin{aligned}
&\sum_{k=0}^{N/n-1}e\left(\frac{b(p+kn)+a(\frac{pt_nr+qrn-knrt_n}{(n,t)})}{N}\right)\\
&\quad=\sum_{k=0}^{N/n-1}e\left(\frac{bp(n,t)+bkn(n,t)+aprt_n+aqrn+aknrt_n}{N(n,t)}\right)\\
&\quad=e\left(\frac{bp(n,t)+aprt_n+aqrn}{N(n,t)}\right)\sum_{k=0}^{N/n-1}e\left(k\frac{bn(n,t)+anrt_n}{N(n,t)}\right)
\end{aligned}$$

The last sum vanishes if and only if the numerator $bn(n,t)+anrt_n$ is nonzero modulo $N(n,t)$, and by fixing a, a short calculation shows that this is precisely when $(a,b)\notin B'$. □

Generalized Moonshine for cyclic groups was proved in [Dong-Li-Mason-2000] up to a scalar ambiguity. We can eliminate the ambiguity by using a central extension, essentially given by unrolling the corresponding abelian intertwining algebra.

Definition 2.3.5. For any $g\in\mathbb{M}$, and let N be any positive integer such that the McKay-Thompson series $T_g(\tau)=\mathrm{Tr}(gq^{L_0-1}|V^\natural)$ is $\Gamma_0(N)$-invariant. By [Carnahan-2012] Corollary 3.25, we may define the vector-valued function $\hat{F}^g=\{\hat{F}^g_{i,j}\}_{i,j=0}^{N-1}$ by $\hat{F}^g_{i,j}=T_{g^{(i,j,N)}}(\tau)|_0A$ for $A=\begin{pmatrix}\frac{*}{(i,j,N)} & \frac{*}{(i,j,N)}\\ \frac{i}{(i,j,N)} & \frac{j}{(i,j,N)}\end{pmatrix}\in SL_2(\mathbb{Z}/\frac{N}{(i,j,N)}\mathbb{Z})$. Similarly, we may define the vector valued function $F^g_{i,j}=$

$\{F^g_{i,j}\}_{i,j=0}^{N-1}$ by the discrete Fourier transform $F^g_{i,j}(\tau) = \frac{1}{N}\sum_{k\in\mathbb{Z}/N\mathbb{Z}} e\left(\frac{-jk}{N}\right)\hat{F}^g_{i,k}(\tau)$. This is a vector-valued modular function for the Weil representation $\rho_{II_{1,1}(1/N)}$. We write $c^g_{i,j}(n)$ for the q^n coefficient of $F^g_{i,j}$.

Corollary 2.3.6. *For any $g \in \mathbb{M}$, hypothesis B_g from section 4.4 of [Carnahan-2012] is satisfied, i.e., if $T_g(\tau)$ has level N, then there are actions $\{\psi_i\}_{i=0}^{N-1}$ of $\mathbb{Z}/N\mathbb{Z}$ on the twisted modules $V^\natural(g^i)$, compatible with the action of $\langle g\rangle$ on $V^\natural$, such that $\mathrm{Tr}(\psi^i(j)q^{L_0-1}|\dim V^\natural(g^i)) = \hat{F}^g_{i,j}(\tau)$, and $\psi_i(i) = e(L_0)$.*

Proof. The actions ψ_i given in Proposition 2.3.4 satisfy these identities, and $V^\natural$ satisfies the hypotheses of that proposition by Theorem 2.2.9. □

2.4. **Adding a spacetime torus.** In order to produce a Lie algebra from a quantization functor, we need a conformal vertex algebra of central charge 26. Our abelian intertwining algebras ${}^gV$ and ${}^g_NV$ have central charge 24, so we have to increase the central charge by two and cancel the braiding. Our functors amount to taking a graded tensor product with the abelian intertwining algebra attached to a well-chosen 2-dimensional hyperbolic lattice.

Definition 2.4.1. Let L be an integer lattice of signature $(1,1)$, and let (D,q) be the discriminant form $L^\vee/L$. We define the "add a torus" functor AT_L from $AIA^c_{(D,q)}$ to the category of L-graded conformal vertex algebras of central charge $c+2$ as follows:

(1) For any object $W = \bigoplus_{a\in D} W_a$ in $AIA^c_{(D,q)}$ with abelian 3-cocycle (F,Ω), we choose an abelian 3-coboundary to endow $V_{L^\vee(-1)}$ with the abelian 3-cocycle $(\frac{1}{F},\frac{1}{\Omega})$, and take the graded tensor product $W\otimes^D V_{L^\vee(-1)} = \bigoplus_{a\in D} W_a\otimes V_{L_{-1}+a}$.
(2) For any isomorphism $W\to W'$ in $AIA^c_{(D,q)}$, the corresponding isomorphism of $L^\vee$-graded conformal vertex algebras is given by taking the tensor product.

For any $\lambda\in L^\vee$, the λ-graded piece is the tensor product $W_{\lambda+L}\otimes\pi_\lambda^{L\otimes\mathbb{C}}$.

Lemma 2.4.2. *Let V be a holomorphic C_2-cofinite vertex operator algebra of central charge c, let g be an automorphism of order n such that $V(g^i)$ has strictly positive L_0-spectrum for $g^i\neq 1$, and let ${}^gV$ be the abelian intertwining algebra given in Theorem 2.2.8. Then there is a lattice L of signature $(1,1)$ and an abelian intertwining algebra $V_{L^\vee(-1)}$ whose abelian 3-cocycle cancels that of ${}^gV$. In particular, adding a torus is well-defined on ${}^gV$, and yields an $L^\vee(-1)$-graded conformal vertex algebra of central charge $c+2$.*

Proof. Let L be the lattice identified in [van Ekeren, et al, 2015] by its Gram matrix $\left(\begin{smallmatrix}-2t_n & n\\ n & 0\end{smallmatrix}\right)$. Then the quadratic form on $L^\vee/L$ is precisely q_Ω for ${}^gV$. Thus, there exists a suitable coboundary for $V_{L^\vee(-1)}$ such that the resulting abelian 3-cocycle cancels that of ${}^gV$. □

Proposition 2.4.3. *Adding a torus 2-commutes with both unrolling against hyperbolic lattice embeddings and twisting against abelian 3-coboundaries. That is, different choices of unrolling or coboundaries on an abelian intertwining algebra yield isomorphic conformal vertex algebras.*

Proof. The claim about coboundaries follows from the fact that if one twists against a coboundary for the object in $AIA^c_{(D,q)}$, the opposite coboundary is chosen on $V_{L^\vee(-1)}$. Thus, the graded tensor product is unchanged.

Suppose we are given an embedding $L\subset M$ of integral lattices of signature $(1,1)$, and let $(A,q_A) = M^\vee/M$ and $(B,q_B) = L^\vee/L$. We have the diagram $M^\vee/M \twoheadleftarrow M^\vee/L \hookrightarrow L^\vee/L$ of morphisms of quadratic groups, so the unrolling functor yields an abelian intertwining algebra on $L^\vee/L$ that is supported on $M^\vee/L$. Adding a torus to the A-graded object yields an $M^\vee(-1)$-graded conformal vertex algebra, while adding a torus to the B-graded object yields an $L^\vee(-1)$-graded

conformal vertex algebra that is supported on the sublattice $M^\vee(-1)$. Furthermore, the homogeneous components are isomorphic, and the multiplication laws match, since the multiplication law on the $L^\vee(-1)$-graded vertex algebra is given by a tensor product of a lattice algebra with a multiplication rule pulled back from the A-graded abelian intertwining algebra. □

Definition 2.4.4. For any $g \in \mathbb{M}$, we write ${}^gV^\natural$ for the abelian intertwining algebra in $AIA^{24}_{q_D}$ given by Theorem 2.2.8 and Theorem 2.2.9. We write ${}^g_NV^\natural$ for the "unrolled" abelian intertwining algebra in $AIA^{24}_{q_B}$ given by Proposition 2.3.4.

Recall that by forgetting some multiplicative structure, we may view ${}^gV^\natural$ as a direct sum of irreducible g^i-twisted $V^\natural$-modules, as g^i ranges over the cyclic group generated by g. We note that similarly, ${}^g_NV^\natural$ can be viewed as $\bigoplus_{i=0}^{N-1} V^\natural(g^i)$.

Corollary 2.4.5. *For any $g \in \mathbb{M}$, and any multiple N of the level of $T_g(\tau)$, any choice of lattice embedding $II_{1,1}(N) \hookrightarrow L$ (for $L^\vee/L$ the discriminant form for ${}^gV^\natural$) induces an isomorphism $AT_L({}^gV^\natural) \cong AT_{II_{1,1}(N)}({}^g_NV^\natural)$ of $II_{1,1}(-1/N)$-graded conformal vertex algebras of central charge 26.*

Proof. Theorem 2.2.9 implies Proposition 2.4.3 is applicable, and the result immediately follows. □

2.5. **Projective action of centralizer.** We show that for any finite order automorphism g of a holomorphic C_2-cofinite vertex operator algebra V of CFT type, there is a natural action by a central extension of the centralizer of g by homogeneous automorphisms on the abelian intertwining algebra ${}^gV$, the unrolled object ${}^g_NW$, and the conformal vertex algebra given by adding a torus.

Lemma 2.5.1. *Let V be a holomorphic C_2-cofinite vertex operator algebra of CFT type, and let g be an automorphism of order n such that $V(g^i)$ has strictly positive L_0-spectrum for $g^i \neq 1$. Let $h \in C_{\mathrm{Aut}\,V}(g)$. Then the set of homogeneous automorphisms of the abelian intertwining algebra ${}^gV$ that restrict to h on V has precisely n elements, and a natural simply transitive action of $\mu_n(\mathbb{C})$.*

Proof. If $\tilde{h}$ is a homogeneous automorphism of the abelian intertwining algebra ${}^gV$ that restricts to h on V, then for each $g^i \in \langle g\rangle$, the restriction of $\tilde{h}$ to $V(g^i)$ is an element of $\mathrm{Aut}\,V(g^i)$ over h. Now, suppose we choose arbitrary elements $h_i \in \mathrm{Aut}\,V(g^i)$ over h for all i. By Remark 2.1.5, the set of lifts of h to an automorphism of $V(g^i)$ has a simply transitive action of $\mathbb{C}^\times$, so we can measure the failure of $\{h_i\}_{i=0}^{n-1}$ to form an automorphism of ${}^gV$ by setting $\lambda_{i,j} \in \mathbb{C}^\times$ to be the unique constant such that $Y(h_ia,z)h_jb = \lambda_{i,j}h_{i+j}Y(a,z)b$ for all $a \in V(g^i)$ and $b \in V(g^j)$. By the associativity and commutativity of fusion, $\lambda = \{\lambda_{i,j}\}$ is a normalized symmetric 2-cocycle on $\mathbb{Z}/n\mathbb{Z}$ with values in $\mathbb{C}^\times$, hence it defines an abelian central extension E of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{C}^\times$. Because $\mathbb{C}^\times$ is divisible, the defining exact sequence splits, and $E \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{C}^\times$. In particular, the cocycle λ is a coboundary, and the set of 1-cochains in the preimage of λ under the differential is a torsor under the group $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \cong \mu_n(\mathbb{C})$ of 1-cocycles. Concretely, this means any normalized symmetric 2-cocycle λ is attainable by a suitable choice of $\{h_i\}$, and those $\{h_i\}$ that yield an automorphism of ${}^gV$ (namely those for which $\lambda = 1$) form a torsor under $\mu_n(\mathbb{C})$, where the action is by rescaling: For each $\zeta \in \mu_n(\mathbb{C})$, one replaces h_i with $\zeta^i h_i$. □

Proposition 2.5.2. *Let V be a holomorphic C_2-cofinite vertex operator algebra of CFT type, and let g be an automorphism of order n such that $V(g^i)$ has strictly positive L_0-spectrum for $g^i \neq 1$. Then there is a central extension $\widetilde{C_{\mathrm{Aut}\,V}(g)}$ of $C_{\mathrm{Aut}\,V}(g)$ by $\mu_n(\mathbb{C})$ with a natural action on the abelian intertwining algebra ${}^gV$, the unrolled object ${}^g_NW$, and the conformal vertex algebra given by adding a torus.*

Proof. By Lemma 2.5.1, the set of all lifts of elements of $C_{\mathrm{Aut}\,V}(g)$ is a $\mu_n(\mathbb{C})$-torsor over $C_{\mathrm{Aut}\,V}(g)$, so in order to prove it is a central extension, we need to show that it is closed under multiplication, the torsor structure map is a homomorphism, and all lifts of identity are central. All lifts of identity are central, because the restriction of a lift of identity to any $V(g^i)$ is a scalar, and all lifts

of elements are sums of linear automorphisms of the twisted modules $V(g^i)$. The remaining claims about the multiplicative structure follow from the following calculation: if $\tilde{h} = \{h_i\}$ and $\tilde{k} = \{k_i\}$ are lifts of $h, k \in C_{\mathrm{Aut}\,V}(g)$, then for all $a \in V(g^i), b \in V(g^j)$,

$$\begin{aligned} h_{i+j}k_{i+j}Y(a,z)b &= h_{i+j}Y(k_i a, z)k_j b \\ &= Y(h_i k_i u, a)h_j k_j v b \end{aligned}$$

Therefore, $\tilde{h}\tilde{k}$ is a lift of hk to ${}^gV$.

We thus obtain a natural action of $\widetilde{C_{\mathrm{Aut}\,V}(g)}$ on ${}^gV$. Since unrolling and adding a torus are functors, we get natural actions of the same group on the resulting objects. □

Lemma 2.5.3. *For any N divisible by the level of $T_g(\tau)$, the $II_{1,1}(1/N)$-homogeneous conformal vertex algebra isomorphism between $AT_L({}^gV^\natural)$ and $AT_{II_{1,1}(N)}({}^g_NV^\natural)$ given in Corollary 2.4.5 can be made $\widetilde{C_{\mathbb{M}}(g)}$-equivariant.*

Proof. Promotion to $\widetilde{C_{\mathbb{M}}(g)}$-equivariance follows from the structure of the unrolling functor - the corresponding graded pieces are identified as $\widetilde{C_{\mathbb{M}}(g)}$-modules, and the multiplication maps are equivariant. □

We summarize the results of this section.

Theorem 2.5.4. *Let g be an automorphism of $V^\natural$, i.e., an element of the Monster simple group $\mathbb{M}$, and suppose the McKay-Thompson series $T_g(\tau)$ has level N. Let $B = \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with the quadratic form $q_B(a,b) = e(ab/N)$. Then there is a central extension $\widetilde{C_{\mathbb{M}}(g)}$ of the centralizer $C_{\mathbb{M}}(g)$ by $\mu_{|g|}(\mathbb{C})$ equipped with a distinguished lift $\tilde{g}$ of g, and an abelian intertwining algebra ${}^g_NV^\natural \in AIA^{24}_{q_B}$ with a homogeneous action of $\widetilde{C_{\mathbb{M}}(g)}$ by autmorphisms, satisfying the following properties:*

1. *${}^g_NV^\natural$ decomposes into $\bigoplus_{a=0}^{N-1} V^\natural(g^a)$ as a direct sum of twisted $V^\natural$-modules.*
2. *The action of $\tilde{g}$ restricted to $V^\natural(g)$ is given by $e(L_0)$.*
3. *The (a,b)-graded component of ${}^g_NV^\natural$ is $\{v \in V^\natural(g^a) | \tilde{g}v = e(b/N)v\}$. This is nonzero if and only if $(a,b) \in B'$ (where $B' \subseteq B$ is a subgroup uniquely determined by the symmetries of T_g), and in that case it is a simple current as a $(V^\natural)^g$-module, and it is endowed with a $\widetilde{C_{\mathbb{M}}(g)}$-action.*
4. *The graded dimension of the (a,b)-graded component ${}^g_NV^\natural$ is given by the component form $F^g_{a,b}(\tau)$ given in Definition 2.3.5, and in particular, the coefficient $c^g_{a,b}(n)$ is equal to the dimension of $\{v \in V^\natural(g^a) | \tilde{g}v = e(b/N)v, L_0 v = (n+1)v\}$. That is, the vector-valued graded dimension of ${}^g_NV^\natural$ is equal to F^g, a vector valued modular function for the Weil representation attached to the lattice $II_{1,1}(N)$.*

The conformal vertex algebra $AT_{II_{1,1}(N)}({}^g_NV^\natural)$ has central charge 26, and is graded by $II_{1,1}(-1/N)$, with an action of $\widetilde{C_{\mathbb{M}}(g)}$ by homogeneous automorphisms.

Proof. As before, Theorem 2.2.9 allows us to use the general results for a holomorphic C_2-cofinite vertex operator algebra V of CFT type equipped with an automorphism g for which $V(g^i)$ has strictly positive L_0-spectrum whenever $g^i \neq 1$.

By Lemma 2.5.1, a lift of g to an automorphism of ${}^g_NV^\natural$ is uniquely determined by what it does to $V^\natural(g)$, and by the $\tau \mapsto \tau + 1$ property given in Proposition 2.3.4, setting $\psi_1 = e(L_0)$ yields a lift $\tilde{g}$ such that the (a,b)-graded component of ${}^g_NV^\natural$ is $\{v \in V^\natural(g^a) | \tilde{g}v = e(b/N)v\}$. The claim that the (a,b)-component is nonzero if and only if $(a,b) \in B'$ is straightforward from the definition of unrolling, and the simple current assertion is given by the first listed claim in Proposition 2.3.4.

The claims about graded dimension follow from the properties of the functions $T(a,b;\tau)$ and $T_{(a,b)}(\tau)$ given in Proposition 2.3.4. In particular, [Carnahan-2012] Lemma 3.6 implies the functions

$\{T(a,b,\tau)\}_{a,b\in\mathbb{Z}/N\mathbb{Z}}$ and $\{F^g_{a,b}(\tau)\}_{a,b\in\mathbb{Z}/N\mathbb{Z}}$ on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ are uniquely determined by $T(0,m,\tau)$ and $F^g_{0,m}(\tau)$ for m ranging over divisors of N, and in both cases, these are given by the McKay-Thompson series $T_{g^m}(\tau)$. The discrete Fourier transform then gives an equality between $T_{(a,b)}(\tau)$ and $F^g_{a,b}(\tau)$, matching the coefficient $c^g_{a,b}(n)$ with the dimension of the stated eigenspace.

The remaining claims follow from Proposition 2.5.2 and Corollary 2.4.5. □

3. Lie algebras

In this section we construct the monstrous Lie algebras. Our construction employs the physical principle first articulated in [Lovelace-1971] and demonstrated in [Goddard-Thorn-1972] and [Brower-1972], that in the critical dimension $d = 26$ of bosonic string theory, Virasoro-type Ward identities cancel two full sets of oscillators. This principle now has at least three methods of realization in physics (cf. [Green-Schwarz-Witten-1987] chapters 2,3): Covariant quantization, passage to the light-cone gauge (also called "construction of the transversal space"), and BRST cohomology. The use of the BRST method for quantizing strings was pioneered in [Kato-Ogawa-1983].

The use of this cancellation principle in mathematics is also far from new. For example, BRST cohomology was used in [Frenkel-Garland-Zuckerman-1986] to compute the character of the space of physical states for a bosonic string propagating in Lorentzian 26-space. The covariant quantization method is also used in the analysis of three Lie algebras, each of which was named the "Monster Lie algebra" upon construction:

(1) The first example is the Kac-Moody Lie algebra whose simple roots are those of the reflection group of the even unimodular Lorentzian lattice $II_{1,25}$. The existence of a Monster action was suggested in [Borcherds-Conway-Queen-Sloane-1984], and Frenkel [Frenkel-1985] used the no-ghost theorem to bound its root multiplicities by coefficients of the modular form $1/\Delta$.
(2) The second example is the Borcherds-Kac-Moody Lie algebra whose real simple roots are those given in the first example, but it has additional imaginary simple roots of multiplicity 24 on positive multiples of the Weyl vector. This was constructed in [Borcherds-1986], and is now known as the "fake Monster Lie algebra". In [Borcherds-1990], Borcherds showed that the root multiplicities saturate the $1/\Delta$ bound given by Frenkel.
(3) The third and last example is the generalized Kac-Moody Lie algebra constructed in [Borcherds-1992]. This is the only example known to have a faithful Monster action.

Supersymmetric variants of this construction have been used, e.g., in [Scheithauer-2000].

3.1. Virasoro preliminaries.

Recall the Virasoro algebra vir is the complex vector space with a distinguished basis given by symbols $K, \{L_n\}_{n\in\mathbb{Z}}$, with the Lie bracket defined by $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}K$ for all $m, n \in \mathbb{Z}$, and $[L_n, K] = 0$ for all $n \in \mathbb{Z}$. We write vir^+ for the subalgebra spanned by $\{L_n\}_{n\geq 1}$, vir^- for the subalgebra spanned by $\{L_n\}_{n\leq -1}$, and vir^0 for the 2-dimensional subalgebra spanned by L_0 and K.

Definition 3.1.1. Let V be a representation of the Virasoro algebra, i.e., a complex vector space equipped with operators $\{L_n\}_{n\in\mathbb{Z}}$ and K that satisfy the Virasoro relations.

(1) We say that V is positive-energy if:
  (a) L_0 acts diagonalizably, i.e., V splits into a direct sum of L_0-eigenspaces.
  (b) vir^+ acts locally nilpotently, i.e., for all $v \in V$, there exists $N > 0$ such that for all sequences $n_1, \ldots, n_k$ of positive integers satisfying $n_1 + \cdots + n_k > N$, we have $L_{n_1} \cdots L_{n_k} v = 0$.
(2) We say that V has central charge $c \in \mathbb{C}$ if $Kv = cv$ for all $v \in V$.
(3) A bilinear or sesquilinear form $(,)$ on V is Virasoro-invariant if for each $n \in \mathbb{Z}$, L_n is adjoint to L_{-n}.

(4) We say that V is unitarizable if it admits a positive-definite hermitian Virasoro-invariant form.
(5) A vector $v \in V$ is:
  (a) primary (or physical) if $L_i v = 0$ for all $i > 0$, i.e., if vir^+ acts trivially.
  (b) spurious if it has the form $L_{-i}w$, for some $w \in V$ and some $i > 0$.
  (c) null if it is primary and spurious.
  We write P^r to denote the subspace of primary vectors v such that $L_0 v = rv$. We may write P^r_V when there is more than one representation under consideration. We write $null^r_V$ to denote the subspace of null vectors of weight r in V.

Given $c, h \in \mathbb{C}$, we define the Verma module $M_{h,c}$ as the induced module $vir \otimes_{vir^+ \oplus vir^0} \mathbb{C}_{h,c}$, where $\mathbb{C}_{h,c}$ is the one dimensional vector space with trivial action of vir^+, such that L_0 has eigenvalue h, and K has eigenvalue c. We write $L_{h,c}$ for the unique irreducible quotient of $M_{h,c}$ with highest weight h.

We note that any Verma module $M_{h,c}$ is positive-energy and admits a canonical invariant bilinear form, called the Shapovalov form, which is nonsingular if and only if $M_{h,c}$ is irreducible (see [Feigin-Fuchs-1984] or [Kac-Raina-1987]). In the following lemma, we set $c = 24$, because that is the case we will use. Similar results hold for other central charges in the open interval $(1, 25) \subset \mathbb{R}$.

Lemma 3.1.2. *Let V be a positive energy vir-module of central charge $c = 24$. Then V is unitarizable if and only if the L_0-spectrum of V is non-negative and L_{-1} annihilates the zero weight subspace $V_0 = \operatorname{Ker} L_0$. If these conditions hold, then V decomposes as an orthogonal direct sum of irreducible vir-modules as follows:*

$$V \cong \bigoplus_{h \geq 0} P^h_V \otimes L_{h,24} \cong (V_0 \otimes L_{0,24}) \oplus \bigoplus_{h>0} P^h_V \otimes M(c,h)$$

Proof. The Kac determinant formula yields the following unitarizability and irreducibility results (where detailed proofs can be found in the first four sections of [Kac-Raina-1987]):

(1) If $h \neq 0$, then $M_{h,24}$ is irreducible (i.e., equal to $L_{h,24}$), and the category of lowest weight vir-modules with central charge 24 and lowest weight h is completely reducible, with $L_{h,24}$ as its unique irreducible object.
(2) For the case $h = 0$, $M_{0,24}$ has a unique proper submodule $M_{1,24}$, and $L_{0,24} = M_{0,24}/M_{1,24}$.
(3) $M_{h,24}$ is unitarizable if and only if $h > 0$. $L_{h,24}$ is unitarizable if and only if $h \geq 0$.

Given V unitarizable, we choose a Virasoro-invariant Hermitian form. By self-adjointness of L_0, any two L_0-eigenspaces with different eigenvalues are orthogonal. For any coset $h + \mathbb{Z}$, we let $V_{h+\mathbb{Z}}$ denote the subspace spanned by L_0-eigenvectors with eigenvalues in $h + \mathbb{Z}$. Then any disjoint cosets of $\mathbb{Z}$ yield orthogonal subspaces of V, so we may consider one coset at a time. Within each coset, the decomposition follows from induction by L_0-eigenvalue, noting that within each L_0-eigenspace the primary subspace is orthogonal to the spurious subspace. The necessary and sufficient condition then follows immediately. □

Lemma 3.1.3. *Let g be an automorphism of $V^\natural$. Then the irreducible g-twisted module $V^\natural(g)$ is unitarizable. If $g \neq 1$, then $V^\natural(g)$ decomposes as a direct sum of Virasoro submodules of the form $M(c,h)$ as h ranges over a multiset of positive rational numbers of the form $\frac{k}{|g|^2}$.*

Proof. For the case $g = 1$, it suffices to show that $V^\natural$ is unitarizable - this is well-known, and follows straightforwardly from its conformal structure. By Theorem 2.2.9, if $g \neq 1$, then the lowest weight space of $V^\natural(g)$ has positive L_0-eigenvalue, so by Lemma 3.1.2, we may decompose $V^\natural(g)$ as a sum of unitarizable Verma modules. □

Lemma 3.1.4. *Let $\lambda \in \mathbb{R}^{r,s}$, and let $\pi^{r,s}_\lambda$ denote the corresponding irreducible Heisenberg module, endowed with the Virasoro action arising from the conformal element $\frac{1}{2}\sum_\mu a^\mu_{-1} b^\mu_{-1} 1 \in \pi^{r,s}_0$. Then*

$\pi^{r,s}_\lambda$ is a positive-energy vir-module of central charge 2 that admits a natural Virasoro-invariant Hermitian form for which b^μ_n is adjoint to b^μ_{-n}, for all $n \in \mathbb{Z}$ and all vectors $b^\mu \in \mathbb{R}^{r,s}$.

Proof. By irreducibility, the conditions on the Hermitian form determine it up to a normalization. See, e.g., section 2.2 of [Frenkel-Garland-Zuckerman-1986]. □

Lemma 3.1.5. *Let V be a Virasoro representation equipped with a Virasoro-invariant bilinear form $(,)$.*

1. *The L_0-eigenspaces in V are pairwise orthogonal with respect to the form.*
2. *$L_{-1}P^0 \subseteq P^1$. In particular, $L_{-1}P^0 \subseteq null^1_V$.*
3. *A vector in P^1 is orthogonal to all other vectors in P^1 if and only if it lies in $null^1_V$.*

Proof. All claims follow from straightforward calculations. □

There is a stronger notion of invariance for vertex algebras (found, e.g., in [Frenkel, et al, 1993]).

Definition 3.1.6. Let V be a conformal vertex algebra. A bilinear form $(,)$ on V is invariant if for all $u, v, w \in V$, the identity $(Y(u,z)v, w) = (v, Y(e^{zL_1}(-z^{-2})^{L_0}u, z^{-1})w)$ holds. Equivalently, if u has conformal weight i, then the adjoint of the operator u_n is $(-1)^i \sum_{j\geq 0}(L_1^j(u))_{2i-j-n-2}/j!$.

Lemma 3.1.7. *Let V be a conformal vertex algebra. Then any invariant bilinear form is Virasoro-invariant.*

Proof. This follows from evaluating the defining identity with u chosen to be the Virasoro vector ω. □

Proposition 3.1.8. *Let V be a holomorphic C_2-cofinite vertex operator algebra of CFT type, and let g be a finite order automorphism of V such that the twisted modules $V(g^i)$ have strictly positive L_0-spectrum for all $g^i \neq 1$. Let L be a hyperbolic lattice such that $L^\vee/L$ has the discriminant form of ${}^gV$, let $M \subset L$ be a hyperbolic sublattice of finite index, and consider the conformal vertex algebra W given by applying AT_M to the $M^\vee/M$-graded abelian intertwining algebra given by unrolling ${}^gV$ (by Proposition 2.4.3, this is isomorphic to $AT_L({}^gV)$ viewed as an $M^\vee(-1)$-graded vertex algebra supported on $L^\vee(-1)$). Let $\langle,\rangle_W$ be the bilinear form on W induced by the contragradient pairings between the component modules $V^{(i,j)}$ in ${}^gV$ and the contragradient pairings between the modules $\pi^{1,1}_\lambda$ and $\pi^{1,1}_{-\lambda}$ for the Heisenberg vertex algebra for $\mathbb{R}^{1,1}$, normalized such that*

$$\langle \mathbf{1}, Y(e^{zL_1}(-z^2)^{-L_0}u, z^{-1})v\rangle_W = \langle Y(u,z)\mathbf{1}, v\rangle_W$$

for all $u \in ({}^gV)^\lambda \otimes \pi^{1,1}_\lambda$ and $v \in ({}^gV)^{-\lambda} \otimes \pi^{1,1}_{-\lambda}$. Then $\langle,\rangle_W$ is an invariant bilinear form on W satisfying the following conditions:

1. *For any $i, j \in L^\vee$ satisfying $i + j \neq 0$, we have $\langle u, v\rangle_W = 0$ for any $u \in ({}^gV)^i$ and $v \in ({}^gV)^j$.*
2. *If $L_0u = ru$ and $L_0v = sv$, and $r \neq s$, then $\langle u, v\rangle_W = 0$.*
3. *If $h \in \widetilde{C_{\mathrm{Aut}\,V}(g)}$, then $\langle hu, hv\rangle_W = \langle u, v\rangle_W$.*

Proof. By Theorem 2.13 in [Scheithauer-1998] (which generalizes earlier results like [Li-1994] Theorem 3.1 to the setting we need), if A is a conformal vertex algebra on which L_1 acts locally nilpotently, the dual space $\mathrm{Hom}(A_0/L_1A_1, \mathbb{C})$ is isomorphic to the space of invariant bilinear forms on A, and an isomorphism is given by sending a map f to the form $(u, v) = (f \circ \pi)(u^*_{-1}v)$, where

1. π is the projection from A to the weight zero space A_0.
2. $u^*_n = (-1)^i \sum_{j\geq 0}(\frac{L_1^j}{j!}u)_{2i-j-n-2}$ when $L_0u = iu$.

We consider the unique functional f on W_0 that sends the unit $\mathbf{1}$ to 1, and annihilates all weight zero $a \otimes b$ for $a \in \pi^{1,1}_\lambda$ and $b \in ({}^gV)^\lambda$ when $\lambda \in L^\vee(-1) \setminus \{0\}$. Since f annihilates all vectors in the image of L_1 on the weight one subspace, it corresponds to an invariant bilinear form $(,)_f$. Since $({}^gV)^0 \otimes \pi^{1,1}_0$ has a unique invariant bilinear form up to normalization, the contragradient pairing

coincides with $(,)_f$ on this subspace. It is clear that both $\langle,\rangle_W$ and $(,)_f$ satisfy the homogeneity and $\widetilde{C_{\mathrm{Aut}\,V}(g)}$-invariance conditions listed in the statement of the proposition. It remains to show that $(,)_f$ coincides with $\langle,\rangle_W$ in general, and for that it suffices to restrict our view to a pairing between $u \in ({}^gV)^\lambda \otimes \pi^{1,1}_\lambda$ and $v \in ({}^gV)^{-\lambda} \otimes \pi^{1,1}_{-\lambda}$. In this case,

$$\begin{aligned}(u,v)_f &= (f \circ \pi)(u^*_{-1}v)\\ &= (\mathbf{1}, u^*_{-1}v)_f\\ &= \langle \mathbf{1}, u^*_{-1}v\rangle_W\end{aligned}$$

By our normalization, we have the equality

$$\langle \mathbf{1}, Y(e^{zL_1}(-z^2)^{-L_0}u, z^{-1})v\rangle_W = \langle Y(u,z)\mathbf{1}, v\rangle_W$$

of elements of $\mathbb{C}[z]$. Isolating the constant term yields $\langle \mathbf{1}, u^*_{-1}v\rangle_W = \langle u, v\rangle_W$. We conclude that $(u,v)_f = \langle u,v\rangle_W$, so $\langle,\rangle_W$ is invariant.

□

3.2. **Old covariant quantization.** In covariant quantization, one begins with a naïve space of states, that has redundancies due to internal symmetry. In the case of internal Virasoro symmetry, to obtain the relevant space of states that couple, one considers the "physical" subspace of states that are invariant under the action of vir^+, and quotients by the "spurious" subspace, which is spanned by the image of the augmentation ideal in $U(vir^-)$ together with the radical of the induced Hermitian form.

An enhancement of the original covariant approach was used in [Borcherds-1992] to construct the Monster Lie algebra as the space of physical states arising from the vertex algebra $V^\natural \otimes V_{II_{1,1}(-1)}$. We will use a variant of this method to construct our monstrous Lie algebras.

Definition 3.2.1. Let $\mathrm{Rep}^+_c(vir)$ denote the groupoid whose objects are positive energy Virasoro representations of central charge c with Virasoro-invariant bilinear forms, and whose morphisms are Virasoro-equivariant isometries. We define the functor OCQ, from $\mathrm{Rep}^+_c(vir)$ to the groupoid of vector spaces with bilinear forms, as $V \mapsto P^1_V/null^1_V$.

Lemma 3.2.2. *We list some relatively straightforward facts:*

(1) *If W is a conformal vertex algebra that is positive-energy, then $P^1/L_{-1}P^0$ has a Lie algebra structure induced by setting $[u,v] = u_0v$, i.e., the coefficient of z^{-1} in $Y(u,z)v$. If a group G acts on W by conformal vertex algebra automorphisms, then G acts on $P^1/L_{-1}P^0$ by Lie algebra automorphisms.*

(2) *If W is a positive-energy conformal vertex algebra equipped with an invariant bilinear form (in particular, with L_i adjoint to L_{-i} for each $i \in \mathbb{Z}$), then the Lie algebra $P^1/L_{-1}P^0$ has an invariant inner product induced from W, and the radical of that inner product is an ideal. In particular, $OCQ(W) = (P^1/L_{-1}P^0)/rad(,)$ is a Lie algebra equipped with a nondegenerate invariant inner product. If a group G acts on W by conformal vertex algebra automorphisms, and the invariant bilinear form is G-invariant, then $OCQ(W)$ has a natural G action by Lie algebra automorphisms, and the induced bilinear form is G-invariant.*

Proof. The first sentence of the first claim is given in section 3 of [Borcherds-1992]. The Lie algebra structure of $W/L_{-1}W$ is known from [Borcherds-1986], so it suffices to show that a_0b is primary if a and b are. This follows from the formula $L_n(a_0b) = a_0L_nb + \sum_{m\geq -1}\binom{n+1}{m+1}(L_ma)_{n-m}b$ for $n \geq -1$ together with the L_{-1}-derivative rule. The second sentence follows from the fact that automorphisms preserve the Virasoro action (hence the property of being primary, the grading, and the L_{-1} operator), and the Lie bracket $(u,v) \mapsto u_0v$.

The first sentence of the second claim is given in the proof of Theorem 6.1 of [Borcherds-1992]. Invariance follows immediately from the adjunction formula $u^*_n = (-1)^i\sum_{j\geq 0}(\frac{L_1^j}{j!}u)_{2i-j-n-2}$ when

$n = 0$, $i = 1$, and $L_1 u = 0$. The ideal property follows immediately from invariance. The part about group actions follows from the same reason as in the first claim. □

The following proposition seems to capture the core of Lovelace's principle of oscillator cancellation. It is sometimes given the name "No Ghost Theorem", because it immediately implies that quantization yields a positive-definite space of states. That is, if we choose a unitary structure on V, then transport of structure along the component isomorphisms yields a unitary structure on the output. Here, the word "ghost" means negative-norm state, whose absence is important in physics, since the existence of such ghosts obstructs reasonable formulations of causality. In the next section, we will consider a different type of ghost, named after Faddeev-Popov, and they get their name from their violation of the spin-statistics theorem.

Proposition 3.2.3. *Let V be a unitarizable Virasoro representation of central charge 24 equipped with a non-degenerate Virasoro-invariant bilinear form, let $\pi_\alpha^{1,1}$ be the Heisenberg module attached to the linear function α on $\mathbb{R}^{1,1}$, and let $W = V \otimes \pi_\alpha^{1,1}$. Then:*

1. *If $\alpha \neq 0$, then $OCQ(W)$ is isomorphic to the subspace $V_{1-(\alpha,\alpha)}$ of V on which L_0 acts by $1 - (\alpha, \alpha)$, where $(,)$ is the induced bilinear form on the dual space of $\mathbb{R}^{1,1}$.*
2. *If $\alpha = 0$, then $OCQ(W)$ is isomorphic to $V_0 \otimes (\pi_0^{1,1})_1 \oplus V_1 \otimes (\pi_0^{1,1})_0 \cong V_0 \oplus V_0 \oplus V_1$, where V_1 is the subspace of V on which L_0 acts by identity.*

Proof. One may find a full proof of a slightly altered version of this claim in the appendix of [Jurisich-1998]. It is an expansion of the sketch of a proof for Theorem 5.1 in [Borcherds-1992], which is in turn a minor alteration of the argument in [Goddard-Thorn-1972]. In order to pass from the proof in [Jurisich-1998] to a proof of our claim, we need to make the following changes:

1. The vector α here is r there, and we remove the restriction that it lie in the lattice $II_{1,1}$, since that restriction is never used in the proof.
2. We remove the unnecessary assumption that V is a vertex operator algebra, as the proof does not use any extra structure on the vector space V except the Virasoro action.
3. The space W here is $\mathcal{H}$ there.
4. Although it isn't necessary for this paper, we remove the assumption that the weight 0 subspace V_0 of V, defined as the kernel of L_0, is one dimensional. One should therefore ignore the paragraph concerning the case $r = 0$ at the end of the proof in [Jurisich-1998]. Instead, we note that all weight one elements of $V \otimes \pi_0^{1,1}$ are primary, so $P^1 \cong V_0 \oplus V_0 \oplus V_1$. The fact that L_{-1} annihilates P^0 follows from the classification of unitarizable Virasoro representations of central charge 24, given in Lemma 3.1.2. By our assumption on V and the fact that $(\pi_0^{1,1})_1$ has a natural real structure identified with $\mathbb{R}^{1,1}$, the bilinear form on the quotient $P_W^1 / L_{-1} P_W^0$ is nondegenerate, i.e., the radical vanishes.

□

Definition 3.2.4. For each $g \in \mathbb{M}$, we define:

$$\mathfrak{m}_g = OCQ \circ AT_{II_{1,1}(N)}({}^g_N V^\natural)$$

where N is the level of $T_g(\tau)$. This is a Lie algebra, by Lemma 3.2.2, and we call it the monstrous Lie algebra attached to g.

Remark 3.2.5. By Proposition 2.4.3, $\mathfrak{m}_g$ is isomorphic to $OCQ \circ AT_L({}^g V^\natural)$ for any L such that $L^\vee / L$ is the discriminant form for ${}^g V^\natural$. Furthermore, we may replace N with any multiple rN, and get an isomorphic object (although it is now a $II_{1,1}(-1/rN)$-graded Lie algebra supported on the sublattice $II_{1,1}(-1/N)$).

Proposition 3.2.6. *Let $g \in \mathbb{M}$. Then the Lie algebra $\mathfrak{m}_g$ satisfies the following properties:*

(1) *$\mathfrak{m}_g$ is graded by $II_{1,1}(-1/N)$, and its degree (0,0) subspace is $\mathbb{C}^2$, identified with the weight one space $(\pi_0^{1,1})_1$ in the Heisenberg module. Furthermore, the degree is given by the eigenvalue under the adjoint action of the degree $(0,0)$-subspace.*

(2) *$\mathfrak{m}_g$ has a nondegenerate invariant bilinear form, and if u and v are homogeneous vectors of degree $a, b \in II_{1,1}(-1/N)$ such that $a + b \neq 0$, then u and v are orthogonal.*

(3) *$\mathfrak{m}_g$ has a canonical action of $\widetilde{C_{\mathbb{M}}(g)}$ by homogeneous Lie algebra automorphisms that preserve the bilinear form.*

(4) *For each nonzero $(a, b) \in II_{1,1}(-1/N)$, the degree (a, b) subspace is isomorphic as a $\widetilde{C_{\mathbb{M}}(g)}$-module to the subspace of $V^\natural(g^a)$ on which $\tilde{g}$ acts by $e(\frac{b}{N})$ and L_0 acts by $1 + \frac{ab}{N}$. The dimension of this subspace is equal to the coefficient $c^g_{a,b}(ab/N)$ of the vector-valued modular function $F^g(\tau)$ given in Definition 2.3.5. In particular, we have a graded vector space isomorphism between $\mathfrak{m}_g$ and the Lie algebra L_g defined in [Carnahan-2012] section 4.2 (written W_g in Proposition 4.4 of that paper).*

Proof. We prove the claims in the order they are stated.

(1) The grading property follows from the fact that $AT_{II_{1,1}(N)}({}^g_N V^\natural)$ is graded by $II_{1,1}(-1/N)$. Because $({}^gV^0)_1 = 0$ and $({}^gV^0)_0 = \mathbb{C}\mathbf{1}$, Proposition 3.2.3 identifies the degree $(0,0)$ space with $V_0 \otimes (\pi_0^{1,1})_1 = (\pi_0^{1,1})_1$. The eigenvalue assertion follows from the identification of the degree $(0,0)$ subalgebra, and the fact that for $u \in (\pi_0^{1,1})_1$ and $v \in \pi_\alpha^{1,1}$, the z^{-1}-coefficient of $Y(u,z)v$ is $\alpha(u)v$.

(2) The fact that the form is nondegenerate and invariant is given in Lemma 3.2.2. The degree restriction follows from the corresponding property of $AT_{II_{1,1}(N)}({}^g_N V^\natural)$, given in Proposition 3.1.8.

(3) By Proposition 2.5.2 (and Theorem 2.2.9), there is a canonical homogeneous action of $\widetilde{C_{\mathbb{M}}(g)}$ on ${}^gV^\natural$ by homogeneous automorphisms, and this endows $AT_{II_{1,1}(N)}({}^g_N V^\natural)$ with a corresponding action. By the third claim of Lemma 3.2.2, homogeneous automorphisms of $AT_{II_{1,1}(N)}({}^g_N V^\natural)$ are taken to homogeneous Lie algebra automorphisms of $\mathfrak{m}_g$.

(4) The identification of $\widetilde{C_{\mathbb{M}}(g)}$-modules follows from Proposition 3.2.3. By the third listed claim of Theorem 2.5.4, the dimension of this space is equal to the Fourier coefficient $c^g_{a,b}(ab/N)$ of F^g. To compare with the graded parts of L_g, we note that by [Carnahan-2012] Proposition 4.4, the $(a, b/N)$ root space of L_g also has dimension equal to the Fourier coefficient $c^g_{a,b}(ab/N)$ of the vector-valued modular form F^g.

□

3.3. **BRST cohomology.** In addition to covariant quantization, there is the "modern" method of Brecchi, Rouet, and Stora (with Tyutin's independent unpublished work often credited), where one extracts the relevant space of states as a cohomology group. This has the advantage of being "very functorial", but the disadvantage of being rather complicated - one extracts the BRST current and the resulting differential by way of a super-variational principle. In the string-theoretic context, BRST quantization was pioneered in [Kato-Ogawa-1983]. One may find string-theoretic expositions of this method in chapter 3 of [Green-Schwarz-Witten-1987], Chapter 4 of [Polchinski-1998], and section 7 of [D'Hoker-1997], the last of which is intended for a more mathematical audience.

In the mathematical world, BRST quantization of Virasoro representations was introduced using semi-infinite cohomology in [Feigin-1984], and generalized in [Frenkel-Garland-Zuckerman-1986]. Proofs can be found in [Lian-Zuckerman-1991]. Borcherds's use of the no-ghost theorem for moonshine was reinterpreted in terms of BRST cohomology in [Lian-Zuckerman-1995]. We will give a modified presentation that is applicable to the vertex algebra objects we have constructed in the previous section.

We introduce the ghost vertex superalgebra V_{ghost} of central charge -26. By boson-fermion correspondence, for any half-integer λ, there is an isomorphism between the bosonic vertex superalgebra attached to the odd positive definite lattice $\mathbb{Z}$ with conformal vector shifted by λ, and the ghost bc-system of weight $\frac{1}{2}+\lambda$. With the notation of [Frenkel, et al, 1988], the bc system of weight $\frac{1}{2}+\lambda$ is generated by the "anti-ghost" field $b(z)$ that is identified with the bosonic lattice field $z^{-\lambda}X(1,z)$ of conformal weight $\frac{1}{2}-\lambda$, and the "ghost" field $c(z)$ identified with $z^{\lambda}X(-1,z)$ of conformal weight $\frac{1}{2}+\lambda$. The conformal vector is then given by $\frac{1}{2}h(-1)^2+\lambda h(-2)$, and it has central charge $1-12\lambda^2$. For V_{ghost}, we set $\lambda=\frac{3}{2}$.

We briefly describe some cohomology functors and some key properties. There is nothing original in this subsection except perhaps the method of presentation.

Definition 3.3.1. Given a Virasoro ("matter") representation V^m of central charge D and energy-momentum tensor $T^m(z)=\sum_n L^m(n)z^{-n-2}$, we define the Virasoro module $C^{\infty/2+*}(vir,V^m)=V^m\otimes V_{ghost}$, which has central charge $D-26$. The old and new BRST currents are defined to be $j^{BRST,old}=cT^m+\frac{1}{2}:cT^{bc}:$ and $j^{BRST,new}=j^{BRST,old}+\frac{3}{2}\partial^2 c$. The $U(1)_{bc}$ current is $j^{bc}=:bc:$. We consider the following operations on $C^{\infty/2+*}(vir,V^m)$:

(1) The old and new BRST operators

$$\begin{aligned}Q^{old}&=j_0^{BRST,old}=\frac{1}{2\pi i}\oint dz j^{BRST,old}(z)\\&=\sum_n L^m(n)c_{-n}+\sum_{m<n}(m-n):b_{m+n}c_{-m}c_{-n}:\\Q^{new}&=j_0^{BRST,new}=\frac{1}{2\pi i}\oint dz j^{BRST,new}(z)\\&=\sum_{r\in\mathbb{Z}}(L_r^{(m)}-\delta_{r,0})c_{-r}-\frac{1}{2}\sum_{r,s\in\mathbb{Z}}(r-s):c_{-r}c_{-s}b_{r+s}:\end{aligned}$$

(2) The old and new ghost number operators

$$\begin{aligned}U^{old}&=j_0^{bc}=\frac{1}{2}(c_0b_0-b_0c_0)+\sum_{n=1}^{\infty}(c_{-n}b_n-b_{-n}c_n)\\U^{new}&=U^{old}+\frac{3}{2}=1+c_0b_0+\sum_{n=1}^{\infty}(c_{-n}b_n-b_{-n}c_n)\end{aligned}$$

An eigenvector of U with eigenvalue k is said to have ghost number k.

When $Q^2=0$, we call $C^{\infty/2+*}(vir,V^m)$ the "BRST complex". We also define the "relative subcomplex" ([Zuckerman-1989], section 4) to be the subspace of $V\otimes V_{ghost}$ annihilated by b_0.

Remark 3.3.2. We have "old" and "new" versions of Q and U, to reflect the use in the literature. Papers from the 1980s and early 1990s (e.g., [Kato-Ogawa-1983], [Frenkel-Garland-Zuckerman-1986], [Green-Schwarz-Witten-1987], [Lian-Zuckerman-1991]) use the old versions, while the sources starting from the mid-1990s (e.g., [Polchinski-1998], [D'Hoker-1997], [Lian-Zuckerman-1995]) add an additional $\frac{3}{2}\partial^2 c$ to the BRST current and a constant 3/2 to the ghost number. This change impacts the output of various functors in the following ways:

(1) The addition of the derivative term makes the new BRST current a "$(1,0)$-form as a quantum operator". Concretely, this means the OPE with the total stress-energy tensor T^m+T^{bc} has the form $T(z)j^{BRST}(0)\sim\frac{1}{z^2}j^{BRST}(0)+\frac{1}{z}\partial j^{BRST}(0)$. In the absence of the derivative, one has a pole of order 3 as the leading term. The extra simplicity makes many calculations easier, especially when quantizing in higher genus settings. However, it is irrelevant for the case at hand, since we just want to compute the cohomology spaces.

(2) When we work out the space of physical states, we find that it lies in ghost degree $-1/2$ in the old convention and 1 in the new convention. In particular, the space of interest was written $H^{-1/2}_{BRST}$ in older work, but is written H^1_{BRST} now. The older convention has the convenient property that Poincaré duality gives a symmetry around zero (see [Frenkel-Garland-Zuckerman-1986] Theorem 1.6 and the remark before Theorem 2.5), while the newer convention is more manageable when considering Gerstenhaber or BV structures on cohomology ([Lian-Zuckerman-1993]).

The calculations relevant to us are unchanged, because a total derivative integrates to zero when evaluating charges. That is, the old and new BRST cohomology functors are naturally isomorphic. To avoid unnecessary confusion, we will write H^1_{BRST} even when referring to results that use the old convention.

Lemma 3.3.3. *Let V^m be a positive-energy representation of Virasoro (called the "matter representation" in the physics literature). We have the following operator calculations in $V^m \otimes V_{ghost}$:*

(1) *$Q^2 = 0$ if and only if $D = 26$. That is, $C^{\infty/2+*}(V^m)$ is a complex with differential Q if and only if V^m has central charge 26.*
(2) *$[U, Q] = Q$, so Q increases ghost number by one.*
(3) *$[U, L_0] = 0$, so ghost number and L_0-eigenvalue provide a bigrading.*

Now, suppose V^m has central charge $D = 26$, so Q gives $V^m \otimes V^{ghost}$ the structure of a complex of vector spaces.

(1) *$[Q, L_0] = 0$, so the BRST complex and cohomology are graded by L_0-eigenvalues.*
(2) *$[Q, b_0] = L_0$. This implies the relative subcomplex is in fact a subcomplex, and all BRST-closed states with nonzero L_0-eigenvalue are BRST-exact. In particular, the cohomology is supported in weight zero.*

Proof. These claims follow from standard OPE calculations. One may find them enumerated in section 4 of [Zuckerman-1989].

□

Lemma 3.3.4. *If V is a conformal vertex algebra of central charge 26, then $H^1_{BRST}(V)$ is a Lie algebra, and morphisms of conformal vertex algebras are taken to maps of Lie algebras under this functor. In particular, if a group G acts on V by conformal vertex algebra automorphisms, then G naturally acts on $H^1_{BRST}(V)$ by Lie algebra automorphisms.*

Proof. For the Lie algebra claim, see [Lian-Zuckerman-1993] Theorem 2.2. While the authors do not make the explicit claim that morphisms of conformal vertex algebras are taken to Lie algebra homomorphisms, the statement follows from the explicit formulas in their proof together with the faithfulness of the "underlying vector space" functors. The claim about group actions then follows from the functoriality. □

We now have the BRST version of oscillator cancellation.

Proposition 3.3.5. *Let $\alpha \in \mathbb{R}^{1,1}$, and let $\pi^{1,1}_\alpha$ be the Heisenberg representation attached to α. If V is a unitarizable Virasoro representation of central charge 24, then $H^1_{BRST}(V \otimes \pi^{1,1}_\alpha)$ is isomorphic to the subspace $V_{1-\alpha^2}$ of V on which L_0 acts with eigenvalue $1 - \alpha^2$ (if $\alpha \neq 0$), and $V_0 \oplus V_0 \oplus V_1$ (if $\alpha = 0$). In particular, any positive definite Hermitian form on V induces a positive definite Hermitian form on $H^1_{BRST}(V \otimes \pi^{1,1}_\alpha)$.*

Proof. The case $\alpha \neq 0$ is covered in [Zuckerman-1989] Theorem 4.9 under the following translation of notation:

(1) $\pi^{1,1}_\alpha$ is written $\mathcal{H}(D, p)$, with $D = 2$ and $p = \alpha$.
(2) V is written $\mathcal{K}$.

(3) The BRST differential on $\mathcal{H}(D,p)\otimes\mathcal{K}\otimes V^{gh}$ is written Q_{mod}.
(4) $H^{u+3/2}_{BRST}(V\otimes\pi^{1,1}_{\alpha})$ is written $H^{u}_{Q_{mod}}(p)$ for the same values of u, and $p=\alpha$.
(5) The character $Tr(q^{L_0}|V)$ of $\mathcal{K}$ is written $\chi(\mathcal{K},q)=trq^{J_0}$.

The conclusion of the theorem is that $H^{-1/2}_{Q_{mod}}(p)\cong H^0_{rel}(p,Q_{mod})$, and that $\dim H^0_{rel}(p,Q_{mod})$ is the coefficient of $q^{-(p,p)/2}$ in the shifted character $Tr(q^{L_0-1}|V)$. (The statement given in the paper contains an erroneous division by Δ rather than q - this can be checked by comparison with the free boson case in Theorem 3.4, or using Corollary 2.27 in [Lian-Zuckerman-1991], which is proved in more detail.) This completes the case $\alpha\neq 0$.

The case $\alpha=0$ follows from an explicit computation of cocycle representatives, following the example of Proposition 2.9 in [Frenkel-Garland-Zuckerman-1986]: One obtains a basis of BRST cohomology given by $\{x\otimes L'_{-1}\wedge L'_{-2}\wedge\cdots\}$ as x ranges over a basis of $(\pi^{1,1}_0\otimes V)_1=((\pi^{1,1}_0)_1\otimes V_0)\oplus((\pi^{1,1}_0)_0\otimes V_1)\cong V_0\oplus V_0\oplus V_1$.

As we remarked shortly before the statement of Proposition 3.2.3, the positive definite Hermitian form on $H^1_{BRST}(V\otimes\pi^{1,1}_{\alpha})$ is inherited from V by the oscillator cancellation.

□

We conclude with a natural isomorphism between the two quantization functors. Let $\mathbf{1}\in V_{ghost}$ be the ghost vacuum vector - it is the unique (up to scaling) vector that is annihilated by $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n>0}$, and it has ghost number one.

Lemma 3.3.6. *The map $W\hookrightarrow W\otimes V_{ghost}$ given by $w\mapsto w\otimes\mathbf{1}$ induces a natural isomorphism $OCQ\Rightarrow H^1_{BRST}$, as the input W ranges over $\mathrm{Rep}^+_{26}(vir)$.*

Proof. This is proved at the end of section 4.4 in [Polchinski-1998]. Partial arguments can be found in section 3.2.1 of [Green-Schwarz-Witten-1987] and section 4.6 of [D'Hoker-1997] (which is in lecture 7 part F on the IAS web site).

When W is restricted to be a direct sum of representations of the form $V\otimes\pi^{r,1}_{\lambda}$ for V unitarizable of central charge $25-r$ and $r\leq 25$, one may instead appeal to the no-ghost theorem together with complete reducibility in the V part. This is the argument of Lemma 6.7 in [Lian-Zuckerman-1995].

□

3.4. **Borcherds-Kac-Moody property.** We discuss a class of infinite dimensional Lie algebras defined by Borcherds in [Borcherds-1988], where he called them "generalized Kac-Moody algebras". Borcherds-Kac-Moody algebras are quite similar to Kac-Moody Lie algebras, since they are defined by generators and relations encoded by a generalized Cartan matrix, and they admit representation-theoretic data like Weyl character formulas. They differ in the sense that the conditions defining a suitable generalized Cartan matrix are weaker, and most importantly, simple roots are allowed to be imaginary. We will show that the Monstrous Lie algebras defined earlier in this section are Borcherds-Kac-Moody algebras, using a minor variant of the characterization from [Borcherds-1995], Theorem 1.

The definitions of Borcherds-Kac-Moody algebras (or generalized Kac-Moody algebras) are not uniform in the literature. We will use the definition given in [Jurisich-1996] and [Jurisich-2004], because that is the definition that is both general enough to account for the Lie algebras we need and for which there is a published proof of a homological description of the denominator formula. We needed the homological description in order to produce twisted denominator identities in [Carnahan-2012].

Definition 3.4.1. If I is a countable index set, a matrix $A=(a_{i,j})_{i,j\in I}$ of real numbers is called a generalized Cartan matrix if it satisfies the following conditions:

(1) A is symmetrizable, i.e., there is a diagonal matrix Q whose diagonal entries $Q_{i,i}=q_i$ are positive real numbers, such that QA is symmetric.

(2) $a_{i,j} < 0$ if $i \neq j$.
(3) If $a_{i,i} > 0$, then $a_{i,i} = 2$ and for all $j \in I$, $a_{i,j} \in \mathbb{Z}$.

Given a generalized Cartan matrix $(a_{i,j})_{i,j\in I}$, its universal Borcherds-Kac-Moody algebra $\mathfrak{g}(A)$ is the Lie algebra with generators $\{h_i, e_i, f_i\}_{i,j\in I}$, and the following relations:

(1) $\mathfrak{sl}_2$ relations: $[h_i, e_k] = a_{i,k}e_k$, $[h_i, f_k] = -a_{i,k}f_k$, $[e_i, f_j] = \delta_{i,j}h_i$.
(2) Serre relations: If $a_{i,i} > 0$, then $\mathrm{ad}(e_i)^{1-2a_{i,j}}(e_j) = \mathrm{ad}(f_i)^{1-2a_{i,j}}(f_j) = 0$.
(3) Orthogonality: If $a_{i,j} = 0$, then $[e_i, e_j] = [f_i, f_j] = 0$.

A Borcherds-Kac-Moody algebra is a Lie algebra of the form $(\mathfrak{g}(A)/C).D$, where C is a central ideal, and D is a commutative Lie algebra of outer derivations.

From these starting data, one can build up a theory that closely mirrors that of finite dimensional semisimple Lie algebras, but with some complications. The general theory is substantially developed with proofs in [Jurisich-1996].

There are two main differences between this definition and others:

(1) Other definitions may start with the symmetrized matrix QA instead of A. This means one often sees the condition that if $a_{i,i} > 0$, then for all $j \in I$, $2a_{i,j}/a_{i,i} \in \mathbb{Z}$.
(2) Other definitions may have central generators $h_{i,j}$ for $i \neq j$, with the relation $[e_i, f_j] = \delta_{i,j}h_i$ replaced with $[e_i, f_j] = h_{i,j}$. As we will see in the next lemma, these necessarily vanish in the cases of interest to us. That is, if we were to use the more general definitions, these generators would end up in the central ideal C.

Most of the Borcherds-Kac-Moody algebras that have naturally appeared in mathematical practice have infinitely many simple roots, but a finite dimensional Cartan subalgebra, i.e., I is infinite, but one chooses C to have finite codimension in the span of $\{h_i\}_{i\in I}$. We give a method to recognize most of the interesting cases:

Lemma 3.4.2. *Any complex Lie algebra $\mathfrak{g}$ satisfying the following conditions is Borcherds-Kac-Moody:*

(1) *$\mathfrak{g}$ admits a nonsingular invariant symmetric bilinear form, i.e., $(x, y) = (y, x)$, $(x, [y, z]) = ([x, y], z)$ for all $x, y, z \in \mathfrak{g}$, and if $(x, y) = 0$ for all $y \in \mathfrak{g}$, then $x = 0$.*
(2) *$\mathfrak{g}$ has a self-centralizing subalgebra H, called a Cartan subalgebra, such that $\mathfrak{g}$ is the sum of eigenspaces under the action of H, and the nonzero eigenvalues, called roots, have finite multiplicity.*
(3) *H has a totally real structure $H_{\mathbb{R}} \subset H$ (i.e., such that $H = H_{\mathbb{R}} \oplus \sqrt{-1}H_{\mathbb{R}}$), such that the restriction of the bilinear form is real-valued, and all roots are elements of the dual $H_{\mathbb{R}}^{\vee}$.*
(4) *There exists an element $h \in H_{\mathbb{R}}$, such that:*
   (a) *the centralizer of h in $\mathfrak{g}$ is H,*
   (b) *for any $M \in \mathbb{R}$, there exist at most finitely many roots α such that $|\alpha(h)| < M$.*
   This vector h is called a regular element. If $\alpha(h) < 0$ then we say that α is negative, and if $\alpha(h) > 0$ then we say that α is positive.
(5) *The norms of roots under the inner product $(,)$ are bounded above.*
(6) *Any two roots of non-positive norm that are both positive or both negative have inner product at most zero, and if the inner product is zero, then their root spaces commute.*

Proof. This is essentially theorem 1 in [Borcherds-1995], but we remove the hypothesis that $\mathfrak{g}$ is defined over the real numbers. Borcherds's proof still works, because the reconstruction of $\mathfrak{g}$ by recursively adding simple roots following increasing values of $\alpha(h)$ only uses the real structure on H, not on $\mathfrak{g}$. To match Borcherds's definition with ours, we would need to allow the generators $h_{i,j}$, but Borcherds shows in the course of his proof that $h_{i,j} = [e_i, f_j] = 0$ when $i \neq j$. Thus, the proof shows that we obtain a Borcherds-Kac-Moody algebra in our restricted sense. □

Recall that an element $g \in \mathbb{M}$ is Fricke if and only if the McKay-Thompson series T_g is invariant under the level N Fricke involution $\tau \mapsto \frac{-1}{N\tau}$ for some N.

Proposition 3.4.3. *For each Fricke element $g \in \mathbb{M}$, the Lie algebra $\mathfrak{m}_g$ from Definition 3.2.4 is a Borcherds-Kac-Moody Lie algebra.*

Proof. It suffices to check that the conditions in Lemma 3.4.2 hold.

(1) The bilinear form is induced from the contragradient form, as shown in Proposition 3.1.8.
(2) We define H to be the $(0,0)$-graded piece of $\mathfrak{m}_g$. By the first claim in Proposition 3.2.6, this is isomorphic to $\mathbb{C}^2$. Furthermore, the left multiplication by H is given by the Heisenberg action on the tensor product vertex algebra, and their eigenvalues are given by degree. The fact that anything not in H has nonzero eigenvalue implies the self-centralizing condition is satisfied.
(3) H is naturally identified with $II_{1,1}(-1/N) \otimes_{\mathbb{Z}} \mathbb{C}$, which has a natural real structure $H_{\mathbb{R}} = II_{1,1}(-1/N) \otimes_{\mathbb{Z}} \mathbb{R}$. The inner product on $H_{\mathbb{R}}$ is real valued, and the roots, viewed as elements of the lattice, lie in $H_{\mathbb{R}}^{\vee}$.
(4) By identifying $H_{\mathbb{R}}$ with $II_{1,1}(-1/N) \otimes_{\mathbb{Z}} \mathbb{R}$, we let h be any negative norm element whose inner product with any root is nonzero. A standard choice is $(1,2)$, under the identification $II_{1,1}(-1/N) \cong \mathbb{Z} \oplus \frac{1}{N}\mathbb{Z} \subset \mathbb{R}^{1,1}$.
(5) If we write the norm of a root at $r \in II_{1,1}(-1/N)$ as r^2, then the fourth claim of Proposition 3.2.6 implies the root space at r is identified with a subspace of ${}^g_N V^\natural$ whose vectors have L_0-eigenvalue equal to $1 - r^2$. The L_0-eigenvalues in ${}^g_N V^\natural$ are bounded below by zero, so the norms of roots are bounded above by 1.
(6) Because H has a real Lorentzian structure, we can identify the real span of the root space with $\mathbb{R}^{1,1}$, and any pair of imaginary positive (resp. negative) roots lies in a single quadrant, where their inner products are bounded above by zero. Two such roots are orthogonal if and only if they are positive multiples of the same norm zero vector. We claim that because g is Fricke, there are no positive roots of norm zero. Such a root would have the form $(a,0)$ or $(0,b/N)$ in $II_{1,1}(-1/N)$ for $a,b>0$, and by the last claim in Proposition 3.2.6, the multiplicities of those root spaces are given by the coefficients $c^g_{a,0}(0)$ and $c^g_{0,b}(0)$ of F^g. By Corollary 3.25 of [Carnahan-2012], these coefficients are in turn the exponents attached to $(1-p^a q^0)$ and $(1-p^0 q^{b/N})$ in the product expansion
$$T_g(\sigma) - T_g(-1/\tau) = p^{-1} \prod_{m>0, n\in\frac{1}{N}\mathbb{Z}} (1-p^m q^n)^{c^g_{m,Nn}(mn)}.$$
Because g is Fricke, we may rewrite the left side as $T_g(\sigma) - T_g(\tau/N)$, and we find that the product expansion has the form $p^{-1}(1-pq^{-1/N})$ times a product for which m and n are strictly positive. Thus, the coefficients vanish, and there are no roots of norm zero.

□

When g is non-Fricke, the last condition is somewhat more complicated to verify due to the presence of norm zero simple roots. Since the non-Fricke case is not needed in this paper, we postpone it to future work.

The following lemma will be useful in the next section, when we consider concrete quantitative questions. It is well-known to experts, but we couldn't find a statement in the literature that was both precise and concise.

Lemma 3.4.4. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Borcherds-Kac-Moody Lie algebras with finite dimensional Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$. Given an inner-product-preserving isomorphism $f : \mathfrak{h}_1 \to \mathfrak{h}_2$, there exists an extension to a Lie algebra isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ if and only if the root multiplicities are identical under the isometry f.*

Proof. It is clear that an isomorphism of Lie algebras that restricts to an isometry of Cartan subalgebras preserves root multiplicities.

Now, assume $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have identical root multiplicities under the isometry $\mathfrak{h}_1 \to \mathfrak{h}_2$. It is well-known that the Weyl-Kac-Borcherds denominator formula may be used to uniquely determine the simple roots of a Borcherds-Kac-Moody algebra from the root multiplicities. Indeed, the set of values of $\alpha(h)$ as α ranges over positive roots is a discrete subset of the positive reals (in particular, well-ordered), and this allows us to recursively match the truncated evaluations of both sides of the Weyl-Kac-Borcherds denominator formula by adding simple roots as necessary. Indeed, this is essentially the same reconstruction process as the one sees in the proof of Lemma 3.4.2. Thus, under a choice of matching regular elements, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have the same simple roots, i.e., the simple roots that are identified under the isometry $\mathfrak{h}_1 \to \mathfrak{h}_2$ have equal multiplicity. Abstractly, this implies that both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic to the Borcherds-Kac-Moody Lie algebra given by generators and relations from the same generalized Cartan matrix. More concretely, the simple root spaces are isomorphic vector spaces, and any choice of isomorphism between simple root spaces that are identified through the isometry $\mathfrak{h}_1 \to \mathfrak{h}_2$ extends uniquely to a Lie algebra isomorphism. □

Remark 3.4.5. Theorem 7.2 of [Borcherds-1992] is essentially an application of the previous lemma to identify the Monster Lie algebra $M = \mathfrak{m}_1$ defined by applying the no-ghost theorem to $V^\natural$ with the abstract Lie algebra $N = L_1$ defined by a specification of simple roots.

4. Comparison theorems

4.1. **Lie algebra comparison.** In [Carnahan-2012] section 4, we defined for each $g \in \mathbb{M}$ a Borcherds-Kac-Moody Lie algebra L_g (written W_g in Proposition 4.4 of that paper), whose Weyl-Kac-Borcherds denominator identity is an infinite product expansion of $T_g(\sigma) - T_g(-1/\tau)$. We now show that the quantization functor yields an isomorphic Lie algebra.

Proposition 4.1.1. *Let $g \in \mathbb{M}$ be a Fricke element. Then there exists a homogeneous Lie algebra isomorphism $\mathfrak{m}_g \to L_g$.*

Proof. By Proposition 3.4.3, $\mathfrak{m}_g$ is a Borcherds-Kac-Moody Lie algebra whose Cartan subalgebra has a real Lorentzian structure. By Theorem 4.2 of [Carnahan-2012], L_g is a Borcherds-Kac-Moody Lie algebra whose Cartan subalgebra has a real Lorentzian structure. By Lemma 3.4.4, if we are given two Borcherds-Kac-Moody Lie algebras and an inner-product-preserving isomorphism between their Cartan subalgebras, this isomorphism can be extended to an isomorphism of the full Lie algebras if and only if the root multiplicities are equal.

If $T_g(\tau)$ has level N, then the root lattices of both Lie algebras are identified with $II_{1,1}(-1/N)$, so it suffices to compare dimensions of the homogeneous spaces of nonzero degree. The dimensions are equal by the last claim in Proposition 3.2.6, which gives a $II_{1,1}(-1/N)$-graded vector space isomorphism $\mathfrak{m}_g \to L_g$. Thus, we have equality of dimensions of root spaces, and hence an isomorphism of Lie algebras. □

Corollary 4.1.2. *For any Fricke $g \in \mathbb{M}$, hypothesis C_g from section 4.4 of [Carnahan-2012] is satisfied, i.e., there is an action of $\widetilde{C_{\mathbb{M}}(g)}$ on L_g by Lie algebra automorphisms, such that the action on the $(a, b/N)$ root space coincides with the action on the subspace of $V^\natural(g^a)$ on which $\tilde{g}$ acts by $e(b/N)$ and L_0 acts by $1 + ab/N$.*

Proof. If we fix a Lie algebra isomorphism $\mathfrak{m}_g \to L_g$, then the action of $\widetilde{C_{\mathbb{M}}(g)}$ on L_g is given by *transport de structure*. The isomorphism between $\widetilde{C_{\mathbb{M}}(g)}$-module structures for the root spaces and the subspaces of twisted modules then follows from the corresponding fact for $\mathfrak{m}_g$, which holds by the fourth listed claim of Proposition 3.2.6. □

4.2. **Main theorem.** The following theorem is the target of this series of papers.

Theorem 4.2.1. *If g is a Fricke element, then for any $h \in C_{\mathbb{M}}(g)$, there exists a discrete group $\Gamma_{g,h} \subset PSL_2(\mathbb{R})$ containing a congruence subgroup $\Gamma(K)$ for some $K > 0$ (hence commensurable with $SL_2(\mathbb{Z})$), such that for any lift $\tilde{h}$ of h to a linear transformation on $V(g)$, the power series $Tr(\tilde{h}q^{L_0-1}|V^\natural(g))$ is the q-expansion of a modular function on $\mathcal{H}$ that is invariant under $\Gamma_{g,h}$, and generates the function field of the quotient $\mathcal{H}/\Gamma_{g,h}$.*

Proof. By Corollary 4.1.2, hypothesis C_g holds, and in particular the positive subalgebra E_g of L_g is Fricke-compatible for the data $(\widetilde{C_{\mathbb{M}}(g)}, \tilde{g}, \{V_k^{i,j} = V^\natural(g^i)^{\tilde{g}=e(j/N), L_0-1=k}\})$ in the sense of [Carnahan-2012] section 4.2. That is:

(1) E_g is graded by $\mathbb{Z}_{>0} \times \frac{1}{N}\mathbb{Z}$, and decomposes as a sum $\bigoplus_{n>0, m\in\frac{1}{N}\mathbb{Z}} E_{i,j}p^iq^j$ of finite dimensional homogeneous components, where p and q denote degree shifts by $(0,1)$ and $(1,0)$.
(2) E_g admits an action of $\widetilde{C_{\mathbb{M}}(g)}$ by homogeneous Lie algebra automorphisms, such that $E_{i,j} \cong V_{1+ij}^{i,j}$ as $\widetilde{C_{\mathbb{M}}(g)}$-modules.
(3) The homology of E_g is given by
   - $H_0(E) = \mathbb{C}$
   - $H_1(E) = \bigoplus_{n\in\frac{1}{N}\mathbb{Z}} V_{1+n}^{1,n} pq^n$
   - $H_2(E) = p\bigoplus_{m=1}^{\infty} V_{1-1/N}^{1,-1/N} \otimes V_{1+m/N}^{m,1/N} p^m$
   - $H_i(E) = 0$ for $i > 2$.
(4) $E_{1,-1/N} \cong V_{1-1/N}^{1,-1/N}$ is one dimensional.

By Proposition 6.3 in [Carnahan-2010], if E_g is Fricke-compatible for $(\widetilde{C_{\mathbb{M}}(g)}, \tilde{g}, \{V_k^{i,j}\})$, then for any $\tilde{h} \in \widetilde{C_{\mathbb{M}}(g)}$, the power series $Tr(\tilde{h}q^{L_0-1}|V^\natural(g))$ is the q-expansion of a Hauptmodul whose invariance group contains a principal congruence subgroup. □

By combining our main theorem with previously known results, we obtain a positive resolution of Norton's conjecture.

Theorem 4.2.2. *(Generalized Moonshine for twisted modules of $V^\natural$) Any rule that assigns*

- *to each $g \in \mathbb{M}$ the irreducible g-twisted $V^\natural$-module $V^\natural(g)$, with its canonical projective $C_{\mathbb{M}}(g)$-action, and*
- *to each commuting pair (g,h) the function $Z(g,h;\tau) = \mathrm{Tr}(\tilde{h}q^{L_0-1}|V^\natural(g))$ for some lift $\tilde{h}$ of h to a linear transformation on $V^\natural(g)$*

satisfies the following conditions:

(1) *The formal series defining $Z(g,h;\tau)$ is the q-expansion of a holomorphic function on the upper half-plane $\mathfrak{H}$.*
(2) *The function $(g,h) \mapsto Z(g,h;\tau)$ is invariant under simultaneous conjugation on the pair (g,h) in $\mathbb{M}$, up to rescaling.*
(3) *$Z(g,h;\tau)$ is either a constant or a Hauptmodul.*
(4) *For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and any commuting pair (g,h) in $\mathbb{M}$, $Z(g,h,\frac{a\tau+b}{c\tau+d})$ is proportional to $Z(g^ah^c, g^bh^d, \tau)$.*
(5) *$Z(g,h;\tau)$ is proportional to $J(\tau)$ if and only if $g = h = 1$.*

Proof. The first claim is well-known: As a statement about normal convergence of power series, it follows from Theorem 8.1 of [Dong-Li-Mason-2000].

The second claim is also well-known, and follows by general nonsense from the Dong-Li-Mason proof of existence and uniqueness for irreducible twisted $V^\natural$-modules: For any $k \in \mathbb{M}$, k-conjugation

on twisted modules induces an equivalence from the category of g-twisted $V^\natural$-modules to the category of kgk^{-1}-twisted $V^\natural$-modules, taking the projective $C_{\mathbb{M}}(g)$-action on $V^\natural(g)$ to the projective $C_{\mathbb{M}}(kgk^{-1})$-action on $V^\natural(kgk^{-1})$ by conjugation. This preserves characters up to rescaling.

The third claim is relatively straightforward in light of Theorem 4.2.1 combined with the $SL_2(\mathbb{Z})$ claim. If $g^a h^c$ is Fricke for some a, c satisfying $(a, c) = 1$, then $Z(g^a h^c, g^b h^d, \tau)$ is a Hauptmodul for all b, d such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. The $SL_2(\mathbb{Z})$ claim then implies $Z(g, h, \tau)$ is proportional to a Möbius transform of $Z(g^a h^c, g^b h^d, \tau)$, hence is a Hauptmodul. On the other hand, if all $g^a h^c$ for $(a, c) = 1$ are non-Fricke, then the twisted modules $V^\natural(g^a h^c)$ have L_0-spectrum bounded below by 1, so $Z(g^a h^c, g^b h^d, \tau)$ is regular at infinity for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then the $SL_2(\mathbb{Z})$ claim implies $Z(g, h, \tau)$ is regular at all cusps, so the maximum modulus principle implies it is a constant function.

As we remarked in the introduction, the $SL_2(\mathbb{Z})$ claim follows immediately from combining Theorem 10.1 in [Dong-Li-Mason-2000] (which reduces the question to a g-rationality condition) with Corollary 5.26 of [Carnahan-Miyamoto-2016] (which asserts g-rationality).

The fifth claim also follows from [Dong-Li-Mason-2000]. In particular, the positivity of the L_0-spectrum of $V^\natural(g)$ for $g \neq 1$ (as given in Theorem 2.2.9) eliminates all nontrivial twistings. For the trivial twisting, the fact that the McKay-Thompson series $T_h(\tau) = Z(1, h, \tau)$ is equal to J if and only if $h = 1$ follows from the faithfulness of the monster action on $V^\natural$ as shown in [Frenkel, et al, 1988]. □

References

[Borcherds-Conway-Queen-Sloane-1984] R. E. Borcherds, J. H. Conway, L. Queen, N. J. A. Sloane, *A Monster Lie algebra?* Adv. in Math. **53** (1984) 75–79.

[Borcherds-1986] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster* Proc. Nat. Acad. Sci. USA, **83** no. 10 (1986) 3068–3071.

[Borcherds-1988] R. Borcherds, *Generalized Kac-Moody algebras* J. Algebra, **115** no. 2 (1988) 501–512.

[Borcherds-1990] R. Borcherds, *The Monster Lie algebra* Adv. Math. **83** no. 1 (1990) 30–47.

[Borcherds-1992] R. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras* Invent. Math. **109** (1992), 405–444.

[Borcherds-1995] *A characterization of generalized Kac-Moody algebras* J. Algebra, **174** no. 3 (1995) 1073-1079.

[Brower-1972] R. Brower, *Spectrum generating algebra and the no-ghost theorem for the dual model* Phys. Rev. **D6** (1972) 1655–1662.

[Carnahan-2007] S. Carnahan, *Monstrous Lie algebras and Generalized Moonshine* Ph.D. Dissertation, U.C. Berkeley, May 2007.

[Carnahan-2010] S. Carnahan, *Generalized Moonshine I: Genus zero functions* Algebra and Number Theory **4** no.6 (2010) 649–679. Available as `http://arxiv.org/abs/0812.3440`

[Carnahan-2012] S. Carnahan, *Generalized Moonshine II: Borcherds products* Duke Math J. **161** no. 5 (2012) 893–950. Available as `http://arxiv.org/abs/0908.4223`

[Carnahan≥2016] S. Carnahan, *Generalized Moonshine III: Equivariant intertwining operators* In preparation.

[Carnahan-Miyamoto-2016] S. Carnahan, M. Miyamoto, *Regularity of fixed-point vertex operator subalgebras* ArXiv preprint `http://arxiv.org/abs/1603.05645`

[Conway-Norton-1979] J. Conway, S. Norton, *Monstrous Moonshine* Bull. Lond. Math. Soc. **11** (1979) 308–339.

[Cummins-Gannon-1997] C. Cummins, T. Gannon, *Modular equations and the genus-zero property of modular functions* Invent. Math. **129** (1997) 413–443.

[D'Hoker-1997] E. D'Hoker, *String theory* Quantum Fields and Strings: a course for mathematicians, volume 2, ed. Deligne, *et al*, AMS, Providence, RI (1999) 807–1012.

[Dijkgraaf-Pasquier-Roche-1990] R.Dijkgraaf, V. Pasquier, P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models* Nucl. Phys. B (Proc. Suppl.) **18** B (1990) 60–72.

[Dixon-Ginsparg-Harvey-1988] L. Dixon, P. Ginsparg, J. Harvey, *Beauty and the Beast: Superconformal Symmetry in a Monster Module* Commun. Math. Phys. **119** (1988) 221–241.

[Dong-1994] C. Dong, *Representations of the moonshine module vertex operator algebra* Contemporary Math. **175** (1994) 27–36.

[Dong-Lepowsky-1993] C. Dong, J. Lepowsky, *Generalized vertex algebras and relative vertex operators* Progress in Mathematics **112** Birkhäuser Boston, Inc., Boston, MA, (1993).

[Dong-Li-Mason-2000] C. Dong, H. Li, G. Mason, *Modular invariance of trace functions in orbifold theory* Comm. Math. Phys. **214** (2000) no. 1, 1–56. Available as http://arxiv.org/abs/q-alg/9703016.

[Duncan-Frenkel-2011] J. F. R. Duncan, I. B. Frenkel, *Rademacher sums, moonshine and gravity* Commun. Number Theory Phys. **5** (2011), no. 4, 1–128. Available as http://arxiv.org/abs/0907.4529

[Eilenberg-Mac Lane-1954] S. Eilenberg, S. Mac Lane, *On the groups $H(\Pi, n)$, II, Methods of computation* Ann. Math. (2) **60** no. 1 (1954), 49–139.

[van Ekeren, et al, 2015] J. van Ekeren, S. Möller, N. Scheithauer, *Construction and classification of vertex operator algebras* ArXiv preprint http://arxiv.org/abs/1507.08142

[Feigin-1984] B. L. Feigin, *The semi-infinite homology of Kac-Moody and Virasoro Lie algebras* Uspekhi Mat. Nauk, **39**:2 (236) (1984), 195–196. Available at http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=rm&paperid=2354

[Feigin-Fuchs-1984] B. L. Feigin, D. B. Fuchs, *Verma modules over the Virasoro algebra* Topology, Proc. Internat. Topol. Conf. Leningrad 1982, L. D. Faddeev (ed.) A. A. Mal'tsev (ed.), Lect. notes in math. 1060, Springer (1984) 230–245.

[Frenkel-1985] I. Frenkel, *Representations of Kac-Moody algebras and dual resonance models* Applications of group theory in theoretical physics, Lect. Appl. Math. **21** A.M.S. (1985) 325–353.

[Frenkel-Ben-Zvi-2004] E. Frenkel, D. Ben-Zvi, *Vertex algebras and Algebraic Curves* Mathematical Surveys and Monographs **88** American Mathematical Society, Providence, RI, (2004).

[Frenkel-Garland-Zuckerman-1986] I. Frenkel, H. Garland, G. Zuckerman, *Semi-infinite cohomology and string theory* Proc. Nat. Acad. Sci. USA, **83** (1986) 8442–8446.

[Frenkel, et al, 1993] I. Frenkel, Y. Huang, J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules* Mem. Amer. Math. Soc. **104** (1993) no. 494.

[Frenkel, et al, 1988] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster* Pure and Applied Mathematics **134** Academic Press, Inc., Boston, MA, (1988).

[Gaberdiel, et al, 2012] M. Gaberdiel, D. Persson, H.Ronellenfitsch, R.Volpato, *Generalized Mathieu moonshine* Commun.Num.Theor.Phys. **7** (2013) 145–223. Available at http://arxiv.org/abs/1211.7074

[Gaberdiel-Persson-Volpato-2013] M. R. Gaberdiel, D. Persson, R. Volpato, *Generalised Moonshine and Holomorphic Orbifolds* Proceedings, String-Math 2012, Bonn, Germany, July 16-21, 2012, (2013) 73–86. Available at http://arxiv.org/abs/1302.5425

[Ganter-2009] N. Ganter, *Hecke operators in equivariant elliptic cohomology and generalized moonshine* Groups and symmetries: from the Neolithic Scots to John McKay, CRM Proc. Lecture Notes **47** Amer. Math. Soc., Providence, RI (2009) 173–209. Available at http://arxiv.org/abs/0706.2898

[Goddard-Thorn-1972] P. Goddard, C. Thorn, *Compatibility of the dual pomeron with unitarity and the absence of ghosts in the dual resonance model* Physics Letters **40B** no. 2, (1972) 235–238.

[Green-Schwarz-Witten-1987] M. Green, J. Schwarz, E. Witten, *Superstring theory* Cambridge University Press, Cambridge, UK (1987).

[Höhn-2003b] G. Höhn, *Generalized moonshine for the Baby Monster* Preprint (2003)

[Huang-2008] Y.-Z. Huang, *Vertex operator algebras and the Verlinde conjecture* Commun. Contemp. Math., **10** no. 1 (2008) 103–154. Available at http://arxiv.org/abs/math/0406291

[Ivanov-Tuite-2002] R. I. Ivanov, M. P. Tuite, *Rational generalized moonshine and abelian orbifolds* Nucl. Phys. B **635** (2002) 435–472. Available at http://arxiv.org/abs/math/0106027

[Jurisich-1996] E. Jurisich, *An exposition of generalized KacMoody algebras* Lie algebras and their representations (Seoul, 1995), Contemporary Mathematics, **194** American Mathematical Society, Providence, RI (1996) 121–159.

[Jurisich-1998] E. Jurisich, *Generalized Kac-Moody Lie algebras, free Lie algebras and the structure of the Monster Lie algebra* J. Pure and Appl. Alg. **126** (1998) 233–266. Available at http://arxiv.org/abs/1311.3258

[Jurisich-2004] E. Jurisich, A resolution for standard modules of Borcherds Lie algebras *J. Pure Appl. Alg.* **192** (2004) 149–158.

[Kac-1997] V. Kac, *Vertex Algebras for Beginners* University Lecture Series **10**, Amer. Math, Soc., Providence, RI (1997)

[Kac-Raina-1987] V. Kac, A. Raina, *Bombay Lectures on highest weight representations of infinite dimensional Lie algebras* Advanced Series in Mathematical Physics **2**, World Scientific, Singapore, (1987).

[Kato-Ogawa-1983] M. Kato, K. Ogawa, *Covariant quantization of string based on BRS invariance* Nucl. Phys. B212 (1983) 443–460.

[Li-1994] H. Li *Symmetric invariant bilinear forms on vertex operator algebras* J. Pure Appl. Algebra, **96** no.3 (1994) 279–297.

[Lian-Zuckerman-1991] B. Lian, G. Zuckerman, *BRST Cohomology and highest weight vectors I* Comm. Math. Phys. **135** (1991) 547–580.

[Lian-Zuckerman-1993] B. Lian, G. Zuckerman, *New perspectives on the BRST-algebraic structure of string theory* Comm. Math. Phys. **154** (1993) 613–646. Available as http://arxiv.org/abs/hep-th/9211072

[Lian-Zuckerman-1995] B. Lian, G. Zuckerman, *Moonshine Cohomology* RIMS Proceedings on Vertex Operator Algebras and the Monster (1995). Available as http://arxiv.org/abs/q-alg/9501025

[Lovelace-1971] C. Lovelace *Pomeron form factors and dual regge cuts* Phys. Lett. **34B** (1971) 500–506.

[MacLane-1952] S. Mac Lane, *Cohomology theory of abelian groups* Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R.I., (1952) 8–14.

[Miyamoto-2013] *C_2-cofiniteness of cyclic-orbifold models* ArXiv preprint: http://arxiv.org/abs/1306.5031

[Norton-1987] S. Norton, *Generalized moonshine* Proc. Sympos. Pure Math. **47** Part 1, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 209–210, Amer. Math. Soc., Providence, RI (1987).

[Norton-2001] S. Norton, *From Moonshine to the Monster* Proceedings on Moonshine and related topics (Montréal, QC, 1999), 163–171, CRM Proc. Lecture Notes **30**, Amer. Math. Soc., Providence, RI (2001).

[Polchinski-1998] J. Polchinski, *String Theory* Cambridge University Press, Cambridge, UK (1998).

[Queen-1981] L. Queen, *Modular Functions arising from some finite groups* Mathematics of Computation **37** (1981) No. 156, 547–580.

[Scheithauer-1998] N. Scheithauer, *Vertex Algebras, Lie Algebras, and Superstrings* J. Algebra **200** (1998) 363–403. Available at http://arxiv.org/abs/hep-th/9802058

[Scheithauer-2000] N. Scheithauer, *The Fake Monster Superalgebra* Adv. Math. **151** (2000) 226–269. Available at http://arxiv.org/abs/math/9905113

[Zhu-1996] Y. Zhu, *Modular invariance of characters of vertex operator algebras* J. Amer, Math. Soc. **9** (1996) 237–302.

[Zuckerman-1989] G. Zuckerman, *Modular forms, strings, and ghosts* Proceedings of Symposia in Pure Mathematics, **49** I (1989) 273–284.

Division of Mathematics, University of Tsukuba
E-mail address: carnahan@math.tsukuba.ac.jp

arXiv:1211.7074v3 [hep-th] 16 Jan 2014

GENERALISED MATHIEU MOONSHINE

MATTHIAS R. GABERDIEL, DANIEL PERSSON, HENRIK RONELLENFITSCH, AND ROBERTO VOLPATO

Abstract. The Mathieu twisted twining genera, i.e. the analogues of Norton's generalised Moonshine functions, are constructed for the elliptic genus of K3. It is shown that they satisfy the expected consistency conditions, and that their behaviour under modular transformations is controlled by a 3-cocycle in $H^3(M_{24}, U(1))$, just as for the case of holomorphic orbifolds. This suggests that a holomorphic VOA may be underlying Mathieu Moonshine.

Contents

1. Introduction 2
1.1. Monstrous Moonshine 2
1.2. Mathieu Moonshine 3
1.3. Generalised Monstrous Moonshine 4
1.4. Generalised Mathieu Moonshine 4
1.5. Outline 5
2. Twisted Twining Genera 6
2.1. Properties of the twisted twining genera 6
2.2. Cohomological obstructions 9
2.3. Classification of independent twisted twining genera 10
2.4. Modular properties of the twisted twining genera 12
2.5. Explicit twining genera 13
3. Holomorphic Orbifolds and Group Cohomology 15
3.1. Review of holomorphic orbifolds 16
3.2. Application to Mathieu Moonshine 18
3.3. Determining the cohomology class 20
3.4. Computation of the twisted twining characters: an example 24
3.5. Projective representations 27
4. K3 Orbifolds 27
4.1. The relabelling phenomenon 28

4.2. An example 30
4.3. Computation of a twisted twining genus 32
5. Conclusions 34
5.1. Summary 34
5.2. Open problems and future work 35
Acknowledgements 37
Appendix A. Details on the unobstructed twisted twining genera 37
A.1. The characters $\phi_{2\mathrm{B},4\mathrm{A}_2}$ (group 13) $\phi_{4\mathrm{B},4\mathrm{A}_3}$ (group 23) and $\phi_{4\mathrm{B},4\mathrm{A}_4}$ (group 24) 38
A.2. The cases $\phi_{3\mathrm{A},3\mathrm{A}_3}$ (group 33) and $\phi_{3\mathrm{A},3\mathrm{B}_1}$ (group 34) 40
Appendix B. Some group cohomology 42
Appendix C. Projective representations of finite groups 43
C.1. Central extension and orbifolds 45
Appendix D. Centralisers $C_{M_{24}}(g)$ and (projective) character tables 46
D.1. Character Tables 47
Appendix E. Decompositions of twisted sectors 61
References 68

1. Introduction

1.1. **Monstrous Moonshine.** In mathematics and physics the word 'Moonshine' has come to represent various surprising and deep connections between a priori unrelated fields, such as number theory, representation theory of finite groups, algebra and quantum field theory. The first and most well-known example of such a connection is, of course, Conway and Norton's *Monstrous Moonshine conjecture* [1]. Their starting point was the observation of McKay that the first few Fourier coefficients of the modular invariant J-function $J(\tau)$ are dimensions of representations of the Monster group $\mathbb{M}$, the largest finite simple sporadic group. Conway and Norton then conjectured that to each element $g \in \mathbb{M}$ of the Monster, one can associate a function $T_g(\tau)$ (the so-called McKay-Thompson series) on the upper half-plane $\mathbb{H}_+$ that is invariant under some subgroup $\Gamma_g \subset SL(2,\mathbb{R})$. They also conjectured that the invariance group Γ_g must be genus zero, meaning that, as a Riemann surface, the quotient $\Gamma_g \backslash \mathbb{H}_+$ is topologically a sphere. Furthermore, they conjectured that $T_g(\tau)$ is in fact the Hauptmodul for Γ_g.

Subsequently, Frenkel, Lepowsky and Meurman (FLM) [2] constructed a graded $\mathbb{M}$-module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$, such that the dimension of the graded subspaces $a(n) = \dim V_n^\natural$ reproduce

precisely the Fourier coefficients in the q-expansion of the J-function, $J(\tau) = \sum_{n=-1}^{\infty} a(n)q^n$, $q = e^{2\pi i\tau}$. The coefficients $a_g(n)$ in the Fourier expansion of the McKay-Thompson series are then identified with the characters $\mathrm{Tr}_{V_n^\natural}(g)$. Physically, the FLM-module is the $\mathbb{Z}_2$-orbifold of the open bosonic string compactified on the Leech torus $(\mathbb{R}^{24}/\Lambda_{\mathrm{Leech}})$, where Λ_{Leech} is the Leech lattice. The FLM-construction provided important clues into the moonshine conjectures, but fell short of proving them. The final proof was found by Borcherds [3], making heavy use of the FLM-module $V^\natural$, but also introducing yet another set of ingredients to the story, namely generalised Kac-Moody algebras and automorphic denominator identities. Thus, by the time the original conjectures were proven, Monstrous Moonshine encompassed not only the realms of finite groups and modular forms, but also string theory and infinite-dimensional algebras, for a nice review see [4].

1.2. **Mathieu Moonshine.** Recently, a very interesting new moonshine phenomenon was conjectured by Eguchi, Ooguri and Tachikawa (EOT) [5]: they observed that the first few Fourier coefficients of the elliptic genus of K3 are dimensions of representations of the largest Mathieu group M_{24}. This suggests the existence of a new moonshine-type connection between Mathieu groups, Jacobi forms and K3 surfaces. The analogues of the McKay-Thompson series, the *twining elliptic genera* $\phi_g(\tau, z)$, $g \in M_{24}$, were constructed in a series of papers [6, 7, 8, 9], and it was shown that they are all weak Jacobi forms of weight 0 and index 1 (with multiplier system) for subgroups $\Gamma_0(N_g) \subset SL(2,\mathbb{Z})$, where N_g is the order of the element g. (The elliptic genus ϕ_{K3} itself corresponds to taking g to be the identity element.) The compatibility with M_{24}-representations was checked up to the first 600 coefficients [8, 9]; according to Gannon [10] this is sufficient to prove that all Fourier coefficients of the elliptic genus of K3 are dimensions of M_{24} representations. It is also shown in [10] that the multiplicities of the real M_{24} representations are always even, see also [11].

Although in a certain sense the above results establish the Mathieu Moonshine conjecture, there are many aspects of it that are much less understood compared to Monstrous Moonshine. For instance, the genus zero property of Monstrous Moonshine does not seem to hold for the M_{24}-twining genera since some of the invariance groups $\Gamma_0(N_g)$ are not genus zero.[*] More importantly, the analogue of the FLM module $V^\natural$ is not yet known for Mathieu Moonshine, i.e. we do not know of any CFT with automorphism group M_{24} whose elliptic

[*]It was proposed in [12] that the correct generalisation of the genus zero property is a certain Rademacher-summability condition, which is indeed satisfied in all cases.

genus reproduces the elliptic genus of K3. In particular, none of the K3-sigma models have this property since M_{24} is not contained in the automorphism group of any of them [13].

1.3. **Generalised Monstrous Moonshine.** A few years after the original Monstrous Moonshine conjectures, Norton proposed a generalisation that he dubbed *Generalised Moonshine* [14]. He suggested that for each commuting pair of Monster group elements $g, h \in \mathbb{M}$ there exists a function $f(g, h; \tau)$ that is also invariant under a genus zero subgroup of $SL(2, \mathbb{R})$. These functions generalise the McKay-Thompson series to which they reduce for $g = e$. Furthermore, they transform into one another under a simultaneous action of $SL(2, \mathbb{Z})$ on $(\tau; g, h) \in \mathbb{H}_+ \times \mathcal{P}(\mathbb{M})$, where $\mathcal{P}(\mathbb{M}) \subset \mathbb{M} \times \mathbb{M}$ is the set of pairs of commuting elements of $\mathbb{M}$. Finally, the Fourier coefficients in the q-expansion of $f(g, h; \tau)$ are conjectured to be dimensions of (projective) representations of the centraliser of g in $\mathbb{M}$, $C_{\mathbb{M}}(g) = \{k \in \mathbb{M} : gk = kg\}$. Although this conjecture has been proven in special cases [15, 16, 17], the general case is still open.[†]

The Generalised Moonshine conjecture was given a physical interpretation by Dixon, Ginsparg and Harvey [24] that was later elaborated upon by Ivanov and Tuite [17, 18, 19]. They showed that the Norton series $f(g, h; \tau)$ arises naturally as the character in the twisted sector $V_g^\natural$ of an orbifold of the Monster CFT $V^\natural$ by the element $g \in \mathbb{M}$, 'twined' by the group element h; in standard CFT language, they can therefore be interpreted as

$$f(g, h; \tau) = {}_g\underset{h}{\square} = \mathrm{Tr}_{V_g^\natural}\left(h\, q^{L_0 - 1}\right) \ . \tag{1}$$

Many of the properties conjectured by Norton can be proved from holomorphic orbifold considerations [15].

1.4. **Generalised Mathieu Moonshine.** In this paper, we show that Norton's generalisation also applies to Mathieu Moonshine. More specifically, for every pair of commuting group elements $g, h \in M_{24}$, we construct 'twisted twining genera' $\phi_{g,h} : \mathbb{H}_+ \times \mathbb{C} \to \mathbb{C}$ that either vanish or are weak Jacobi forms of weight 0 and index 1 for some $\Gamma_{g,h} \subset SL(2, \mathbb{Z})$. For $g = e$, they reduce to the twining genera of [6, 7, 8, 9], and they transform under the modular group $SL(2, \mathbb{Z})$ into one another. The multiplier phases that appear in these transformations behave as though these twisted twining genera were twisted twining characters of a holomorphic orbifold; in particular, they are controlled by a 3-cocycle $\alpha \in H^3(M_{24}, U(1))$ via a formula that was first written down by Dijkgraaf and Witten in [25]. Furthermore, the Fourier coefficients of the twisted elliptic genera $f(g, e; \tau)$ equal dimensions of projective

[†]Carnahan has announced a series of papers [20, 21, 22, 23] which he claims will lead to a complete proof.

representations of the centraliser $C_{M_{24}}(g)$, and the central extension that characterises the projective representation is again determined by the cohomology class $\alpha \in H^3(M_{24}, U(1))$.

The idea for using the cohomology group $H^3(M_{24}, U(1))$ and the formalism of Dijkgraaf and Witten in order to understand the multiplier phases of the twisted twining genera was first suggested to us by Terry Gannon in 2011 [26], see also [10]. Mason has also speculated [27] that $H^3(\mathbb{M}, U(1))$ should play a similar role in the context of Generalised Monstrous Moonshine, although this has, to our knowledge, not yet been worked out (partially, because $H^3(\mathbb{M}, U(1))$ is unknown).

We should also mention that Mason has previously proposed a version of Generalised Moonshine for M_{24} [28], where the role of the Norton series $f(g, h; \tau)$ is played by products of eta functions $\eta(\tau)$. The precise relation between our twisted twining genera and Mason's eta products will be explained elsewhere [29].

1.5. **Outline.** The paper is organised as follows. In section 2 we introduce the twisted twining genera $\phi_{g,h}$, and discuss their general properties, in particular the expected behaviour under modular transformations. Based on these properties we then explain how many independent twisted twining genera there are, and list all of them explicitly, see table 3. The detailed derivation of these genera is illustrated in section 3. First, we review the structure of holomorphic orbifolds and explain in particular, the role of the 3-cocycle in characterising the various transformation properties of the twisted twining characters. We then postulate that the transformation properties of the twisted twining genera are similarly constrained. In support of this assumption, we prove that there exists a unique 3-cocycle in $H^3(M_{24}, U(1))$ compatible with the known properties of the (untwisted) twining genera of [6, 7, 8, 9]. This then allows us to find all twisted twining genera explicitly. We also check that, up to the first 500 levels in each twisted sector, the resulting functions are compatible with the requirement that they arise from the appropriate projective representation of the centraliser. In section 4 we subject our results to two independent consistency checks. First, for group elements whose orbifold leads again to a K3 sigma-model, we calculate the twining genera of the orbifold from the twisted twining genera of the original theory, and show that we reproduce answers from [8, 9]. However, as it turns out, the relevant group element is sometimes different from what one would have expected, and we explain this 'relabelling' phenomenon in quite some detail in section 4.1, see also section 4.2 for an explicit example. Secondly, we explain the vanishing of some of the twisted twining genera from a geometrical point of view, see section 4.3. Finally we end with some conclusions and open problems in

section 5. We have relegated some of the more technical material to various appendices to which we refer throughout the main body of the paper. Moreover, the appendices contain the character tables of the centralisers and the decompositions of the twisted sectors, up to the first 20 levels. The decompositions up to 500 levels, as well as the details of the computations requiring computer support, are collected in the ancillary files in the arXiv repository of this paper.

2. Twisted Twining Genera

Let us begin by introducing the twisted twining genera that are the main object of this paper. Suppose we are given a K3 sigma-model $\mathcal{H}$, i.e. a CFT describing string propagation on K3, whose automorphism group contains two commuting elements $g, h \in M_{24}$. Then we can consider the orbifold of the sigma-model by g, and in particular, define the g-twisted sector $\mathcal{H}_g$. Since g and h commute, h gives rise to an action on $\mathcal{H}_g$, and we can define the *twisted twining genus* $\phi_{g,h}(\tau, z)$ by

$$\phi_{g,h}(\tau, z) = \operatorname{Tr}_{\mathcal{H}_g}\left(h(-1)^{J_0+\bar{J}_0} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} y^{J_0}\right) , \tag{2}$$

where $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$. We expect that $\phi_{g,h} : \mathbb{H}_+ \times \mathbb{C} \to \mathbb{C}$ is holomorphic in both τ and z. Furthermore, since the elliptic genus is independent of the choice of the underlying K3 sigma-model, we expect that the same is true for these twisted twining genera, i.e. we expect that (2) does not depend on the choice of $\mathcal{H}$ (as long as g and h are automorphisms of $\mathcal{H}$). By construction, for $g = e$ the identity element in M_{24}, the twisted twining genus $\phi_{e,h}$ agrees with the twining genus ϕ_h considered in [6, 7, 8, 9]; in particular, for $g = h = e$ $\phi_{e,e}$ is just the elliptic genus of K3.

Unfortunately, while for some commuting pairs (g, h) an actual K3 sigma-model for which both g and h are automorphisms can be found, this is not true in general [13]; in fact, this problem already arises for the usual twining genera, i.e. for the pairs (e, h). In the spirit of the EOT conjecture, we shall nevertheless assume that twisted twining genera $\phi_{g,h}$ can be defined — albeit not directly by a formula of the form (2) — for *all* commuting pairs $g, h \in M_{24}$. The fact that our construction will be successful is an a posteriori justification for this assumption.

2.1. Properties of the twisted twining genera.

The definition of the twisted twining genera $\phi_{g,h}$ in terms of (2) suggests that they should satisfy the following properties:

(A) Elliptic and modular properties:

$$\phi_{g,h}(\tau, z+\ell\tau+\ell') = e^{-2\pi i(\ell^2\tau+2\ell z)}\,\phi_{g,h}(\tau,z) \qquad \ell,\ell'\in\mathbb{Z} \tag{3}$$

$$\phi_{g,h}\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) = \chi_{g,h}\big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big)\, e^{2\pi i\frac{cz^2}{c\tau+d}}\,\phi_{h^c g^a,h^d g^b}(\tau,z)\ , \quad \big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big)\in SL(2,\mathbb{Z}) \tag{4}$$

for a certain multiplier $\chi_{g,h} : SL(2,\mathbb{Z}) \to U(1)$. In particular, each $\phi_{g,h}$ is a weak Jacobi form of weight 0 and index 1 with multiplier $\chi_{g,h}$ under a subgroup $\Gamma_{g,h}$ of $SL(2,\mathbb{Z})$ (see [30] for the definitions).

(B) Invariance under conjugation of the pair g,h in M_{24},

$$\phi_{g,h}(\tau,z) = \xi_{g,h}(k)\,\phi_{k^{-1}gk,k^{-1}hk}(\tau,z)\ , \qquad k\in M_{24}\ , \tag{5}$$

where $\xi_{g,h}(k)$ is a phase.

(C) If $g\in M_{24}$ has order N, the twisted twining genera $\phi_{g,h}$ have an expansion of the form

$$\phi_{g,h}(\tau,z) = \sum_{\substack{r\in\lambda_g+\mathbb{Z}/N\\ r\geq 0}} \mathrm{Tr}\,_{\mathcal{H}_{g,r}}\big(\rho_{g,r}(h)\big)\,\mathrm{ch}_{h=\frac14+r,\ell}(\tau,z)\ , \tag{6}$$

where $\lambda_g\in\mathbb{Q}$, and $\mathrm{ch}_{h,\ell}(\tau,z)$ are elliptic genera of Ramond representations of the $\mathcal{N}=4$ superconformal algebra at central charge $c=6$. (Here $\ell=\frac12$, except possibly for $h=\frac14$, where $\ell=0$ is also possible — if both $\ell=0,\frac12$ appear for $r=0$, it is understood that there are two such terms in the above sum. We use the same conventions for the elliptic genera as e.g. in [31].) Furthermore, each vector space $\mathcal{H}_{g,r}$ is finite dimensional, and it carries a projective representation $\rho_{g,r}$ of the centraliser $C_{M_{24}}(g)$ of g in M_{24}, such that

$$\rho_{g,r}(g) = e^{2\pi i r}\ , \qquad \rho_{g,r}(h_1)\,\rho_{g,r}(h_2) = c_g(h_1,h_2)\,\rho_{g,r}(h_1h_2)\ , \tag{7}$$

for all $h_1,h_2\in C_{M_{24}}(g)$. Here $c_g : C_{M_{24}}(g)\times C_{M_{24}}(g)\to U(1)$ is independent of r, and satisfies the cocycle condition

$$c_g(h_1,h_2)\,c_g(h_1h_2,h_3) = c_g(h_1,h_2h_3)\,c_g(h_2,h_3) \tag{8}$$

for all $h_1,h_2,h_3\in C_{M_{24}}(g)$.

(D) For $g=e$, where e is the identity element of M_{24}, the functions $\phi_{e,h}$ correspond to the twining genera considered in [6]–[9]. In particular, $\phi_{e,e}$ is the K3 elliptic genus.

Since the elliptic genus behaves essentially like the character of a holomorphic CFT — in particular, it is modular invariant and holomorphic by itself — it is natural to believe that the same will be true for the twisted twining genera, i.e. that they will be analogous

to twisted twining characters of a holomorphic CFT. As we will review in more detail in section 3, the modular properties of the twisted twining characters of a holomorphic CFT are controlled by a 3-cocycle $\alpha : G \times G \times G \to U(1)$ representing a cohomology class in the third cohomology group $H^3(G, U(1))$. (Some background material about group cohomology can be found in appendix B.) We will therefore postulate that

(E) The multipliers $\chi_{g,h}$, the phases $\xi_{g,h}$, and the 2-cocycles c_g associated with the projective representations $\rho_{g,r}$ are completely determined (by the same formulas as for holomorphic orbifolds) in terms of a 3-cocycle α representing a class in $H^3(M_{24}, U(1))$.

The third cohomology group of M_{24} was only recently computed with the result [32][‡]

$$H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12} \ . \tag{9}$$

The fact that this group is known explicitly plays a crucial role in our analysis. The specific cohomology class $[\alpha] \in H^3(M_{24}, U(1))$ that is relevant in our context is uniquely determined by the condition that it reproduces the multiplier system for the twining genera $\phi_{e,h}$ as described in [8], namely

$$\chi_{e,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = e^{\frac{2\pi i c d}{o(h)\ell(h)}} \ , \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(o(h)) \ . \tag{10}$$

Here, $o(h)$ is the order of h and $\ell(h)$ is the length of the smallest cycle, when $h \in M_{24}$ is regarded as a permutation of 24 symbols [12]. Indeed, since $\ell(12\mathrm{B}) = 12$, it follows that α must correspond to the primitive generator of $H^3(M_{24}, U(1))$. The main result of our paper can now be stated as follows:

There exists a unique set of functions $\phi_{g,h}$ (unique up to redefinitions by (g,h)-dependent but otherwise constant phases) and a unique cohomology class $[\alpha] \in H^3(M_{24}, U(1))$ such that all conditions (A)–(E) are satisfied (condition (C) has been verified only for the first 500 representations $\rho_{g,r}$ for each $g \in M_{24}$).

When $g = e$ is the trivial element of M_{24}, the existence of all representations $\rho_{e,r}$ fitting eq. (6) has been recently proven in [10]. When $g \neq e$, condition (C) is, strictly speaking, still a conjecture. However, we provide compelling numerical evidence in favour of it, by

[‡]Note that for a finite group G one has the isomorphisms

$$H_{n-1}(G, \mathbb{Z}) \cong H^n(G, \mathbb{Z}), \qquad H^n(G, \mathbb{Z}) \cong H^{n-1}(G, U(1)) \ ,$$

which in particular imply that $H_3(M_{24}, \mathbb{Z}) \cong H^3(M_{24}, U(1))$.

verifying eq. (6) for the first 500 representations $\rho_{g,r}$ in each twisted sector. A complete proof of (6) should be possible along the lines of [10].

In the following we want to illustrate how the set of functions $\phi_{g,h}$ can be determined, and which explicit form it takes. As it turns out, a surprisingly large number of the twisted twining genera vanish, as we shall now explain.

2.2. **Cohomological obstructions.** For some pairs of commuting elements $g, h \in M_{24}$, the transformation properties above can only be satisfied if $\phi_{g,h}$ vanishes identically. In this case, we will say that the corresponding twisted twining genus is *obstructed*. As will be shown in more detail in section 3, these obstructions are also controlled by the cohomology class of the cocycle $\alpha \in H^3(M_{24}, U(1))$.[§]

In order to understand the origin of these obstructions, let us first derive some general consequences of our assumptions (A)–(E). First we note that the $SL(2,\mathbb{Z})$ action on the twisted twining characters can be extended to $GL(2,\mathbb{Z})$ by setting

$$\phi^*_{g,h}(\tau, z) = \chi_{g,h}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \phi_{g,h^{-1}}(\tau, z) \ , \tag{11}$$

where $\phi^*_{g,h}(\tau, z)$ is obtained by taking the complex conjugate of all Fourier coefficients of $\phi_{g,h}(\tau, z)$

$$\phi^*_{g,h}(\tau, z) = \overline{\phi_{g,h}(-\bar{\tau}, -\bar{z})} \ . \tag{12}$$

The identity (11) follows from eq. (6), together with the observation that, for any projective representation R of a finite group G, $\mathrm{Tr}_R(h^{-1})$ equals $\overline{\mathrm{Tr}_R(h)}$ up to a phase, which we have denoted by $\chi_{g,h}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. As we will see, this phase also depends on the 3-cocycle α. We also note that since the $\mathcal{N} = 4$ characters are invariant under $z \to -z$, we have the identity

$$\phi_{g,h}(\tau, -z) = \phi_{g,h}(\tau, z) \ . \tag{13}$$

With these preparations we can now describe two possible kinds of obstructions.

Obstruction 1: Let us consider pairwise commuting g, h, k. Then, by (5), we have

$$\phi_{g,h}(\tau, z) = \xi_{g,h}(k)\, \phi_{g,h}(\tau, z) \ , \tag{14}$$

and if

$$\xi_{g,h}(k) \neq 1 \ , \tag{15}$$

it follows that $\phi_{g,h}(\tau, z) = 0$.

[§]The fact that such obstructions might exist was first suggested to us by T. Gannon.

Obstruction 2: Suppose there are $g, h, k \in M_{24}$, with g and h commuting and

$$k^{-1}g^{-1}k = g \ , \qquad k^{-1}h^{-1}k = h \ , \tag{16}$$

i.e. the commuting pair (g, h) is conjugate within M_{24} to the pair (g^{-1}, h^{-1}). Then, by eqs. (51) and (42) (see section 3 for details), we obtain the relations

$$\phi_{g,h}(\tau, z) = \chi_{g,h}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \phi_{g^{-1},h^{-1}}(\tau, -z) \tag{17}$$

$$= \chi_{g,h}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \xi_{g^{-1},h^{-1}}(k)\, \phi_{g,h}(\tau, -z) \ . \tag{18}$$

Therefore, if

$$\chi_{g,h}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \xi_{g^{-1},h^{-1}}(k) \neq 1 \ , \tag{19}$$

eq. (13) implies $\phi_{g,h}(\tau, z) = 0$.

As we shall see, these two obstructions are responsible for the fact that most twisted twining genera vanish.

2.3. **Classification of independent twisted twining genera.** Our next aim is to enumerate all possible independent twisted twining genera. Let us denote the set of commuting group elements by

$$\mathcal{P} = \{(g, h) \in M_{24} \times M_{24} \mid gh = hg\} \ . \tag{20}$$

This set carries an action of $GL(2, \mathbb{Z}) \times M_{24}$, given by

$$(g, h) \mapsto (k^{-1}(g^a h^c)k, k^{-1}(g^b h^d)k) \ , \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z}) \ , \quad k \in M_{24} \ , \tag{21}$$

and the twisted twining genera associated to different (g, h)'s in the same orbit are related to one another by modular transformations, see eqs. (4), (5), and/or by complex conjugation, see eq. (11). We therefore want to describe the set of orbits of $\mathcal{P}$ under the action (21).

First we note that the $GL(2, \mathbb{Z})$ orbit of a pair (g, h) consists of all possible pairs of generators of the abelian group $\langle g, h\rangle \subset M_{24}$. Thus the orbits of $GL(2, \mathbb{Z}) \times M_{24}$ are in one-to-one correspondence with the conjugacy classes of abelian subgroups of M_{24} generated by two elements, i.e.

$$\bar{\mathcal{P}} = \mathcal{P}/(GL(2, \mathbb{Z}) \times M_{24}) = \{M_{24} - \text{conjugacy classes of groups } \langle g, h\rangle \subset M_{24}, gh = hg\} \ . \tag{22}$$

This description allows a complete classification of the orbits $[g, h] \in \bar{\mathcal{P}}$: there are 55 such orbits, 21 of which correspond to cyclic subgroups, i.e. to subgroups of the form $[e, h]$. The associated twisted twining genera are therefore just the twining genera $\phi_{e,h}$, for which explicit expressions were already derived in [8, 9]. Thus we only need to construct

#	Structure	Elements	$\lvert C(g,h)\rvert$	$\frac{\lvert N(g,h)\rvert}{\lvert C(g,h)\rvert}$	Orbits on **24**	Max subgr.
1.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2A)^3$	1536	6	$1^8\cdot 4^4$	
2.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2A)^3$	1536	6	2^{12}	
3.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2A)^3$	128	6	$1^4\cdot 2^6\cdot 4^2$	
4.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2B)^3$	3840	6	4^6	
5.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2B)^3$	96	6	4^6	
6.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2B)^3$	64	6	4^6	
7.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2A)^2(2B)$	256	2	$2^8\cdot 4^2$	
8.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2A)(2B)^2$	512	2	$2^4\cdot 4^4$	
9.	$\mathbb{Z}_2\times\mathbb{Z}_2$	$(2A)(2B)^2$	128	2	$2^4\cdot 4^4$	
10.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)^3(4A)^4$	64	8	$2^4\cdot 8^2$	1
11.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)^3(4A)^4$	64	8	4^6	2
12.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)^2(2B)(4A)^2(4B)^2$	32	2	$2^2\cdot 4^3\cdot 8^1$	7
13.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)(2B)^2(4A)^4$	64	8	$4^2\cdot 8^2$	8
14.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)(2B)^2(4A)^4$	32	8	$4^2\cdot 8^2$	8
15.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)(2B)^2(4C)^4$	32	4	$4^2\cdot 8^2$	8
16.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)(2B)^2(4C)^4$	16	4	$4^2\cdot 8^2$	9
17.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)^3(4B)^4$	64	8	$1^4\cdot 2^2\cdot 8^2$	1
18.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)^3(4B)^4$	64	8	$2^4\cdot 4^4$	2
19.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)^3(4B)^4$	16	8	$1^2\cdot 2^3\cdot 4^2\cdot 8^1$	3
20.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)(2B)^2(4B)^4$	64	8	$2^4\cdot 8^2$	8
21.	$\mathbb{Z}_2\times\mathbb{Z}_4$	$(2A)(2B)^2(4B)^4$	16	8	$2^4\cdot 8^2$	9
22.	$\mathbb{Z}_4\times\mathbb{Z}_4$	$(2A)(2B)^2(4A)^4(4C)^8$	16	16	$8^1\cdot 16^1$	$13, 15, 15$
23.	$\mathbb{Z}_4\times\mathbb{Z}_4$	$(2A)^3(4A)^8(4B)^4$	16	32	$2^2\cdot 4^1\cdot 16^1$	$10, 10, 17$
24.	$\mathbb{Z}_4\times\mathbb{Z}_4$	$(2A)^3(4A)^8(4B)^4$	16	32	$4^2\cdot 8^2$	$11, 11, 18$
25.	$\mathbb{Z}_4\times\mathbb{Z}_4$	$(2A)^3(4B)^{12}$	16	96	$1^4\cdot 4^1\cdot 16^1$	$17, 17, 17$
26.	$\mathbb{Z}_4\times\mathbb{Z}_4$	$(2A)^3(4B)^{12}$	16	96	4^6	$18, 18, 18$
27.	$\mathbb{Z}_2\times\mathbb{Z}_8$	$(2A)(2B)^2(4B)^4(8A)^8$	16	8	$2^2\cdot 4^1\cdot 16^1$	20
28.	$\mathbb{Z}_2\times\mathbb{Z}_6$	$(2A)^3(3A)^2(6A)^6$	12	12	$1^2\cdot 3^2\cdot 4^1\cdot 12^1$	1
29.	$\mathbb{Z}_2\times\mathbb{Z}_6$	$(2A)^3(3A)^2(6A)^6$	12	12	$2^3\cdot 6^3$	2
30.	$\mathbb{Z}_2\times\mathbb{Z}_6$	$(2B)^3(3B)^2(6B)^6$	12	12	12^2	4
31.	$\mathbb{Z}_2\times\mathbb{Z}_6$	$(2B)^3(3B)^2(6B)^6$	12	12	12^2	5
32.	$\mathbb{Z}_2\times\mathbb{Z}_{10}$	$(2B)^3(5A)^4(10A)^{12}$	20	12	$4^1\cdot 20^1$	4
33.	$\mathbb{Z}_3\times\mathbb{Z}_3$	$(3A)^8$	9	48	$1^3\cdot 3^4\cdot 9^1$	
34.	$\mathbb{Z}_3\times\mathbb{Z}_3$	$(3A)^2(3B)^6$	9	12	$3^2\cdot 9^2$	

TABLE 1. The 34 conjugacy classes of abelian subgroups of rank 2 in M_{24}.

the remaining 34 genuinely twisted twining genera. We have tabulated the corresponding conjugacy classes of groups $\langle g,h\rangle$ in table 1. There we have described their structure as an abelian group, i.e. as $\mathbb{Z}_m \times \mathbb{Z}_n$, the M_{24} classes of all its elements (excluding the identity), the order of the centraliser $C(g,h)$, the index $|N(g,h)|/|C(g,h)|$ of the centraliser in the normaliser of $\langle g,h\rangle$ in M_{24}, and the lengths of the orbits of $\langle g,h\rangle \subset M_{24}$ when acting as a group of permutations of 24 objects. Finally, the last column gives the conjugacy classes of the non-cyclic maximal subgroups. For example, group 24 has three distinct non-cyclic maximal subgroups, all of the form $\mathbb{Z}_2 \times \mathbb{Z}_4$; two of them are conjugated to group 11 and one is conjugated to group 18.

For M_{24} it turns out that the orbits of $\mathcal{P}$ under $SL(2,\mathbb{Z}) \times M_{24}$ are exactly the same as those under $GL(2,\mathbb{Z}) \times M_{24}$, i.e.[¶]

$$\bar{\mathcal{P}} = \mathcal{P}/(GL(2,\mathbb{Z}) \times M_{24}) = \mathcal{P}/(SL(2,\mathbb{Z}) \times M_{24}) \ . \tag{23}$$

Thus the twisted twining genera in each orbit in $\bar{\mathcal{P}}$ are just related by phases, see eqs. (4) and (5), and we do not need to invoke eq. (11).

2.4. **Modular properties of the twisted twining genera.** For each commuting pair $(g,h) \in \mathcal{P}$, let us denote by $\tilde{\Gamma}_{g,h} \subset SL(2,\mathbb{Z}) \times M_{24}$ the group of elements (γ,k) that leave the pair (g,h) fixed or map it to its inverse (g^{-1},h^{-1})

$$\tilde{\Gamma}_{g,h} = \left\{ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), k \right) \in SL(2,\mathbb{Z}) \times M_{24} \mid \left(k^{-1}(g^a h^c)k,\ k^{-1}(g^b h^d)k\right) = (g,h) \text{ or } (g^{-1},h^{-1}) \right\} \ . \tag{24}$$

It follows from (4) and (5), together with (13) as well as (51) below, that the corresponding twisted twining genus $\phi_{g,h}$ will be invariant (up to a phase) under $\Gamma_{g,h} \equiv \pi(\tilde{\Gamma}_{g,h})$, where π denotes the projection of $SL(2,\mathbb{Z}) \times M_{24}$ onto its first factor, i.e. onto $SL(2,\mathbb{Z})$.

Each twisted twining genus $\phi_{g,h}$ belongs to a (finite-dimensional) vector representation of $SL(2,\mathbb{Z})$, that is spanned by the functions $\{\phi_{(g,h)\gamma}(\tau,z)\}$, where $\gamma \in SL(2,\mathbb{Z})$. Since every $\gamma \in \Gamma_{g,h}$ acts trivially (i.e. up to a phase) on $\phi_{g,h}$, the $SL(2,\mathbb{Z})$ representation is spanned by $\{\phi_{(g,h)\gamma}(\tau,z)\}_{\gamma\in\Gamma_{g,h}\backslash SL(2,\mathbb{Z})}$, where γ runs over a set of representatives for the cosets in $\Gamma_{g,h}\backslash SL(2,\mathbb{Z})$. Each of these functions $\phi_{(g,h)\gamma}$ is a weak Jacobi form of weight 0 and index 1 for the congruence subgroup $\gamma^{-1}\Gamma_{g,h}\gamma \subseteq SL(2,\mathbb{Z})$. We have tabulated for each orbit $[g,h] \in \bar{\mathcal{P}}$ the functions $\phi_{(g,h)\gamma}$ (starting from $\phi_{g,h}$) and the invariance group $\Gamma_{g,h}$ in table 2. (Note, however, that some of these functions vanish identically; in particular, this will be

[¶]For a generic group G different from M_{24}, some $SL(2,\mathbb{Z}) \times G$ orbits might be strictly smaller than the $GL(2,\mathbb{Z}) \times G$ orbits.

the case if there is an obstruction.) Each $\phi_{g,h}$ is denoted by the M_{24} class of g and the $C_{M_{24}}(g)$-class of h, named as in appendix D.

Group 27 is the only case where the pairs (g,h) and (g^{-1},h^{-1}) are not conjugated within M_{24}. Thus, charge conjugation gives the identities $\phi_{2\mathrm{B},8\mathrm{A}_1} = \phi_{2\mathrm{B},8\mathrm{A}_2}$, $\phi_{8\mathrm{A},2\mathrm{B}_1} = \phi_{8\mathrm{A},2\mathrm{B}_2}$, and so on, and the respective functions are denoted in table 2 by $\phi_{2\mathrm{B},8\mathrm{A}_{1,2}}$, $\phi_{8\mathrm{A},2\mathrm{B}_{1,2}}$, etcetera. Because of (11), most of the functions satisfy a 'reality' condition of the form

$$\phi^*_{g,h} = \zeta\, \phi_{g,h} \ , \tag{25}$$

for some constant phase ζ, which implies that the Fourier coefficients of $\zeta^{1/2}\phi_{g,h}$ are all real. Note that even if $\phi^*_{g,h}$ and $\phi_{g,h}$ are not proportional to one another, they are necessarily related by a modular transformation because of (23); if this is the case, they are listed as distinct functions in the same orbit in $\bar{\mathcal{P}}$.

Most of the modular groups $\Gamma_{g,h}$ are of the form $\Gamma(1) = SL(2,\mathbb{Z})$ or

$$\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid c \equiv 0 \mod N \} \ , \tag{26}$$

or conjugates of $\Gamma_0(N)$ in $SL(2,\mathbb{Z})$. The exceptions are the group in case 32, where

$$\Gamma_{2\mathrm{B},10\mathrm{A}} = \bigcup_{i\in\mathbb{Z}/3\mathbb{Z}, j\in\mathbb{Z}/4\mathbb{Z}} \left(\begin{smallmatrix} 1 & 1 \\ -5 & -4 \end{smallmatrix}\right)^i \left(\begin{smallmatrix} -3 & -1 \\ 10 & 3 \end{smallmatrix}\right)^j \Gamma_{2,10} \ , \tag{27}$$

is a subgroup of index 12 in $SL(2,\mathbb{Z})$ and

$$\Gamma_{2,10} := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid a \equiv 1,\ b \equiv 0 \mod 2,\ c \equiv 0,\ d \equiv 1 \mod 10 \} \ , \tag{28}$$

is the group of elements $\gamma \in SL(2,\mathbb{Z})$ such that $(g,h)\cdot\gamma = (g,h)$; the group in case 12, with

$$\Gamma_{2\mathrm{A},4\mathrm{A}} = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid b \equiv 0 \mod 2, c \equiv 0 \mod 4 \} \ , \tag{29}$$

which is a conjugate of $\Gamma_0(8)$ in $SL(2,\mathbb{R})$; and the group in case 22, with

$$\Gamma_{4\mathrm{A},4\mathrm{C}} = \langle \left(\begin{smallmatrix} -1 & 1 \\ -2 & 1 \end{smallmatrix}\right) \ , \ \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) \ , \ \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) \ , \ \left(\begin{smallmatrix} 3 & -2 \\ -4 & 3 \end{smallmatrix}\right) \rangle \ . \tag{30}$$

2.5. **Explicit twining genera.** We have now reduced our problem to finding 34 functions $\phi_{g,h}$ that are weak Jacobi forms with respect to $\Gamma_{g,h}$ up to some multiplier phases. As will be explained in the following section 3, our assumption (E) allows us to determine the precise form of these multipliers explicitly. Then most of the 34 functions vanish because of the obstructions, see table 3. For those that do not, the modular properties are strong enough to determine them explicitly, see also table 3. These explicit results are one of the main results of this paper, and their derivation is sketched in section 3, see in particular section 3.4, as well as appendix A. As will be explained below in section 3.5, these functions

#	Structure	Functions	$\Gamma_{g,h}$
1.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{A},2\mathrm{A}_2}$	$\Gamma(1)$
2.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{A},2\mathrm{A}_3}$	$\Gamma(1)$
3.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{A},2\mathrm{A}_5}$	$\Gamma(1)$
4.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{B},2\mathrm{B}_2}$	$\Gamma(1)$
5.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{B},2\mathrm{B}_4}$	$\Gamma(1)$
6.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{B},2\mathrm{B}_6}$	$\Gamma(1)$
7.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{B},2\mathrm{A}_1},\ \phi_{2\mathrm{A},2\mathrm{B}_3},\ \phi_{2\mathrm{A},2\mathrm{A}_4}$	$\Gamma_0(2)$
8.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{A},2\mathrm{B}_1},\ \phi_{2\mathrm{B},2\mathrm{A}_2},\ \phi_{2\mathrm{B},2\mathrm{B}_1}$	$\Gamma_0(2)$
9.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2\mathrm{A},2\mathrm{B}_2},\ \phi_{2\mathrm{B},2\mathrm{A}_3},\ \phi_{2\mathrm{B},2\mathrm{B}_5}$	$\Gamma_0(2)$
10.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{A}_2},\ \phi_{4\mathrm{A},2\mathrm{A}_2},\ \phi_{4\mathrm{A},4\mathrm{A}_3}$	$\Gamma_0(2)$
11.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{A}_3},\ \phi_{4\mathrm{A},2\mathrm{A}_3},\ \phi_{4\mathrm{A},4\mathrm{A}_7}$	$\Gamma_0(2)$
12.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{A}_4}, \phi_{2\mathrm{A},4\mathrm{B}_4}, \phi_{2\mathrm{B},4\mathrm{A}_1}, \phi_{2\mathrm{B},4\mathrm{B}_1}, \phi_{4\mathrm{A},2\mathrm{A}_1}, \phi_{4\mathrm{A},2\mathrm{B}_1},$ $\phi_{4\mathrm{B},2\mathrm{A}_4}, \phi_{4\mathrm{B},2\mathrm{B}_2}, \phi_{4\mathrm{B},4\mathrm{A}_1}, \phi_{4\mathrm{B},4\mathrm{A}_2}, \phi_{4\mathrm{A},4\mathrm{B}_3}, \phi_{4\mathrm{A},4\mathrm{B}_4}$	$\Gamma_{2\mathrm{A},4\mathrm{A}}$
13.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{B},4\mathrm{A}_2},\ \phi_{4\mathrm{A},2\mathrm{B}_3},\ \phi_{4\mathrm{A},4\mathrm{A}_5}$	$\Gamma_0(2)$
14.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{B},4\mathrm{A}_3},\ \phi_{4\mathrm{A},2\mathrm{B}_2},\ \phi_{4\mathrm{A},4\mathrm{A}_2}$	$\Gamma_0(2)$
15.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{C}_1}, \phi_{4\mathrm{C},2\mathrm{A}_2}, \phi_{4\mathrm{C},2\mathrm{B}_3}, \phi_{2\mathrm{B},4\mathrm{C}_2}, \phi_{4\mathrm{C},4\mathrm{C}_4}, \phi_{4\mathrm{C},4\mathrm{C}_6}$	$\Gamma_0(4)$
16.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{C}_2}, \phi_{4\mathrm{C},2\mathrm{A}_1}, \phi_{4\mathrm{C},2\mathrm{B}_2}, \phi_{2\mathrm{B},4\mathrm{C}_3}, \phi_{4\mathrm{C},4\mathrm{C}_3}, \phi_{4\mathrm{C},4\mathrm{C}_5}$	$\Gamma_0(4)$
17.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{B}_2},\ \phi_{4\mathrm{B},2\mathrm{A}_2},\ \phi_{4\mathrm{B},4\mathrm{B}_8}$	$\Gamma_0(2)$
18.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{B}_3},\ \phi_{4\mathrm{B},2\mathrm{A}_5},\ \phi_{4\mathrm{B},4\mathrm{B}_9}$	$\Gamma_0(2)$
19.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{A},4\mathrm{B}_5},\ \phi_{4\mathrm{B},2\mathrm{A}_3},\ \phi_{4\mathrm{B},4\mathrm{B}_3}$	$\Gamma_0(2)$
20.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{B},4\mathrm{B}_2},\ \phi_{4\mathrm{B},2\mathrm{B}_3},\ \phi_{4\mathrm{B},4\mathrm{B}_1}$	$\Gamma_0(2)$
21.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2\mathrm{B},4\mathrm{B}_3},\ \phi_{4\mathrm{B},2\mathrm{B}_1},\ \phi_{4\mathrm{B},4\mathrm{B}_5}$	$\Gamma_0(2)$
22.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4\mathrm{A},4\mathrm{C}_1}, \phi_{4\mathrm{A},4\mathrm{C}_2}, \phi_{4\mathrm{C},4\mathrm{A}_1}, \phi_{4\mathrm{C},4\mathrm{A}_2}, \phi_{4\mathrm{C},4\mathrm{C}_7}, \phi_{4\mathrm{C},4\mathrm{C}_8}$	$\Gamma_{4\mathrm{A},4\mathrm{C}}$
23.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4\mathrm{B},4\mathrm{A}_3},\ \phi_{4\mathrm{A},4\mathrm{B}_1},\ \phi_{4\mathrm{A},4\mathrm{A}_1}$	$\Gamma_0(2)$
24.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4\mathrm{B},4\mathrm{A}_4},\ \phi_{4\mathrm{A},4\mathrm{B}_2},\ \phi_{4\mathrm{A},4\mathrm{A}_4}$	$\Gamma_0(2)$
25.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4\mathrm{B},4\mathrm{B}_4}$	$\Gamma(1)$
26.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4\mathrm{B},4\mathrm{B}_7}$	$\Gamma(1)$
27.	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\phi_{2\mathrm{B},8\mathrm{A}_{1,2}}, \phi_{8\mathrm{A},2\mathrm{B}_{1,2}}, \phi_{4\mathrm{B},8\mathrm{A}_{2,3}}, \phi_{8\mathrm{A},4\mathrm{B}_{1,3}}, \phi_{8\mathrm{A},8\mathrm{A}_{2,8}}, \phi_{8\mathrm{A},8\mathrm{A}_{6,7}}$	$\Gamma_0(4)$
28.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2\mathrm{A},6\mathrm{A}_2}, \phi_{6\mathrm{A},2\mathrm{A}_1}, \phi_{6\mathrm{A},6\mathrm{A}_1}, \phi_{6\mathrm{A},6\mathrm{A}_2}$	$\Gamma_0(3)$
29.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2\mathrm{A},6\mathrm{A}_3}, \phi_{6\mathrm{A},2\mathrm{A}_2}, \phi_{6\mathrm{A},6\mathrm{A}_3}, \phi_{6\mathrm{A},6\mathrm{A}_4}$	$\Gamma_0(3)$
30.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2\mathrm{B},6\mathrm{B}_2}, \phi_{6\mathrm{B},2\mathrm{B}_1}, \phi_{6\mathrm{B},6\mathrm{B}_2}, \phi_{6\mathrm{B},6\mathrm{B}_5}$	$\Gamma_0(3)$
31.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2\mathrm{B},6\mathrm{B}_3}, \phi_{6\mathrm{B},2\mathrm{B}_2}, \phi_{6\mathrm{B},6\mathrm{B}_1}, \phi_{6\mathrm{B},6\mathrm{B}_3}$	$\Gamma_0(3)$
32.	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\phi_{2\mathrm{B},10\mathrm{A}_1}, \phi_{2\mathrm{B},10\mathrm{A}_3}, \phi_{10\mathrm{A},10\mathrm{A}_1}, \phi_{10\mathrm{A},10\mathrm{A}_2}, \phi_{10\mathrm{A},10\mathrm{A}_3}, \phi_{10\mathrm{A},10\mathrm{A}_5},$ $\phi_{10\mathrm{A},10\mathrm{A}_7}, \phi_{10\mathrm{A},10\mathrm{A}_9}, \phi_{10\mathrm{A},10\mathrm{A}_{10}}, \phi_{10\mathrm{A},10\mathrm{A}_{11}}, \phi_{10\mathrm{A},2\mathrm{B}_2}, \phi_{10\mathrm{A},2\mathrm{B}_3}$	$\Gamma_{2\mathrm{B},10\mathrm{A}}$
33.	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\phi_{3\mathrm{A},3\mathrm{A}_3}$	$\Gamma(1)$
34.	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\phi_{3\mathrm{A},3\mathrm{B}_1}, \phi_{3\mathrm{B},3\mathrm{A}_1}, \phi_{3\mathrm{B},3\mathrm{B}_3}, \phi_{3\mathrm{B},3\mathrm{B}_4}$	$\Gamma_0(3)$

TABLE 2. The independent functions and the group $\Gamma_{g,h}$ for each $[g,h] \in \bar{P}$.

#	$\phi_{g,h}$	Obstr.	#	$\phi_{g,h}$	Obstr.
1.	0	1	18.	0	1
2.	0	1	19.	0	1
3.	0	1	20.	0	2
4.	0	2	21.	0	1
5.	0	2	22.	0	2
6.	0	1	23.	$\phi_{4\mathrm{B},4\mathrm{A}_3} = 2\sqrt{2}\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no
7.	0	1	24.	$\phi_{4\mathrm{B},4\mathrm{A}_4} = 2\sqrt{2}\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no
8.	0	1	25.	0	2
9.	0	1	26.	0	2
10.	0	1	27.	$\phi_{2\mathrm{B},8\mathrm{A}_{1,2}} = 2\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no
11.	0	1	28.	0	2
12.	0	1	29.	0	2
13.	$\phi_{2\mathrm{B},4\mathrm{A}_2} = 4\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no	30.	0	2
14.	0	1	31.	0	2
15.	0	1	32.	0	2
16.	0	1	33.	$\phi_{3\mathrm{A},3\mathrm{A}_3} = 0$	no
17.	0	1	34.	$\phi_{3\mathrm{A},3\mathrm{B}_1} = 0$	no

TABLE 3. For each of the 34 cases we give here the explicit result for the twisted twining genus, as well as where applicable, the obstruction that it responsible for its vanishing.

are also compatible with the expected projective representation of the centraliser $C_{M_{24}}(g)$ on the g-twisted sector, see property (C) in section 2.

3. Holomorphic Orbifolds and Group Cohomology

In this section we review the modular properties of holomorphic orbifolds since, according to our assumption (E), these are also relevant for the twisted twining genera of K3. We then explain how, in our context, the known multiplier phases of the twining genera determine the underlying 3-cocycle $\alpha \in H^3(M_{24}, U(1))$ uniquely, and how to obtain from this our explicit results given in table 3. Finally, we shall check that these formulae indeed give rise to the appropriate projective representations of the centraliser $C_{M_{24}}(g)$.

3.1. **Review of holomorphic orbifolds.** Suppose $\mathcal{C}$ is a self-dual VOA, and G is a group of $\mathcal{C}$-automorphisms. We are interested in the 'orbifold' of $\mathcal{C}$ by G. To this end we consider the G-invariant sub-VOA of $\mathcal{C}$,

$$\mathcal{C}^G = \{\psi \in \mathcal{C} : g\psi = \psi, \forall g \in G\} \ . \tag{31}$$

Each representation of $\mathcal{C}$ gives rise to a representation of $\mathcal{C}^G$. In addition, there are new representations of $\mathcal{C}^G$ that appear in the different twisted sectors $\mathcal{C}_A$; since $\mathcal{C}$ is holomorphic there is a unique twisted sector $\mathcal{C}_A$ for each conjugacy class A in G.

3.1.1. *Twisted twining characters.* Each twisted sector $\mathcal{C}_A$ defines a representation of the G-invariant sub-VOA $\mathcal{C}^G$, but it is typically not irreducible. In particular, for each g_A in the conjugacy class $g_A \in [A]$, we can project $\mathcal{C}_{g_A}$ onto a $\mathcal{C}^G$-invariant subspace using any (projective) character χ of $C_G(g_A)$. Thus we can label the irreducible representations of $\mathcal{C}^G$ by pairs (A, χ). In physics, by the orbifold of $\mathcal{C}$ by G one usually means the theory where one chooses $\chi = 1$ in each twisted sector, i.e. the full partition function takes the form

$$Z_{\text{orb}}(\tau) = \sum_A \frac{1}{|C_G(A)|} \sum_{h \in C_G(g_A)} Z_{g_A,h}(\tau) \ , \tag{32}$$

where

$$Z_{g,h}(\tau) \equiv {}_g\underset{h}{\square} = \text{Tr}_{\mathcal{C}_g}\left(\rho_g(h)\, q^{L_0 - \frac{c}{24}}\right) \ , \tag{33}$$

and ρ_g denotes the action of $C_G(g)$ on $\mathcal{C}_g$. We should mention that this action is usually *not* canonically defined; for example, what is usually referred to as discrete torsion [33] can be thought of as being an ambiguity in the definition of ρ_g. Incidentally, the fact that the action is not canonically defined will be important to us later, see in particular section 4.1. For the moment though, we want to assume that such an action has been chosen.

3.1.2. *Projective representations.* As alluded to above, the twisted sectors $\mathcal{C}_g$ typically do not form genuine representations of $C_G(g)$ but only *projective representations*. A projective representation ρ of a finite group G is characterised by the relation (see also appendix C for some introductory comments)

$$\rho(g_1)\,\rho(g_2) = c(g_1, g_2)\,\rho(g_1 g_2) \ , \tag{34}$$

where $c : G \times G \to U(1)$ is a (normalised) 2-cocycle representing a cohomology class in $H^2(G, U(1))$. Genuine representations of finite groups are classified, up to equivalence, by their characters $\text{Tr}_\rho(h)$. This is, however, no longer true for projective representations since taking the trace over a projective representation does not lead to a class function,

i.e. a function that is invariant under conjugation by elements in G. Instead, for projective representations one has

$$\mathrm{Tr}_\rho(hgh^{-1}) = \frac{c(h,h^{-1}gh)}{c(g,h)}\,\mathrm{Tr}_\rho(g)\ , \tag{35}$$

as follows from the defining relation (34) together with the cyclicity of the trace (see e.g. [34] for a nice discussion). For the case of the holomorphic orbifold, this then results in the conjugation relation

$$Z_{g,h}(\tau) = \frac{c_g(h,k)}{c_g(k,k^{-1}hk)}\,Z_{k^{-1}gk,k^{-1}hk}(\tau)\ , \tag{36}$$

where the 2-cocycle c_g represents a class in $H^2(C_G(g),U(1))$ specifying the projective representation of the centralizer $C_G(g)$ on $\mathcal{C}_g$, and we have assumed that k and g commute so that both expressions are evaluated in the same twisted sector. (In fact, (36) is even true if k and g do not commute, see [25].) If in addition k commutes with h, (36) reduces to

$$Z_{g,h}(\tau) = \frac{c_g(h,k)}{c_g(k,h)}\,Z_{g,h}(\tau)\ , \tag{37}$$

which therefore implies that $Z_{g,h}$ vanishes unless the 2-cocycle is *regular*,

$$c_g(h,k) = c_g(k,h)\ . \tag{38}$$

This is the holomorphic orbifold version of the first type of obstruction discussed in section 2.2 above.

3.1.3. *The underlying cohomology.* The possible choices for the 2-cocycles c_g in the various twisted sectors are constrained; for example this follows from demanding consistency of the OPE involving fields from different twist sectors.[‖] In particular, it was argued in [25] that the various consistent choices are in one-to-one correspondence with elements in the third cohomology group $[\alpha]\in H^3(G,U(1))$, see also [35, 36, 37, 38] for subsequent work. The 3-cocycle α determines distinguished elements $c_g\in H^2(C_G(g),U(1))$ via the formula [25, 35]

$$c_g(h_1,h_2) = \frac{\alpha(g,h_1,h_2)\,\alpha(h_1,h_2,(h_1h_2)^{-1}g(h_1h_2))}{\alpha(h_1,h_1^{-1}gh_1,h_2)}\ . \tag{39}$$

Since α fixes all 2-cocylces c_g via (39), it is clear that the choice of α determines the behaviour of the twisted twining characters under conjugation (36). Furthermore, α also

[‖]This origin of the constraint was suggested in [36].

determines all the multiplier phases that appear under modular transformations, since it follows from [35] that**

$$\begin{aligned} Z_{g,h}(\tau+1) &= c_g(g,h)\, Z_{g,gh}(\tau)\ , \\ Z_{g,h}(-1/\tau) &= \overline{c_h(g,g^{-1})}\, Z_{h,g^{-1}}(\tau)\ . \end{aligned} \tag{40}$$

The choice of a particular representative α in the class $[\alpha] \in H^3(G, U(1))$ corresponds to a choice of normalisation for the action of h on the g-twisted sector. To see this, let $\tilde{\alpha}$ represent a 3-cocycle in the same cohomology class $[\alpha] \in H^3(G, U(1))$, so that $\tilde{\alpha} = \alpha \cdot \partial\beta$ (see eq. (158)) for some 2-cochain β. Then the associate 2-cocycles c_g and $\tilde{c}_g$ differ (for each $g \in G$) only by a 2-coboundary i.e. $\tilde{c}_g = c_g \cdot \partial\gamma_g$, with $\gamma_g(h) = \frac{\beta(h,g)}{\beta(g,h)}$, see appendix B. Thus, the 3-cocycle $\tilde{\alpha}$ is associated with the modular properties of the functions $\tilde{Z}_{g,h} = \gamma_g(h) Z_{g,h}$, which, by (33), simply correspond to a different normalisation $\tilde{\rho}_g(h) = \gamma_g(h)\rho_g(h)$ for the action of h on the g-twisted sector.

3.2. **Application to Mathieu Moonshine.** Given what we said before, it is now natural to postulate that for the twisted twining genera of K3, the various multiplier phases also come from a 3-cocycle $\alpha \in H^3(M_{24}, U(1))$. This is to say, we postulate that the projective representation appearing in (C) is the one determined by c_g, given by (39). Furthermore, $\chi_{g,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in (4) is determined by (40), and $\xi_{g,h}(k)$ in (5) agrees with (37), where in both cases c_g is again the function determined by (39) for a fixed $\alpha \in H^3(M_{24}, U(1))$. More explicitly, we therefore propose that

$$\begin{aligned} &\phi_{g,h}\left(-\tfrac{1}{\tau}, \tfrac{z}{\tau}\right) = \overline{c_h(g,g^{-1})}\, e^{2\pi i \frac{z^2}{\tau}}\, \phi_{h,g^{-1}}(\tau, z)\ , \\ &\phi_{g,h}(\tau+1, z) = c_g(g,h)\, \phi_{g,gh}(\tau, z)\ , \end{aligned} \tag{41}$$

with $c_g(h_1, h_2)$ given by (39), and

$$\phi_{g,h}(\tau, z) = \frac{c_g(h,k)}{c_g(k, k^{-1}hk)}\, \phi_{k^{-1}gk, k^{-1}hk}(\tau, z)\ , \qquad k \in M_{24}\ . \tag{42}$$

Moreover, we postulate that

$$\phi_{g,h^{-1}}(\tau, z) = c_g(h, h^{-1})\, \phi^*_{g,h}(\tau, z)\ , \tag{43}$$

where $\phi^*_{g,h}(\tau, z)$ is defined in (12).

**Note that (40) is obtained from [35] upon working with the inverse 3-cohomology element. We also thank Terry Gannon for explaining these formulae to us. It is believed (and we have verified this in simple cases) that (40) follows from the transformation rules for the irreducible orbifold characters, see [37, eqs. (5.23) and (5.24)], but we have not proven this in complete generality.

Combining modular transformations with conjugation one obtains a whole set of relations among the twisted twining genera, where the multipliers are given by

$$\chi_{g,h}(\gamma_1\gamma_2) = \chi_{g,h}(\gamma_1)\,\chi_{(g,h)\gamma_1}(\gamma_2)\ , \qquad \gamma_1,\gamma_2 \in SL(2,\mathbb{Z})\ , \tag{44}$$

$$\chi_{g,h}\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) = \frac{1}{c_h(g,g^{-1})}\ , \tag{45}$$

$$\chi_{g,h}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) = c_g(g,h)\ , \tag{46}$$

$$\chi_{g,h}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right) = \frac{1}{c_g(h,h^{-1})}\ , \tag{47}$$

$$\xi_{g,h}(k) = \frac{c_g(h,k)}{c_g(k,k^{-1}hk)}\ . \tag{48}$$

Since $SL(2,\mathbb{Z})$ is generated by $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ with the only relations being $S^4 = 1$ and $(ST)^3 = S^2$, it follows that eqs. (44) – (47) determine the multiplier $\chi_{g,h}(\gamma)$, $\gamma \in SL(2,\mathbb{Z})$, provided that the following consistency conditions are satisfied

$$\chi_{g,h}(S^4) = 1\ , \qquad \chi_{g,h}((ST)^3) = \chi_{g,h}(S^2)\ . \tag{49}$$

Furthermore, since the actions of $SL(2,\mathbb{Z})$ and M_{24} on the set $\mathcal{P}$ commute with one another, there are two more consistency conditions

$$\xi_{g,h}(k)\,\chi_{k^{-1}gk,k^{-1}hk}(\gamma) = \chi_{g,h}(\gamma)\,\xi_{(g,h)\gamma}(k) \qquad \xi_{g,h}(k)\,\xi_{k^{-1}gk,k^{-1}hk}(k') = \chi_{g,h}(kk')\ , \tag{50}$$

for all $\gamma \in SL(2,\mathbb{Z})$ and $k,k' \in M_{24}$. In fact, conditions (49) and (50) hold for a generic group G, and can be easily checked using standard identities for the cocycle c_g, in particular, eqs. (3.2.5)–(3.2.9) of [36] (see also [35]).

Note that the charge conjugation operator $C = S^2$ acts on the genera $\phi_{g,h}(\tau,z)$ by flipping the sign of the second argument z. In fact, writing $\tau = -\frac{1}{\tilde\tau}$ and $z = \frac{\tilde z}{\tilde\tau}$, we get from twice applying (41)

$$\begin{aligned}\phi_{g,h}(\tau,z) &= \phi_{g,h}(-\tfrac{1}{\tilde\tau},\tfrac{\tilde z}{\tilde\tau}) = \frac{e^{2\pi i\frac{\tilde z^2}{\tilde\tau}}}{c_h(g,g^{-1})}\,\phi_{h,g^{-1}}(\tilde\tau,\tilde z) \\ &= \frac{e^{-2\pi i\frac{z^2}{\tau}}}{c_h(g,g^{-1})}\,\phi_{h,g^{-1}}(-\tfrac{1}{\tau},-\tfrac{z}{\tau}) = \frac{1}{c_h(g,g^{-1})c_{g^{-1}}(h,h^{-1})}\,\phi_{g^{-1},h^{-1}}(\tau,-z)\ ,\end{aligned} \tag{51}$$

so that

$$\chi_{g,h}\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right) = \frac{1}{c_h(g,g^{-1})c_{g^{-1}}(h,h^{-1})}\ . \tag{52}$$

Together with (13), this then leads to

$$\phi_{g,h}(\tau,z) = \frac{1}{c_h(g,g^{-1})\,c_{g^{-1}}(h,h^{-1})}\,\phi_{g^{-1},h^{-1}}(\tau,z)\ , \tag{53}$$

thus reproducing (17). This identity played a key role in the derivation of obstruction 2 in section 2.2.

While our proposal about the multiplier phases is certainly natural, we do not have a direct proof for it. However, the fact that with this assumption we shall find consistent answers is in our opinion very convincing evidence in favour of this ansatz.

3.3. **Determining the cohomology class.** In order to have explicit formulae for the multipliers, the next step consists of determining $[\alpha] \in H^3(M_{24}, U(1))$. In this subsection, we will prove (with some computer support) that there exists a unique cohomology class $[\alpha]$ that reproduces (10) for all the *untwisted* twining genera $\phi_{e,g}(\tau, z)$. Since this is an overdetermined problem, the existence of such class already represents a non-trivial consistency check for our proposal. In the following subsections, we will then explain how the remaining twisted twining characters can be computed, once the class $[\alpha]$ is known.

First we note that the normalisation condition for α, see eq. (159), implies that $c_e(g, h) = 1$ for all $g, h \in M_{24}$. Thus $\phi_{e,g}(\tau + 1, z) = \phi_{e,g}(\tau, z)$, $\phi_{e,g^{-1}}(\tau, z) = \phi_{e,g}(\tau, z)$ and $\phi_{e,g}(\tau, z) = \phi_{e,k^{-1}gk}(\tau, z)$ for all $k \in M_{24}$, as expected. Furthermore, the representation ρ_e in the decomposition (6) is a genuine representation of $C_{M_{24}}(e) = M_{24}$.

Next we observe that eq. (41) implies

$$\phi_{g^a,g^b}(\tau + o(g^a), z) = \kappa(g^a, g^b)\, \phi_{g^a,g^b}(\tau, z) \ , \tag{54}$$

where

$$\kappa(g^a, g^b) := \prod_{k=1}^{o(g^a)} c_{g^a}(g^a, g^{b+ak}) = \prod_{k=1}^{o(g^a)} \alpha(g^a, g^{b+ak}, g^a) \ . \tag{55}$$

More precisely, the phases $\kappa(g^a, g^b)$ are the multipliers of $\phi_{e,g}$ under certain parabolic elements of the form

$$\gamma \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \gamma^{-1} \in \Gamma_0(o(g)) \ , \tag{56}$$

where $\gamma \in SL(2, \mathbb{Z})$ is such that $(e, g)\gamma = (g^a, g^b)$ and $n = o(g^a)$. By comparing such a multiplier with the one expected from (10), one obtains the following condition

$$\kappa(g^a, g^b) = e^{-\frac{2\pi i}{\ell(g^a)}} \ , \tag{57}$$

for all $a, b \in \mathbb{Z}$, where $\ell(g^a)$ is the length of the smallest cycle of g^a when considered as a permutation of 24 objects. It is easy to verify that $\kappa(g^a, g^b)$ only depends on the cohomology class of the 3-cocycle α. Thus we have to prove that there is a unique class $[\alpha] \in H^3(M_{24}, U(1))$ such that (57) is satisfied for all $g \in M_{24}$, $a, b \in \mathbb{Z}$.

In order to show that such an α exists, we first need two basic facts about group cohomology. Given two finite groups G and H and a homomorphism $\iota : H \to G$, we can define the induced map ι^* from $U(1)$-cochains α for G to $U(1)$-cochains $\iota^*(\alpha)$ for H

$$\iota^*(\alpha)(g_1, \ldots, g_n) = \alpha(\iota(g_1), \ldots, \iota(g_n)) \ , \tag{58}$$

which induces a well-defined map on cohomologies ([43], Chapter III.8). A particular case is when $H = G$ and the map $H \to G$ is conjugation $\iota_k(g) = k^{-1}gk$ by an element $k \in G$. In this case, the induced map on the cohomology ι_k^* acts as the identity, i.e. $[\iota^*(\alpha)] = [\alpha]$ (see [43], Chapter III Proposition 8.3). A second interesting case is when H is a subgroup of G with ι the natural inclusion; in this case, $\mathrm{Res} \equiv \iota^*$ is called a restriction map. In particular, if H is a Sylow p-subgroup $\mathrm{Syl}_p\, G$ of G,[††] then the image of the restriction map

$$\mathrm{Res}_p : H^n(G, U(1)) \to H^n(\mathrm{Syl}_p(G), U(1)) \tag{59}$$

is isomorphic to the p-part of $H^n(G, U(1))$ ([43], Chapter III Theorem 10.3). For $G = M_{24}$, using that $H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12}$ [32], this implies

$$\mathrm{Res}_2(H^3(M_{24}, U(1))) \cong \mathbb{Z}_4 \ , \qquad \mathrm{Res}_3(H^3(M_{24}, U(1))) \cong \mathbb{Z}_3 \ , \tag{60}$$

while $\mathrm{Res}_p(H^3(M_{24}, U(1)))$ is trivial for all primes $p > 3$.

With these preparations we can now turn to the proof that a unique class $[\alpha] \in H^3(M_{24}, U(1))$ satisfying (57) exists. From the invariance of the cohomology classes under the conjugation map ι_k^*, it follows that the multiplier in (57) only depends on the conjugacy class of g. Next we note that in order to compute $\kappa(g^a, g^b)$, it is sufficient to consider the restriction of the cocycle α to the group $\langle g \rangle$ generated by g. For a cyclic group $\mathbb{Z}_N$, the third cohomology $H^3(\mathbb{Z}_N, U(1))$ is isomorphic to $\mathbb{Z}_N$, and a set of cocycle representatives for the cohomology classes are given by [44, 37]

$$\omega_q(g^a, g^b, g^c) = e^{2\pi i \frac{qa}{o(g)^2}([b]+[c]-[b+c])} \ , \qquad q = 0, \ldots, o(g) - 1 \ , \tag{61}$$

where $[\cdot] : \mathbb{Z} \to \{0, \ldots, o(g) - 1\}$ is the reduction modulo $o(g)$.

Let us assume, without loss of generality, that $1 \leq a < o(g) - 1$ and $0 \leq b < a$, and that a divides $o(g)$; in fact, if this was not true, we can rewrite both g^a and g^b as powers of another

[††]Recall that a p-group (p prime) is a finite group with order a power of p; a Sylow p-subgroup of a group G is a p-subgroup of G that is maximal, i.e. that is not properly contained in any other p-subgroup. For each p, all Sylow p-subgroups of G are isomorphic and conjugate to one another inside G.

$\tilde{g}$ in $\langle g\rangle$ for which this is then true. The multiplier corresponding to a 3-cocycle ω_q is

$$\kappa(g^a,g^b)=\prod_{k=0}^{o(g^a)-1}\omega_q(g^a,g^{b+ak},g^a)=e^{2\pi i\frac{qa}{o(g)^2}\sum_{k=0}^{o(g^a)-1}([b+ak]+[a]-[b+a(k+1)])} \tag{62}$$
$$=e^{2\pi i\frac{qa}{o(g)^2}([b]+o(g^a)[a]-[b+ao(g^a)])}=e^{2\pi i\frac{qa}{o(g)}}\ ,$$

where we have used that $o(g^a)=o(g)/a$ when a divides $o(g)$. Thus, $\kappa(g^a,g^b)\equiv\kappa(g^a)$ is independent of b, and $\kappa(g^a)=\kappa(g)^a$ whenever a divides $o(g)$. The latter property is consistent with (57) thanks to the identity

$$\frac{1}{\ell(g^a)}\equiv\frac{a}{\ell(g)}\mod 1\ , \tag{63}$$

that holds for all elements $g\in M_{24}$ and for all divisors a of $o(g)$.*

It therefore follows that it is sufficient to prove (57) for $a=1$ and $b=0$, and for one representative g for each conjugacy class. Furthermore, thanks to $\kappa(g^a)=\kappa(g)^a$, we can restrict to classes whose order is a prime power. In other words, we only need to check (57) for a Sylow p-subgroup $\mathrm{Syl}_p(M_{24})$ and for each prime p.

Using the software `GAP` [41] with the package `HAP` [42] implemented, we have computed a basis of cocycle representatives for $H^3(\mathrm{Syl}_2(M_{24}),U(1))$ and $H^3(\mathrm{Syl}_3(M_{24}),U(1))$. Note that any 3-cocycle in the image of Res_p has the property that, for any $g\in\mathrm{Syl}_p(M_{24})$, the corresponding multiplier $\kappa(g)$ depends only on the conjugacy class of g in M_{24}. We have verified that there is a unique $\mathbb{Z}_4$ subgroup of $H^3(\mathrm{Syl}_2(M_{24}),U(1))$ and a unique $\mathbb{Z}_3$ subgroup of $H^3(\mathrm{Syl}_3(M_{24}),U(1))$ satisfying this property. Therefore these subgroups must correspond to the restrictions $\mathrm{Res}_p(H^3(M_{24},U(1)))$, for $p=2$ and $p=3$, respectively. Finally, for both $p=2$ and $p=3$, we have verified that there is a unique cohomology class $[\alpha_p]\in\mathrm{Res}_p(H^3(M_{24},U(1)))$ that satisfies (57) for all g in $\mathrm{Syl}_p(M_{24})$. Thus, the unique class $[\alpha]\in H^3(M_{24},U(1))$ that has the property that $\mathrm{Res}_p([\alpha])=[\alpha_p]$ for all primes p, satisfies (57) for all $g\in M_{24}$.

We have thus proven that there exists a unique class $[\alpha]\in H^3(M_{24},U(1))$ satisfying (57). It remains to show that this class exactly reproduces (10) for all $\phi_{e,g}$. First, we notice that the multiplier (10) of $\phi_{e,g}$ is trivial whenever the class of g restricts to a class in M_{23}. Since $H^3(M_{23},U(1))$ is trivial, this is consistent with any choice of the cocycle α. Thus, we only need to check (10) for the classes that do not restrict to M_{23}. The corresponding characters

*Note that this property is special for M_{24}, and is not true for a generic permutation.

$\phi_{e,g}$ are Jacobi forms under $\Gamma_0(o(g))$, where the possible orders $o(g)$ are $2,3,4,6$ and 12. For $N=2,3,4$, the group $\Gamma_0(N)$ is generated by

$$T=\begin{pmatrix}1&1\\0&1\end{pmatrix}\,,\qquad C=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\,,\qquad \begin{pmatrix}1&0\\-N&1\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}T^N\begin{pmatrix}0&-1\\1&0\end{pmatrix} \tag{64}$$

while $\Gamma_0(6)$ is generated by T, C, together with

$$\begin{pmatrix}1&0\\-6&1\end{pmatrix}\,,\qquad \begin{pmatrix}-5&3\\-12&7\end{pmatrix}=\begin{pmatrix}1&0\\2&1\end{pmatrix}T^3\begin{pmatrix}1&0\\-2&1\end{pmatrix}\,. \tag{65}$$

The correct modular properties (10) under T and C are automatic for any normalised cocycle α. The remaining generators are of the form (56) with $(e,g)\gamma=(g^a,g^b)$ and $n=o(g^a)$, and hence (10) follows again from (57).

The group $\Gamma_0(12)$ is generated by T,C and[†]

$$\alpha_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&12\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\,,\qquad \alpha_2=\begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}1&4\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-3&1\end{pmatrix}\,, \tag{66}$$

$$\alpha_3=\begin{pmatrix}1&0\\6&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-6&1\end{pmatrix}\,,\qquad \alpha_4=\begin{pmatrix}1&0\\2&1\end{pmatrix}\begin{pmatrix}1&3\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-2&1\end{pmatrix}\,. \tag{67}$$

The generators $\alpha_1,\ldots,\alpha_4$ are of the form (56) with $(e,g)\gamma=(g^a,g^b)$, but only for α_1,α_2 is the condition $n=o(g^a)$ satisfied, and thus only in these cases is condition (57) sufficient to establish (10). The generators α_3,α_4 satisfy $(e,g)\alpha_i=(e,g^7)$, $i=3,4$, so one needs to conjugate by an element $k\in M_{24}$ such that $k^{-1}g^7k=g$. In order to compute the multiplier under these generators, one needs to determine the restriction of $[\alpha]\in H^3(M_{24},U(1))$ to the subgroup of M_{24} generated by g and k. Using `GAP`, we computed a basis for the third cohomology of the group $\langle g,k\rangle$ and looked for those classes that satisfy condition (57) for all elements of the group. It turns out that there is only one such class, and hence it must correspond to the restriction of $[\alpha]\in H^3(M_{24},U(1))$ to the cohomology of $\langle g,k\rangle$. Then it is straightforward to check that this class reproduces indeed the multipliers (10) under α_3 and α_4; this is true for both the M_{24} conjugacy classes of order 12. This completes the proof.

The `GAP`-files containing the computations described above are available online [45].

[†]It is straightforward to check that $T,C,\alpha_1,\ldots,\alpha_4$ actually generate $\Gamma_0(12)$ by simply comparing them to a minimal set of generators obtained by some standard method, such as the Farey symbol algorithm.

3.4. **Computation of the twisted twining characters: an example.** In the previous subsection, we proved that there is a unique cohomology class $[\alpha] \in H^3(M_{24}, U(1))$ that reproduces the multipliers (10) for all the *untwisted* twining genera $\phi_{e,g}$. In this section, we shall explain how the remaining twisted twining genera $\phi_{g,h}$ can be determined, leading to the results collected in section 2.5.

The first step is to compute the obstructions and the multipliers for a given twisted twining genus $\phi_{g,h}$. To this end we note that it is sufficient to consider the restriction of the cocycle α to the normaliser $N(g,h)$ of $\langle g,h\rangle$. If this normaliser is contained in some Sylow p-subgroup, one can simply use the cocycle α_p determined in the previous subsection. Otherwise, one can use `GAP` to compute a basis for the cohomology of $N(g,h)$ and check which cohomology classes reproduce the correct multipliers (and in particular (57)) for all genera of the form $\phi_{e,k}$, $k \in N(g,h)$. In all cases we considered, there was only one cohomology class with this property, which must therefore be the restriction of the class $[\alpha] \in H^3(M_{24}, U(1))$ to the cohomology of $N(g,h)$. Once the restriction of $[\alpha]$ to $N(g,h)$ is determined, one can check the existence of obstructions for all 34 groups of table 1. The results are collected in table 3 and the `GAP`-files containing these computations are available online [45].

For the six unobstructed twisted twining genera $\phi_{g,h}$, one can compute the precise multiplier system. It turns out that this information is sufficient to determine $\phi_{g,h}$ up to normalisation. The normalisation can be fixed (up to a phase) by requiring that a decomposition (6) exists. In the remainder of this subsection, we illustrate how one of these non-zero twisted twining genera, namely the one associated to $g \in 2\mathrm{B}$ and $h \in 8\mathrm{A}$, can be determined explicitly; the other five cases are completely analogous and are dealt with in appendix A. Let us work with the conventions that the three permutations

$$(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23)(24)\ ,$$
$$(3,17,10,7,9)(4,13,14,19,5)(8,18,11,12,23)(15,20,22,21,16)(1)(2)(6)(24)\ ,$$
$$(1,24)(2,23)(3,12)(4,16)(5,18)(6,10)(7,20)(8,14)(9,21)(11,17)(13,22)(15,19)$$

generate $M_{24} \subset S_{24}$. As representative for $g \in 2\mathrm{B}$ we can then take

$$g = (1,10)(2,14)(3,8)(4,5)(6,22)(7,20)(9,18)(11,23)(12,24)(13,19)(15,16)(17,21)\ , \quad (68)$$

while for $h \in 8\mathrm{A}$ we consider

$$h = (2,14)(3,9,8,18)(4,6,21,19,15,24,20,11)(5,22,17,13,16,12,7,23)(1)(10)\ . \quad (69)$$

The elements g, h generate the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_8$ corresponding to case 27 of table 1. There are two interesting conjugation relations, namely

$$(k^{-1}gh^4k\,,\,k^{-1}h^{-1}k) = (g,h)\ , \qquad (r^{-1}gr\,,\,r^{-1}ghr) = (g,h)\ , \tag{70}$$

where

$$\begin{aligned} k &= (1,10)(2,14)(3,18)(4,11)(5,13)(6,20)(7,12)(8,9)(15,19)(16,23)(17,22)(21,24)\ , \\ r &= (1,2)(3,9)(6,22)(8,18)(10,14)(11,23)(12,24)(13,19)(4)(5)(7)(15)(16)(17)(20)(21)\ . \end{aligned} \tag{71}$$

Thus, we can use the general formulae (41) and their inverses to deduce that[‡]

$$\phi_{g,h}(\tau+1) = c_g(g,h)\phi_{g,gh}(\tau) = c_g(g,h)\frac{c_g(gh,r)}{c_g(r,h)}\phi_{g,h}(\tau) = i\phi_{g,h}(\tau)\ , \tag{72}$$

where we have used the explicit form of the multiplier factors from section 3.3. We also have (recall that $g = g^{-1}$)

$$\begin{aligned} \phi_{g,h}(\tfrac{\tau}{-4\tau+1}) =& \frac{\phi_{h,g}(4-\frac{1}{\tau})}{c_h(g,g)} = \frac{\prod_{i=1}^3 c_h(h,gh^i)}{c_h(g,g)}\phi_{h,gh^4}(-\tfrac{1}{\tau}) = \frac{\prod_{i=1}^3 c_h(h,gh^i)}{c_h(g,g)c_{gh^4}(h,h^{-1})}\phi_{gh^4,h^{-1}}(\tau) \\ =& \frac{\prod_{i=1}^3 c_h(h,gh^i)}{c_h(g,g)c_{gh^4}(h,h^{-1})}\frac{c_{g^4h}(h^{-1},k)}{c_{g^4h}(k,h)}\phi_{g,h}(\tau) = -\phi_{g,h}(\tau)\ . \end{aligned} \tag{73}$$

This implies that $\phi_{g,h}(\tau,z)$ is a Jacobi form under the group $\Gamma_0(4)$ with the above multipliers. Let us consider now the dependence of $\phi_{g,h}(\tau,z)$ on z, for a fixed $\tau \in \mathbb{H}_+$. The characters $\mathrm{ch}_{h,\ell}(\tau,z)$ of the $\mathcal{N}=4$ algebra are even functions of z and $\mathrm{ch}_{h,\ell}(\tau,z=0)$ is independent of τ (in particular, $\mathrm{ch}_{h,\ell}(\tau,0)=0$ for $h>1/4$) [31]. By (6), all twisted-twining genera must satisfy the same properties. Since the multiplier system for $\phi_{g,h}$ is non-trivial, the constant $\phi_{g,h}(\tau,0)$ must vanish and by $\phi_{g,h}(\tau,z) = \phi_{g,h}(\tau,-z)$ it follows that $\phi_{g,h}$ has a double zero at $z=0$ (mod $\mathbb{Z}+\tau\mathbb{Z}$). Every (weak) Jacobi form of index 1 has exactly two zeros (counting multiplicity) in each fundamental domain of $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ [30], so that these are the only zeroes of $\phi_{g,h}(\tau,z)$. On the other hand, the function

$$\psi(\tau,z) \equiv \frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2} \tag{74}$$

is also a Jacobi form weight 0 and index 1 under $\Gamma_0(4)$ (with trivial multiplier) and has only a double zero at $z=0$ (mod $\mathbb{Z}+\tau\mathbb{Z}$). It therefore follows that

$$F(\tau) \equiv \frac{\phi_{g,h}(\tau,z)}{\psi(\tau,z)} \tag{75}$$

[‡]For brevity of presentation we omit the dependence of z, as well as the factors $e^{2\pi i\frac{cz^2}{c\tau+d}}$, in the following formulae.

is independent of z and is a modular function of weight 0 under $\Gamma_0(4)$ (with multiplier system (72)–(73)) that is holomorphic in the interior of $\mathbb{H}_+$. Let us consider the asymptotic behaviour of $F(\tau)$ at the boundary $\Gamma_0(4)\backslash(\mathbb{Q}\cup\{\infty\})$ of the compactification $\Gamma_0(4)\backslash\bar{\mathbb{H}}_+$. This boundary consists of three points (cusps) of the form $\gamma_c\cdot\infty$, where $\gamma_\infty=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\gamma_0=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $\gamma_{1/2}=\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$. The twisted twining genus $\phi_{g,h}$ has the Fourier expansion at these cusps

$$\phi_{g,h}(\gamma_\infty(\tau,z))=\phi_{g,h}(\tau,z)=A_\infty\,q^{1/4}(2-y-y^{-1})+O(q^{5/4}) \tag{76}$$

$$\phi_{g,h}(\gamma_0(\tau,z))\sim\phi_{h,g}(\tau,z)=A_0\,q^{1/8}(2-y-y^{-1})+O(q^{3/8}) \tag{77}$$

$$\phi_{g,h}(\gamma_{1/2}(\tau,z))\sim\phi_{gh^2,h}(\tau,z)=A_{1/2}\,q^{1/4}(2-y-y^{-1})+O(q^{3/4})\ , \tag{78}$$

where $\sim$ denotes equality up to a phase, and we have included the lowest non-negative powers of q^r that are compatible with the multiplier system. It follows from (6) that the constants A_∞, A_0, $A_{1/2}$ equal

$$A_\infty=\operatorname{Tr}_{\mathcal{H}_{g,r=\frac14}}(\rho_{g,\frac14}(h))\ ,\quad A_0=\operatorname{Tr}_{\mathcal{H}_{h,r=\frac18}}(\rho_{h,\frac18}(g))\ ,\quad A_{1/2}=\operatorname{Tr}_{\mathcal{H}_{gh^2,r=\frac14}}(\rho_{gh^2,\frac14}(h))\ , \tag{79}$$

and from the Fourier expansion of $\phi_{h,e}$ we know that $\dim\mathcal{H}_{h,r=\frac18}=1$, so that $|A_0|=1$. On the other hand, at the cusps $\psi(\tau,z)$ satisfies

$$\psi(\gamma_\infty(\tau,z))=\psi(\gamma_{1/2}(\tau,z))=\frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2}=\frac14(2-y-y^{-1})+O(q) \tag{80}$$

$$\psi(\gamma_0(\tau,z))=\frac{\vartheta_1(\tau,z)^2}{\vartheta_4(\tau,0)^2}=q^{1/4}(2-y-y^{-1})+O(q^{3/4})\ , \tag{81}$$

and we therefore obtain

$$F(\gamma_\infty\cdot\tau)=4\,A_\infty\,q^{1/4}+O(q^{5/4}) \tag{82}$$

$$F(\gamma_0\cdot\tau)\sim A_0\,q^{-1/8}+O(q^{1/8}) \tag{83}$$

$$F(\gamma_{1/2}\cdot\tau)\sim 4\,A_{1/2}\,q^{1/4}+O(q^{3/4})\ . \tag{84}$$

Up to a phase, there is a unique modular form with the correct modular properties and the expected Fourier expansion at the cusps, namely

$$F(\tau)=8\frac{\eta(2\tau)^6}{\eta(\tau)^6}\ . \tag{85}$$

Thus we conclude that, up to an overall phase, we have

$$\phi_{2\mathrm{B},8\mathrm{A}_{1,2}}(\tau,z)=8\frac{\eta(2\tau)^6}{\eta(\tau)^6}\frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2}=2\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2\ , \tag{86}$$

where we used the identity $\vartheta_2(\tau,0)^2 = 4\frac{\eta(2\tau)^4}{\eta(\tau)^2}$. The other non-trivial cases can be worked out similarly; see appendix A for the details.

3.5. **Projective representations.** As we have argued above, see property (C) in section 2, the twisted sector $\mathcal{H}_g$ should carry a projective representation of the centraliser $C_{M_{24}}(g)$, whose 2-cocycle is given by c_g as determined in (39). Given our explicit knowledge of all twisted twining genera as well as c_g, this can now be tested as in [8] (where the corresponding analysis was performed for the case of the twining genera): if each $\mathcal{H}_{g,r}$ is a projective representation of $C_{M_{24}}(g)$, we can decompose it as

$$\mathcal{H}_{g,r} = \bigoplus_j h_{g,r}^{(j)} R_j \ , \tag{87}$$

where $h_{g,r}^{(j)}$ is the multiplicity with which the irreducible projective representation R_j (whose projectivity is characterised by c_g) appears in $\mathcal{H}_{g,r}$. On the other hand, using the orthogonality of group characters (see appendix C), we can calculate $h_{g,r}^{(j)}$ from the knowledge of the twisted twining genera explicitly. The consistency condition is then that all $h_{g,r}^{(j)}$ are indeed (non-negative) integers. We have done this analysis for all twisted sectors and for the first 500 levels (i.e. the first 500 values for r), and the multiplicities are indeed always non-negative integers. The explicit results for the first 20 levels as well as the (projective) character tables of all non-isomorphic centralisers $C_{M_{24}}(g)$ are given in appendix D. The decompositions for all g and for the first 500 levels are available online [45].

4. K3 Orbifolds

There are at least two further consistency checks on our proposal that can be fairly easily analysed. Suppose we consider a K3 sigma-model $\mathcal{C}$ whose automorphism group contains the cyclic subgroup generated by g. Then we can consider the orbifold of $\mathcal{C}$ by $G = \langle g \rangle$. A priori, it is not guaranteed that this orbifold is consistent — indeed, since these symmetries effectively act asymmetrically, they generically suffer from the level-matching problem, and hence may be inconsistent [39]. However, as was already mentioned in [40], the level-matching condition is satisfied provided that the twining genus ϕ_g has a trivial multiplier system. This is in particular the case if $g \in M_{23}$. In that case, the resulting orbifold theory $\widehat{\mathcal{C}} = \mathcal{C}/G$ is again a K3 sigma model, as was also shown in [40].

Suppose then that the original $\mathcal{C}$ has, in addition to $g \in M_{23}$, another commuting group element h in its automorphism group. Then h also gives rise to a symmetry $\hat{h}$ of the orbifold

theory $\widehat{\mathcal{C}}$, and we can calculate the twining genus of $\hat{h}$ by the usual orbifold formula

$$\phi_{e,\hat{h}} = \frac{1}{o(g)} \sum_{i,j=0}^{o(g)-1} \phi_{g^i,hg^j} \ . \tag{88}$$

This leads to a non-trivial consistency condition: given our explicit knowledge of all twisted twining genera, we can calculate the right-hand-side explicitly, and this must agree, for every h, with *one* of the twining genera of [8, 9]. We have checked that this is indeed true; an example is illustrated in section 4.2.

Naively, one may have guessed that $\hat{h}$ should simply agree with h, i.e. that the left-hand-side of (88) is the twining genus associated to the *same* group element $h \in M_{24}$. However, this is in general not the case, as will be explained in the following section 4.1, see also section 4.2 for an explicit example. Whenever this happens we will say that the group element h has been *relabelled* in the orbifold theory.

The other consistency check is even more obvious. As was mentioned at the beginning of section 2, there exist K3 sigma-models with a commuting pair of automorphisms (g, h). For them (2) can be calculated directly, and thus compared to our answers. We shall perform this computation for some simple cases in section 4.3. In particular, we shall concentrate on examples where our obstruction analysis predicts that the associated twisted twining genus must vanish. The fact that we can reproduce this result using elementary methods is a good consistency check on our analysis.

4.1. **The relabelling phenomenon.** Suppose that a K3 sigma-model $\mathcal{C}$ contains in its automorphism group an element g such that the orbifold of $\mathcal{C}$ by g, $\hat{\mathcal{C}} = \mathcal{C}/\langle g \rangle$ is consistent. In general, each g^r-twisted sector carries a projective representation of $H \equiv C_{M_{24}}(g)$. Equivalently, we can think of the untwisted and twisted sectors as carrying genuine representations of some central extension $\tilde{H}$ of H. In fact, the central extension can be chosen to be of the form (see also appendix C.1 for more details)

$$1 \to \langle \tilde{Q} \rangle \cong \mathbb{Z}_N \to \tilde{H} \xrightarrow{\rho} H \to 1 \ . \tag{89}$$

Here ρ is the representation in the untwisted sector, i.e. on the spectrum of $\mathcal{C}$, N is the order $o(g)$ of g and the generator $\tilde{Q}$ (that we call the *quantum symmetry*) of the central $\mathbb{Z}_N$ acts by $e^{2\pi i r/N}$ on the g^r-twisted sector. Note that there is always a lift $\tilde{g} \in \tilde{H}$ of g that acts by $e^{2\pi i k/N}$ on the states of conformal weight $h - 1/4 = k/N$.

In the orbifold theory $\hat{\mathcal{C}}$, all states are invariant under the action of $\tilde{g}$ — this is just the precise way of imposing the usual requirement that the orbifold theory consists of 'g-invariant' states

only. Since $\tilde{g}$ is in the center of $\tilde{H}$, the spectrum of $\hat{\mathcal{C}}$ carries a well-defined representation $\hat{\rho}$ of $\tilde{H}$, such that $\hat{\rho}(\tilde{g})$ is the identity. In fact, it is easy to see that $\ker\hat{\rho} = \langle\tilde{g}\rangle$, because $\tilde{Q}$ acts non-trivially on the twisted sectors, and each non-trivial element in $H/\langle g\rangle$ lifts to an element of $\tilde{H}$ that acts non-trivially on the g-invariant untwisted sector, which in turn is part of the spectrum of $\hat{\mathcal{C}}$. Thus, we have an exact sequence

$$1 \to \langle\tilde{g}\rangle \cong \mathbb{Z}_N \to \tilde{H} \xrightarrow{\hat{\rho}} \hat{H} \to 1 \ . \tag{90}$$

The group $\hat{H} \equiv \tilde{H}/\langle\tilde{g}\rangle$ defines a full-fledged symmetry of the orbifold CFT $\hat{\mathcal{C}}$. For the case at hand, both the original theory and the orbifold are K3 sigma models, and thus all of these symmetry groups should be subgroups of M_{24},[§]

$$\hat{H} \subseteq M_{24} \ . \tag{91}$$

This assumption allows us to determine $\hat{H}$ (at least as an abstract group). In fact, the order of $\hat{H}$ is given by

$$|\hat{H}| = \frac{|\tilde{H}|}{N} = |H| = |C_{M_{24}}(g)| \ . \tag{92}$$

The quantum symmetry $Q \equiv \hat{\rho}(\tilde{Q})$ must be in the same M_{24}-conjugacy class as g, since it has the same eigenvalues on the 24-dimensional representation of RR ground states. Since all elements of $\hat{H}$ commute with Q, we can conclude by (92) that

$$\hat{H} = C_{M_{24}}(Q) \cong C_{M_{24}}(g) = H \ . \tag{93}$$

This construction allows us to give a precise interpretation of (88). For each symmetry $h \in H$ in the model $\mathcal{C}$, one chooses a lift $\tilde{h} \in \tilde{H}$, with $\rho(\tilde{h}) = h$ (this amounts to choosing the phases of the twisted twining genera $\phi_{g,h}$). Then the formula for the twining genus of the symmetry $\hat{\rho}(\tilde{h})$ in the orbifold model $\hat{\mathcal{C}}$ takes the form

$$\phi_{e,\hat{\rho}(\tilde{h})} = \frac{1}{o(g)} \sum_{i,j=0}^{o(g)-1} \phi_{g^i,\tilde{h}\tilde{g}^j} \ . \tag{94}$$

After these preparations we can now explain the relabelling phenomenon. While it follows from (93) that $\hat{H}$ and H are isomorphic, they are not canonically isomorphic; in fact, we only have a canonical isomorphism between

$$H/\langle g\rangle \cong \tilde{H}/\langle\tilde{g},\tilde{Q}\rangle \cong \hat{H}/\langle Q\rangle \ , \tag{95}$$

[§]We are assuming here that the relevant K3 sigma model is 'non-exceptional' in the terminology of [40].

but this does not necessarily lift to a canonical isomorphism $\Phi : \hat{H} \to H$, with $\Phi(Q) = g$. Put differently, in general it is not possible to find an isomorphism $\Phi : \hat{H} \to H$, with $\Phi(Q) = g$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \tilde{H} \arrow[dl, "\hat{\rho}"'] \arrow[dr, "\rho"] & \\
\hat{H} \arrow[rr, dashed, "\Phi"] \arrow[dr] & & H \arrow[dl] \\
 & \tilde{H}/\langle \tilde{g}, \tilde{Q} \rangle &
\end{tikzcd}$$

For example, this will be the case if for any lift $\tilde{h}$ with $\rho(\tilde{h}) = h$, the corresponding symmetry $\hat{\rho}(\tilde{h})$ is not in the same M_{24} conjugacy class as h. In that case, the twining genus obtained by formula (94) does not agree with $\phi_{e,h}$. Whenever this is the case, we shall say that the conjugacy class of h has been *relabelled* in the orbifold theory.

4.2. **An example.** As an explicit example of the relabelling phenomenon let us consider the case where g is in the class 2A of M_{24} (so that $N = o(g) = 2$). Then the group $\tilde{H}$ is a central extension of H by the $\mathbb{Z}_2$ group generated by $\tilde{Q}$. We denote by $\tilde{g}$ the lift of g in $\tilde{H}$, such that $\tilde{g}$ and $\tilde{Q}$ have the following eigenvalues

$\tilde{Q}$	$\tilde{g}$	sector	model
$+1$	$+1$	untwisted	$\mathcal{C}, \hat{\mathcal{C}}$
$+1$	-1	untwisted	$\mathcal{C}$
-1	$+1$	twisted, $q^{\mathbb{Z}}$	$\hat{\mathcal{C}}$
-1	-1	twisted, $q^{\mathbb{Z}+\frac{1}{2}}$	$-$

(96)

As expected from the discussion in the previous section, the group $\tilde{H}/\tilde{Q}$ is isomorphic to the centraliser in M_{24} of an element of class 2A.

Now consider an element $h \in H$ in class 8A, with $h^4 = g$. The trace of h over the 24-dimensional representation of ground states in the original model $\mathcal{C}$ is $+2$. Furthermore, since hg is in the same class, also the trace over hg equals $+2$, from which we conclude that on the ground states of $\mathcal{C}$ we have

$$\mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=1,\tilde{g}=1}}(h) = +2 \ , \qquad \mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=1,\tilde{g}=-1}}(h) = 0 \ , \tag{97}$$

where $\mathcal{H}^0_{\tilde{Q}=\pm1,\tilde{g}=\pm1}$ are the eigenspaces of $\tilde{Q}$ and $\tilde{g}$ at conformal weight $h_L - \frac{1}{4} = 0$. The lifts $\tilde{h}$ and $\tilde{Q}\tilde{h}$ in $\tilde{H}$ are such that $\tilde{h}^4 = (\tilde{Q}\tilde{h})^4 = \tilde{g}$. On dimensional grounds, the twisted sector at $h_L - 1/4 = 0$ must correspond to a certain 8-dimensional irreducible representation of $\tilde{H}$, where the lifts $\tilde{h}$ and $\tilde{Q}\tilde{h}$ have trace ± 2. We notice that, up to a sign, this trace is independent of the choice of the central extension, and is in agreement with the expectation from the twisted twining genus $\phi_{g,h} \equiv \phi_{h^4,h}$. On the ground states in the orbifold theory $\hat{\mathcal{C}}$, the trace of $\tilde{h}$ is now

$$\mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=1,\tilde{g}=1}}(\tilde{h}) + \mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=-1,\tilde{g}=1}}(\tilde{h}) = 2 \pm 2 \ , \tag{98}$$

and thus there is no choice for $\tilde{h}$ such that the trace is equal to $+2$, as required for an element in class 8A. In fact, the images $\hat{h}$ and $Q\hat{h}$ in $\hat{H} \cong C_{M_{24}}(Q)$ have order 4 and belong to the classes 4A and 4B. In particular, while the pair of commuting elements g, h in the original model generate the cyclic group

$$\langle g, h\rangle = \langle h\rangle \cong \mathbb{Z}_8 \ , \tag{99}$$

both the groups $\langle Q, \hat{h}\rangle$ and $\langle Q, Q\hat{h}\rangle$ correspond to group 12 in our general list. Thus, there cannot exist an isomorphism $\Phi : \hat{H} \to H$ such that $\Phi(\hat{\rho}(\tilde{h})) = h$ or $\Phi(\hat{\rho}(\tilde{Q}\tilde{h})) = h$.

Let us verify the consistency of eq. (94) for this case, see the comments at the beginning of section 4. The twining genera $\phi_{e,\hat{\rho}(\tilde{h})}$ and $\phi_{e,\hat{\rho}(\tilde{Q}\tilde{h})}$ are given by

$$\phi_{e,\hat{\rho}(\tilde{h})}(\tau, z) = \frac{1}{2}(\phi_{e,h}(\tau, z) + \phi_{e,h^5}(\tau, z) + \phi_{h^4,h}(\tau, z) + \phi_{h^4,h^5}(\tau, z)) \ , \tag{100}$$

$$\phi_{e,\hat{\rho}(\tilde{Q}\tilde{h})}(\tau, z) = \frac{1}{2}(\phi_{e,h}(\tau, z) + \phi_{e,h^5}(\tau, z) - \phi_{h^4,h}(\tau, z) - \phi_{h^4,h^5}(\tau, z)) \ . \tag{101}$$

Using the identity

$$\phi_{e,8\mathrm{A}}(\tau, z) = \frac{1}{2}(\phi_{e,4\mathrm{B}}(\tau, z) + \phi_{e,4\mathrm{A}}(\tau, z)) \ , \tag{102}$$

it is easy to verify that

$$\phi_{e,\hat{\rho}(\tilde{h})} = \phi_{e,4\mathrm{B}}, \qquad \phi_{e,\hat{\rho}(\tilde{Q}\tilde{h})} = \phi_{e,4\mathrm{A}} \ , \tag{103}$$

as expected. This example also shows that the consistency of eq. (94) gives, in general, highly non-trivial conditions on the twisted twining genera.

We have verified explicitly that a similar phenomenon also occurs for the orbifold by an element g in class 4B.

4.3. **Computation of a twisted twining genus.** In this section we shall calculate some twisted twining genera directly. In particular, we shall concentrate on some simple cases where our general analysis predicts that they must vanish (because of some obstruction). As we shall see, we can reach the same conclusion from first principles.

Let us consider a K3 σ-model $\mathcal{C}$ whose automorphism group contains two commuting elements g, h of order m and n, respectively, generating the non-cyclic group $G = \langle g, h\rangle$. If these symmetries have a geometric origin, i.e. if they are induced by automorphisms of the K3 target space that fix the holomorphic 2-form, then the group G must be a subgroup of M_{23} with at least five orbits over the 24-dimensional representation, as follows from the Mukai theorem [46]. These conditions are satisfied by the groups 1 and 3 ($\mathbb{Z}_2 \times \mathbb{Z}_2$), groups 17 and 19 ($\mathbb{Z}_2 \times \mathbb{Z}_4$), group 25 ($\mathbb{Z}_4 \times \mathbb{Z}_4$), group 28 ($\mathbb{Z}_2 \times \mathbb{Z}_6$) and group 33 ($\mathbb{Z}_3 \times \mathbb{Z}_3$) of our table 1. In all these cases, the twisted twining genus vanishes, $\phi_{g,h} = 0$, see table 3. In the following we want to verify this independently.

In the general $\mathbb{Z}_n \times \mathbb{Z}_m$ case, the twisted twining genus $\phi_{g,h}$ is expected to be a weak Jacobi form of weight 0 and index 1 under the congruence subgroup

$$\Gamma_{m,n} \equiv \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid a \equiv 1, b \equiv 0 \mod m,\ c \equiv 0, d \equiv 1 \mod n\} \ . \tag{104}$$

Note that this group is in general smaller than the group $\Gamma_{g,h}$ defined below (124), because we are not using the identifications under conjugation (5). Since both g and h are elements of M_{23}, the multiplier system for $\phi_{g,h}$ must be trivial (since $H^3(M_{23}, U(1)) \cong 0$).

Because of the triviality of the multiplier system, it follows that any modular image $\phi_{g',h'}$ of $\phi_{g,h}$, where $(g', h') = (g, h) \cdot \gamma$ for some $\gamma \in SL(2,\mathbb{Z})$, has a Fourier expansion of the form

$$\phi_{g',h'}(\tau, z) = \sum_{n=0}^{\infty} \sum_{\ell} c_{g',h'}(n, \ell)\, q^{\frac{n}{o(g')}}\, y^{\ell} \ , \tag{105}$$

where $o(g')$ is the order of g'. Let us assume

$$c_{g',h'}(0, \ell) = 0 \quad \forall \ell \in \mathbb{Z} \ , \tag{106}$$

for any modular image $(g', h') = (g, h) \cdot \gamma$; we will prove this below. Then it follows that

$$\phi_{g,h}(\tau, 0) = \sum_{\ell} c_{g,h}(0, \ell) = 0 \ . \tag{107}$$

Because of (13), $\phi_{g,h}(\tau, z) = \phi_{g,h}(\tau, -z)$, and we can conclude that $\phi_{g,h}$ has a double zero at $z = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}$. Since Jacobi forms of index 1 have only two zeroes on each fundamental domain of $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ [30], $\phi_{g,h}$ is nowhere else vanishing. Next we recall that $\vartheta_1(\tau, z)^2/\eta(\tau)^2$ is a Jacobi form of weight 0 and index 1 under $SL(2,\mathbb{Z})$ (with a non-trivial

multiplier). It has a double zero at $z = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}$, and is non-vanishing elsewhere, so that

$$F_{g,h}(\tau) = \frac{\eta(\tau)^2 \phi_{g,h}(\tau, z)}{\vartheta_1(\tau, z)^2} \tag{108}$$

is a modular function of weight 0 under $\Gamma_{m,n}$ (with non-trivial multiplier), which is holomorphic in the interior of the upper-half plane $\mathbb{H}_+$. Let us consider the asymptotics of $F_{g,h}$ at the boundary $\Gamma_{m,n}\backslash(\mathbb{Q} \cup \infty)$ of the compactification $\Gamma_{m,n}\backslash\bar{\mathbb{H}}_+$. The boundary is the union of points (cusps) of the form $\gamma(\infty)$, where γ runs through the representatives of the double cosets $\Gamma_{m,n}\backslash SL(2,\mathbb{Z})/\Gamma^\infty$. Here, Γ^∞ is the parabolic subgroup of $SL(2,\mathbb{Z})$ that fixes $\infty \in \bar{\mathbb{H}}_+$ and is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The Jacobi form $\vartheta_1(\tau, z)^2/\eta(\tau)^2$ has the same asymptotic behaviour at each cusp, namely

$$\frac{\vartheta_1(\gamma \cdot (\tau, z))^2}{\eta(\gamma \cdot \tau)^2} \overset{\tau \to \infty}{\sim} q^{1/6}(2 - y - y^{-1}) + O(q^{7/6}) \ , \tag{109}$$

up to a constant coefficient. By (105) and (106), it then follows that

$$F_{g,h}(\gamma \cdot \tau) \sim \frac{\eta(\tau)^2 \phi_{g',h'}(\tau, z)}{\vartheta_1(\tau, z)^2} \overset{\tau \to \infty}{\sim} O(q^{\frac{1}{o(g')} - \frac{1}{6}}) \ , \tag{110}$$

where $(g', h') = (g, h)\gamma$ and $o(g')$ is the order of g'. For all the groups we are interested in, we have $o(g') \leq 6$, and the inequality is strict for at least one g'. It follows that $F_{g,h}$ has no poles and at least one zero at the boundary of $\Gamma_{m,n}\backslash\bar{\mathbb{H}}_+$. Since it is holomorphic in the interior, it must therefore vanish identically, and we conclude that

$$\phi_{g,h}(\tau, z) = 0 \ , \tag{111}$$

as predicted by our results.

It therefore only remains to prove (106). The coefficient $c_{g,h}(0, \ell)$ corresponds to the trace of h (or, more generally, of the lift $\tilde{h}$ to a central extension of the group $\langle g, h\rangle$) on the RR g-twisted states with conformal weight $h_L = h_R = 1/4$ and weight ℓ under the $\mathfrak{su}(2)_L$ subalgebra of the left-moving $\mathcal{N} = 4$ algebra. The g-twisted ground states are g-invariant (the conformal weight satisfies $h_L - 1/4 \in \mathbb{Z}$), so they belong to the spectrum of the orbifold theory $\mathcal{C}/\langle g\rangle$.

Any sigma-model whose target space is a K3 manifold X contains 24 RR ground states corresponding to the harmonic forms on X. Thus, the real vector space spanned by these states can be identified with the real cohomology $H^*(X, \mathbb{R})$ on X. The orbifold model $\hat{\mathcal{C}} = \mathcal{C}/\langle g\rangle$ corresponds to a non-linear sigma-model on the geometric orbifold $\hat{X} = X/\langle g\rangle$, which is a singular K3 surface. In particular, for each point of X that is fixed by some non-trivial subgroup $\langle g^r\rangle \subseteq \langle g\rangle$, the orbifold $\hat{X}$ has a singularity of type A_{n-1}, where n

is the order of g^r. By blowing up all these singularities, one obtains again a smooth K3 manifold.

The RR ground states in the untwisted sector of $\hat{\mathcal{C}}$ are obtained by projecting onto the g-invariant subspace of $H^*(X,\mathbb{R})$, while the ground states in the twisted sectors are the Poincaré duals of the cycles corresponding to the exceptional divisors in the blow-up of $\hat{X}$. More precisely, the resolution of each A_{n-1} singularity of $\hat{\mathcal{C}}$ gives rise to $n-1$ rational curves $\mathbb{P}^1$, representing elements in the homology of the blow-up. The cohomology classes dual to these cycles span an $(n-1)$-dimensional subspace of RR ground states for $\hat{\mathcal{C}}$, one in each g^{ir} twisted sector, for $i = 1, \ldots, n-1$. Thus, we arrive at the usual statement that the ground states of the g^k-twisted sector are in one to one correspondence with the g^k-fixed points of the target space.

Any other symplectic automorphism h of X that commutes with g acts as a permutation on the g^r-fixed points. This action induces a permutation on the singularities of $\hat{X}$, and therefore on the twisted RR ground states dual to the exceptional cycles.[¶] Therefore, the trace of h over the g-twisted sector only gets a non-zero contribution from the singularities of $\hat{X}$ that are fixed by h, i.e. only from the points of X that are fixed by both g and h. It is known (see, for example Proposition 1.5 of [46]) that a finite group of symplectic automorphisms of a K3 manifold X that fixes at least one point must be a subgroup of $SL(2,\mathbb{C})$. On the other hand, all finite abelian subgroups of $SL(2,\mathbb{C})$ are cyclic. Thus we conclude that if $\langle g, h\rangle$ is abelian but not cyclic, the trace of h over the g-twisted ground fields is 0 and (106) follows.

Our results also agree with the analysis of [47], where it was shown that the twisted twining genus $\phi_{g,h}$ for a pair of commuting symplectic automorphisms g, h of a K3 manifold X is given by a sum of contributions corresponding to the points fixed by both g and h. If the abelian group $\langle g, h\rangle$ is not cyclic, then there are no fixed points and $\phi_{g,h} = 0$.

5. Conclusions

5.1. **Summary.** Inspired by the generalised Monstrous Moonshine idea of Norton [14] we have in this paper established 'generalised Mathieu Moonshine' for the elliptic genus of K3. More specifically, we have found, for each commuting pair $(g, h) \in M_{24}$, explicit expressions for the twisted twining genera $\phi_{g,h} : \mathbb{H}_+ \times \mathbb{C} \to \mathbb{C}$, and we have verified that these functions

[¶]More precisely, one should choose a lift $\tilde{h}$ of h to a central extension of the group of symmetries of $\mathcal{C}$; then, $\tilde{h}$ acts on the g-twisted sector by a permutation times an overall phase that depends on the particular lift. Since the phase is not important for our argument, we will neglect this subtlety.

satisfy conditions (A), (B), (D) of section 2.1, and given convincing evidence that condition (C) is also met. In particular, the twisted twining genera $\phi_{g,h}(\tau, z)$ are weak Jacobi forms of weight 0 and index 1 for certain subgroups $\Gamma_{g,h} \subset SL(2,\mathbb{Z})$ with a multiplier system $\chi_{g,h} : \Gamma_{g,h} \to U(1)$.

One of the key insights of our work is that these multiplier phases are all determined in terms of a class $[\alpha] \in H^3(M_{24}, U(1))$ via a formula (see eq. (39)) that was first derived by Dijkgraaf and Witten in the context of holomorphic orbifolds [25]. We have also shown that the twisted twining genera $\phi_{g,h}$ are compatible with a decomposition of the g-twisted sector into (projective) representations of the centraliser $C_{M_{24}}(g)$, up to the first 500 levels. The particular central extension of $C_{M_{24}}(g)$ associated with this projective representation is also determined by the cohomology class $[\alpha] \in H^3(M_{24}, U(1))$.

As it turned out, many of the twisted twining genera vanish, and in most cases this follows from the structure of the class $[\alpha] \in H^3(M_{24}, U(1))$. While these sorts of cohomological obstructions should also arise in the context of other holomorphic orbifolds (in particular in Monstrous Moonshine), our results provide, to our knowledege, the first example where they have been explicitly verified. We have also confirmed the vanishing of some of these twisted twining genera independently, using geometrical arguments.

5.2. **Open problems and future work.** The elliptic genus of K3 is closely related to the physics of 1/4 BPS-dyons in $\mathcal{N} = 4$ string theory. The generating function of elliptic genera of symmetric products of K3's is the famous Igusa cusp form Φ_{10}, whose inverse is the partition function of $\mathcal{N} = 4$ dyons [48, 49]. This fact lies at the heart of the recent progress in understanding wall-crossing of multi-centered BPS-states in $\mathcal{N} = 4$ string theory using mock modular forms [50]. It is therefore natural to wonder about the corresponding physical interpretation of the twisted twining genera $\phi_{g,h}$. In the special case of a trivial twist $g = e$, it has been shown that for some elements h the twining genera $\phi_{e,h}$ correspond to BPS-indices in CHL-orbifolds [6, 51, 52, 53]. We may therefore expect that the twisted twining genera $\phi_{g,h}$ should be related to the counting of 'twisted dyons' in CHL-models. Indeed for some pairs (g, h) we have verified this by showing that the twisted CHL-indices of Sen [47] are compatible with our $\phi_{g,h}$. It would be interesting to determine more generally whether $\phi_{g,h}$ have a physical interpretation in terms of CHL-models or some generalisation thereof.

In Borcherds' proof of the Monstrous Moonshine conjecture he introduced a new class of infinite-dimensional (super-)Lie algebras that he called generalised Kac-Moody algebras

(GKM) [3]. The key idea was that the multiplicative lift of the modular J-function could be interpreted as the denominator formula for a specific GKM, now commonly called the Monster Lie algebra, whose root system carries an action of the Monster group $\mathbb{M}$. The generalised moonshine ideas of Norton [14] suggest that similar GKMs should exist for each class $[g] \in \mathbb{M}$ [20, 21, 54]. In the context of Mathieu Moonshine the role of generalised Kac-Moody algebras has so far not been understood. Although there is a GKM associated with the elliptic genus of K3 [55], this algebra does not seem to carry any natural action of M_{24} due to the fact that its denominator formula is $1/\sqrt{\Phi_{10}}$ rather than $1/\Phi_{10}$. For some conjugacy classes $[g] \in M_{24}$ the associated second-quantised twining genera Φ_g also give rise to denominator formulas of GKMs [56], which should presumably be identified with the wall-crossing algebras found in CHL-models [57, 58, 59, 60, 51]. Although these observations are very suggestive, the precise role of the GKMs for Mathieu Moonshine remains to be understood.

A crucial point in our analysis was that the multiplier phases are controlled by a class $[\alpha] \in H^3(M_{24}, U(1))$, from which also the various obstructions could be read off. It follows from the work of [25, 37] that similarly a cohomology class in $H^3(\mathbb{M}, U(1))$ should underly generalised Monstrous Moonshine, and it would be very interesting to see whether this point of view may lead to new insights. In particular, some of Norton's generalised Moonshine functions $f(g, h; \tau)$ are known to vanish, and it would be natural to expect that this is due to some cohomological obstructions similar to those we have found in this paper.[‖] Some related ideas were already put forward some time ago by Mason [27], but very little has been done explicitly, mainly because the group $H^3(\mathbb{M}, U(1))$ seems to be poorly understood.

In a similar vein, it would also be interesting to investigate the generalised versions of the recently discovered Umbral Moonshine observations [61], of which the M_{24}/K3-moonshine is a special case. In particular, one might expect that there is a relation between the Rademacher summability condition of [12] and the cohomological obstructions we have described here.

The fact that there is a class in $H^3(M_{24}, U(1))$ determining the properties of the twisted twining genera $\phi_{g,h}$ represents strong evidence for the idea that a holomorphic VOA $\mathcal{H}$ underlies Mathieu Moonshine. Indeed, the original motivation for suspecting the relevance of the cohomology group $H^3(M_{24}, U(1))$ was based on the formal analogy with holomorphic orbifolds [25, 37]. In a sense we are therefore in a similar situation now as for Monstrous

[‖]We thank Terry Gannon for suggestions along these lines, see also [10].

Moonshine before the discovery of the Monster VOA $V^\natural$ [2]. However, the story appears to be more subtle since the natural expectation that $\mathcal{H}$ would arise as a superconformal σ-model on K3 does not hold [13]. On the other hand, this is perhaps not too surprising, given the fact that the modular J-function corresponds to the *full* partition function of a CFT, while the elliptic genus of K3 only receives contribution from a *subset* of the physical states of the $\mathcal{N} = (4,4)$ superconformal theory on K3. Hence it is tempting to speculate that $\mathcal{H}$ should correspond to something like the algebra of BPS-states of string theory on K3 (in the sense of Harvey and Moore [62, 63]). In particular, one might therefore hope that $\mathcal{H}$ can be constructed using some kind of topological sigma model, e.g. the 'half-twisted' topological σ-model on K3 [64]. Mathematically, this sigma model (or rather its large radius limit) can be viewed as a bundle of VOAs known as the chiral de Rham complex, Ω_X^{ch} [65], and in [66] it was shown that the elliptic genus of K3 can be obtained by taking the graded trace over the corresponding cohomology $H^*(\Omega_X^{ch})$. It is therefore tempting to speculate that one may be able to construct an action of M_{24} on the cohomology of Ω_X^{ch}, at least for some choice of K3 surface X.

We hope to return to these and related issues in future work.

Acknowledgements

We are indebted to Terry Gannon for the suggestion that the 3rd cohomology of M_{24} should control the modular properties of the twisted twining genera, as well as for many useful explanations and directions to the relevant literature. We are furthermore grateful to Mathieu Dutour and Graham Ellis for helpful correspondence on using the computer package `HAP` for calculating the cohomology of finite groups. We also thank Nils Carqueville, Miranda Cheng, Jeff Harvey, Stefan Hohenegger, Axel Kleinschmidt, Sameer Murthy, Daniel Roggenkamp, Ashoke Sen and Don Zagier for useful discussions, as well as the anonymous referee for many useful suggestions that considerably improved the exposition of the paper. D.P. would like to thank the Albert Einstein Institute in Golm for hospitality while part of this work was carried out. The work of M.R.G. is partially supported by the Swiss National Science Foundation, and the visit of Henrik Ronellenfitsch at ETH was supported by the Pauli Center.

Appendix A. Details on the unobstructed twisted twining genera

In this appendix, following the example of section 3.4, we will determine the twisted twining genera $\phi_{g,h}$ for the groups 13, 23, 24, 33 and 34 in Table 1. The genus $\phi_{2\mathrm{B},8\mathrm{A}_{1,2}}$ for group

27 was obtained in section 3.4, while for all the other groups $\phi_{g,h}$ vanishes due to some obstruction.

A.1. **The characters $\phi_{2\mathrm{B},4\mathrm{A}_2}$ (group 13) $\phi_{4\mathrm{B},4\mathrm{A}_3}$ (group 23) and $\phi_{4\mathrm{B},4\mathrm{A}_4}$ (group 24).** In this section we will consider the twisted twining genera $\phi_{g,h}$ for groups 13, 23 and 24; as we will see, it turns out that these three characters have exactly the same modular properties. For group 13 g is in class 2B and h in class 4A, and we can choose

$$g = (1,10)(2,14)(3,8)(4,16)(5,15)(6,12)(7,21)(9,18)(11,13)(17,20)(19,23)(22,24) \ ,$$
$$h = (1,2)(3,18)(4,7,15,17)(5,20,16,21)(6,19,24,11)(8,9)(10,14)(12,23,22,13) \ . \quad (112)$$

For the groups 27 and 28, g is in class 4B and h in class 4A and we can choose

$$g = (3,8)(4,21,15,20)(5,17,16,7)(6,19,24,11)(9,18)(12,23,22,13) \ , \quad (113)$$

for both groups (from now on, we drop the trivial cycles). For group 23 we can take

$$h = (1,10)(2,14)(3,9)(4,11,5,13)(6,17,12,21)(7,22,20,24)(8,18)(15,19,16,23) \ . \quad (114)$$

and for group 24

$$h = (1,2,10,14)(3,9,8,18)(4,5,15,16)(6,13)(7,21,17,20)(11,22)(12,19)(23,24) \ , \quad (115)$$

In all the three cases we have the relations

$$(k^{-1}gk, k^{-1}ghk) = (g,h) \ , \qquad (r^{-1}gh^{-2}r, r^{-1}hr) = (g,h) \ , \quad (116)$$

where, for group 13

$$k = (1,9,2,3)(6,13,22,19)(7,21)(8,10,18,14)(11,24,23,12)(17,20) \ , \quad (117)$$
$$r = (3,8)(4,11,5,13)(6,20,12,7)(9,18)(15,19,16,23)(17,24,21,22) \ , \quad (118)$$

for group 23

$$k = (1,2,10,14)(3,9,8,18)(5,16)(6,11,24,19)(7,17)(12,23,22,13) \ , \quad (119)$$
$$r = (1,2)(3,8)(6,12)(7,20)(9,18)(10,14)(17,21)(22,24) \ , \quad (120)$$

and for group 24

$$k = (3,9)(4,12,5,6)(7,11,20,13)(8,18)(15,22,16,24)(17,19,21,23) \ , \quad (121)$$
$$r = (1,3)(2,9)(4,20)(5,7)(8,10)(14,18)(15,21)(16,17) \ . \quad (122)$$

Using the explicit knowledge of the 3-cocycle α we can derive the modular properties of the twisted twining genus $\phi_{g,h}$, including the multiplier system. It turns out that in all these three cases $\phi_{g,h}$ is invariant under $\Gamma_0(2)$ with the same multiplier

$$\phi_{g,h}(\tau+1) = c_g(g,h)\frac{c_g(gh,k)}{c_g(k,h)}\phi_{g,h}(\tau) = i\phi_{g,h}(\tau) \ , \tag{123}$$

$$\begin{aligned}\phi_{g,h}(\frac{\tau}{2\tau+1}) &= \frac{1}{c_h(h,h^{-1}g)c_h(h,h^{-2}g)}\frac{c_h(h^{-2}g,r)}{c_h(r,g)}\phi_{g,h}(\tau) \\ &= -i\phi_{g,h}(\tau) \ . \end{aligned} \tag{124}$$

By arguments similar to the ones in section 4.3, $\phi_{g,h}(\tau,z)$ has only a double zero at $z=0$ (mod $\mathbb{Z}+\tau\mathbb{Z}$). This implies that $\phi_{g,h}$ can be expressed as

$$\phi_{g,h}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2}F(\tau) \tag{125}$$

for some $\Gamma_0(2)$-invariant function $F(\tau)$ that is holomorphic in $\mathbb{H}_+$. The group $\Gamma_0(2)$ has two cusps at ∞ and 0. To proceed we study the asymptotic behavior at infinity

$$\phi_{g,h}(\tau,z) = A_\infty\, q^{1/4}(2-y-y^{-1}) + \cdots \tag{126}$$

$$\frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2} = \frac{1}{4}\,(2-y-y^{-1}) + \cdots \ , \tag{127}$$

which implies that $F(\tau) \sim 4A_\infty\, q^{1/4} + \cdots$ has a zero at $\tau\to\infty$. The behavior at $\tau=0$ is

$$\phi_{g,h}(\gamma_0(\tau,z)) \sim \phi_{h,g^{-1}}(\tau,z) = A_0\, q^{1/8}\,(2-y-y^{-1}) + \cdots \tag{128}$$

$$\frac{\vartheta_1(\tau,z)^2}{\vartheta_4(\tau,0)^2} = q^{1/4}\,(2-y-y^{-1}) + \cdots \ , \tag{129}$$

implying that $F(\gamma_0\cdot\tau) \sim A_0\, q^{-1/8} + \cdots$ has a pole at 0. A function with the correct modular properties and behaviour at the cusps is

$$F(\tau) = 4A_\infty\,\frac{\eta(2\tau)^6}{\eta(\tau)^6} \ , \tag{130}$$

so that

$$\phi_{g,h}(\tau,z) = 4A_\infty\frac{\eta(2\tau)^6}{\eta(\tau)^6}\frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,z)^2} = A_\infty\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2 \ . \tag{131}$$

The constants A_∞ for the three cases are determined (up to the phase) by ensuring a decomposition into projective characters of $C_{M_{24}}(g)$. In particular, $A_\infty = 4$ for group 13 and $A_\infty = 2\sqrt{2}$ for groups 23 and 24.

A.2. **The cases $\phi_{3A,3A_3}$ (group 33) and $\phi_{3A,3B_1}$ (group 34).** In this section we show that the unobstructed twisted twining genera $\phi_{3A,3A_3}$ and $\phi_{3A,3B_1}$ must vanish by modular arguments.

Let us consider first the group $\langle g,h\rangle = \langle 3A, 3B_1\rangle$ (type 34), generated by

$$g = (2,23,11)(3,20,10)(5,14,9)(6,16,13)(7,19,22)(12,24,18) \tag{132}$$

in class 3A and

$$h = (1,15,17)(2,19,16)(3,18,5)(4,21,8)(6,11,7)(9,10,24)(12,14,20)(13,23,22) \tag{133}$$

in class 3B. The following conjugation relation holds

$$(k^{-1}gk, k^{-1}ghk) = (g,h)\ , \tag{134}$$

where

$$k = (2,7,16)(3,20,10)(4,21,8)(5,9,14)(6,11,22)(13,23,19)\ . \tag{135}$$

The modular properties of the associated twisted twining genus are

$$\phi_{g,h}(\tau+1) = c_g(g,h)\frac{c_g(gh,k)}{c_g(k,h)}\phi_{g,h}(\tau) = e^{\frac{4\pi i}{3}}\phi_{g,h}(\tau)\ , \tag{136}$$

and

$$\phi_{g,h}(\frac{\tau}{-3\tau+1}) = \prod_{i=0}^{2} c_h(h,h^i g^{-1})\phi_{g,h}(\tau) = e^{\frac{4\pi i}{3}}\phi_{g,h}(\tau)\ , \tag{137}$$

where we have used the explicit form for the multiplier phases as determined in terms of the 3-cocycle $\alpha \in H^3(M_{24}, U(1))$. We conclude that, up to the multiplier, $\phi_{g,h}$ is invariant under $\Gamma_0(3)$. Furthermore, as in the previous subsection and in section 4.3, for each fixed $\tau \in \mathbb{H}_+$, $\phi_{g,h}(\tau,z)$ has only a double zero at $z=0$ (mod $\mathbb{Z}+\tau\mathbb{Z}$). Since $\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^2}$ is a weak Jacobi form of weight 0 and index 1 with non-trivial multiplier under $SL(2,\mathbb{Z})$ and only a double zero at $z=0$, we have

$$\phi_{g,h}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^2}F_{g,h}(\tau) \tag{138}$$

for some function $F_{g,h}(\tau)$ that is $\Gamma_0(3)$-invariant (up to a multiplier). The asymptotics for $\tau\to\infty$ is

$$\phi_{g,h}(\tau,z) \sim A_\infty\, q^{\frac{2}{3}} + \cdots\ , \tag{139}$$

for some (possibly vanishing) constant A_∞, while the asymptotic behavior at 0 is given by

$$\phi_{h,g}(\tau,z) \sim A_0\, q^{\frac{2}{9}} + \cdots\ . \tag{140}$$

Since both at ∞ and 0 we have

$$\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^2} \sim q^{1/6}(2-y-y^{-1}) + \cdots \ , \tag{141}$$

we deduce that $F_{g,h}$ has a zero at both 0 and ∞ and no poles; hence it must vanish. We therefore conclude

$$\phi_{3\mathrm{A},3\mathrm{B}_1}(\tau,z) = 0 \ , \tag{142}$$

as claimed.

The $(3\mathrm{A}, 3\mathrm{A}_3)$-case (group 33) is analogous. Here, the group is generated by

$$g = (2,23,11)(3,20,10)(5,14,9)(6,16,13)(7,19,22)(12,24,18) \tag{143}$$

and

$$h = (2,22,13)(4,21,8)(5,14,9)(6,23,7)(11,19,16)(12,18,24) \ , \tag{144}$$

and we have the relations

$$(k^{-1}gk, k^{-1}ghk) = (g,h) \ , \qquad (r^{-1}hr, r^{-1}g^{-1}r) = (g,h) \tag{145}$$

where

$$k = (1,17,15)(3,9,24)(5,18,20)(6,16,13)(7,22,19)(10,14,12) \ , \tag{146}$$

$$r = (3,4,10,21)(5,18)(6,16,19,7)(8,20)(9,24,14,12)(11,22,23,13) \ . \tag{147}$$

The modular properties of $\phi_{g,h}$ are

$$\phi_{g,h}(\tau+1) = c_g(g,h)\,\frac{c_g(gh,k)}{c_g(k,h)}\,\phi_{g,h}(\tau) = e^{\frac{4\pi i}{3}}\,\phi_{g,h}(\tau) \ , \tag{148}$$

$$\phi_{g,h}(-1/\tau) = \frac{1}{c_h(g,g^{-1})}\frac{c_h(g^{-1},r)}{c_h(r,h)}\phi_{g,h}(\tau) = \phi_{g,h}(\tau) \ . \tag{149}$$

Thus $\phi_{g,h}$ is invariant (up to a multiplier) under $SL(2,\mathbb{Z})$, and $\phi_{g,h} \sim A\,q^{2/3}$ at ∞, so that

$$F_{3\mathrm{A},3\mathrm{A}_3}(\tau) \equiv \phi_{g,h}(\tau,z)\frac{\eta(\tau)^2}{\vartheta_1(\tau,z)^2} \sim A\,q^{1/2} + \cdots \ . \tag{150}$$

Thus, $F_{3\mathrm{A},3\mathrm{A}_3}$ has a zero and no poles and therefore must vanish, thus leading to

$$\phi_{3\mathrm{A},3\mathrm{A}_3}(\tau,z) = 0 \ . \tag{151}$$

Appendix B. Some group cohomology

In this section, we will review the basic definitions and properties of group cohomology, see for example [43] for an introduction to the subject.

In general, group cohomology is defined in terms of a finite group G and a G-module A, i.e. an abelian group with an action of G satisfying certain compatibility properties. For our purposes, it is sufficient to consider $A = U(1)$ with the trivial action of G. An n-cochain ψ (with $n \geq 0$) for G with values in $U(1)$ is simply a function from $G \times \ldots \times G$ (n times) to $U(1)$; a 0-cochain is just an element of $U(1)$. On the space $C^n(G, U(1))$ of n-cochains one can define a *coboundary* operator $\partial_n : C^n(G, U(1)) \to C^{n+1}(G, U(1))$ by

$$\partial_n \psi\left(g_1, \ldots, g_{n+1}\right) = \psi(g_2, \ldots, g_{n+1}) \times \prod_{i=1}^{n} \psi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1})^{(-1)^i} \psi(g_1, \ldots, g_n)^{(-1)^{n+1}} \ . \quad (152)$$

The cochains in the kernel of ∂_n are called n-cocycles (or co-closed n-cochains), while the cochains in the image of ∂_{n-1} are called n-coboundaries. The coboundary operator satisfies $\partial_{n+1} \circ \partial_n = 1$ (the trivial element of $U(1)$)), so that we can define the cohomology, as usual, as the quotient of the space of cocycles modulo coboundaries

$$H^n(G, U(1)) = \frac{\ker \partial_n}{\mathrm{Im} \partial_{n-1}} \ , \quad (153)$$

where $n > 0$, and we use the convention that $H^0(G, U(1)) = U(1)$, and that every 0-cochain is coclosed, ie. $\partial_0 = 1$.

In particular, for a 1-cochain γ, the condition $\partial_1 \gamma = 1$ is simply $\gamma(g)\gamma(h) = \gamma(gh)$ for all $g, h \in G$, so that $H^1(G, U(1))$ is just the group of homomorphism from G to $U(1)$. A 2-cochain $\beta : G \times G \to U(1)$ is co-closed (and hence defines a cocycle) provided it satisfies

$$\beta(g_1, g_2 g_3)\, \beta(g_2, g_3) = \beta(g_1 g_2, g_3)\, \beta(g_1, g_2) \quad (154)$$

for $g_1, g_2, g_3 \in G$. The second cohomology $H^2(G, U(1))$ then consists of the co-closed 2-cochains, modulo the ambiguity

$$\beta(g_1, g_2) \to \beta(g_1, g_2) \frac{\gamma(g_1)\gamma(g_2)}{\gamma(g_1 g_2)} \ , \quad (155)$$

where γ is an arbitrary 1-cochain, i.e. an arbitrary function $\gamma : G \to U(1)$. The second cohomology of a group G classifies the inequivalent projective representations of G, see appendix C. A 3-cochain α,

$$\alpha : G \times G \times G \to U(1) \quad (156)$$

is closed provided it satisfies

$$\alpha(g_1,g_2,g_3)\,\alpha(g_1,g_2g_3,g_4)\,\alpha(g_2,g_3,g_4)=\alpha(g_1g_2,g_3,g_4)\,\alpha(g_1,g_2,g_3g_4)\ . \tag{157}$$

In the cohomology group $H^3(G,U(1))$ the 3-cocycles are then identified modulo

$$\alpha(g_1,g_2,g_3)\to\alpha(g_1,g_2,g_3)\,\frac{\beta(g_1,g_2g_3)\beta(g_2,g_3)}{\beta(g_1g_2,g_3)\beta(g_1,g_2)}\ . \tag{158}$$

Note that the multiplying factor is trivial if β is closed, i.e. if it satisfies the 2-cocycle condition (154). We shall always work with normalised cocycles, i.e. we shall assume that

$$\alpha(e,g_1,g_2)=\alpha(g_1,e,g_2)=\alpha(g_1,g_2,e)=1 \tag{159}$$

for all $g_1,g_2\in G$, where e denotes the identity element of G.

Given a 3-cocycle α, we can define, for any $h\in G$, a 2-cochain $c_h:G\times G\to U(1)$ via

$$c_h(g_1,g_2)=\frac{\alpha(h,g_1,g_2)\,\alpha(g_1,g_2,(g_1g_2)^{-1}h(g_1g_2))}{\alpha(g_1,g_1^{-1}hg_1,g_2)}\ . \tag{160}$$

It is shown in [25] that c_h defines a 2-cocycle of the centraliser $C_G(h)\subset G$ (i.e. the subgroup of all elements g_1,g_2 which commute with h). When $g_1,g_2\in C_G(h)$, we have the simplified expression

$$c_h(g_1,g_2)=\frac{\alpha(h,g_1,g_2)\,\alpha(g_1,g_2,h)}{\alpha(g_1,h,g_2)}\ ,\qquad g_1,g_2\in C_G(h)\ . \tag{161}$$

Under the 'gauge transformation' (158), c_h transforms as

$$c_h(g_1,g_2)\to c_h(g_1,g_2)\frac{\gamma_h(g_1)\gamma_h(g_2)}{\gamma_h(g_1g_2)}\ ,\qquad g_1,g_2\in C_G(h)\ , \tag{162}$$

where we defined the 1-cochain γ_h by

$$\gamma_h(g)\equiv\frac{\beta(g,h)}{\beta(h,g)}\ . \tag{163}$$

This is indeed of the form (155), and hence, for all $h\in G$, c_h defines a map

$$c_h\ :\ H^3(G,U(1))\ \to H^2(C_G(h),U(1))\ . \tag{164}$$

Appendix C. Projective representations of finite groups

This section follows [37, 67]. A projective representation is a map $\rho:G\to\mathrm{End}(V)$ from a group G to the endomorphism group of some complex vector space V satisfying

$$\rho(g)\,\rho(h)=\beta(g,h)\,\rho(gh)\ ,\quad \beta(g,h)\in U(1)\ . \tag{165}$$

It follows from the associativity of the group multiplication that the phase $\beta(g,h)$ satisfies the 2-cocycle condition

$$\beta(g,h)\,\beta(gh,k)=\beta(h,k)\,\beta(g,hk)\ . \tag{166}$$

There is the freedom of rescaling the maps $\rho(g)$ as

$$\rho(g) \mapsto \gamma(g)\rho(g) \ , \quad \gamma(g) \in U(1) \ . \tag{167}$$

For the 2-cocylce β this leads to the rescaling by a coboundary

$$\beta(g,h) \to \beta(g,h)\,\gamma(gh)\,\gamma(g)^{-1}\,\gamma(h)^{-1} \ . \tag{168}$$

In particular, one can always choose a rescaling such that $\rho(e)$ is the identity matrix in $\mathrm{End}(V)$; with this choice, the cocycle β is normalised. Two projective representations whose 2-cocycles differ only by a coboundary are called projectively equivalent, since their representation matrices are related by a rescaling. Therefore, the inequivalent projective representations of G are captured by the second group cohomology $H^2(G, U(1))$.

As for genuine linear representations, one may define the character of a projective representation as the trace

$$\chi(g) = \mathrm{Tr}_V(\rho(g)) \ . \tag{169}$$

Due to the modified composition law, projective characters are in general not class functions, but differ by a phase on different representatives of one conjugacy class. However, it is always possible to choose a cocycle representative such that the characters are indeed class functions.

Assuming that $gh = hg$, we have, using the projective composition law,

$$\chi(g) = \chi(h^{-1}gh) = \frac{\beta(gh, h^{-1})}{\beta(h^{-1}, gh)}\,\chi(g) \ . \tag{170}$$

Using the cocycle condition, it is easy to see that this condition on the characters is equivalent to the obstuction of the first kind (14). Also, the obstruction does not depend on the specific cocycle representative $\beta(g,h)$ or the class representative g. Classes whose projective characters vanish are called β-irregular. It may be shown that to each cocycle $\beta(g,h)$, there exist exactly as many linearly inequivalent irreducible projective representations as there are β-regular conjugacy classes.

Two irreducible characters belonging to representations with the same cocycle $\beta(g,h)$ satisfy an orthogonality relation of the form

$$\sum_{[g]} |K_g|^{-1}\,\chi(g)^*\,\tilde{\chi}(g) = \begin{cases} 1 & \chi = \tilde{\chi} \\ 0 & \text{otherwise,} \end{cases} \tag{171}$$

where the sum is taken over a set of (β-regular) class representatives of G, and $|K_g|$ is the size of the conjugacy class of g.

There is a natural connection between projective representations and central extensions of a group G. We say that $\tilde{G}$ is a central extention of G if there is an abelian subgroup $A \subset Z(\tilde{G})$ of the center $Z(\tilde{G})$ of $\tilde{G}$ such that the short sequence

$$0 \to A \to \tilde{G} \to G \to 0 \tag{172}$$

is exact, or equivalently that $G = \tilde{G}/A$. The simplest central extension is of course $\tilde{G} = A \times G$, but this is in general not the only possibility. It can be shown that the non-isomorphic central extensions of G by A are also classified by $H^2(G, A)$. Each set of projectively inequivalent projective representations of G corresponds to a set of genuine representations of some extension $\tilde{G}$.

In some sense there exists a maximal central extension S for each G, the Schur cover or 'Darstellungsgruppe'. It is characterised by the property that any projective representation of G corresponds to a genuine representation of S. The Schur cover is, in general, not unique, unless G is a perfect group, i.e. it coincides with the commutator subgroup. Otherwise, the Schur covers are related by isoclinism.

C.1. **Central extension and orbifolds.** Let $\mathcal{C}$ be a K3 sigma-model and suppose that its automorphism group contains an element g such that the orbifold of $\mathcal{C}$ by g, $\hat{\mathcal{C}} = \mathcal{C}/\langle g \rangle$ is consistent. In this subsection, we will show that all projective representations of $H \equiv C_{M_{24}}(g)$ on the g^r-twisted sectors are equivalent to genuine representations of a central extension $\tilde{H}$ of the form described in section 4.1, eq. (89).

By (54) and (62), the level-matching condition is satisfied if and only if the restriction of the class $[\alpha] \in H^3(M_{24}, U(1))$ to $H^3(\langle g \rangle, U(1))$ is trivial. Each g^r-twisted sector carries a, in general projective, representation ρ_{g^r} of $H \equiv C_{M_{24}}(g)$, with underlying 2-cocycle c_{g^r} given by (39). The various cocycles c_{g^r} are related to one another by the identity

$$c_{g^r}(h,k) = \frac{f_{g,r}(h) f_{g,r}(k)}{f_{g,r}(hk)} \, c_g(h,k)^r \ , \qquad h,k \in C(g) \ , \tag{173}$$

where

$$f_{g,r}(h) = \prod_{i=1}^{r-1} c_h(g, g^i) \ . \tag{174}$$

Here we have used the definition (39) of c_g and applied repeatedly the condition that the 3-cocycle α is closed. Thanks to the level-matching condition, we can choose the representative cocycle α to be identically 1 when restricted to $\langle g \rangle$, so that $f_{g,r}(g^i) = 1$ for all i. Furthermore,

by changing α by a coboundary $\partial\beta$ as in (158), the functions $f_{g,r}(h)$ transform as

$$f_{g,r}(h) \to f_{g,r}(h)\,\frac{\beta(g^r,h)\beta(h,g)^r}{\beta(h,g^r)\beta(g,h)^r}\ . \tag{175}$$

Thus, we can choose β in such a way that $f_{g,r}(h)=1$ for all $h\in C(g)$, so that

$$c_{g^r}(h,k) = c_g(h,k)^r\ , \qquad h,k\in C(g)\ . \tag{176}$$

In particular, since $c_e(h,k)=1$, $c_g(h,k)$ is always an N-th root of unity, where $N=o(g)$.

Appendix D. Centralisers $C_{M_{24}}(g)$ and (projective) character tables

This section describes the projective character tables of the centraliser $H\equiv C_{M_{24}}(g)$ for each conjugacy class of M_{24}, corresponding to the projective equivalence class determined by the 2-cocycle c_g. The procedure to compute these tables is as follows.

Let Schur(H) denote a Schur cover of H, i.e. a central extension of H fitting into the exact sequence

$$1\to M(H)\to \mathrm{Schur}(H)\to H\to 1\ , \tag{177}$$

where $M(H)=H_2(H,\mathbb{Z})=(H^2(H,U(1)))^*$ is the Schur multiplier. The Schur cover Schur(H) and its character table can be easily computed (we used `GAP` [41] for this). Any irreducible projective representation ρ of H is equivalent to a genuine irreducible representation $\tilde\rho$ of Schur(H) and vice versa. Thus, one can associate a class of $H^2(H,U(1))$ to each irreducible representation of Schur(H). The projective character table we are interested in is obtained from the one of Schur(H) by keeping only the representations associated to the class $[c_g]$, and by choosing a lift in Schur(H) of each conjugacy class of H (different choices of the lifts lead to projectively equivalent representations).

It only remains to determine which irreducible representations of Schur(H) correspond to the class $[c_g]$. Consider an irreducible projective representation ρ of H associated with the 2-cocycle c_g. For any commuting pair $h,k\in H$, $hk=kh$, we have

$$\rho(h)\rho(k)\rho(h)^{-1}\rho(k)^{-1} = \frac{c_g(h,k)}{c_g(k,h)}\ . \tag{178}$$

(Note that $\frac{c_g(h,k)}{c_g(k,h)}$ only depends on the class $[c_g]$ and not on the representative cocycle c_g.) Thus, any representation σ of Schur(H) in the class $[c_g]$ must satisfy

$$\sigma(\tilde h\tilde k\tilde h^{-1}\tilde k^{-1}) = \frac{c_g(h,k)}{c_g(k,h)}\ , \tag{179}$$

where $\tilde h,\tilde k$ are some lifts of h,k in Schur(H). It is easy to see that the commutator $\tilde h\tilde k\tilde h^{-1}\tilde k^{-1}$ is an element of $M(H)$, and that it does not depend of the choice of the lifts. In fact, the

commutators of this form generate $M(H)$, and if a representation σ satisfies (179) for a set of generators of $M(H) = (H^2(H, U(1)))^*$, then it must be associated with the class $[c_g]$. This condition can be easily tested for each irreducible representation of $\mathrm{Schur}(H)$, given our explicit knowledge of the cocycle c_g.

D.1. **Character Tables.** In this section we collect some information about the centralisers $C_{M_{24}}(g)$ of representatives of each M_{24} conjugacy class $[g]$, in particular a (rough) description of the group structure and the group order. Our notation is as follows:

$G = N.Q$	group G containing a normal subgroup N and such that $G/N \cong Q$
$N \rtimes Q$	(left) semidirect product of N and Q
$\mathbb{Z}_n$	cyclic group of order n
S_n	group of permutations (symmetric group) of n objects
A_n	group of even permutations (alternating group) of n objects
D_n	dihedral group of order n
$PSL_n(q)$	projective special linear group over the n-dim. vector space over the finite field $\mathbb{F}_q$

Furthermore, for each class $[g]$, we specify whether the representation carried by the g-twisted sector $\mathcal{H}_g$ is (projectively equivalent to) a linear representation, or whether it is genuinely projective; in the latter case we describe a minimal central extension with respect to which the representation is linear. Finally, we provide the character table of $C_{M_{24}}$ for the relevant projective equivalence class. The names of the conjugacy classes are chosen to correspond to the ATLAS names of M_{24} classes, with additional indices in case of redundancy. We adopt the following notation for algebraic numbers

$$\varepsilon_n = e^{\frac{2\pi i}{n}} \ , \qquad\qquad e_p^{\pm} = \frac{1}{2}(1 \pm i\sqrt{p}) \ .$$

Classes 11A, 12A, 12B, 14AB, 15AB, 21AB, 23AB

When g is in one of these classes, the centralizer $C_{M_{24}}(g)$ is the cyclic group $\langle g \rangle$ and all representations in the g-twisted sector are (projectively equivalent to) linear representations. Thus, all irreducible representations are one dimensional, and a useful choice for the corresponding (projective) characters χ_k, $k = 1, \ldots, o(g)$ is

$$\chi_k(g^a) = e^{2\pi i \frac{a}{o(g)}(k - \frac{1}{\ell(g)})} \ , \tag{180}$$

$a = 0, \ldots, o(g) - 1$, where $\ell(g)$ is the length of the shortest cycle of g when considered as a permutation of 24 objects. Indeed, the characters (180) correspond to the standard definition $\rho_g(g^a) = e^{2\pi i a(L_0 - \tilde{L}_0)}$ for the action of $\langle g \rangle$ in the g-twisted sector.

Class 2A

$C_{M_{24}}(2A) \cong \mathbb{Z}_2^4.(\mathbb{Z}_2^3 \rtimes PSL_3(2))$ Order: $2^{10} \cdot 3 \cdot 7 = 21504$

Genuinely projective rep. Central extension: $\mathbb{Z}_2.C_{M_{24}}(2A)$

	21504	21504	1536	1536	384	512	256	256	64	128	128	64	128	64	32	64	32	32	16	16	16	24	24	12	12	12	14	14	14	14
	1A	$2A_1$	$2A_2$	$2A_3$	$4A_1$	$2B_1$	$2A_4$	$2B_3$	$4B_2$	$2A_5$	$2B_2$	$4A_2$	$4B_1$	$4A_3$	$4C_1$	$4B_3$	$4B_4$	$4A_4$	$4B_5$	$4C_2$	$8A_1$	$3A_1$	$6A_1$	$6A_2$	$6A_3$	$12A_1$	$7A_1$	$14A_1$	$7B_1$	$14B_1$
χ_1	8	-8	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	2	-2	2	0	0	0	-1	1	-1	1
χ_2	8	8	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	-2	0	0	-2	-1	-1	-1	-1
χ_3	8	8	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	$\sqrt{3}$	$\sqrt{3}$	1	-1	-1	-1	-1
χ_4	8	8	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	$-\sqrt{3}$	$-\sqrt{3}$	1	-1	-1	-1	-1
χ_5	24	-24	0	0	0	0	0	0	0	0	0	0	-4	0	0	0	0	0	0	0	2	0	0	0	0	0	e_7^-	$-e_7^-$	e_7^+	$-e_7^+$
χ_6	24	-24	0	0	0	0	0	0	0	0	0	0	-4	0	0	0	0	0	0	0	2	0	0	0	0	0	e_7^+	$-e_7^+$	e_7^-	$-e_7^-$
χ_7	24	24	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	e_7^-	e_7^-	e_7^+	e_7^+
χ_8	24	24	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	e_7^+	e_7^+	e_7^-	e_7^-
χ_9	48	-48	0	0	0	0	0	0	0	0	0	0	8	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1
χ_{10}	56	-56	0	0	0	0	0	0	0	0	0	0	-4	0	0	0	0	0	0	0	-2	-2	2	0	0	0	0	0	0	0
χ_{11}	56	56	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	-2	0	0	2	0	0	0	0
χ_{12}	56	56	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	$-\sqrt{3}$	$\sqrt{3}$	-1	0	0	0	0
χ_{13}	56	56	0	0	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	$\sqrt{3}$	$-\sqrt{3}$	-1	0	0	0	0
χ_{14}	64	-64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	-2	0	0	0	-1	1	-1	1

TABLE 4. Projective character table for the centralizer of class 2A.

Class 2B

$C_{M_{24}}(2B) \cong \mathbb{Z}_2^6 \rtimes S_5$ Order: $2^9 \cdot 3 \cdot 5 = 7680$

Genuinely projective rep. Central extension: $\mathbb{Z}_2.C_{M_{24}}(2B) \cong (\mathbb{Z}_2.\mathbb{Z}_2^6) \rtimes S_5$

	7680	512	256	3840	512	7680	96	32	96	32	24	24	12	12	12	128	64	32	32	64	64	128	20	20	20	20	16	16	16	16
	1A	$2B_1$	$2A_1$	$2B_2$	$2A_2$	$2B_3$	$2B_4$	$4A_1$	$4C_1$	$4C_2$	$3B_1$	$6B_1$	$6B_2$	$6B_3$	$12B_1$	$2B_5$	$4A_2$	$4B_1$	$4A_3$	$4B_2$	$2B_6$	$2A_3$	$5A_1$	$10A_1$	$10A_2$	$10A_3$	$4C_3$	$8A_1$	$4B_3$	$8A_2$
χ_1	4	0	0	4	0	−4	2	0	−2	0	−1	1	−1	−1	1	0	2	0	0	−2	0	0	−1	1	1	−1	0	$2e_1^-$	0	$2e_1^+$
χ_2	4	0	0	−4	0	−4	2	0	2	0	−1	1	1	−1	−1	0	2	0	0	2	0	0	−1	−1	1	1	0	$-2e_1^+$	0	$-2e_1^-$
χ_3	4	0	0	4	0	−4	−2	0	2	0	−1	1	−1	1	−1	0	2	0	0	−2	0	0	−1	1	1	−1	0	$-2e_1^-$	0	$-2e_1^+$
χ_4	4	0	0	−4	0	−4	−2	0	−2	0	−1	1	1	1	1	0	2	0	0	2	0	0	−1	−1	1	1	0	$2e_1^+$	0	$2e_1^-$
χ_5	16	0	0	16	0	−16	4	0	−4	0	−1	1	−1	1	−1	0	0	0	0	0	0	0	1	−1	−1	1	0	0	0	0
χ_6	16	0	0	−16	0	−16	4	0	4	0	−1	1	1	1	1	0	0	0	0	0	0	0	1	1	−1	−1	0	0	0	0
χ_7	16	0	0	16	0	−16	−4	0	4	0	−1	1	−1	−1	1	0	0	0	0	0	0	0	1	−1	−1	1	0	0	0	0
χ_8	16	0	0	−16	0	−16	−4	0	−4	0	−1	1	1	−1	−1	0	0	0	0	0	0	0	1	1	−1	−1	0	0	0	0
χ_9	16	0	0	0	0	16	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	1	$\sqrt{5}$	1	$\sqrt{5}$	0	0	0	0
χ_{10}	16	0	0	0	0	16	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	1	$-\sqrt{5}$	1	$-\sqrt{5}$	0	0	0	0
χ_{11}	20	0	0	20	0	−20	−2	0	2	0	1	−1	1	1	−1	0	2	0	0	−2	0	0	0	0	0	0	0	$2e_1^-$	0	$2e_1^+$
χ_{12}	20	0	0	−20	0	−20	−2	0	−2	0	1	−1	−1	1	1	0	2	0	0	2	0	0	0	0	0	0	0	$-2e_1^+$	0	$-2e_1^-$
χ_{13}	20	0	0	20	0	−20	2	0	−2	0	1	−1	1	−1	1	0	2	0	0	−2	0	0	0	0	0	0	0	$-2e_1^-$	0	$-2e_1^+$
χ_{14}	20	0	0	−20	0	−20	2	0	2	0	1	−1	−1	−1	−1	0	2	0	0	2	0	0	0	0	0	0	0	$2e_1^+$	0	$2e_1^-$
χ_{15}	24	0	0	24	0	−24	0	0	0	0	0	0	0	0	0	0	−4	0	0	4	0	0	−1	1	1	−1	0	0	0	0
χ_{16}	24	0	0	−24	0	−24	0	0	0	0	0	0	0	0	0	0	−4	0	0	−4	0	0	−1	−1	1	1	0	0	0	0
χ_{17}	32	0	0	0	0	32	0	0	0	0	−2	−2	0	0	0	0	0	0	0	0	0	0	2	0	2	0	0	0	0	0
χ_{18}	48	0	0	0	0	48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−2	0	−2	0	0	0	0	0

TABLE 5. Projective character table for the centralizer of class 2B.

Class 3A

$C_{M_{24}}(3A) \cong \mathbb{Z}_3.A_6$ Order: $2^3 \cdot 3^3 \cdot 5 = 1080$ Linear rep.

	1080	1080	1080	24	24	24	9	9	12	12	12	15	15	15	15	15	15
	1A	$3A_1$	$3A_2$	$6A_1$	$6A_2$	$2A_1$	$3A_3$	$3B_1$	$12A_1$	$12A_2$	$4A_1$	$15A_1$	$5A_1$	$15B_1$	$5A_2$	$15A_2$	$15B_2$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	$-3e_3^+$	$-3e_3^-$	e_3^+	e_3^-	-1	0	0	$-e_3^+$	$-e_3^-$	1	α	e_{-5}^+	$\bar{\alpha}$	e_{-5}^-	β	$\bar{\beta}$
χ_3	3	$-3e_3^+$	$-3e_3^-$	e_3^+	e_3^-	-1	0	0	$-e_3^+$	$-e_3^-$	1	$\bar{\beta}$	e_{-5}^-	β	e_{-5}^+	$\bar{\alpha}$	α
χ_4	3	$-3e_3^-$	$-3e_3^+$	e_3^-	e_3^+	-1	0	0	$-e_3^-$	$-e_3^+$	1	β	e_{-5}^-	$\bar{\beta}$	e_{-5}^+	α	$\bar{\alpha}$
χ_5	3	$-3e_3^-$	$-3e_3^+$	e_3^-	e_3^+	-1	0	0	$-e_3^-$	$-e_3^+$	1	$\bar{\alpha}$	e_{-5}^+	α	e_{-5}^-	$\bar{\beta}$	β
χ_6	5	5	5	1	1	1	-1	2	-1	-1	-1	0	0	0	0	0	0
χ_7	5	5	5	1	1	1	2	-1	-1	-1	-1	0	0	0	0	0	0
χ_8	6	$-6e_3^+$	$-6e_3^-$	$-2e_3^+$	$-2e_3^-$	2	0	0	0	0	0	$-e_3^-$	1	$-e_3^+$	1	$-e_3^+$	$-e_3^-$
χ_9	6	$-6e_3^-$	$-6e_3^+$	$-2e_3^-$	$-2e_3^+$	2	0	0	0	0	0	$-e_3^+$	1	$-e_3^-$	1	$-e_3^-$	$-e_3^+$
χ_{10}	8	8	8	0	0	0	-1	-1	0	0	0	e_{-5}^+	e_{-5}^+	e_{-5}^+	e_{-5}^-	e_{-5}^-	e_{-5}^-
χ_{11}	8	8	8	0	0	0	-1	-1	0	0	0	e_{-5}^-	e_{-5}^-	e_{-5}^-	e_{-5}^+	e_{-5}^+	e_{-5}^+
χ_{12}	9	9	9	1	1	1	0	0	1	1	1	-1	-1	-1	-1	-1	-1
χ_{13}	9	$-9e_3^+$	$-9e_3^-$	$-e_3^+$	$-e_3^-$	1	0	0	$-e_3^+$	$-e_3^-$	1	e_3^-	-1	e_3^+	-1	e_3^+	e_3^-
χ_{14}	9	$-9e_3^-$	$-9e_3^+$	$-e_3^-$	$-e_3^+$	1	0	0	$-e_3^-$	$-e_3^+$	1	e_3^+	-1	e_3^-	-1	e_3^-	e_3^+
χ_{15}	10	10	10	-2	-2	-2	1	1	0	0	0	0	0	0	0	0	0
χ_{16}	15	$-15e_3^+$	$-15e_3^-$	e_3^+	e_3^-	-1	0	0	e_3^+	e_3^-	-1	0	0	0	0	0	0
χ_{17}	15	$-15e_3^-$	$-15e_3^+$	e_3^-	e_3^+	-1	0	0	e_3^-	e_3^+	-1	0	0	0	0	0	0

TABLE 6. Character table for the centralizer of class 3A. Here, $\alpha = -e^{2\pi i \frac{2}{15}} - e^{2\pi i \frac{8}{15}}$ and $\beta = -e^{2\pi i \frac{1}{15}} - e^{2\pi i \frac{4}{15}}$.

Class 3B

$C_{M_{24}}(3B) \cong \mathbb{Z}_3 \times PSL_3(2)$ Order: $2^3 \cdot 3^2 \cdot 7 = 504$ Linear rep.

	504	504	504	9	9	9	21	21	21	21	21	21	24	24	24	12	12	12
	$1A_1$	$3B_1$	$3B_2$	$3A_1$	$3B_3$	$3B_4$	$7A_1$	$21A_1$	$21A_2$	$7B_1$	$21B_1$	$21B_2$	$2B_1$	$6B_1$	$6B_2$	$4C_1$	$12B_1$	$12B_2$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	$-e_3^+$	$-e_3^-$	1	$-e_3^+$	$-e_3^-$	1	$-e_3^+$	$-e_3^-$	1	$-e_3^+$	$-e_3^-$	1	$-e_3^+$	$-e_3^-$	1	$-e_3^+$	$-e_3^-$
χ_3	1	$-e_3^-$	$-e_3^+$	1	$-e_3^-$	$-e_3^+$	1	$-e_3^-$	$-e_3^+$	1	$-e_3^-$	$-e_3^+$	1	$-e_3^-$	$-e_3^+$	1	$-e_3^-$	$-e_3^+$
χ_4	3	3	3	0	0	0	$-e_7^+$	$-e_7^+$	$-e_7^+$	$-e_7^-$	$-e_7^-$	$-e_7^-$	-1	-1	-1	1	1	1
χ_5	3	3	3	0	0	0	$-e_7^-$	$-e_7^-$	$-e_7^-$	$-e_7^+$	$-e_7^+$	$-e_7^+$	-1	-1	-1	1	1	1
χ_6	3	$-3e_3^+$	$-3e_3^-$	0	0	0	$-e_7^+$	α	β	$-e_7^-$	$\bar{\beta}$	$\bar{\alpha}$	-1	e_3^+	e_3^-	1	$-e_3^+$	$-e_3^-$
χ_7	3	$-3e_3^+$	$-3e_3^-$	0	0	0	$-e_7^-$	$\bar{\beta}$	$\bar{\alpha}$	$-e_7^+$	α	β	-1	e_3^+	e_3^-	1	$-e_3^+$	$-e_3^-$
χ_8	3	$-3e_3^-$	$-3e_3^+$	0	0	0	$-e_7^+$	β	α	$-e_7^-$	$\bar{\alpha}$	$\bar{\beta}$	-1	e_3^-	e_3^+	1	$-e_3^-$	$-e_3^+$
χ_9	3	$-3e_3^-$	$-3e_3^+$	0	0	0	$-e_7^-$	$\bar{\alpha}$	$\bar{\beta}$	$-e_7^+$	β	α	-1	e_3^-	e_3^+	1	$-e_3^-$	$-e_3^+$
χ_{10}	6	6	6	0	0	0	-1	-1	-1	-1	-1	-1	2	2	2	0	0	0
χ_{11}	6	$-6e_3^+$	$-6e_3^-$	0	0	0	-1	e_3^+	e_3^-	-1	e_3^+	e_3^-	2	$-2e_3^+$	$-2e_3^-$	0	0	0
χ_{12}	6	$-6e_3^-$	$-6e_3^+$	0	0	0	-1	e_3^-	e_3^+	-1	e_3^-	e_3^+	2	$-2e_3^-$	$-2e_3^+$	0	0	0
χ_{13}	7	7	7	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1
χ_{14}	7	$-7e_3^+$	$-7e_3^-$	1	$-e_3^+$	$-e_3^-$	0	0	0	0	0	0	-1	e_3^+	e_3^-	-1	e_3^+	e_3^-
χ_{15}	7	$-7e_3^-$	$-7e_3^+$	1	$-e_3^-$	$-e_3^+$	0	0	0	0	0	0	-1	e_3^-	e_3^+	-1	e_3^-	e_3^+
χ_{16}	8	8	8	-1	-1	-1	1	1	1	1	1	1	0	0	0	0	0	0
χ_{17}	8	$-8e_3^+$	$-8e_3^-$	-1	e_3^+	e_3^-	1	$-e_3^+$	$-e_3^-$	1	$-e_3^+$	$-e_3^-$	0	0	0	0	0	0
χ_{18}	8	$-8e_3^-$	$-8e_3^+$	-1	e_3^-	e_3^+	1	$-e_3^-$	$-e_3^+$	1	$-e_3^-$	$-e_3^+$	0	0	0	0	0	0

TABLE 7. Character table for the centralizer of class 3B. Here, $\alpha = e^{2\pi i \frac{2}{21}} + e^{2\pi i \frac{8}{21}} + e^{2\pi i \frac{11}{21}}$ and $\beta = e^{2\pi i \frac{1}{21}} + e^{2\pi i \frac{4}{21}} + e^{2\pi i \frac{16}{21}}$.

Class 4A

$C_{M_{24}}(4A)$ Order: $2^7 \cdot 3 = 384$

Genuinely projective rep. Central extension: $\mathbb{Z}_2.C_{M_{24}}(4A)$

	384	32	64	16	12	64	64	32	16	16	12	32	64	16	32	12	16	12	32	16	64	32	384	64	384	384
	$1A_1$	$2A_1$	$2A_2$	$4B_1$	$3A_1$	$2B_3$	$2A_3$	$2B_2$	$4C_1$	$4A_1$	$6A_1$	$4A_2$	$4A_3$	$4B_2$	$2B_1$	$12A_1$	$4C_2$	$12A_2$	$4B_3$	$4A_4$	$4A_5$	$4B_4$	$4A_6$	$4A_7$	$4A_8$	$2A_4$
χ_1	2	0	0	$\sqrt{2}$	-1	-2	0	0	0	$i\sqrt{2}$	1	0	0	$\sqrt{2}$	0	i	0	$-i$	0	$i\sqrt{2}$	$-2i$	0	$-2i$	0	$2i$	-2
χ_2	2	0	0	$-\sqrt{2}$	-1	-2	0	0	0	$-i\sqrt{2}$	1	0	0	$-\sqrt{2}$	0	i	0	$-i$	0	$-i\sqrt{2}$	$-2i$	0	$-2i$	0	$2i$	-2
χ_3	2	0	0	$\sqrt{2}$	-1	-2	0	0	0	$i\sqrt{2}$	1	0	0	$-\sqrt{2}$	0	$-i$	0	i	0	$-i\sqrt{2}$	$2i$	0	$2i$	0	$-2i$	-2
χ_4	2	0	0	$-\sqrt{2}$	-1	-2	0	0	0	$-i\sqrt{2}$	1	0	0	$\sqrt{2}$	0	$-i$	0	i	0	$i\sqrt{2}$	$2i$	0	$2i$	0	$-2i$	-2
χ_5	4	0	0	0	1	-4	0	0	0	0	-1	0	0	0	0	$-i$	0	i	0	0	$-4i$	0	$-4i$	0	$4i$	-4
χ_6	4	0	0	0	1	-4	0	0	0	0	-1	0	0	0	0	i	0	$-i$	0	0	$4i$	0	$4i$	0	$-4i$	-4
χ_7	4	0	0	0	1	0	0	0	-2	0	1	0	0	0	0	-1	$-2i$	-1	0	0	0	0	-4	0	-4	4
χ_8	4	0	0	0	1	0	0	0	-2	0	1	0	0	0	0	1	$2i$	1	0	0	0	0	4	0	4	4
χ_9	6	0	0	$-\sqrt{2}$	0	2	0	0	0	$i\sqrt{2}$	0	0	0	$\sqrt{2}$	0	0	0	0	0	$-i\sqrt{2}$	$2i$	0	$-6i$	0	$6i$	-6
χ_{10}	6	0	0	$\sqrt{2}$	0	2	0	0	0	$-i\sqrt{2}$	0	0	0	$-\sqrt{2}$	0	0	0	0	0	$i\sqrt{2}$	$2i$	0	$-6i$	0	$6i$	-6
χ_{11}	4	0	0	0	1	0	0	0	2	0	1	0	0	0	0	-1	$2i$	-1	0	0	0	0	-4	0	-4	4
χ_{12}	4	0	0	0	1	0	0	0	2	0	1	0	0	0	0	1	$-2i$	1	0	0	0	0	4	0	4	4
χ_{13}	6	0	0	$-\sqrt{2}$	0	2	0	0	0	$i\sqrt{2}$	0	0	0	$-\sqrt{2}$	0	0	0	0	0	$i\sqrt{2}$	$-2i$	0	$6i$	0	$-6i$	-6
χ_{14}	6	0	0	$\sqrt{2}$	0	2	0	0	0	$-i\sqrt{2}$	0	0	0	$\sqrt{2}$	0	0	0	0	0	$-i\sqrt{2}$	$-2i$	0	$6i$	0	$-6i$	-6
χ_{15}	8	0	0	0	-1	0	0	0	0	0	-1	0	0	0	0	-1	0	-1	0	0	0	0	8	0	8	8
χ_{16}	8	0	0	0	-1	0	0	0	0	0	-1	0	0	0	0	1	0	1	0	0	0	0	-8	0	-8	8

TABLE 8. Projective character table for the centralizer of class 4A.

Class 4B

$C_{M_{24}}(4B) \cong ((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ Order: 128

Genuinely projective rep. Central extension: $\mathbb{Z}_2.C_{M_{24}}(4B)$

	128	16	128	128	64	64	128	16	16	16	16	32	16	16	32	32	32	16	16	16	16	64	16	64	64	64
	1A	$8A_1$	$4B_2$	$4B_6$	$2B_3$	$4B_1$	$2A_1$	$2B_1$	$4B_3$	$2A_3$	$4B_5$	$2B_2$	$8A_2$	$8A_3$	$4A_2$	$4A_1$	$2A_4$	$4B_7$	$4B_4$	$4A_3$	$4A_4$	$2A_2$	$8A_4$	$4B_9$	$2A_5$	$4B_8$
χ_1	2	-2	-2	-2	2	2	2	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	0	0	0	0	0
χ_2	2	-2	-2	-2	2	2	2	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	0	0	0	0	0
χ_3	2	0	-2	-2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	2	0	0	0
χ_4	2	0	-2	-2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	2	0	0	0
χ_5	2	0	-2	-2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	0	-2	0	0	0
χ_6	2	0	-2	-2	-2	-2	2	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	0	-2	0	0	0
χ_7	2	2	-2	-2	2	2	2	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0	0
χ_8	2	2	-2	-2	2	2	2	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0	0
χ_9	4	0	4	4	-4	4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	4	0	4	4	4	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{11}	4	0	$4i$	$-4i$	0	0	-4	0	0	0	0	0	$2i$	2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{12}	4	0	$4i$	$-4i$	0	0	-4	0	0	0	0	0	$-2i$	-2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{13}	4	0	$-4i$	$4i$	0	0	-4	0	0	0	0	0	2	$2i$	0	0	0	0	0	0	0	0	0	0	0	0
χ_{14}	4	0	$-4i$	$4i$	0	0	-4	0	0	0	0	0	-2	$-2i$	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 9. Projective character table for the centralizer of class 4B

Class 4C

$C_{M_{24}}(4C) \cong \mathbb{Z}_4 \times S_4$ Order: $2^5 \cdot 3 = 96$

Genuinely projective rep. Central extension: $\mathbb{Z}_2.C_{M_{24}}(4C) \cong \mathbb{Z}_4 \times \mathbb{Z}_2.S_4$

	96	16	12	16	32	96	16	12	16	32	96	16	12	16	32	96	16	12	16	32
	1A	$2A_1$	$3B_1$	$4A_1$	$2A_2$	$4C_1$	$4C_3$	$12B_1$	$4C_7$	$4C_4$	$4C_2$	$4C_5$	$12B_2$	$4C_8$	$4C_6$	$2B_1$	$2B_2$	$6B_1$	$4A_2$	$2B_3$
χ_1	2	0	-1	$\sqrt{2}$	0	2	0	-1	$i\sqrt{2}$	0	2	0	-1	$i\sqrt{2}$	0	2	0	-1	$\sqrt{2}$	0
χ_2	2	0	-1	$-\sqrt{2}$	0	2	0	-1	$-i\sqrt{2}$	0	2	0	-1	$-i\sqrt{2}$	0	2	0	-1	$-\sqrt{2}$	0
χ_3	2	0	-1	$\sqrt{2}$	0	-2	0	1	$-i\sqrt{2}$	0	-2	0	1	$-i\sqrt{2}$	0	2	0	-1	$\sqrt{2}$	0
χ_4	2	0	-1	$-\sqrt{2}$	0	-2	0	1	$i\sqrt{2}$	0	-2	0	1	$i\sqrt{2}$	0	2	0	-1	$-\sqrt{2}$	0
χ_5	2	0	-1	$\sqrt{2}$	0	$-2i$	0	i	$\sqrt{2}$	0	$2i$	0	$-i$	$-\sqrt{2}$	0	-2	0	1	$-\sqrt{2}$	0
χ_6	2	0	-1	$-\sqrt{2}$	0	$-2i$	0	i	$-\sqrt{2}$	0	$2i$	0	$-i$	$\sqrt{2}$	0	-2	0	1	$\sqrt{2}$	0
χ_7	2	0	-1	$\sqrt{2}$	0	$2i$	0	$-i$	$-\sqrt{2}$	0	$-2i$	0	i	$\sqrt{2}$	0	-2	0	1	$-\sqrt{2}$	0
χ_8	2	0	-1	$-\sqrt{2}$	0	$2i$	0	$-i$	$\sqrt{2}$	0	$-2i$	0	i	$-\sqrt{2}$	0	-2	0	1	$\sqrt{2}$	0
χ_9	4	0	1	0	0	4	0	1	0	0	4	0	1	0	0	4	0	1	0	0
χ_{10}	4	0	1	0	0	-4	0	-1	0	0	-4	0	-1	0	0	4	0	1	0	0
χ_{11}	4	0	1	0	0	$-4i$	0	$-i$	0	0	$4i$	0	i	0	0	-4	0	-1	0	0
χ_{12}	4	0	1	0	0	$4i$	0	i	0	0	$-4i$	0	$-i$	0	0	-4	0	-1	0	0

TABLE 10. Projective character table for the centralizer of class 4C.

Class 5A

$C_{M_{24}}(5A) \cong \mathbb{Z}_5 \times A_4$ Order: $2^2 \cdot 3 \cdot 5 = 60$ Linear rep.

	60	15	15	20	15	15	15	15	20	20	15	15	60	20	20	15	15	60	60	60
	1A	$3A_1$	$3A_2$	$10A_1$	$15A_1$	$15B_1$	$15A_2$	$15B_2$	$10A_2$	$10A_3$	$15B_3$	$15A_3$	$5A_1$	$2B_1$	$10A_4$	$15A_4$	$15B_4$	$5A_2$	$5A_3$	$5A_4$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	ε_5^4	ε_5^4	ε_5^4	ε_5^2	ε_5^2	ε_5^2	ε_5	ε_5	ε_5	ε_5^2	1	ε_5^3	ε_5^3	ε_5^3	ε_5^4	ε_5^3	ε_5
χ_3	1	1	1	ε_5^3	ε_5^3	ε_5^3	ε_5^4	ε_5^4	ε_5^4	ε_5^2	ε_5^2	ε_5^2	ε_5^4	1	ε_5	ε_5	ε_5	ε_5^3	ε_5	ε_5^2
χ_4	1	1	1	ε_5^2	ε_5^2	ε_5^2	ε_5	ε_5	ε_5	ε_5^3	ε_5^3	ε_5^3	ε_5	1	ε_5^4	ε_5^4	ε_5^4	ε_5^2	ε_5^4	ε_5^3
χ_5	1	1	1	ε_5	ε_5	ε_5	ε_5^3	ε_5^3	ε_5^3	ε_5^4	ε_5^4	ε_5^4	ε_5^3	1	ε_5^2	ε_5^2	ε_5^2	ε_5	ε_5^2	ε_5^4
χ_6	1	$-e_3^+$	$-e_3^-$	1	$-e_3^+$	$-e_3^-$	$-e_3^-$	$-e_3^+$	1	1	$-e_3^-$	$-e_3^+$	1	1	1	$-e_3^-$	$-e_3^+$	1	1	1
χ_7	1	$-e_3^-$	$-e_3^+$	1	$-e_3^-$	$-e_3^+$	$-e_3^+$	$-e_3^-$	1	1	$-e_3^+$	$-e_3^-$	1	1	1	$-e_3^+$	$-e_3^-$	1	1	1
χ_8	1	$-e_3^+$	$-e_3^-$	ε_5^4	ε_{15}^7	ε_{15}^2	ε_{15}^{11}	ε_{15}	ε_5^2	ε_5	ε_{15}^8	ε_{15}^{13}	ε_5^2	1	ε_5^3	ε_{15}^{14}	ε_{15}^4	ε_5^4	ε_5^3	ε_5
χ_9	1	$-e_3^+$	$-e_3^-$	ε_5^3	ε_{15}^4	ε_{15}^{14}	ε_{15}^2	ε_{15}^7	ε_5^4	ε_5^2	ε_{15}^{11}	ε_{15}	ε_5^4	1	ε_5	ε_{15}^8	ε_{15}^{13}	ε_5^3	ε_5	ε_5^2
χ_{10}	1	$-e_3^+$	$-e_3^-$	ε_5^2	ε_{15}	ε_{15}^{11}	ε_{15}^8	ε_{15}^{13}	ε_5	ε_5^3	ε_{15}^{14}	ε_{15}^4	ε_5	1	ε_5^4	ε_{15}^2	ε_{15}^7	ε_5^2	ε_5^4	ε_5^3
χ_{11}	1	$-e_3^+$	$-e_3^-$	ε_5	ε_{15}^{13}	ε_{15}^8	ε_{15}^{14}	ε_{15}^4	ε_5^3	ε_5^4	ε_{15}^2	ε_{15}^7	ε_5^3	1	ε_5^2	ε_{15}^{11}	ε_{15}	ε_5	ε_5^2	ε_5^4
χ_{12}	1	$-e_3^-$	$-e_3^+$	ε_5^4	ε_{15}^2	ε_{15}^7	ε_{15}	ε_{15}^{11}	ε_5^2	ε_5	ε_{15}^{13}	ε_{15}^8	ε_5^2	1	ε_5^3	ε_{15}^4	ε_{15}^{14}	ε_5^4	ε_5^3	ε_5
χ_{13}	1	$-e_3^-$	$-e_3^+$	ε_5^3	ε_{15}^{14}	ε_{15}^4	ε_{15}^7	ε_{15}^2	ε_5^4	ε_5^2	ε_{15}	ε_{15}^{11}	ε_5^4	1	ε_5	ε_{15}^{13}	ε_{15}^8	ε_5^3	ε_5	ε_5^2
χ_{14}	1	$-e_3^-$	$-e_3^+$	ε_5^2	ε_{15}^{11}	ε_{15}	ε_{15}^{13}	ε_{15}^8	ε_5	ε_5^3	ε_{15}^4	ε_{15}^{14}	ε_5	1	ε_5^4	ε_{15}^7	ε_{15}^2	ε_5^2	ε_5^4	ε_5^3
χ_{15}	1	$-e_3^-$	$-e_3^+$	ε_5	ε_{15}^8	ε_{15}^{13}	ε_{15}^4	ε_{15}^{14}	ε_5^3	ε_5^4	ε_{15}^7	ε_{15}^2	ε_5^3	1	ε_5^2	ε_{15}	ε_{15}^{11}	ε_5	ε_5^2	ε_5^4
χ_{16}	3	0	0	-1	0	0	0	0	-1	-1	0	0	3	-1	-1	0	0	3	3	3
χ_{17}	3	0	0	$-\varepsilon_5^2$	0	0	0	0	$-\varepsilon_5$	$-\varepsilon_5^3$	0	0	$3\varepsilon_5$	-1	$-\varepsilon_5^4$	0	0	$3\varepsilon_5^2$	$3\varepsilon_5^4$	$3\varepsilon_5^3$
χ_{18}	3	0	0	$-\varepsilon_5$	0	0	0	0	$-\varepsilon_5^3$	$-\varepsilon_5^4$	0	0	$3\varepsilon_5^3$	-1	$-\varepsilon_5^2$	0	0	$3\varepsilon_5$	$3\varepsilon_5^2$	$3\varepsilon_5^4$
χ_{19}	3	0	0	$-\varepsilon_5^4$	0	0	0	0	$-\varepsilon_5^2$	$-\varepsilon_5$	0	0	$3\varepsilon_5^2$	-1	$-\varepsilon_5^3$	0	0	$3\varepsilon_5^4$	$3\varepsilon_5^3$	$3\varepsilon_5$
χ_{20}	3	0	0	$-\varepsilon_5^3$	0	0	0	0	$-\varepsilon_5^4$	$-\varepsilon_5^2$	0	0	$3\varepsilon_5^4$	-1	$-\varepsilon_5$	0	0	$3\varepsilon_5^3$	$3\varepsilon_5$	$3\varepsilon_5^2$

TABLE 11. Character table for the centralizer of class 5A.

Classes 6A and 6B

$C_{M_{24}}(6A) \cong C_{M_{24}}(6B) \cong \mathbb{Z}_6.\mathbb{Z}_2^2 \cong \mathbb{Z}_3 \times D_8$ Order: 24 Linear rep.

	24	12	12	24	24	12	12	12	12	12	12	12	24	24	24
$C_{M_{24}}(6A)$	1A	$2A_1$	$6A_1$	$3A_1$	$3A_2$	$6A_2$	$2A_2$	$4A_1$	$12A_1$	$6A_3$	$12A_2$	$6A_4$	$6A_5$	$2A_3$	$6A_6$
$C_{M_{24}}(6B)$	1A	$2B_1$	$6B_2$	$3B_2$	$3B_1$	$6B_5$	$2B_2$	$4C_1$	$12B_1$	$6B_3$	$12B_2$	$6B_1$	$6B_4$	$2B_3$	$6B_6$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	-1	-1	1	1	-1	1	-1	1	1	1
χ_3	1	-1	-1	1	1	-1	1	-1	-1	1	-1	1	1	1	1
χ_4	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1
χ_5	1	-1	e_3^-	$-e_3^-$	$-e_3^+$	e_3^+	-1	1	$-e_3^+$	e_3^+	$-e_3^-$	e_3^-	$-e_3^+$	1	$-e_3^-$
χ_6	1	-1	e_3^+	$-e_3^+$	$-e_3^-$	e_3^-	-1	1	$-e_3^-$	e_3^-	$-e_3^+$	e_3^+	$-e_3^-$	1	$-e_3^+$
χ_7	1	-1	e_3^-	$-e_3^-$	$-e_3^+$	e_3^+	1	-1	e_3^+	$-e_3^+$	e_3^-	$-e_3^-$	$-e_3^+$	1	$-e_3^-$
χ_8	1	-1	e_3^+	$-e_3^+$	$-e_3^-$	e_3^-	1	-1	e_3^-	$-e_3^-$	e_3^+	$-e_3^+$	$-e_3^-$	1	$-e_3^+$
χ_9	1	1	$-e_3^+$	$-e_3^+$	$-e_3^-$	$-e_3^-$	-1	-1	e_3^-	e_3^-	e_3^+	e_3^+	$-e_3^-$	1	$-e_3^+$
χ_{10}	1	1	$-e_3^-$	$-e_3^-$	$-e_3^+$	$-e_3^+$	-1	-1	e_3^+	e_3^+	e_3^-	e_3^-	$-e_3^+$	1	$-e_3^-$
χ_{11}	1	1	$-e_3^+$	$-e_3^+$	$-e_3^-$	$-e_3^-$	1	1	$-e_3^-$	$-e_3^-$	$-e_3^+$	$-e_3^+$	$-e_3^-$	1	$-e_3^+$
χ_{12}	1	1	$-e_3^-$	$-e_3^-$	$-e_3^+$	$-e_3^+$	1	1	$-e_3^+$	$-e_3^+$	$-e_3^-$	$-e_3^-$	$-e_3^+$	1	$-e_3^-$
χ_{13}	2	0	0	2	2	0	0	0	0	0	0	0	-2	-2	-2
χ_{14}	2	0	0	$-2e_3^+$	$-2e_3^-$	0	0	0	0	0	0	0	$2e_3^-$	-2	$2e_3^+$
χ_{15}	2	0	0	$-2e_3^-$	$-2e_3^+$	0	0	0	0	0	0	0	$2e_3^+$	-2	$2e_3^-$

TABLE 12. Character table for the centralizer of classes 6AB. NOTE: There is an accidental isomorphism $C_{M_{24}}(6A) \cong C_{M_{24}}(6B)$, therefore the character table is given only once. However, following our conventions, we give different names to the conjugacy classes depending on whether the group is $C_{M_{24}}(6A)$ or $C_{M_{24}}(6B)$.

Classes 7A and 7B

$C_{M_{24}}(7A) \cong C_{M_{24}}(7B) \cong S_3 \times \mathbb{Z}_7$ Order: $2 \cdot 3 \cdot 7 = 42$ Linear rep.

	42	14	14	21	42	14	21	14	21	21	14	14	42	42	42	42	14	21	21	42	21
$C_{M_{24}}(7A)$	1A	$2A_1$	$14B_1$	$21B_1$	$7B_1$	$14B_2$	$21B_2$	$14A_1$	$21A_1$	$21A_2$	$14A_2$	$14A_3$	$7A_1$	$7A_2$	$7A_3$	$7B_2$	$14B_3$	$3B_1$	$21B_3$	$7B_3$	$21A_3$
$C_{M_{24}}(7B)$	1A	$2A_1$	$14A_1$	$21A_1$	$7A_1$	$14A_2$	$21A_2$	$14B_1$	$21B_1$	$21B_2$	$14B_2$	$14B_3$	$7B_1$	$7B_2$	$7B_3$	$7A_2$	$14A_3$	$3B_1$	$21A_3$	$7A_3$	$21B_3$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	-1	1	-1	1	1	-1	-1	1	1	1	1	-1	1	1	1	1
χ_3	1	-1	$-\varepsilon_7$	ε_7	ε_7	$-\varepsilon_7^4$	ε_7^4	$-\varepsilon_7^3$	ε_7^3	ε_7^6	$-\varepsilon_7^6$	$-\varepsilon_7^5$	ε_7^5	ε_7^3	ε_7^6	ε_7^2	$-\varepsilon_7^2$	1	ε_7^2	ε_7^4	ε_7^5
χ_4	1	-1	$-\varepsilon_7^2$	ε_7^2	ε_7^2	$-\varepsilon_7$	ε_7	$-\varepsilon_7^6$	ε_7^6	ε_7^5	$-\varepsilon_7^5$	$-\varepsilon_7^3$	ε_7^3	ε_7^6	ε_7^5	ε_7^4	$-\varepsilon_7^4$	1	ε_7^4	ε_7	ε_7^3
χ_5	1	-1	$-\varepsilon_7^3$	ε_7^3	ε_7^3	$-\varepsilon_7^5$	ε_7^5	$-\varepsilon_7^2$	ε_7^2	ε_7^4	$-\varepsilon_7^4$	$-\varepsilon_7$	ε_7	ε_7^2	ε_7^4	ε_7^6	$-\varepsilon_7^6$	1	ε_7^6	ε_7^5	ε_7
χ_6	1	-1	$-\varepsilon_7^4$	ε_7^4	ε_7^4	$-\varepsilon_7^2$	ε_7^2	$-\varepsilon_7^5$	ε_7^5	ε_7^3	$-\varepsilon_7^3$	$-\varepsilon_7^6$	ε_7^6	ε_7^5	ε_7^3	ε_7	$-\varepsilon_7$	1	ε_7	ε_7^2	ε_7^6
χ_7	1	-1	$-\varepsilon_7^5$	ε_7^5	ε_7^5	$-\varepsilon_7^6$	ε_7^6	$-\varepsilon_7$	ε_7	ε_7^2	$-\varepsilon_7^2$	$-\varepsilon_7^4$	ε_7^4	ε_7	ε_7^2	ε_7^3	$-\varepsilon_7^3$	1	ε_7^3	ε_7^6	ε_7^4
χ_8	1	-1	$-\varepsilon_7^6$	ε_7^6	ε_7^6	$-\varepsilon_7^3$	ε_7^3	$-\varepsilon_7^4$	ε_7^4	ε_7	$-\varepsilon_7$	$-\varepsilon_7^2$	ε_7^2	ε_7^4	ε_7	ε_7^5	$-\varepsilon_7^5$	1	ε_7^5	ε_7^3	ε_7^2
χ_9	1	1	ε_7^6	ε_7^6	ε_7^6	ε_7^3	ε_7^3	ε_7^4	ε_7^4	ε_7	ε_7	ε_7^2	ε_7^2	ε_7^4	ε_7	ε_7^5	ε_7^5	1	ε_7^5	ε_7^3	ε_7^2
χ_{10}	1	1	ε_7^5	ε_7^5	ε_7^5	ε_7^6	ε_7^6	ε_7	ε_7	ε_7^2	ε_7^2	ε_7^4	ε_7^4	ε_7	ε_7^2	ε_7^3	ε_7^3	1	ε_7^3	ε_7^6	ε_7^4
χ_{11}	1	1	ε_7^4	ε_7^4	ε_7^4	ε_7^2	ε_7^2	ε_7^5	ε_7^5	ε_7^3	ε_7^3	ε_7^6	ε_7^6	ε_7^5	ε_7^3	ε_7	ε_7	1	ε_7	ε_7^2	ε_7^6
χ_{12}	1	1	ε_7^3	ε_7^3	ε_7^3	ε_7^5	ε_7^5	ε_7^2	ε_7^2	ε_7^4	ε_7^4	ε_7	ε_7	ε_7^2	ε_7^4	ε_7^6	ε_7^6	1	ε_7^6	ε_7^5	ε_7
χ_{13}	1	1	ε_7^2	ε_7^2	ε_7^2	ε_7	ε_7	ε_7^6	ε_7^6	ε_7^5	ε_7^5	ε_7^3	ε_7^3	ε_7^6	ε_7^5	ε_7^4	ε_7^4	1	ε_7^4	ε_7	ε_7^3
χ_{14}	1	1	ε_7	ε_7	ε_7	ε_7^4	ε_7^4	ε_7^3	ε_7^3	ε_7^6	ε_7^6	ε_7^5	ε_7^5	ε_7^3	ε_7^6	ε_7^2	ε_7^2	1	ε_7^2	ε_7^4	ε_7^5
χ_{15}	2	0	0	-1	2	0	-1	0	-1	-1	0	0	2	2	2	2	0	-1	-1	2	-1
χ_{16}	2	0	0	$-\varepsilon_7^3$	$2\varepsilon_7^3$	0	$-\varepsilon_7^5$	0	$-\varepsilon_7^2$	$-\varepsilon_7^4$	0	0	$2\varepsilon_7$	$2\varepsilon_7^2$	$2\varepsilon_7^4$	$2\varepsilon_7^6$	0	-1	$-\varepsilon_7^6$	$2\varepsilon_7^5$	$-\varepsilon_7$
χ_{17}	2	0	0	$-\varepsilon_7^2$	$2\varepsilon_7^2$	0	$-\varepsilon_7$	0	$-\varepsilon_7^6$	$-\varepsilon_7^5$	0	0	$2\varepsilon_7^3$	$2\varepsilon_7^6$	$2\varepsilon_7^5$	$2\varepsilon_7^4$	0	-1	$-\varepsilon_7^4$	$2\varepsilon_7$	$-\varepsilon_7^3$
χ_{18}	2	0	0	$-\varepsilon_7$	$2\varepsilon_7$	0	$-\varepsilon_7^4$	0	$-\varepsilon_7^3$	$-\varepsilon_7^6$	0	0	$2\varepsilon_7^5$	$2\varepsilon_7^3$	$2\varepsilon_7^6$	$2\varepsilon_7^2$	0	-1	$-\varepsilon_7^2$	$2\varepsilon_7^4$	$-\varepsilon_7^5$
χ_{19}	2	0	0	$-\varepsilon_7^6$	$2\varepsilon_7^6$	0	$-\varepsilon_7^3$	0	$-\varepsilon_7^4$	$-\varepsilon_7$	0	0	$2\varepsilon_7^2$	$2\varepsilon_7^4$	$2\varepsilon_7$	$2\varepsilon_7^5$	0	-1	$-\varepsilon_7^5$	$2\varepsilon_7^3$	$-\varepsilon_7^2$
χ_{20}	2	0	0	$-\varepsilon_7^5$	$2\varepsilon_7^5$	0	$-\varepsilon_7^6$	0	$-\varepsilon_7$	$-\varepsilon_7^2$	0	0	$2\varepsilon_7^4$	$2\varepsilon_7$	$2\varepsilon_7^2$	$2\varepsilon_7^3$	0	-1	$-\varepsilon_7^3$	$2\varepsilon_7^6$	$-\varepsilon_7^4$
χ_{21}	2	0	0	$-\varepsilon_7^4$	$2\varepsilon_7^4$	0	$-\varepsilon_7^2$	0	$-\varepsilon_7^5$	$-\varepsilon_7^3$	0	0	$2\varepsilon_7^6$	$2\varepsilon_7^5$	$2\varepsilon_7^3$	$2\varepsilon_7$	0	-1	$-\varepsilon_7$	$2\varepsilon_7^2$	$-\varepsilon_7^6$

TABLE 13. Character table for the centralizer of classes 7AB.

Class 8A

$C_{M_{24}}(8A) \cong \mathbb{Z}_2 \times \mathbb{Z}_8$ Order: $2^4 = 16$ Linear rep.

	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
	1A	$8A_1$	$8A_2$	$8A_3$	$4B_1$	$8A_4$	$4B_2$	$8A_5$	$8A_6$	$4B_3$	$8A_7$	$4B_4$	$8A_8$	$2A_1$	$2B_1$	$2B_2$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	1	1
χ_3	1	-1	1	-1	-1	-1	1	-1	1	-1	1	1	1	1	-1	-1
χ_4	1	1	-1	1	-1	1	1	1	-1	-1	-1	1	-1	1	-1	-1
χ_5	1	$-i$	$-i$	i	-1	i	-1	$-i$	i	-1	i	-1	$-i$	1	1	1
χ_6	1	i	i	$-i$	-1	$-i$	-1	i	$-i$	-1	$-i$	-1	i	1	1	1
χ_7	1	$-i$	i	i	1	i	-1	$-i$	$-i$	1	$-i$	-1	i	1	-1	-1
χ_8	1	i	$-i$	$-i$	1	$-i$	-1	i	i	1	i	-1	$-i$	1	-1	-1
χ_9	1	$-\varepsilon_8$	$-\varepsilon_8$	ε_8^7	$-i$	$-\varepsilon_8^7$	i	ε_8	$-\varepsilon_8^7$	i	ε_8^7	$-i$	ε_8	-1	-1	1
χ_{10}	1	ε_8^7	ε_8^7	$-\varepsilon_8$	i	ε_8	$-i$	$-\varepsilon_8^7$	ε_8	$-i$	$-\varepsilon_8$	i	$-\varepsilon_8^7$	-1	-1	1
χ_{11}	1	$-\varepsilon_8^7$	$-\varepsilon_8^7$	ε_8	i	$-\varepsilon_8$	$-i$	ε_8^7	$-\varepsilon_8$	$-i$	ε_8	i	ε_8^7	-1	-1	1
χ_{12}	1	ε_8	ε_8	$-\varepsilon_8^7$	$-i$	ε_8^7	i	$-\varepsilon_8$	ε_8^7	i	$-\varepsilon_8^7$	$-i$	$-\varepsilon_8$	-1	-1	1
χ_{13}	1	$-\varepsilon_8$	ε_8	ε_8^7	i	$-\varepsilon_8^7$	i	ε_8	ε_8^7	$-i$	$-\varepsilon_8^7$	$-i$	$-\varepsilon_8$	-1	1	-1
χ_{14}	1	ε_8^7	$-\varepsilon_8^7$	$-\varepsilon_8$	$-i$	ε_8	$-i$	$-\varepsilon_8^7$	$-\varepsilon_8$	i	ε_8	i	ε_8^7	-1	1	-1
χ_{15}	1	$-\varepsilon_8^7$	ε_8^7	ε_8	$-i$	$-\varepsilon_8$	$-i$	ε_8^7	ε_8	i	$-\varepsilon_8$	i	$-\varepsilon_8^7$	-1	1	-1
χ_{16}	1	ε_8	$-\varepsilon_8$	$-\varepsilon_8^7$	i	ε_8^7	i	$-\varepsilon_8$	$-\varepsilon_8^7$	$-i$	ε_8^7	$-i$	ε_8	-1	1	-1

TABLE 14. Character table for the centralizer of class 8A.

Class 10A

$C_{M_{24}}(10A) \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$ Order: 20 Linear rep.

	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20
	1A	$10A_1$	$10A_2$	$10A_3$	$5A_1$	$10A_4$	$2B_1$	$2B_2$	$10A_5$	$10A_6$	$10A_7$	$10A_8$	$5A_2$	$5A_3$	$10A_9$	$10A_{10}$	$2B_3$	$10A_{11}$	$5A_4$	$10A_{12}$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	1	1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	1
χ_3	1	1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1
χ_4	1	-1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	1	1	-1	1	1	-1
χ_5	1	ε_5^3	ε_5^4	ε_5^2	ε_5^4	ε_5^2	1	1	ε_5	ε_5^4	ε_5^2	ε_5^3	ε_5^3	ε_5	ε_5	ε_5^3	1	ε_5^4	ε_5^2	ε_5
χ_6	1	$-\varepsilon_5^3$	$-\varepsilon_5^4$	$-\varepsilon_5^2$	ε_5^4	ε_5^2	1	-1	$-\varepsilon_5$	ε_5^4	$-\varepsilon_5^2$	ε_5^3	ε_5^3	ε_5	$-\varepsilon_5$	$-\varepsilon_5^3$	-1	$-\varepsilon_5^4$	ε_5^2	ε_5
χ_7	1	ε_5^3	ε_5^4	ε_5^2	ε_5^4	$-\varepsilon_5^2$	-1	-1	ε_5	$-\varepsilon_5^4$	$-\varepsilon_5^2$	$-\varepsilon_5^3$	ε_5^3	ε_5	$-\varepsilon_5$	$-\varepsilon_5^3$	1	$-\varepsilon_5^4$	ε_5^2	$-\varepsilon_5$
χ_8	1	$-\varepsilon_5^3$	$-\varepsilon_5^4$	$-\varepsilon_5^2$	ε_5^4	$-\varepsilon_5^2$	-1	1	$-\varepsilon_5$	$-\varepsilon_5^4$	ε_5^2	$-\varepsilon_5^3$	ε_5^3	ε_5	ε_5	ε_5^3	-1	ε_5^4	ε_5^2	$-\varepsilon_5$
χ_9	1	ε_5	ε_5^3	ε_5^4	ε_5^3	ε_5^4	1	1	ε_5^2	ε_5^3	ε_5^4	ε_5	ε_5	ε_5^2	ε_5^2	ε_5	1	ε_5^3	ε_5^4	ε_5^2
χ_{10}	1	$-\varepsilon_5$	$-\varepsilon_5^3$	$-\varepsilon_5^4$	ε_5^3	ε_5^4	1	-1	$-\varepsilon_5^2$	ε_5^3	$-\varepsilon_5^4$	ε_5	ε_5	ε_5^2	$-\varepsilon_5^2$	$-\varepsilon_5$	-1	$-\varepsilon_5^3$	ε_5^4	ε_5^2
χ_{11}	1	ε_5	ε_5^3	ε_5^4	ε_5^3	$-\varepsilon_5^4$	-1	-1	ε_5^2	$-\varepsilon_5^3$	$-\varepsilon_5^4$	$-\varepsilon_5$	ε_5	ε_5^2	$-\varepsilon_5^2$	$-\varepsilon_5$	1	$-\varepsilon_5^3$	ε_5^4	$-\varepsilon_5^2$
χ_{12}	1	$-\varepsilon_5$	$-\varepsilon_5^3$	$-\varepsilon_5^4$	ε_5^3	$-\varepsilon_5^4$	-1	1	$-\varepsilon_5^2$	$-\varepsilon_5^3$	ε_5^4	$-\varepsilon_5$	ε_5	ε_5^2	ε_5^2	ε_5	-1	ε_5^3	ε_5^4	$-\varepsilon_5^2$
χ_{13}	1	ε_5^4	ε_5^2	ε_5	ε_5^2	ε_5	1	1	ε_5^3	ε_5^2	ε_5	ε_5^4	ε_5^4	ε_5^3	ε_5^3	ε_5^4	1	ε_5^2	ε_5	ε_5^3
χ_{14}	1	$-\varepsilon_5^4$	$-\varepsilon_5^2$	$-\varepsilon_5$	ε_5^2	ε_5	1	-1	$-\varepsilon_5^3$	ε_5^2	$-\varepsilon_5$	ε_5^4	ε_5^4	ε_5^3	$-\varepsilon_5^3$	$-\varepsilon_5^4$	-1	$-\varepsilon_5^2$	ε_5	ε_5^3
χ_{15}	1	ε_5^4	ε_5^2	ε_5	ε_5^2	$-\varepsilon_5$	-1	-1	ε_5^3	$-\varepsilon_5^2$	$-\varepsilon_5$	$-\varepsilon_5^4$	ε_5^4	ε_5^3	$-\varepsilon_5^3$	$-\varepsilon_5^4$	1	$-\varepsilon_5^2$	ε_5	$-\varepsilon_5^3$
χ_{16}	1	$-\varepsilon_5^4$	$-\varepsilon_5^2$	$-\varepsilon_5$	ε_5^2	$-\varepsilon_5$	-1	1	$-\varepsilon_5^3$	$-\varepsilon_5^2$	ε_5	$-\varepsilon_5^4$	ε_5^4	ε_5^3	ε_5^3	ε_5^4	-1	ε_5^2	ε_5	$-\varepsilon_5^3$
χ_{17}	1	ε_5^2	ε_5	ε_5^3	ε_5	ε_5^3	1	1	ε_5^4	ε_5	ε_5^3	ε_5^2	ε_5^2	ε_5^4	ε_5^4	ε_5^2	1	ε_5	ε_5^3	ε_5^4
χ_{18}	1	$-\varepsilon_5^2$	$-\varepsilon_5$	$-\varepsilon_5^3$	ε_5	ε_5^3	1	-1	$-\varepsilon_5^4$	ε_5	$-\varepsilon_5^3$	ε_5^2	ε_5^2	ε_5^4	$-\varepsilon_5^4$	$-\varepsilon_5^2$	-1	$-\varepsilon_5$	ε_5^3	ε_5^4
χ_{19}	1	ε_5^2	ε_5	ε_5^3	ε_5	$-\varepsilon_5^3$	-1	-1	ε_5^4	$-\varepsilon_5$	$-\varepsilon_5^3$	$-\varepsilon_5^2$	ε_5^2	ε_5^4	$-\varepsilon_5^4$	$-\varepsilon_5^2$	1	$-\varepsilon_5$	ε_5^3	$-\varepsilon_5^4$
χ_{20}	1	$-\varepsilon_5^2$	$-\varepsilon_5$	$-\varepsilon_5^3$	ε_5	$-\varepsilon_5^3$	-1	1	$-\varepsilon_5^4$	$-\varepsilon_5$	ε_5^3	$-\varepsilon_5^2$	ε_5^2	ε_5^4	ε_5^4	ε_5^2	-1	ε_5	ε_5^3	$-\varepsilon_5^4$

TABLE 15. Character table for the centralizer of class 10A.

Appendix E. Decompositions of twisted sectors

In this section we give the decomposition of the first 20 representation spaces $\mathcal{H}_g^n$ for all non-isomorphic twisted sectors. In each table, the integer in the i'th row and the r'th column is the multiplicity $h_{i,r}$ of the irreducible representation χ_i (numbered as in the corresponding character table) in the representation $\rho_{g,r}$ of eq. (6). When the centraliser $C_{M_{24}}(g)$ is a cyclic group $\mathbb{Z}_N$, $N = o(g)$ (classes 11A, 12A, 12B, 14AB, 15AB, 21AB, 23AB), the multiplicity $h_{i,r}$ of the (projective) irreducible characters χ_i (see eq. (180)) is only non-zero if $Nr + 1/\ell(g) \equiv i \mod N$, where $\ell(g)$ is the length of the shortest cycle of g as a permutation of 24 objects; then, only these values of $h_{i,r}$ are given.

i \ $2r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	4	0	2	0	14	0	18	0	52	0	96	0	204
2	0	0	0	0	0	0	0	0	0	4	0	6	0	14	0	34	0	72	0	132	0
3,4	0	1	0	0	0	0	0	1	0	2	0	6	0	18	0	32	0	72	0	137	0
5,6	0	0	1	0	0	0	2	0	4	0	16	0	28	0	72	0	141	0	306	0	570
7,8	0	0	0	0	0	2	0	2	0	10	0	18	0	50	0	96	0	216	0	408	0
9	0	0	0	0	2	0	2	0	12	0	22	0	68	0	130	0	304	0	578	0	1184
10	0	0	0	0	0	0	6	0	10	0	30	0	72	0	166	0	332	0	702	0	1346
11	0	0	0	2	0	2	0	6	0	20	0	48	0	106	0	240	0	486	0	972	0
12,13	0	0	0	0	0	2	0	8	0	18	0	48	0	108	0	236	0	486	0	978	0
14	0	0	0	0	2	0	4	0	14	0	36	0	80	0	184	0	392	0	788	0	1550

Table 16. Multiplicities of irreducible representations for the 2A-twisted sector.

i \ $4r$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
1	0	0	0	0	0	0	2	0	2	0	8	0	16	0	42	0	76	0	172	0
2,3	1	0	0	0	0	0	1	0	4	0	6	0	21	0	36	0	84	0	160	0
4	0	0	0	0	0	0	2	0	2	0	8	0	16	0	42	0	76	0	172	0
5,8	0	0	1	0	0	0	6	0	10	0	32	0	68	0	162	0	318	0	670	0
6,7	0	0	0	0	2	0	4	0	14	0	26	0	76	0	150	0	334	0	648	0
9,10	0	1	0	0	0	2	0	8	0	18	0	47	0	108	0	224	0	466	0	924
11,14	0	0	0	0	2	0	4	0	16	0	38	0	90	0	192	0	414	0	814	0
12,13	0	0	1	0	2	0	6	0	14	0	40	0	86	0	198	0	406	0	826	0
15,16	0	0	0	0	2	0	6	0	18	0	46	0	106	0	234	0	490	0	986	0
17	0	0	0	2	0	4	0	14	0	40	0	94	0	210	0	456	0	932	0	1836
18	0	0	0	2	0	8	0	22	0	58	0	140	0	318	0	682	0	1396	0	2760

Table 17. Multiplicities of irreducible representations for the 2B-twisted sector.

i \ $3r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	0	0	0	0	0	0	0	2	0	0	2	0	0	10	0	0	16	0	0
2,3	0	1	0	0	0	0	0	2	0	0	6	0	0	20	0	0	41	0	0	104
4,5	0	0	0	0	0	0	1	0	0	6	0	0	10	0	0	34	0	0	74	0
6,7	0	0	0	0	0	0	0	0	6	0	0	14	0	0	42	0	0	92	0	0
8	0	0	0	0	2	0	0	2	0	0	16	0	0	30	0	0	94	0	0	194
9	1	0	0	0	0	0	4	0	0	6	0	0	28	0	0	60	0	0	162	0
10,11	0	0	0	0	0	2	0	0	8	0	0	24	0	0	64	0	0	154	0	0
12	0	0	2	0	0	2	0	0	10	0	0	24	0	0	74	0	0	172	0	0
13	0	0	0	0	2	0	0	6	0	0	20	0	0	50	0	0	134	0	0	298
14	0	0	0	0	0	0	4	0	0	12	0	0	40	0	0	94	0	0	236	0
15	0	0	0	0	0	4	0	0	8	0	0	32	0	0	76	0	0	200	0	0
16	0	0	0	0	2	0	0	10	0	0	30	0	0	88	0	0	216	0	0	510
17	0	0	0	2	0	0	6	0	0	22	0	0	62	0	0	164	0	0	382	0

TABLE 18. Multiplicities of irreducible representations for the 3A-twisted sector.

i \ $9r$	2	5	8	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	62
1	0	0	0	0	0	2	0	0	2	0	0	10	0	0	18	0	0	54	0	0	108
2	0	0	0	0	0	0	0	2	0	0	4	0	0	18	0	0	36	0	0	90	0
3	0	0	0	0	0	0	0	0	0	6	0	0	10	0	0	30	0	0	62	0	0
4,5	0	0	0	0	0	2	0	0	10	0	0	22	0	0	65	0	0	146	0	0	344
6,7	0	0	0	0	2	0	0	5	0	0	18	0	0	44	0	0	118	0	0	254	0
8,9	1	0	0	0	0	0	4	0	0	10	0	0	36	0	0	82	0	0	202	0	0
10	0	0	0	0	0	6	0	0	14	0	0	50	0	0	122	0	0	308	0	0	666
11	0	2	0	0	2	0	0	12	0	0	32	0	0	96	0	0	220	0	0	526	0
12	0	0	0	2	0	0	6	0	0	26	0	0	64	0	0	174	0	0	390	0	0
13	0	0	2	0	0	4	0	0	20	0	0	54	0	0	150	0	0	346	0	0	794
14	0	0	0	0	4	0	0	12	0	0	42	0	0	104	0	0	268	0	0	600	0
15	0	0	0	2	0	0	10	0	0	26	0	0	80	0	0	194	0	0	468	0	0
16	0	0	2	0	0	6	0	0	22	0	0	64	0	0	168	0	0	400	0	0	902
17	0	0	0	0	4	0	0	14	0	0	46	0	0	122	0	0	304	0	0	690	0
18	0	0	0	2	0	0	10	0	0	32	0	0	90	0	0	224	0	0	530	0	0

TABLE 19. Multiplicities of irreducible representations for the 3B-twisted sector.

i \ $8r$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
1,2	1	0	0	0	0	0	0	0	4	0	0	0	10	0	0	0	26	0	0	0
3,4	0	0	0	0	0	0	2	0	0	0	4	0	0	0	16	0	0	0	44	0
5	0	0	0	0	2	0	0	0	6	0	0	0	18	0	0	0	54	0	0	0
6	0	0	0	0	0	0	2	0	0	0	12	0	0	0	32	0	0	0	84	0
7,11	0	1	0	0	0	2	0	0	0	7	0	0	0	26	0	0	0	68	0	0
8,12	0	0	0	0	0	0	0	4	0	0	0	14	0	0	0	40	0	0	0	110
9,10	0	0	0	0	2	0	0	0	8	0	0	0	28	0	0	0	80	0	0	0
13	0	0	2	0	0	0	4	0	0	0	16	0	0	0	48	0	0	0	130	0
14	0	0	0	0	0	0	4	0	0	0	16	0	0	0	48	0	0	0	128	0
15	0	0	0	2	0	0	0	8	0	0	0	28	0	0	0	84	0	0	0	216
16	0	0	0	0	0	4	0	0	0	16	0	0	0	48	0	0	0	136	0	0

TABLE 20. Multiplicities of irreducible representations for the 4A-twisted sector.

i \ $4r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1,6	0	1	0	0	0	3	0	0	0	10	0	0	0	33	0	0	0	90	0	0
2,3,4,5,7,8	0	0	0	0	0	2	0	0	0	10	0	0	0	32	0	0	0	90	0	0
9,10	0	0	0	2	0	0	0	10	0	0	0	36	0	0	0	110	0	0	0	290
11	1	0	0	0	2	0	0	0	16	0	0	0	46	0	0	0	146	0	0	0
12	0	0	0	0	4	0	0	0	12	0	0	0	52	0	0	0	136	0	0	0
13	0	0	0	0	0	0	8	0	0	0	24	0	0	0	88	0	0	0	224	0
14	0	0	2	0	0	0	6	0	0	0	30	0	0	0	80	0	0	0	236	0

TABLE 21. Multiplicities of irreducible representations for the 4B-twisted sector.

i \ $16r$	3	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79
1,2	0	0	0	2	0	0	0	8	0	0	0	30	0	0	0	90	0	0	0	228
3,4	0	0	0	0	0	4	0	0	0	18	0	0	0	52	0	0	0	144	0	0
5,6	0	0	1	0	0	0	7	0	0	0	22	0	0	0	68	0	0	0	184	0
7,8	1	0	0	0	2	0	0	0	12	0	0	0	41	0	0	0	112	0	0	0
9	0	0	0	4	0	0	0	18	0	0	0	60	0	0	0	176	0	0	0	460
10	0	2	0	0	0	8	0	0	0	32	0	0	0	106	0	0	0	288	0	0
11	0	0	2	0	0	0	12	0	0	0	46	0	0	0	136	0	0	0	364	0
12	0	0	0	0	6	0	0	0	24	0	0	0	80	0	0	0	228	0	0	0

TABLE 22. Multiplicities of irreducible representations for the 4C-twisted sector.

i \ $5r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	0	0	0	8	0	0	0	0	20	0	0	0	0	60	0
2	0	0	0	0	0	0	0	2	0	0	0	0	14	0	0	0	0	42	0	0	0
3	1	0	0	0	0	2	0	0	0	0	6	0	0	0	0	28	0	0	0	0	74
4	0	0	0	2	0	0	0	0	4	0	0	0	0	16	0	0	0	0	50	0	0
5	0	0	0	0	0	0	2	0	0	0	0	10	0	0	0	0	30	0	0	0	0
6,7	0	0	0	0	1	0	0	0	0	6	0	0	0	0	20	0	0	0	0	63	0
8,12	0	0	0	0	0	0	0	4	0	0	0	0	12	0	0	0	0	42	0	0	0
9,13	0	0	0	0	0	2	0	0	0	0	7	0	0	0	0	26	0	0	0	0	74
10,14	0	0	0	0	0	0	0	0	4	0	0	0	0	18	0	0	0	0	48	0	0
11,15	0	1	0	0	0	0	2	0	0	0	0	10	0	0	0	0	32	0	0	0	0
16	0	0	0	0	4	0	0	0	0	16	0	0	0	0	64	0	0	0	0	180	0
17	0	0	0	2	0	0	0	0	14	0	0	0	0	46	0	0	0	0	154	0	0
18	0	0	0	0	0	0	8	0	0	0	0	28	0	0	0	0	100	0	0	0	0
19	0	0	2	0	0	0	0	8	0	0	0	0	40	0	0	0	0	118	0	0	0
20	1	0	0	0	0	4	0	0	0	0	24	0	0	0	0	76	0	0	0	0	230

TABLE 23. Multiplicities of irreducible representations for the 5A-twisted sector.

i \ $6r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1,2	0	0	0	0	0	4	0	0	0	0	0	14	0	0	0	0	0	56	0	0
3,4	0	0	0	0	0	2	0	0	0	0	0	16	0	0	0	0	0	52	0	0
5,12	0	1	0	0	0	0	0	5	0	0	0	0	0	25	0	0	0	0	0	76
6,11	0	0	0	1	0	0	0	0	0	10	0	0	0	0	0	34	0	0	0	0
7,10	0	0	0	0	0	0	0	6	0	0	0	0	0	22	0	0	0	0	0	80
8,9	0	0	0	2	0	0	0	0	0	8	0	0	0	0	0	38	0	0	0	0
13	0	0	2	0	0	0	0	0	14	0	0	0	0	0	58	0	0	0	0	0
14	1	0	0	0	0	0	8	0	0	0	0	0	38	0	0	0	0	0	130	0
15	0	0	0	0	4	0	0	0	0	0	24	0	0	0	0	0	88	0	0	0

TABLE 24. Multiplicities of irreducible representations for the 6A-twisted sector.

i \ $36r$	5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137	143
1,2	0	0	0	0	2	0	0	0	0	0	16	0	0	0	0	0	50	0	0	0	0	0	158	0
3,4	0	0	0	0	3	0	0	0	0	0	13	0	0	0	0	0	54	0	0	0	0	0	152	0
5,12	0	0	0	0	0	0	6	0	0	0	0	0	22	0	0	0	0	0	78	0	0	0	0	0
6,11	0	0	2	0	0	0	0	0	8	0	0	0	0	0	36	0	0	0	0	0	106	0	0	0
7,10	1	0	0	0	0	0	4	0	0	0	0	0	24	0	0	0	0	0	74	0	0	0	0	0
8,9	0	0	1	0	0	0	0	0	10	0	0	0	0	0	33	0	0	0	0	0	112	0	0	0
13	0	2	0	0	0	0	0	14	0	0	0	0	0	56	0	0	0	0	0	182	0	0	0	0
14	0	0	0	0	0	8	0	0	0	0	0	36	0	0	0	0	0	126	0	0	0	0	0	366
15	0	0	0	4	0	0	0	0	0	22	0	0	0	0	0	86	0	0	0	0	0	260	0	0

TABLE 25. Multiplicities of irreducible representations for the 6B-twisted sector.

i \ $7r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	0	0	0	0	0	2	0	0	0	0	0	0	8	0	0	0	0	0	0
2	0	0	0	0	0	0	2	0	0	0	0	0	0	10	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	15	0	0	0
4	0	0	0	0	0	2	0	0	0	0	0	0	6	0	0	0	0	0	0	28
5	0	1	0	0	0	0	0	0	2	0	0	0	0	0	0	14	0	0	0	0
6	0	0	0	0	1	0	0	0	0	0	0	6	0	0	0	0	0	0	22	0
7	0	0	0	0	0	0	0	3	0	0	0	0	0	0	10	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	20	0	0
9	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	16	0	0
10	1	0	0	0	0	0	0	2	0	0	0	0	0	0	12	0	0	0	0	0
11	0	0	0	0	2	0	0	0	0	0	0	5	0	0	0	0	0	0	24	0
12	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	12	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	8	0	0	0	0	0	0	26
14	0	0	1	0	0	0	0	0	0	2	0	0	0	0	0	0	18	0	0	0
15	0	0	0	0	0	0	4	0	0	0	0	0	0	16	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	26	0	0	0	0
17	0	0	0	0	0	2	0	0	0	0	0	0	16	0	0	0	0	0	0	52
18	0	0	1	0	0	0	0	0	0	8	0	0	0	0	0	0	32	0	0	0
19	0	0	0	2	0	0	0	0	0	0	10	0	0	0	0	0	0	36	0	0
20	1	0	0	0	0	0	0	4	0	0	0	0	0	0	22	0	0	0	0	0
21	0	0	0	0	2	0	0	0	0	0	0	11	0	0	0	0	0	0	46	0

TABLE 26. Multiplicities of irreducible representations for the 7AB-twisted sectors.

i \ $8r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1,4	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	24	0	0	0	0
2,3	0	0	0	2	0	0	0	0	0	0	0	12	0	0	0	0	0	0	0	46
5	0	0	0	0	0	2	0	0	0	0	0	0	0	16	0	0	0	0	0	0
6	0	1	0	0	0	0	0	0	0	8	0	0	0	0	0	0	0	34	0	0
7	0	0	0	0	0	4	0	0	0	0	0	0	0	16	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	8	0	0	0	0	0	0	0	32	0	0
9,13	1	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	28	0	0	0
10,14	0	0	1	0	0	0	0	0	0	0	9	0	0	0	0	0	0	0	40	0
11,15	0	0	0	0	0	0	4	0	0	0	0	0	0	0	20	0	0	0	0	0
12,16	0	0	0	0	2	0	0	0	0	0	0	0	14	0	0	0	0	0	0	0

TABLE 27. Multiplicities of irreducible representations for the 8A-twisted sector.

i \ $20r$	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41
1,2	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	22	0
3,4	0	0	0	2	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0
5,6	0	0	0	0	2	0	0	0	0	0	0	0	0	0	12	0	0	0	0	0
7,8	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	0	0	24
9,10	1	0	0	0	0	0	0	0	0	0	6	0	0	0	0	0	0	0	0	0
11,12	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	14	0	0	0	0
13,14	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	16	0	0	0
15,16	0	1	0	0	0	0	0	0	0	0	0	7	0	0	0	0	0	0	0	0
17,18	0	0	1	0	0	0	0	0	0	0	0	0	9	0	0	0	0	0	0	0
19,20	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	18	0	0

TABLE 28. Multiplicities of irreducible representations for the 10A-twisted sector.

$11r \equiv i-1$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$h_{i,r}$	2	0	2	2	2	2	4	4	6	6	8	8	12	10	14	16	20	20	26	26

TABLE 29. Multiplicities of irreducible representations for the 11A-twisted sector.

$24r \equiv 2i-1$	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41
$h_{i,r}$	1	2	2	2	2	4	4	4	6	6	8	8	10	12	12	16	18	20	22	24

TABLE 30. Multiplicities of irreducible representations for the 12A-twisted sector.

$144r \equiv 12i-1$	23	35	47	59	71	83	95	107	119	131	143	155	167	179	191	203	215	227	239	251
$h_{i,r}$	2	2	2	2	2	4	4	6	6	6	8	10	10	12	14	16	18	20	24	26

TABLE 31. Multiplicities of irreducible representations for the 12B-twisted sector.

$14r \equiv i-1$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$h_{i,r}$	1	0	1	2	1	2	2	2	3	4	4	4	6	6	7	8	8	10	12	12

TABLE 32. Multiplicities of irreducible representations for the 14AB-twisted sectors.

$15r \equiv i-1$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$h_{i,r}$	1	0	1	1	2	1	2	2	3	2	4	4	5	4	6	6	8	7	10	10

TABLE 33. Multiplicities of irreducible representations for the 15AB-twisted sectors.

$63r \equiv 3i-1$	8	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	62	65
$h_{i,r}$	7	7	13	0	14	7	14	7	14	26	14	14	21	14	28	21	58	28	588	28

TABLE 34. Multiplicities of irreducible representations for the 21AB-twisted sectors.

$23r \equiv i-1$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$h_{i,r}$	1	0	0	1	1	0	1	0	2	1	2	2	2	1	2	2	2	2	3	2

TABLE 35. Multiplicities of irreducible representations for the 23AB-twisted sectors.

References

[1] J.H. Conway and S.P. Norton, "Monstrous Moonshine," Bull. London Math. Soc. **11**, 308 (1979).

[2] I. Frenkel, J. Lepowsky and A. Meurman, "Vertex operator algebras and the Monster," Pure and Applied Mathematics **134**, Academic Press, Inc., Boston, MA, (1988).

[3] R.E. Borcherds, "Monstrous Moonshine and monstrous Lie superalgebras," Invent. Math. **109**, 405 (1992).

[4] T.J. Gannon, "Moonshine beyond the Monster: The bridge connecting algebra, modular forms and physics," Cambridge University Presse (2006).

[5] T. Eguchi, H. Ooguri and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," Exper. Math. **20**, 91 (2011) [arXiv:1004.0956 [hep-th]].

[6] M.C.N. Cheng, "K3 Surfaces, N=4 dyons, and the Mathieu group M_{24}," Commun. Number Theory Phys. **4**, 623 (2010) [arXiv:1005.5415 [hep-th]].

[7] M.R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu twining characters for K3," JHEP **1009**, 058 (2010) [arXiv:1006.0221 [hep-th]].

[8] M.R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," JHEP **1010**, 062 (2010) [arXiv:1008.3778 [hep-th]].

[9] T. Eguchi and K. Hikami, "Note on twisted elliptic genus of K3 surface," Phys. Lett. B **694**, 446 (2011) [arXiv:1008.4924 [hep-th]].

[10] T. Gannon, "Much ado about Mathieu," arXiv:1211.5531 [math.RT].

[11] T. Creutzig, G. Höhn and T. Miezaki, "The McKay-Thompson series of Mathieu Moonshine modulo two," arXiv:1211.3703v1 [math.NT].

[12] M.C.N. Cheng and J.F.R. Duncan, "On Rademacher sums, the largest Mathieu group, and the holographic modularity of Moonshine," arXiv:1110.3859 [math.RT].

[13] M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Symmetries of K3 sigma models," Comm. Number Theory and Phys. **6** (1) (2012) [arXiv:1106.4315 [hep-th]].

[14] S. Norton, "Generalized moonshine," Proc. Sympos. Pure Math. **47**, 209 Amer. Math. Soc., Providence, RI (1987).

[15] C.-y. Dong, H.-s. Li and G. Mason, "Modular invariance of trace functions in orbifold theory," Commun. Math. Phys. **214**, 1 (2000) [arXiv:q-alg/9703016].

[16] G. Höhn, "Generalized moonshine for the baby Monster," Habilitation thesis, Mathematisches Institut, Universität Freiburg (2003), http://www.math.ksu.edu/∼gerald/papers/baby8.ps.

[17] M.P. Tuite, "Generalized moonshine and abelian orbifold constructions," , Contemp. Math. **193**, 353 (1995) [arXiv:hep-th/9412036].

[18] R.I. Ivanov and M.P. Tuite, "Rational generalized moonshine from abelian orbifoldings of the moonshine module," Nucl. Phys. B **635**, 435 (2002) [math/0106027 [math-qa]].

[19] R.I. Ivanov and M.P. Tuite, "Some irrational generalised moonshine from orbifolds," Nucl. Phys. B **635**, 473 (2002) [math/0202275 [math-qa]].

[20] S. Carnahan, "Generalized Moonshine I: Genus zero functions," arXiv:0812.3440 [math.RT].

[21] S. Carnahan, "Generalized Moonshine II: Borcherds products," arXiv:0908.4223 [math.RT].

[22] S. Carnahan, "Generalized Moonshine III: Equivariant intertwining operators," to appear.

[23] S. Carnahan, "Generalized Moonshine IV: Monstrous Lie algebras," arXiv:1208.6254 [math.RT].

[24] L.J. Dixon, P.H. Ginsparg and J.A. Harvey, "Beauty and the beast: superconformal symmetry in a Monster module," Commun. Math. Phys. **119**, 221 (1988).

[25] R. Dijkgraaf and E. Witten, "Topological gauge theories and group cohomology," Commun. Math. Phys. **129**, 393 (1990).

[26] T. Gannon, talk at 'Mathieu Moonshine', ETH Zürich, July 2011.

[27] G. Mason, "Orbifold conformal field theory and the cohomology of the Monster," unpublished.

[28] G. Mason, "On a system of elliptic modular forms attached to the largest Mathieu group," Nagoya Math. J. **118**, 177 (1990).

[29] D. Persson and R. Volpato, "Second Quantized Mathieu Moonshine," arXiv:1312.0622 [hep-th].

[30] M. Eichler and D. Zagier, "The theory of Jacobi forms," Birkhäuser (1985).

[31] T. Eguchi and A. Taormina, "Character formulas for the N=4 superconformal algebra," Phys. Lett. B **200**, 315 (1988).

[32] M. Dutour Sikirić and G. Ellis, "Wythoff polytopes and low-dimensional homology of Mathieu groups," J. Algebra **322**, 4143 (2009).

[33] C. Vafa, "Modular invariance and discrete torsion on orbifolds," Nucl. Phys. B **273**, 592 (1986).

[34] S. Willerton, "The twisted Drinfeld double of a finite group via gerbes and finite groupoids," Algebr. Geom. Topol. **8**, 1419 (2008) [arXiv:math/0503266].

[35] P. Bantay, "Orbifolds and Hopf algebras," Phys. Lett. B **245**, 477 (1990).

[36] R. Dijkgraaf, V. Pasquier and P. Roche, "Quasi-Hopf algebras, group cohomology and orbifold models," Nucl. Phys. Proc. Suppl. **18B**, 60 (1990).

[37] A. Coste, T. Gannon and P. Ruelle, "Finite group modular data," Nucl. Phys. B **581**, 679 (2000) [arXiv:hep-th/0001158].

[38] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, "Defect lines, dualities, and generalised orbifolds," in *XVIth International Congress on Mathematical Physics*, 608–613, World Sci. Publ., Hackensack, NJ (2010) [arXiv:0909.5013 [math-ph]].

[39] K.S. Narain, M.H. Sarmadi and C. Vafa, "Asymmetric orbifolds," Nucl. Phys. B **288**, 551 (1987).

[40] M.R. Gaberdiel and R. Volpato, "Mathieu moonshine and orbifold K3s," arXiv:1206.5143 [hep-th].

[41] The GAP Group, "GAP – Groups, Algorithms, and Programming, version 4.5.6," (2012) http://www.gap-system.org.

[42] G. Ellis, "HAP (Homological Algebra Programming) – a GAP package, version 1.10.8," (2012) http://www.gap-system.org/Packages/hap.html.

[43] K.S. Brown, "Cohomology of groups," Graduate Texts in Mathematics **87**, Springer, New York (1994).

[44] M.D.F. de Wild Propitius, "Topological interactions in broken gauge theories," hep-th/9511195.

[45] See the ancillary files attached with the preprint version of this paper at http://arxiv.org.

[46] S. Mukai, "Finite groups of automorphisms of $K3$ surfaces and the Mathieu group," Invent. Math. **94**, 183 (1988).

[47] A. Sen, "Discrete information from CHL black holes," JHEP **1011**, 138 (2010) [arXiv:1002.3857 [hep-th]].

[48] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, "Counting dyons in N=4 string theory," Nucl. Phys. B **484**, 543 (1997) [arXiv:hep-th/9607026].

[49] D. Shih, A. Strominger and X. Yin, "Recounting dyons in N=4 string theory," JHEP **0610**, 087 (2006) [arXiv:hep-th/0505094].

[50] A. Dabholkar, S. Murthy and D. Zagier, "Quantum black holes, wall crossing, and mock modular forms," arXiv:1208.4074 [hep-th].

[51] S. Govindarajan, "BKM Lie superalgebras from counting twisted CHL dyons," JHEP **1105**, 089 (2011) [arXiv:1006.3472 [hep-th]].

[52] S. Govindarajan, "Brewing moonshine for Mathieu," arXiv:1012.5732 [math.NT].

[53] S. Govindarajan, "Unravelling Mathieu moonshine," Nucl. Phys. B **864**, 823 (2012) [arXiv:1106.5715 [hep-th]].

[54] J.F. Duncan and I.B. Frenkel, "Rademacher sums, moonshine and gravity," Commun. Num. Theor. Phys. **5**, 849 (2011) [arXiv:0907.4529 [math.RT]].

[55] V.A. Gritsenko and V.V. Nikulin, "Siegel automorphic form corrections of some Lorentzian Kac–Moody Lie algebras," [arXiv:alg-geom/9504006].

[56] V.A. Gritsenko and F. Clery, "The Siegel modular forms of genus 2 with the simplest divisor," [arXiv:0812.3962 [math.NT]].

[57] M.C.N. Cheng and E.P. Verlinde, "Wall crossing, discrete attractor flow, and Borcherds algebra," SIGMA **4**, 068 (2008) [arXiv:0806.2337 [hep-th]].

[58] M.C.N. Cheng and A. Dabholkar, "Borcherds-Kac-Moody symmetry of N=4 dyons," Commun. Num. Theor. Phys. **3**, 59 (2009) [arXiv:0809.4258 [hep-th]].

[59] S. Govindarajan and K. Gopala Krishna, "Generalized Kac-Moody algebras from CHL dyons," JHEP **0904**, 032 (2009) [arXiv:0807.4451 [hep-th]].

[60] S. Govindarajan and K. Gopala Krishna, "BKM Lie superalgebras from dyon spectra in Z_N CHL orbifolds for composite N," JHEP **1005**, 014 (2010) [arXiv:0907.1410 [hep-th]].

[61] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, "Umbral moonshine," arXiv:1204.2779 [math.RT].

[62] J.A. Harvey and G.W. Moore, "Algebras, BPS states, and strings," Nucl. Phys. B **463**, 315 (1996) [arXiv:hep-th/9510182].

[63] J.A. Harvey and G.W. Moore, "On the algebras of BPS states," Commun. Math. Phys. **197**, 489 (1998) [arXiv:hep-th/9609017].

[64] A. Kapustin, "Chiral de Rham complex and the half-twisted sigma-model," arXiv:hep-th/0504074.

[65] F. Malikov, V. Schechtman and A. Vaintrob, "Chiral de Rham complex," Commun. Math. Phys. **204**, 439 (1999) [arXiv:math/9803041].

[66] L.A. Borisov and L. Libgober, "Elliptic genera of toric varieties and applications to mirror symmetry," Invent. Math. **140**, 453 (2000) [arXiv:math/9904126].

[67] G. Karpilovsky, "Group representations: volume 2," North-Holland Mathematical Studies **177**, Elsevier (1993).

Institut für Theoretische Physik, ETH Zürich, CH-8093 Zürich, Switzerland
E-mail address: gaberdiel@itp.phys.ethz.ch

Fundamental Physics, Chalmers University of Technology, SE-412 96, Gothenburg, Sweden
E-mail address: daniel.persson@chalmers.se
URL: http://www.danper.se

Max-Planck-Institut für Dynamik und Selbstorganisation (MPIDS), 37077 Göttingen, Germany
E-mail address: henrik.ronellenfitsch@ds.mpg.de

Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, 14476 Golm, Germany
E-mail address: roberto.volpato@aei.mpg.de

arXiv:1307.5793v2 [math.RT] 22 Jul 2014

Umbral Moonshine and the Niemeier Lattices

Miranda C. N. Cheng*
John F. R. Duncan†
Jeffrey A. Harvey‡

July 23, 2014

Abstract

In this paper we relate umbral moonshine to the Niemeier lattices: the 23 even unimodular positive-definite lattices of rank 24 with non-trivial root systems. To each Niemeier lattice we attach a finite group by considering a naturally defined quotient of the lattice automorphism group, and for each conjugacy class of each of these groups we identify a vector-valued mock modular form whose components coincide with mock theta functions of Ramanujan in many cases. This leads to the umbral moonshine conjecture, stating that an infinite-dimensional module is assigned to each of the Niemeier lattices in such a way that the associated graded trace functions are mock modular forms of a distinguished nature. These constructions and conjectures extend those of our earlier paper, and in particular include the Mathieu moonshine observed by Eguchi–Ooguri–Tachikawa as a special case. Our analysis also highlights a correspondence between genus zero groups and Niemeier lattices. As a part of this relation we recognise the Coxeter numbers of Niemeier root systems with a type A component as exactly those levels for which the corresponding classical modular curve has genus zero.

MSC2010: 11F22, 11F37, 11F46, 11F50, 20C34, 20C35

*Université Paris 7, UMR CNRS 7586, Paris, France. *E-mail:* chengm@math.jussieu.fr

†Department of Mathematics, Applied Mathematics and Statistics, Case Western Reserve University, Cleveland, OH 44106, U.S.A. *E-mail:* john.duncan@case.edu

‡Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, IL 60637, U.S.A. *E-mail:* j-harvey@uchicago.edu

Contents

1 Introduction 5
1.1 Notation . . . 9

2 Groups 11
2.1 Root Systems . . . 11
2.2 Lattices . . . 14
2.3 Genus Zero Groups . . . 16
2.4 Umbral Groups . . . 20
2.5 McKay Correspondence . . . 25

3 Automorphic Forms 26
3.1 Mock Modular Forms . . . 27
3.2 Jacobi Forms . . . 28
3.3 The Eichler–Zagier Operators and an ADE Classification . . . 30
3.4 From Meromorphic Jacobi Forms to Mock Modular Forms . . . 35

4 The Umbral Mock Modular Forms 41
4.1 The Umbral Shadows . . . 41
4.2 From Niemeier Lattices to Principal Moduli . . . 46
4.3 From Weight Zero Jacobi Forms to Umbral Mock Modular Forms . . . 49

5 The Umbral McKay–Thompson Series 57
5.1 Shadows . . . 58
5.2 Prime Lambencies . . . 59
5.3 Multiplicative Relations . . . 61
5.4 Mock Theta Functions . . . 65
5.5 Specification . . . 69

6 Conjectures 73
6.1 Modules . . . 73
6.2 Modularity . . . 74
6.3 Moonshine . . . 75
6.4 Discriminants . . . 77

7 Conclusions and Discussion 79

A Special Functions **83**
A.1 Dedekind Eta Function . . . 83
A.2 Jacobi Theta Functions . . . 83
A.3 Higher Level Modular Forms . . . 84
A.4 Weight Zero Jacobi Forms . . . 84

B Characters **87**
B.1 Irreducible Characters . . . 88
B.2 Euler Characters . . . 94

C Coefficient Tables **102**
C.1 Lambency 2 . . . 103
C.2 Lambency 3 . . . 104
C.3 Lambency 4 . . . 106
C.4 Lambency 5 . . . 109
C.5 Lambency 6 . . . 113
C.6 Lambency 6+3 . . . 118
C.7 Lambency 7 . . . 120
C.8 Lambency 8 . . . 123
C.9 Lambency 9 . . . 127
C.10 Lambency 10 . . . 131
C.11 Lambency 10+5 . . . 134
C.12 Lambency 12 . . . 136
C.13 Lambency 12+4 . . . 138
C.14 Lambency 13 . . . 141
C.15 Lambency 14+7 . . . 145
C.16 Lambency 16 . . . 147
C.17 Lambency 18 . . . 149
C.18 Lambency 18+9 . . . 151
C.19 Lambency 22+11 . . . 153
C.20 Lambency 25 . . . 155
C.21 Lambency 30+15 . . . 157
C.22 Lambency 30+6,10,15 . . . 158
C.23 Lambency 46+23 . . . 159

D Decompositions 160

D.1 Lambency 2 . . . 162

D.2 Lambency 3 . . . 163

D.3 Lambency 4 . . . 164

D.4 Lambency 5 . . . 165

D.5 Lambency 6 . . . 166

D.6 Lambency 6+3 . . . 167

D.7 Lambency 7 . . . 168

D.8 Lambency 8 . . . 169

D.9 Lambency 9 . . . 170

D.10 Lambency 10+5 . . . 172

D.11 Lambency 12+4 . . . 173

D.12 Lambency 13 . . . 174

D.13 Lambency 14+7 . . . 176

D.14 Lambency 30+6,10,15 . . . 177

References 178

1 Introduction

In this paper we relate umbral moonshine to the Niemeier lattices. This relation associates one case of umbral moonshine to each of the 23 Niemeier lattices and in particular constitutes an extension of our previous work [1], incorporating 17 new instances. Moreover, this prescription displays interesting connections to genus zero groups (subgroups $\Gamma < SL_2(\mathbb{R})$ that define a genus zero quotient of the upper-half plane) and extended Dynkin diagrams via McKay's correspondence.

To explain this moonshine relation, let us first recall what Niemeier lattices are. In 1973 Niemeier classified the even unimodular positive-definite lattices of rank 24 [2]. There are 24 of them, including the so-called *Leech lattice* discovered almost a decade earlier in the context of optimal lattice sphere packing in 24 dimensions [3]. Niemeier proved that the Leech lattice is the unique even, unimodular and positive-definite lattice of rank 24 with no root vectors (lattice vectors with norm-square 2), while the other 23 even unimodular rank 24-dimensional lattices all have root systems of the full rank 24. Moreover, these 23 lattices are uniquely labelled by their root systems, which are in turn uniquely specified by the following two conditions: first, they are unions of simply-laced root systems with the same Coxeter numbers; second, the total rank is 24.

We will refer to these 23 root systems as the *Niemeier root systems* and the 23 corresponding lattices as the *Niemeier lattices*. In this paper we associate a finite group and a set of vector-valued mock modular forms to each of these 23 Niemeier lattices. The main result of the present paper is then the *umbral moonshine conjecture* relating the two.

To understand this statement let us recall what one means by moonshine. This term was first introduced in mathematics to describe the remarkable *monstrous moonshine* phenomenon. The study of monstrous moonshine was initiated by John McKay's observation that $196883 + 1 = 196884$, where the summands on the left are degrees of irreducible representations of the *Fischer–Griess monster* $\mathbb{M}$ and the number on the right is the coefficient of q in the Fourier expansion of the *normalised elliptic modular invariant*

$$J(\tau) = \sum_{m\geq -1} a(m)\, q^m = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots , \tag{1.1}$$

where we write $q = e^{2\pi i\tau}$. Following Thompson's idea [4] that $J(\tau)$ should be the graded dimension of an infinite-dimensional module for $\mathbb{M}$, this observation was later expanded into the full *monstrous moonshine conjecture* by Conway and Norton [5], conjecturing that the graded

character $T_g(\tau)$ attached to the monster module

$$V = \bigoplus_{m \geq -1} V_m \tag{1.2}$$

and $g \in \mathbb{M}$ should be a principal modulus for a certain genus zero group $\Gamma_g < SL_2(\mathbb{R})$. (When a discrete group $\Gamma < SL_2(\mathbb{R})$ has the property that $\Gamma \backslash \mathbb{H}$ is isomorphic to the Riemann sphere minus finitely many points, there exists a holomorphic function f on $\mathbb{H}$ that generates the field of Γ-invariant functions on $\mathbb{H}$. Such a function f is called a *principal modulus*, or *Hauptmodul*, for Γ.) We refer to [6] or the introduction of [1] for a more detailed account of monstrous moonshine.

In 2010 the study of a new type of moonshine was triggered by an observation of Eguchi–Ooguri–Tachikawa, which constituted an analogue of McKay's observation in monstrous moonshine. In the work of Eguchi–Taormina and Eguchi–Ooguri–Taormina–Yang in the 1980's [7–9], these authors encountered a q-series

$$H^{(2)}(\tau) = 2\, q^{-1/8}(-1 + 45\, q + 231\, q^2 + 770\, q^3 + 2277\, q^4 + \cdots) \tag{1.3}$$

in the decomposition of the elliptic genus of a $K3$ surface into irreducible characters of the $\mathcal{N} = 4$ superconformal algebra. It was later understood by Eguchi–Hikami [10] that the above q-series is a mock modular form. See §3 for the definition of mock modular forms. Subsequently the coincidence between the numbers 45, 231, 770, 2277,... and the dimensions of irreducible representations of M_{24} was pointed out in [11]. This observation was later extended into a *Mathieu moonshine* conjecture in [12–15] by providing the corresponding twisted characters, the mock modular forms $H_g^{(2)}$, and was moreover related in a more general context to the $K3$-compactification of superstring theory in [12]. Very recently, the existence of an infinite-dimensional M_{24}-module underlying the mock modular form (1.3) and those constructed in [12–15] was shown by T. Gannon [16], although the nature of this M_{24}-module remains mysterious. See [17] and [18] for a review of this M_{24}-mock modular relation, and see [19–23] for recent developments in this direction.

Meanwhile, it was found that Mathieu moonshine is but one example of a more general phenomenon, *umbral moonshine*. In [1] we associated a finite group $G^{(\ell)}$ and a vector-valued mock modular form $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$ with $(\ell - 1)$-components for every conjugacy class $[g]$ of $G^{(\ell)}$ to each of the six positive integers ℓ such that $\ell - 1$ divides 12, and conjectured that there exists an infinite-dimensional $G^{(\ell)}$-module, the *umbral module*, with the property that its graded character coincides with the mock modular form $H_g^{(\ell)}$ for every conjugacy class $[g] \subset G^{(\ell)}$.

Despite the discovery of this more general framework of umbral moonshine, encompassing Mathieu moonshine as a special case and displaying various beautiful properties, many questions remained unanswered. For example: why these specific umbral groups $G^{(\ell)}$? Why are they labelled by divisors of the number 12? What is the structure underlying all these instances of moonshine?

In the present paper we provide partial answers to the above questions. We present evidence that there exists an instance of umbral moonshine naturally associated to each of the 23 Niemeier lattices. As a Niemeier lattice is uniquely determined by its root system X, in the main text we shall use X (or equivalently the corresponding *lambency*; see Tables 2-3) to label the instances of umbral moonshine. In particular, we construct in each instance an *umbral group* G^X as the quotient of the automorphism group of the corresponding Niemeier lattice L^X by the normal subgroup generated by refections in root vectors. This property gives a uniform construction as well as a concrete understanding of the umbral groups.

Similarly, we provide a prescription that attaches to each of the 23 Niemeier lattices a distinguished vector-valued modular form—the *umbral mock modular form* H^X—which conjecturally encodes the dimensions of the homogeneous subspaces of the corresponding umbral module. The Niemeier lattice uniquely specifies the shadow of the mock modular form through a map which associates a unary theta series of a specific type to each of the irreducible simply-laced ADE root systems, as well as unions of such root systems where all the irreducible components have the same Coxeter number. As will be explained in §3.3, this map bears a strong resemblance to the ADE classification by Cappelli–Itzykson–Zuber of modular invariant combinations of the characters of the $\widehat{A}_1$ affine Lie algebra [24]. When applied to the Niemeier root systems, we dictate the resulting unary theta series to be the shadow of the corresponding umbral mock modular form. Together with the natural requirement that H^X satisfies an optimal growth condition the specification of the shadow uniquely fixes the desired umbral form (cf. Theorem 4.1 and Corollary 4.2).

By associating a case of umbral moonshine to each Niemeier lattice we extend our earlier work on umbral moonshine to include 17 more instances. In fact, the 6 instances discussed in the earlier paper, labelled by the 6 divisors of 12, correspond to *pure A-type* Niemeier root systems containing only A-type irreducible components. There are 8 pure A-type Niemeier root systems, one for each divisor $\ell-1$ of 24, and they are given simply as the union of $\frac{24}{\ell-1}$ copies of $A_{\ell-1}$. This new proposal relating Niemeier lattices and umbral moonshine can be regarded as a completion of our earlier work [1], in that it includes Niemeier root systems with D- or/and E- components and sheds important light on the underlying structure of umbral moonshine.

More properties of umbral moonshine reveal themselves as new instances are included and

as the structure of umbral moonshine is examined in light of the connection to Niemeier lattices. Recall that in [1] we observed a connection between the (extended) Dynkin diagrams and some of the groups $G^{(\ell)}$ via McKay's correspondence for subgroups of $SU(2)$. In the present paper we observe that the same holds for many of the new instances of umbral moonshine, and the result presents itself as a natural sequence of extended Dynkin diagrams with decreasing rank, starting with $\widehat{E}_8$ and ending with $\widehat{A}_1$. Moreover, we observe an interesting relation between umbral moonshine and the genus zero groups $\Gamma < SL_2(\mathbb{R})$ through the shadows of the former and the principal moduli for the latter. As will be discussed in §2.3 and §4.2, our construction attaches a principal modulus for a genus zero group to each Niemeier lattice. In particular, we recognise the Coxeter numbers of the root systems with an A-type component as exactly those levels for which the corresponding classical modular curve has genus zero.

The outline of this paper is as follows. In §2 we give some background on Niemeier lattices, define the umbral finite groups, and discuss the mysterious relation to extended ADE diagrams and genus zero quotients of the upper-half plane. In §3 we introduce various automorphic objects that play a role in umbral moonshine, including (mock) modular forms and Jacobi forms of the weak, meromorphic, and mock type. For later use we also introduce the Eichler–Zagier (or Atkin–Lehner) map on Jacobi forms, and an ADE classification of such maps. In §4 we focus on the umbral mock modular forms, which are conjecturally the generating functions of the dimensions of the homogeneous subspaces of the umbral modules. In §4.5 we give explicit formulas for these umbral mock modular forms as well as most of the umbral McKay–Thompson series. This is achieved partially with the help of *multiplicative relations*, relating McKay–Thompson series in different instances of umbral moonshine corresponding to Niemeier lattices with one Coxeter number being the multiple of the nother. In §6 we present the main results of the paper, which are the umbral moonshine conjectures relating the umbral groups and umbral mock modular forms, and a counterpart for umbral moonshine of the genus zero property of monstrous moonshine. We also observe certain discriminant properties relating the exponents of the powers of q in the mock modular forms and the imaginary quadratic number fields over which the homogeneous submodules of the umbral modules are defined, extending the discriminant properties observed in [1]. Finally, we present some conclusions and discussions in §7.

To provide the data and evidence in support of our conjectures, this paper also contains four appendices. In Appendix A we describe some modular forms and Jacobi forms which are utilised in the paper. In Appendix B we present tables of irreducible characters as well as the characters of certain naturally defined (signed) permutation representations of the 23 umbral groups. In Appendix C we provide the first few dozen coefficients of all the umbral McKay–Thompson series. In Appendix D, using the tables in Appendix B and C, we explicitly present

decompositions into irreducible representations for the first 10 or so homogeneous subspaces of the umbral modules for all instances of umbral moonshine.

1.1 Notation

We conclude this introduction with a guide to the most important and frequently used notation, and indications as to where the relevant definitions can be found.

- X A root system (cf. §2.1). Usually X is a union of irreducible simply-laced root systems with the same Coxeter number; for example, a Niemeier root system (cf. §2.2).
- m The Coxeter number of an irreducible root system X (cf. §2.1), or the Coxeter number of any irreducible component of X when all such numbers coincide.
- r The rank of a root system X (cf. §2.1).
- π^X The (formal) product of Frame shapes of Coxeter elements of irreducible components of a root system X (cf. §2.1).
- W^X The Weyl group of a root system X (cf. §2.1).
- L^X The Niemeier lattice attached to the Niemeier root system X (cf. §2.2).
- X_A The union of irreducible components of type A in a Niemeier root system X (cf. §2.2). Similarly for X_D and X_E.
- d_A^X The number of irreducible components of type A in the root system X (cf. §2.2). Similarly for d_D^X and d_E^X.
- d^X The total number of irreducible components of the root system X (cf. §2.3).
- Γ^X The genus zero subgroup of $SL_2(\mathbb{R})$ attached to the Niemeier root system X (cf. §2.3).
- T^X A certain principal modulus for Γ^X, for X a Niemeier root system (cf. §2.3).
- f^X A certain modular form of weight 2 for Γ^X, for X Niemeier root system (cf. §§2.3,4.2).
- ℓ A lambency. A symbol that encodes a genus zero group Γ^X attached to a Niemeier root system X, and thereby also X (cf. §2.3).
- G^X The umbral group attached to the Niemeier root system X (cf. §2.4). Also denoted $G^{(\ell)}$ for ℓ the lambency corresponding to X.
- $\bar{G}^X$ A naturally defined quotient of G^X (cf. §2.4). Also denoted $\bar{G}^{(\ell)}$.
- $\widetilde{\chi}^X$ A twisted Euler character. A certain naturally defined character of G^X (cf. §2.4). Similarly for $\widetilde{\chi}^{X_A}$, χ^{X_A}, $\bar{\chi}^{X_A}$, &c.
- $D^{(\ell)}$ The finite subgroup of $SU(2)$ attached to the umbral group $G^{(\ell)}$ for $2 < \ell < 11$ (cf. §2.5).
- $\Delta^{(\ell)}$ The extended Dynkin diagram of rank $11 - \ell$ attached to the umbral group $G^{(\ell)}$ for ℓ as above (cf. §2.5).

m Usually the index of a Jacobi form (cf. §3.2). Often this is chosen to coincide with the Coxeter number m of some root system X, in which case we write m for both (cf. §3.3).

θ_m The vector-valued function whose components are the standard index m theta functions $\theta_{m,r}$ for $r \in \mathbb{Z}/2m\mathbb{Z}$ (cf. §3.2).

Ω^X The $2m \times 2m$ matrix attached to a simply-laced root system X with all irreducible components having Coxeter number m (cf. §3.3).

$\mathcal{W}^X$ The Eichler–Zagier operator on Jacobi forms of index m attached to a simply-laced root system X with all irreducible components having Coxeter number m (cf. §3.3).

ψ^P The polar part of a meromorphic Jacobi form ψ (cf. §3.4).

ψ^F The finite part of a meromorphic Jacobi form ψ (cf. §3.4).

$\mu_{m,j}$ A generalised Appell–Lerch sum of index m (cf. §3.4). The function $\mu_{1,0}$ is a meromorphic Jacobi form of weight 1 and index 1 with vanishing finite part. More generally, scalar multiples of $\mu_{m,0}$ arise as polar parts of certain meromorphic Jacobi forms of weight 1 and index m.

h Usually a vector-valued mock modular form, with components h_r for $r \in \mathbb{Z}/2m\mathbb{Z}$, obtained from the theta expansion of the finite part of a meromorphic Jacobi form of weight 1 and index m (cf. §3.4).

S_m The vector-valued cusp form of weight 3/2 whose components are the unary theta series $S_{m,r}$ for $r \in \mathbb{Z}/2m\mathbb{Z}$ (cf. §3.4), related to θ_m by $S_{m,r}(\tau) = \frac{1}{2\pi i}\partial_z \theta_{m,r}(\tau, z)|_{z=0}$.

S^X The vector-valued cusp form of weight 3/2 attached to a simply-laced root system X with all irreducible components having the same Coxeter number (cf. §4.1). An umbral shadow in case X is a Niemeier root system.

ψ^X The unique meromorphic Jacobi form of weight 1 and index m satisfying the conditions of Theorem 4.1, if such a function exists, where X is a simply-laced root system for which all irreducible components have Coxeter number m (cf. §4.1).

H^X The unique vector-valued mock modular form with shadow S^X whose components furnish the theta expansion of the finite part of ψ^X, if ψ^X exists (cf. §4.1). An umbral mock modular form in case X is a Niemeier root system (cf. §4.3), and, in this situation, also denoted $H^{(\ell)}$ for ℓ the lambency corresponding to X.

σ^X The skew-holomorphic Jacobi cusp form of weight 2 and index m naturally attached to X, where X is a simply-laced root system for which all irreducible components have Coxeter number m (cf. §4.2).

H_g^X The umbral McKay–Thompson series attached to $g \in G^X$ for X a Niemeier root system (cf. §5). A vector-valued mock modular form of weight 1/2. Also denoted $H_g^{(\ell)}$ for ℓ the lambency corresponding to X.

S_g^X The vector-valued cusp form conjectured to be the shadow of H_g^X, for $g \in G^X$ and X a Niemeier root system (cf. §§5.1,6.2).

Ω_g^X The $2m \times 2m$ matrix attached to $g \in G^X$ for X a Niemeier root system with Coxeter number m (cf. §5.1).

K^X The umbral module attached to the Niemeier root system X. A conjectural G^X-module with graded-super-characters given by the H_g^X (cf. §6.1).

n_g The order of the image of an element $g \in G^X$ in the quotient group $\bar{G}^X$ (cf. §6.2).

h_g The unique positive integer such that $n_g h_g$ is the product of the shortest and longest cycle lengths in the cycle shape $\widetilde{\Pi}_g^X$ for $g \in G^X$ and X a Niemeier root system (cf. §6.2).

$\widetilde{\Pi}_g^X$ The cycle shape attached to $g \in G^X$ via the permutation representation of G^X with twisted Euler character $\widetilde{\chi}^X$ (cf. §§2.4,6.2,B.2). Similarly for $\widetilde{\Pi}_g^{X_A}$, $\widetilde{\Pi}_g^{X_D}$, &c.

ν_g^X The multiplier system of H_g^X (cf. §6.2).

2 Groups

2.1 Root Systems

In this subsection we give a brief summary of simply-laced root systems and their corresponding *Dynkin diagrams*. Standard references for this material include [25] and [26].

Let V be a finite-dimensional vector space of rank r over $\mathbb{R}$ equipped with an inner product $\langle \cdot, \cdot \rangle$. For $v \in V$ define the hyperplane H_v to be the set of elements of V orthogonal to v and the reflection in the hyperplane H_v to be the linear map $r_v : V \to V$ defined by

$$r_v(v') = v' - 2\frac{\langle v, v' \rangle}{\langle v, v \rangle} v. \tag{2.1}$$

A finite subset $X \subset V$ of non-zero vectors is a rank r crystallographic *root system* if

- X spans V,
- $r_\alpha(X) \in X$ for all $\alpha \in X$,
- $X \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in X$,
- $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in X$.

Given a root system X we say that X is *irreducible* provided that it can not be partitioned into proper subsets $X = X_1 \cup X_2$ with $\langle \alpha_1, \alpha_2 \rangle = 0$ for all $\alpha_1 \in X_1$ and $\alpha_2 \in X_2$. If X is an irreducible root system then there are at most two values for the length $\langle \alpha, \alpha \rangle^{1/2}$ that occur. If all roots have the same length then the irreducible root system is called *simply-laced*.

It is possible to choose a subset of roots in X that form a basis of V. We define a subset $\Phi = \{f_1, f_2, \cdots, f_{\mathsf{r}}\} \subset X$ to be a set of *simple roots* provided that

- Φ is a basis for V,
- each root $\alpha \in X$ can be written as a linear combination of the f_i with integer coefficients

$$\alpha = \sum_{i=1}^{\mathsf{r}} n_i f_i \tag{2.2}$$

  and with either all $n_i \leq 0$ or all $n_i \geq 0$.

Given a choice of simple roots we define the positive roots of X to be those α for which all $n_i \geq 0$ in (2.2). The negative roots are those for which all $n_i \leq 0$. We also define the *height* of α as in (2.2) by setting

$$\mathrm{ht}(\alpha) = \sum_{i=1}^{\mathsf{r}} n_i. \tag{2.3}$$

To each irreducible root system we can associate a connected Dynkin diagram as follows. We associate a node to each simple root. The nodes associated to two distinct simple roots f_i, f_j are then either not connected if $\langle f_i, f_j \rangle = 0$ or connected by N_{ij} lines with

$$N_{ij} = \frac{2\langle f_i, f_j \rangle}{\langle f_i, f_i \rangle} \frac{2\langle f_j, f_i \rangle}{\langle f_j, f_j \rangle} \in \{1, 2, 3\}. \tag{2.4}$$

The Dynkin diagrams associated to simply-laced irreducible root systems all have $N_{ij} = \{0, 1\}$ and are of type A_n, D_n, E_6, E_7, E_8 as shown in Figure 1. Here the subscript indicates the rank of the associated root system, and in the figure we choose a specific enumeration of simple roots for later use in §2.4.

The height function defines a $\mathbb{Z}$-gradation on the set of roots. Every irreducible root system has a unique root θ of largest height with respect to a given set of simple roots $\Phi = \{f_i\}$ with an expansion

$$\theta = \sum_{i=1}^{\mathsf{r}} a_i f_i \tag{2.5}$$

where the a_i are a set of integers known as the *Coxeter labels* of the root system or Dynkin diagram. If we append the negative of this highest root (the lowest root) to the simple roots of the simply-laced root system, we obtain the extended Dynkin diagrams of type $\widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7, \widehat{E}_8$. These are shown in Figure 2, where we indicate the lowest root with a filled in circle and the simple roots with empty circles.

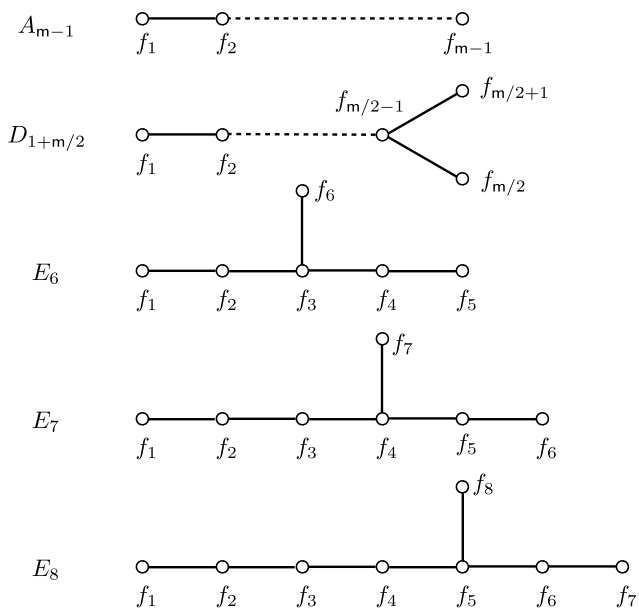


Figure 1: The ADE Dynkin diagrams

Given an irreducible root system X, its *Coxeter number* $\mathsf{m} = \mathsf{m}(X)$ is the sum

$$\mathsf{m} = 1 + \sum_{i=1}^{\mathsf{r}} a_i. \tag{2.6}$$

An equivalent definition of the Coxeter number may be given in terms of the Weyl group of X. The *Weyl group* W^X is the group generated by the reflections r_α for $\alpha \in X$. The product $w = r_1 r_2 \dots r_{\mathsf{r}}$ of reflections $r_i := r_{f_i}$ in simple roots $f_i \in \Phi$ is called a *Coxeter element* of W^X and is uniquely determined up to conjugacy in W^X, meaning that different choices of simple roots and different orderings of the simple roots chosen lead to conjugate elements of W^X. The Coxeter number $\mathsf{m} = \mathsf{m}(X)$ is then the order of any Coxeter element of X.

We obtain a finer invariant of X by considering the eigenvalues of a Coxeter element of W^X. Say $u_1, \dots, u_{\mathsf{r}}$ are the *Coxeter exponents* of X if a Coxeter element w has eigenvalues $e^{2\pi i u_1/\mathsf{m}}, \dots, e^{2\pi i u_{\mathsf{r}}/\mathsf{m}}$ (counting multiplicity). This data is conveniently recorded using the notion of *Frame shape*, whereby a formal product $\prod_i n_i^{k_i}$ (with $n_i, k_i \in \mathbb{Z}$ and $n_i > 0$) serves as a shorthand for the rational polynomial $\prod_i (x^{n_i} - 1)^{k_i}$. For each Coxeter element there is a Frame shape π^X—the *Coxeter Frame shape* of X—such that the corresponding polynomial function coincides with the characteristic polynomial $\prod_{i=1}^{\mathsf{r}} (x - e^{2\pi i u_i/\mathsf{m}})$ of w. These Frame shapes will play a prominent role in what follows. They are given along with the corresponding Coxeter

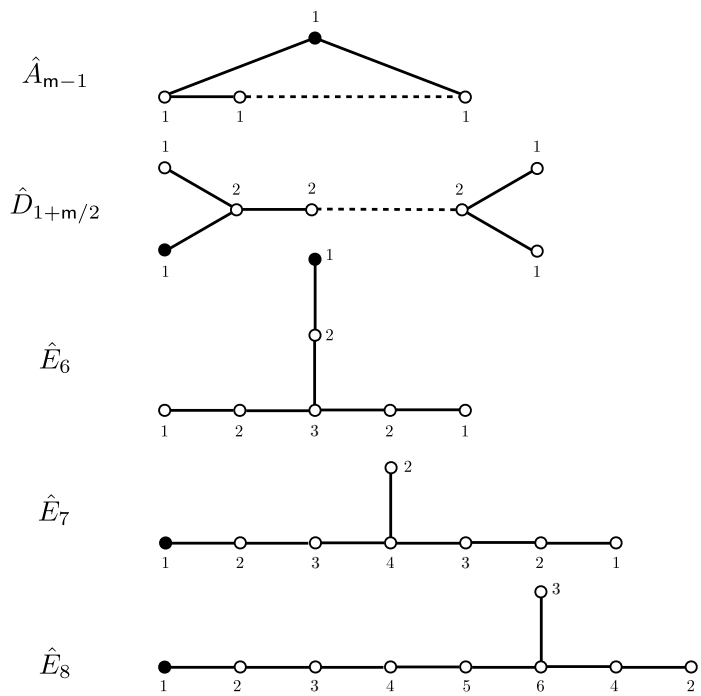


Figure 2: The extended ADE Dynkin diagrams

numbers in Table 1.

2.2 Lattices

A *lattice* is a free $\mathbb{Z}$-module equipped with a symmetric bilinear form $\langle \cdot\,, \cdot \rangle$. We say that a lattice L is positive-definite if $\langle \cdot\,, \cdot \rangle$ induces a positive-definite inner product on the vector space $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$. Since L is a free $\mathbb{Z}$-module the natural map $L \to L_{\mathbb{R}}$ is an embedding and we may identify L with its image in $L_{\mathbb{R}}$. Say that L is integral if we have $\langle \lambda, \mu \rangle \in \mathbb{Z}$ for all $\lambda, \mu \in L$ and say that L is even if we have $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ for each $\lambda \in \Lambda$. (An even lattice is necessarily integral.) The dual of L is the lattice $L^* \subset L_{\mathbb{R}}$ defined by setting

$$L^* = \{\lambda \in L_{\mathbb{R}} \mid \langle \lambda, L \rangle \subset \mathbb{Z}\}. \tag{2.7}$$

Clearly, if L is integral then L^* contains L. In the case that L^* coincides with (the image of) L (in $L_{\mathbb{R}}$) we say that L is unimodular. For an even lattice L we call $L_2 = \{\lambda \in L \mid \langle \lambda, \lambda \rangle = 2\}$ the set of *roots* of L.

	$A_{\mathsf{m}-1}$	$D_{1+\mathsf{m}/2}$	E_6	E_7	E_8
Coxeter number	m	m	12	18	30
Coxeter exponents	$1, 2, 3, \ldots, \mathsf{m}-1$	$1, 3, 5, \ldots, \mathsf{m}-1,$ $\mathsf{m}/2$	1,4,5, 7, 8, 11	1,5,7,9, 11,13,17	1,7,11,13, 17,19,23,29
Coxeter Frame shapes	$\frac{\mathsf{m}}{1}$	$\frac{2.\mathsf{m}}{1.(\mathsf{m}/2)}$	$\frac{2.3.12}{1.4.6}$	$\frac{2.3.18}{1.6.9}$	$\frac{2.3.5.30}{1.6.10.15}$

Table 1: Coxeter numbers, exponents, and Frame shapes

The Leech lattice is the unique (up to isomorphism) even, unimodular, positive-definite lattice of rank 24 with no roots [27], and is named for its discoverer, John Leech [3,28]. Shortly after Leech's work, the unimodular even positive-definite lattices of rank 24 were classified by Niemeier [2]; we refer to those with non-empty root sets as the Niemeier lattices. There are exactly 23 Niemeier lattices up to isomorphism, and if L is such a lattice then its isomorphism type is determined by its root set L_2, which is a union of irreducible simply-laced root systems (cf. §2.1). Say a root system X is a Niemeier root system if it occurs as L_2 for some Niemeier lattice L. The Niemeier root systems are precisely the 23 root systems satisfying the two conditions that first, they are unions of simply-laced root systems with the same Coxeter numbers, and second, the total rank is 24. Explicitly, they are

$$A_1^{24},\ A_2^{12},\ A_3^{8},\ A_4^{6},\ A_5^{4}D_4,\ A_6^{4},\ A_7^{2}D_5^{2},\ A_8^{3},\ A_9^{2}D_6,\ A_{11}D_7E_6,\ A_{12}^{2},\ A_{15}D_9,\ A_{17}E_7,\ A_{24}, \tag{2.8}$$

$$D_4^{6},\ D_6^{4},\ D_8^{3},\ D_{10}E_7^{2},\ D_{12}^{2},\ D_{16}E_8,\ D_{24}, \tag{2.9}$$

$$E_6^{4},\ E_8^{3}. \tag{2.10}$$

In (2.8) we list the Niemeier root systems containing a type A component, in (2.9) we list the root systems containing a type D component but no type A component, and the remaining two root systems, having only type E components, appear in (2.10). We will call them the *A-type*, *D-type*, and the *E-type* Niemeier root systems, respectively. We say that a Niemeier root system X has *Coxeter number* m if m is the Coxeter number of any simple component of X.

Since all the simple components of a Niemeier root system have the same Coxeter number all the type A components appearing have the same rank, and similarly for components of type D and E. So we can write

$$X = X_A X_D X_E \tag{2.11}$$

where $X_A = A_{\mathsf{m}-1}^{d_A^X}$ for some non-negative integer d_A^X (or $X_A = \emptyset$), and m the Coxeter number of X, and similarly for X_D and X_E. For example,

$$\text{if } X = A_7^2 D_5^2 \text{ then } \mathsf{m} = 8,\ X_A = A_7^2,\ X_D = D_5^2,\ d_A^X = d_D^X = 2 \text{ and } X_E = \emptyset. \tag{2.12}$$

Before finishing this subsection we will comment on the relation between the Niemeier lattices and the Leech lattice. The covering radius of the Leech lattice is $\sqrt{2}$ according to [29], meaning that $R = \sqrt{2}$ is the minimal positive R such that the 24-dimensional vector space $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is covered by placing a closed ball of radius R at each point of Λ,

$$\sqrt{2} = \sup_{x \in \Lambda_{\mathbb{R}}} \inf_{\lambda \in \Lambda} \|x - \lambda\|. \tag{2.13}$$

A point $x \in \Lambda_{\mathbb{R}}$ that realizes the maximum value $\sqrt{2} = \inf_{\lambda \in \Lambda} \|x - \lambda\|$ is called a *deep hole* of Λ. Let $x \in \Lambda_{\mathbb{R}}$ be a deep hole and let V_x be the set of vertices of x,

$$V_x = \left\{ \lambda \in \Lambda \mid \|x - \lambda\| = \sqrt{2} \right\}. \tag{2.14}$$

It is shown in [29] that if $\lambda, \lambda' \in V_x$ with $\lambda \neq \lambda'$ then $\|\lambda - \lambda'\|^2 \in \{4, 6, 8\}$. Following [29] define the hole diagram attached to x by joining vertices $\lambda, \lambda' \in V_x$ with a single edge if $\|\lambda - \lambda'\|^2 = 6$, and by joining them with a double edge if $\|\lambda - \lambda'\|^2 = 8$. The vertices λ and λ' are disjoined in case $\|\lambda - \lambda'\|^2 = 4$. Then the diagram so obtained is the extended Dynkin diagram corresponding to a Niemeier root system, and all Niemeier root systems arise in this way [29]. Conversely, from each Niemeier lattice one can obtain a different "holy" construction of the Leech lattice [30].

2.3 Genus Zero Groups

In this section we attach a genus zero subgroup of $SL_2(\mathbb{R})$ to each of the 23 Niemeier root systems.

If Γ is a discrete subgroup of $SL_2(\mathbb{R})$ that is commensurable with the modular group $SL_2(\mathbb{Z})$ then its natural action on the boundary $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\mathbf{i}\infty\}$ of the upper half plane $\mathbb{H}$ restricts to $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\mathbf{i}\infty\}$. The orbits of Γ on $\widehat{\mathbb{Q}}$ are called the *cusps* of Γ, and the quotient space

$$X_\Gamma = \Gamma \backslash \mathbb{H} \cup \widehat{\mathbb{Q}} \tag{2.15}$$

is naturally a compact Riemann surface (cf. e.g. [31, §1.5]). We adopt the common practice of saying that Γ has *genus zero* in case X_Γ is a genus zero surface.

For n a positive integer the *Hecke congruence group* of level n, denoted $\Gamma_0(n)$, is defined by

setting

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ cn & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\, ad - bcn = 1 \right\}. \tag{2.16}$$

Say e is an *exact divisor* of n, and write $e \| n$, if $e|n$ and $(e, n/e) = 1$. According to [5] the normaliser $N(\Gamma_0(n))$ of $\Gamma_0(n)$ in $SL_2(\mathbb{R})$ is commensurable with $SL_2(\mathbb{Z})$ and admits the description

$$N(\Gamma_0(n)) = \left\{ \frac{1}{\sqrt{e}} \begin{pmatrix} ae & b/h \\ cn/h & de \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\, e\|n/h,\, ade - bcn/eh^2 = 1 \right\} \tag{2.17}$$

where h is the largest divisor of 24 such that h^2 divides n. So if $e\|n$ then we obtain a coset $W_n(e)$ for $\Gamma_0(n)$ in its normaliser by setting

$$W_n(e) = \left\{ \frac{1}{\sqrt{e}} \begin{pmatrix} ae & b \\ cn & de \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\, e\|n,\, ade - bcn/e = 1 \right\}. \tag{2.18}$$

Observe that the product of any two elements of $W_n(e)$ lies in $W_n(1) = \Gamma_0(n)$. More generally, the operation $e * f = ef/(e,f)^2$ equips the set of exact divisors of n with a group structure isomorphic to that obtained by multiplication of Atkin–Lehner involutions, $W_n(e)W_n(f) = W_n(e * f)$. So for S a subgroup of the group of exact divisors of n we may define a group $\Gamma_0(n) + S$, containing and normalizing $\Gamma_0(n)$, by taking the union of the Atkin–Lehner cosets $W_n(e)$ for $e \in S$. It is traditional [5] to simplify this notation by writing $\Gamma_0(n) + e, f, \ldots$ in place of $\Gamma_0(n) + \{1, e, f, \ldots\}$. Thus we have

$$\Gamma_0(n) + S = \left\{ \frac{1}{\sqrt{e}} \begin{pmatrix} ae & b \\ cn & de \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\, e \in S,\, ade - bcn/e = 1 \right\}. \tag{2.19}$$

The positive integers occurring as Coxeter numbers of the A-type Niemeier root systems (cf. (2.8)) are

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25. \tag{2.20}$$

Observe that these are exactly the positive integers $n > 1$ for which the Hecke congruence group $\Gamma_0(n)$ has genus zero (cf. e.g. [32]). The Coxeter numbers of the D-type Niemeier root systems

(cf. (2.9)) are

$$6, 10, 14, 18, 22, 30, 46, \tag{2.21}$$

and these are exactly the even integers $2n$ such that the group $\Gamma_0(2n)+n$ has genus zero [32]. We will demonstrate momentarily that the root system E_6^4, having Coxeter number 12, is naturally attached to the genus zero group $\Gamma_0(12)+4$, and E_8^3, having Coxeter number 30, is naturally attached to the genus zero group $\Gamma_0(30)+6, 10, 15$. As such, we obtain a correspondence between the 23 Niemeier root systems and the genus zero groups of the form

$$\Gamma_0(n), \quad \Gamma_0(2n)+n, \quad \Gamma_0(12)+4, \quad \Gamma_0(30)+6, 10, 15. \tag{2.22}$$

Write Γ^X for the genus zero subgroup of $SL_2(\mathbb{R})$ associated in this way to a Niemeier root system X. Write T^X for the unique principal modulus for Γ^X that has an expansion

$$T^X = q^{-1} - d^X + O(q) \tag{2.23}$$

about the infinite cusp, where d^X denotes the number of irreducible components of X. Then we may recover T^X, and hence also Γ^X, directly, as follows, from the Coxeter Frame shapes (cf. §2.1) of the irreducible components of X.

Define the Coxeter Frame shape π^X of an arbitrary root system X to be the product of Coxeter Frame shapes of the irreducible components of X. Next, for a Frame shape $\pi = \prod_i n_i^{k_i}$ define the associated *eta product* η_π by setting

$$\eta_\pi(\tau) = \prod_i \eta(n_i\tau)^{k_i}, \tag{2.24}$$

and observe that if X is simply laced and π^X is the Coxeter Frame shape of X then

$$\frac{1}{\eta_{\pi^X}(\tau)} = q^{-\mathsf{r}/24}(1 - d^X q + O(q^2)) \tag{2.25}$$

where r denotes the rank of X and d^X is the number of irreducible components. We may also consider the *lambda sum* λ_π attached to a Frame shape π, which is the function

$$\lambda_\pi(\tau) = \sum_i k_i \lambda_{n_i}(\tau) \tag{2.26}$$

where $\lambda_n(\tau)$ is defined in (A.9). Observe that if $\pi = \prod_i n_i^{k_i}$ is such that $\sum_i k_i = 0$ then

Table 2: Niemeier Root Systems and Principal Moduli

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
π^X	$\frac{2^{24}}{1^{24}}$	$\frac{3^{12}}{1^{12}}$	$\frac{4^8}{1^8}$	$\frac{5^6}{1^6}$	$\frac{2^1 6^5}{1^5 3^1}$	$\frac{7^4}{1^4}$	$\frac{2^2 8^4}{1^4 4^2}$
Γ^X	$2B$	$3B$	$4C$	$5B$	$6E$	$7B$	$8E$

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
π^X	$\frac{9^3}{1^3}$	$\frac{2^1 10^3}{1^3 5^1}$	$\frac{2^2 3^1 12^3}{1^3 4^1 6^2}$	$\frac{13^2}{1^2}$	$\frac{2^1 16^2}{1^2 8^1}$	$\frac{2^1 3^1 18^2}{1^2 6^1 9^1}$	$\frac{25^1}{1^1}$
Γ^X	$9B$	$10E$	$12I$	$13B$	$16B$	$18D$	$(25Z)$

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
π^X	$\frac{2^6 6^6}{1^6 3^6}$	$\frac{2^4 10^4}{1^4 5^4}$	$\frac{2^3 14^3}{1^3 7^3}$	$\frac{2^3 3^2 18^3}{1^3 6^2 9^3}$	$\frac{2^2 22^2}{1^2 11^2}$	$\frac{2^2 3^1 5^1 30^2}{1^2 6^1 10^1 15^2}$	$\frac{2^1 46^1}{1^1 23^1}$
Γ^X	$6C$	$10B$	$14B$	$18C$	$22B$	$30G$	$46AB$

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
π^X	$\frac{2^4 3^4 12^4}{1^4 4^4 6^4}$	$\frac{2^3 3^3 5^3 30^3}{1^3 6^3 10^3 15^3}$
Γ^X	$12B$	$30A$

$\lambda_\pi = q\partial_q \log \eta_\pi$.

The Coxeter Frame shapes of the Niemeier root systems are given in Table 2. By inspection we obtain the following result.

Proposition 2.1. *If X is a Niemeier root system and π^X is the Coxeter Frame shape of X then*

$$T^X = \frac{1}{\eta_{\pi^X}} \tag{2.27}$$

is the unique principal modulus for Γ^X satisfying $T^X = q^{-1} - d^X + O(q)$ as $\tau \to i\infty$.

Remark 2.2. Niemeier's classification of even unimodular positive-definite lattices of rank 24 together with Proposition 2.1 imply that if X is the root system of an even unimodular positive-

definite lattice of rank 24 then the eta product of the Coxeter Frame shape of X is a principal modulus for a genus zero subgroup of $SL_2(\mathbb{R})$. It would be desirable to have a conceptual proof of this fact.

The relation between the T^X and umbral moonshine will be discussed in §4.2, where the weight two Eisenstein forms

$$f^X = \lambda_{\pi^X} \tag{2.28}$$

will play a prominent role. (Cf. Table 6.) We have $\sum_i k_i = 0$ when $\pi^X = \prod_i n_i^{k_i}$ for every Niemeier root system X, so the functions f^X and T^X are related by

$$f^X = \lambda_\pi = q\partial_q \log \eta_\pi = -q\partial_q \log T^X \tag{2.29}$$

for $\pi = \pi^X$.

It is interesting to note that all of the Γ^X, except for $X = A_{24}$, appear in monstrous moonshine as genus zero groups for whom monstrous McKay–Thompson series serve as principal moduli. Indeed, all of the Frame shapes π^X, except for $X = A_{24}$, are Frame shapes of elements of Conway's group Co_0, the automorphism group of the Leech lattice (cf. [5, §7]). We observe that for the cases that π^X is the Frame shape of an element in Co_0 the corresponding centralizer in Co_0 typically contains a subgroup isomorphic to G^X.

We include the ATLAS names [33] (see also [5]) for the monstrous conjugacy classes corresponding to the groups Γ^X via monstrous moonshine in the rows labelled Γ^X in Table 2. Extending the notation utilised in [1] we assign *lambencies* ℓ—now symbols rather than integers—to each Niemeier system X according to the prescription of Table 2. The lambencies then serve to name the groups Γ^X also, according to the convention that n corresponds to $\Gamma_0(n)$, and $12+4$ corresponds to $\Gamma_0(12)+4$, &c. It will be convenient in what follows to sometimes use (ℓ) in place of X, writing $G^{(\ell)}$, $H^{(\ell)}$, &c., to label the finite groups and mock modular forms associated to the corresponding Niemeier root system.

2.4 Umbral Groups

Given a Niemeier root system X we may consider the automorphism group of the associated Niemeier lattice L^X. The reflections in roots of L^X generate a normal subgroup of the full automorphism group of L^X—the Weyl group of X—which we denote W^X. We define G^X to

be the corresponding quotient,

$$G^X = \mathrm{Aut}(L^X)/W^X. \tag{2.30}$$

The particular groups G^X arising in this way are displayed in Table 3. Observe[1] that the group $G^{(\ell)}$ of [1] appears here as G^X for X the unique root system with a component $A_{\ell-1}$. In fact, the $G^{(\ell)}$ of [1] are exactly those G^X for which X is of the form $X = A_{\ell-1}^d$ with even d. It will develop that, for every Niemeier root system X, the representation theory of G^X is intimately related to a set of vector-valued mock modular forms H_g^X, to be introduced in §4-5.

Table 3: Umbral Groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

As mentioned in §2.3, it will often be useful to use the lambencies to label the groups and mock modular forms associated to a given Niemeier root system. To this end we define $G^{(n)} = G^X$ in case $\Gamma_0(n)$ has genus zero and X is the unique A-type Niemeier root system with

[1] We are grateful to George Glauberman for first alerting us to this fact.

Coxeter number n (cf. (2.8)). We define $G^{(2n+n)} = G^X$ when $\Gamma_0(2n) + n$ has genus zero and X is the unique D-type Niemeier root system with Coxeter number $2n$ (cf. (2.9)). We write $G^{(12+4)}$ for G^X when $X = E_6^4$ and we write $G^{(30+6,10,15)}$ for G^X when $X = E_8^3$.

Observe that the subgroup $\widehat{W}^X < \mathrm{Aut}(L^X)$ consisting of automorphisms of L^X that stabilize the irreducible components of X is also normal in $\mathrm{Aut}(L^X)$. Define $\bar{G}^X$ to be the corresponding quotient,

$$\bar{G}^X = \mathrm{Aut}(L^X)/\widehat{W}^X, \tag{2.31}$$

so that $\bar{G}^X$ is precisely the group of permutations of the irreducible components of X induced by automorphisms of L^X, and is a quotient of G^X (viz., the quotient by $\widehat{W}^X/W^X$) since $W^X < \widehat{W}^X$. It turns out that $\widehat{W}^X/W^X$ has order 2 when $X_A \neq \emptyset$ or $X = X_E = E_6^4$, has order 3 when $X = X_D = D_4^6$, and is trivial otherwise.

Remark 2.3. In terms of the notation of [30] we have $W^X = G_0$, $\widehat{W}^X/W^X \simeq G_1$, $\bar{G}^X \simeq G_2$ and $G^X \simeq G_1 G_2$.

The groups G^X and $\bar{G}^X$ come naturally equipped with various permutation representations. To see this choose a set Φ of simple roots for L^X, meaning a set which is the union of sets of simple roots for each irreducible root sublattice of L^X. Then Φ constitutes a basis for the 24-dimensional space $L^X_{\mathbb{R}}$, and for each $g \in G^X$ there is a unique element in the pre-image of g under $\mathrm{Aut}(L^X) \to G^X$ that belongs to the subgroup $\mathrm{Aut}(L^X, \Phi)$, consisting of automorphisms of L^X that stabilize Φ as a set (i.e. act as permutations of the irreducible root subsystems followed by permutations—corresponding to Dynkin diagram automorphisms—of simple roots within irreducible root subsystems). Thus we obtain a section $G^X \to \mathrm{Aut}(L^X)$ of the projection $\mathrm{Aut}(L^X) \to G^X$ whose image is $\mathrm{Aut}(L^X, \Phi)$, and composition with the natural map $\mathrm{Aut}(L^X, \Phi) \to \mathit{Sym}_\Phi$ defines a permutation representation of G^X on Φ. Write $\Phi = \Phi_A \cup \Phi_D \cup \Phi_E$ where Φ_A contains the roots in Φ belonging to type A components of X, and similarly for Φ_D and Φ_E. Then the decomposition $\Phi = \Phi_A \cup \Phi_D \cup \Phi_E$ is stable under G^X, since $\mathrm{Aut}(L^X)$ cannot mix roots that belong to non-isomorphic root systems, so we obtain maps $G^X \to \mathit{Sym}_{\Phi_A}$, $G^X \to \mathit{Sym}_{\Phi_D}$ and $G^X \to \mathit{Sym}_{\Phi_E}$. Write $g \mapsto \widetilde{\chi}^X_g$ for the character of G^X attached to the representation $G^X \to \mathit{Sym}_\Phi$, write $g \mapsto \widetilde{\chi}^{X_A}_g$ for that attached to $G^X \to \mathit{Sym}_{\Phi_A}$, and interpret $\widetilde{\chi}^{X_D}$ and $\widetilde{\chi}^{X_E}$ similarly. Observe that $\widetilde{\chi}^X$ (and hence also $\widetilde{\chi}^{X_A}$, $\widetilde{\chi}^{X_D}$ and $\widetilde{\chi}^{X_E}$) are independent of the choice of Φ. We set $\widetilde{\chi}^{X_A} = 0$ in case Φ_A is empty, and similarly for $\widetilde{\chi}^{X_D}$ and $\widetilde{\chi}^{X_E}$. We have

$$\widetilde{\chi}^X = \widetilde{\chi}^{X_A} + \widetilde{\chi}^{X_D} + \widetilde{\chi}^{X_E}. \tag{2.32}$$

The characters $\widetilde{\chi}^{X_A}$, &c., are naturally decomposed further as follows. Suppose that $X_A \neq \emptyset$. Then $X_A = A_{\mathsf{m}-1}^{d_A}$ for $d_A = d_A^X$ (cf. (2.11)) and we may write

$$\Phi_A = \left\{ f_j^i \mid 1 \leq i \leq d_A,\, 1 \leq j \leq \mathsf{m}-1 \right\} \tag{2.33}$$

where the superscript indicates the $A_{\mathsf{m}-1}$ component to which the simple root f_j^i belongs, and the inner products between the f_j^i for varying j are as described by the labeling in Figure 1 (so that $\langle f_j^i, f_k^i \rangle$ is -1 or 0 according as the nodes labelled f_j and f_k are joined by an edge or not). Then for fixed j the vectors $\{f_j^i + f_{\mathsf{m}-j}^i \mid 1 \leq i \leq d_A\}$ define a permutation representation of degree d_A for G^X. We denote the corresponding character $g \mapsto \bar{\chi}_g^{X_A}$ since the isomorphism type of the representation is evidently independent of the choice of j. Observe that $\bar{\chi}^{X_A}$ is generally not a faithful character since permutations of Φ_A arising from diagram automorphisms, exchanging f_j^i with $f_{\mathsf{m}-j}^i$ for some i, act trivially. The vectors $\{f_j^i - f_{\mathsf{m}-j}^i \mid 1 \leq i \leq d_A\}$ also span G^X-invariant subspaces of $\mathrm{Span}_{\mathbb{R}}\, \Phi_A < L_{\mathbb{R}}^X$, with different j in the range $0 < j < \mathsf{m}/2$ furnishing isomorphic (signed permutation) representations; we denote the corresponding character $g \mapsto \chi_g^{X_A}$. Since the f_j^i are linearly independent we can conclude that

$$\widetilde{\chi}^{X_A} = \left\lceil \frac{\mathsf{m}-1}{2} \right\rceil \bar{\chi}^{X_A} + \left\lfloor \frac{\mathsf{m}-1}{2} \right\rfloor \chi^{X_A} \tag{2.34}$$

by counting the possibilities for j in each case.

If Φ_D is non-empty then m is even and $X_D = D_{\mathsf{m}/2+1}^{d_D}$ for $d_D = d_D^X$ (cf. (2.11)). Write now

$$\Phi_D = \left\{ f_j^i \mid 1 \leq i \leq d_D,\, 1 \leq j \leq \mathsf{m}/2+1 \right\} \tag{2.35}$$

where, similar to the above, the superscript indicates the $D_{\mathsf{m}/2+1}$ component to which the simple root f_j^i belongs, and the inner products between the f_j^i for varying j are as described in Figure 1. Suppose first that $\mathsf{m} > 6$. Then $\mathsf{m}/2+1 > 4$ and the only non-trivial diagram automorphism of $D_{\mathsf{m}/2+1}$ has order 2 and interchanges $f_{\mathsf{m}/2}^i$ and $f_{\mathsf{m}/2+1}^i$. So we find that for $1 \leq j < \mathsf{m}/2$ the sets $\{f_j^i \mid 1 \leq i \leq d_D\}$ serve as bases for isomorphic permutation representations of degree d_D for G^X, as does $\{f_{\mathsf{m}/2}^i + f_{\mathsf{m}/2+1}^i \mid 1 \leq i \leq d_D\}$; we denote the character of this (i.e. any one of these) permutation representation(s) by $g \mapsto \bar{\chi}_g^{X_D}$. We define χ^{X_D} to be the (signed permutation) character of the representation spanned by the vectors $\{f_{\mathsf{m}/2}^i - f_{\mathsf{m}/2+1}^i \mid 1 \leq i \leq d_D\}$, and we have

$$\widetilde{\chi}^{X_D} = \frac{\mathsf{m}}{2} \bar{\chi}^{X_D} + \chi^{X_D} \tag{2.36}$$

when $X_D \neq \emptyset$ and $\mathsf{m} > 6$. In case $\mathsf{m} = 6$ the group of diagram automorphisms of $D_{\mathsf{m}/2+1} = D_4$ is a copy of S_3, acting transitively on the sets $\{f_1^i, f_3^i, f_4^i\}$ (for fixed i), so we define $\bar{\chi}^{X_D}$ to be the character attached to the (permutation) representation of G^X spanned by $\{f_2^i \mid 1 \leq i \leq d_D\}$ (or equivalently, $\{f_1^i + f_3^i + f_4^i \mid 1 \leq i \leq d_D\}$) and define $\check{\chi}^{X_D}$ to be the character of the representation spanned by the vectors $\{f_1^i - f_3^i, f_1^i - f_4^i \mid 1 \leq i \leq d_D\}$. Evidently

$$\widetilde{\chi}^{X_D} = 2\bar{\chi}^{X_D} + \check{\chi}^{X_D} \tag{2.37}$$

in case $\mathsf{m} = 6$. In preparation for §5.1, where the characters defined here will be used to specify certain vector-valued cusp forms of weight 3/2, we define $\chi_g^{X_D} = \mathrm{sgn}_g^{X_D} \bar{\chi}_g^{X_D}$ for $g \in G^X$ when $X = A_4^5 D_4$ or $X = D_4^6$—the two cases for which X involves D_4—where $\mathrm{sgn}_g^{X_D} = \pm 1$ is the function defined as follows. Write the image of $g \in G^X$ in Sym_{Φ_D} as a product $g_d \circ g_p$ where $g_p \cdot f_j^i = f_j^{\pi(i)}$ for all $j \in \{1, 2, 3, 4\}$, for some permutation $\pi \in Sym_{d_D^X}$, and $g_d \cdot f_j^i = f_{\sigma_i(j)}^i$ for some permutations $\sigma_i \in Sym_4$. Then set $\mathrm{sgn}_g^{X_D} = \prod_{i=1}^{d_D^X} \mathrm{sgn}(\sigma_i)$.

If $\Phi_E = E_n^{d_E}$ for $d_E = d_E^X > 0$ then we may identify $f_j^i \in \Phi_E$ such that

$$\Phi_E = \left\{ f_j^i \mid 1 \leq i \leq d_E,\, 1 \leq j \leq n \right\} \tag{2.38}$$

and, as above, the superscripts enumerate simple components of X_E and the subscripts indicate inner products for simple vectors within a component as per Figure 1. Define $\bar{\chi}^{X_E}$ to be the character of G^X attached to the permutation representation spanned by the set $\{f_3^i \mid 1 \leq i \leq d_E\}$ (for example). In case $n = 6$ write χ^{X_E} for the character of G^X attached to the representation afforded by $\mathrm{Span}_{\mathbb{R}}\{f_1^i - f_5^i \mid 1 \leq i \leq d_E\}$. We have

$$\widetilde{\chi}^{X_E} = n\bar{\chi}^{X_E} \tag{2.39}$$

when $n \in \{7, 8\}$ since for each $1 \leq j \leq n$ the set $\{f_j^i \mid 1 \leq i \leq d_E\}$ spans a representation with character $\bar{\chi}^{X_E}$ in these cases, and

$$\widetilde{\chi}^{X_E} = 4\bar{\chi}^{X_E} + 2\chi^{X_E} \tag{2.40}$$

when $n = 6$, the invariant subspace with character $2\chi^{X_E}$ being spanned by the vectors $f_1^i - f_5^i$ and $f_2^i - f_4^i$ for $1 \leq i \leq d_E$.

We call the functions $\bar{\chi}_g^{X_A}$, $\chi_g^{X_A}$, $\bar{\chi}_g^{X_D}$, $\chi_g^{X_D}$, &c., the *twisted Euler characters* attached to G^X. They are given explicitly in the tables of §B.2. As mentioned above, we will use them to attach a vector-valued cusp form S_g^X to each $g \in G^X$ for X a Niemeier root system in §5.1.

2.5 McKay Correspondence

The *McKay correspondence* [34] relates finite subgroups of $SU(2)$ to the extended Dynkin diagrams of ADE type by associating irreducible representations of the finite groups to nodes of the corresponding diagrams. A beautiful explanation for this can be given in terms of resolutions of simple singularities $\mathbb{C}^2/G$ for $G < SU(2)$ [35,36]. In §3.5 of [1] we observed a curious connection between the umbral groups $G^{(\ell)}$ and certain finite subgroups $D^{(\ell)} < SU(2)$, for the cases $\ell \in \{3,4,5,7\}$, such that the lambency ℓ and the rank r of the Dynkin diagram attached to $D^{(\ell)}$ via McKay's correspondence are related by $\ell + \mathsf{r} = 11$. In this section we describe an extension of this observation, relating the umbral group $G^{(\ell)}$ to a finite subgroup $D^{(\ell)} < SU(2)$, for each ℓ in $\{3,4,5,6,7,8,9,10\}$.

In [1] it was observed that a Dynkin diagram of rank $11-\ell$ may be attached to each $G^{(\ell)}$ for $\ell \in \{3,4,5,7\}$ in the following manner. If $p = (25-\ell)/(\ell-1)$ then p is a prime and there is a unique (up to conjugacy) subgroup $\bar{L}^{(\ell)} < \bar{G}^{(\ell)}$ such that $\bar{L}^{(\ell)}$ is isomorphic to $PSL_2(p)$ and acts transitively in the degree $24/(\ell-1)$ permutation representation of $G^{(\ell)}$ defined in [1, §3.3]. Now $\bar{L}^{(\ell)}$ has a unique (up to isomorphism) subgroup $\bar{D}^{(\ell)}$ of index p in $\bar{L}^{(\ell)}$—a fact which is peculiar to the particular p arising—and $\bar{D}^{(\ell)}$ is a finite subgroup of $SO(3)$ whose preimage $D^{(\ell)}$ in $SU(2)$ realises the extended diagram $\Delta^{(\ell)}$ corresponding (cf. Figure 2) to a Dynkin diagram of rank $11-\ell$ via McKay's correspondence. In the present setting, with groups $G^{(\ell)}$ defined for all ℓ such that $\Gamma_0(\ell)$ has genus zero, and in particular for $3 \leq \ell \leq 10$, it is possible to extend this correspondence as follows.

Table 4: The McKay Correspondence in Umbral Moonshine

X	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$	A_8^3	$A_9^2D_6$
ℓ	3	4	5	6	7	8	9	10
p	11	7	5	4	3	2	2	2
$G^{(\ell)}$	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4	Dih_6	4
$D^{(\ell)}$	$2.Alt_5$	$2.Sym_4$		Dih_6	Q_8	4	3	2
$\bar{G}^{(\ell)}$	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2	$PSL_2(2)$	2
$\bar{L}^{(\ell)}$	$PSL_2(11)$	$PSL_2(7)$	$PSL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2	$PSL_2(2)$	2
$\bar{D}^{(\ell)}$	Alt_5	Sym_4	Alt_4	Sym_3	2^2	2	3	1
$\Delta^{(\ell)}$	$\widehat{E}_8$	$\widehat{E}_7$	$\widehat{E}_6$	$\widehat{D}_5$	$\widehat{D}_4$	$\widehat{A}_3$	$\widehat{A}_2$	$\widehat{A}_1$

Since $(25-\ell)/(\ell-1)$ is not an integer for $\ell \in \{6,8,10\}$ we seek a new definition of p. Armed with the Niemeier root systems attached to each $G^{(\ell)}$ we set $p = d_A - 1$ in case $X = X_A = A_{\ell-1}^{d_A}$

has only A-type components, and set $p = d_A$ otherwise. This definition yields values coincident with the former one when $(25-\ell)/(\ell-1)$ is an integer. Next we seek a subgroup $\bar{L}^{(\ell)} < \bar{G}^{(\ell)}$ acting transitively on the irreducible components of X_A and X_D that has a unique up to isomorphism index p subgroup $\bar{D}^{(\ell)}$,

$$[\bar{L}^{(\ell)} : \bar{D}^{(\ell)}] = p. \tag{2.41}$$

Such $\bar{L}^{(\ell)}$ and $\bar{D}^{(\ell)}$ exist for each $3 \leq \ell \leq 10$ and are given explicitly in Table 4. In the new cases $\ell \in \{6, 8, 9, 10\}$ the groups $\bar{L}^{(\ell)}$ and $\bar{G}^{(\ell)}$ coincide.

The main observation of this section is the following.

> *For every $3 \leq \ell \leq 10$ the group $\bar{D}^{(\ell)}$ is the image in $SO(3)$ of a finite subgroup $D^{(\ell)} < SU(2)$ that is attached, via McKay's correspondence, to the extended diagram $\Delta^{(\ell)}$ corresponding to a Dynkin diagram of rank $11 - \ell$.*

The group $D^{(\ell)}$ is even a subgroup of $G^{(\ell)}$—the pre image under the natural map $G^X \to \bar{G}^X$—except in the case that $\ell = 5$. (We refer to [1, §3.4] for a discussion of this exceptional case.) To aid in the reading of Table 4 we note here the exceptional isomorphisms

$$PGL_2(5) \simeq Sym_5, \; PSL_2(5) \simeq Alt_5, \tag{2.42}$$

$$PGL_2(3) \simeq Sym_4, \; PSL_2(3) \simeq Alt_4, \tag{2.43}$$

$$PGL_2(2) \simeq PSL_2(2) \simeq Sym_3. \tag{2.44}$$

In [1] we used the common abbreviation $L_n(q)$ for $PSL_n(q)$.

Recall from §3.5 of [1] the following procedure for obtaining a length 8 sequence of Dynkin diagrams. Start with the (finite type) E_8 Dynkin diagram, being star shaped with three branches, and construct a sequence of diagrams iteratively by removing the end node from a branch of maximal length at each iteration. In this way we obtain E_8, E_7, E_6, D_5, D_4, A_3, A_2, A_1, and it is striking to observe that our list $\Delta^{(\ell)}$, obtained by applying the McKay correspondence to distinguished subgroups of the $G^{(\ell)}$, is exactly the sequence obtained from this by replacing (finite type) Dynkin diagrams with their corresponding extended diagrams.

3 Automorphic Forms

In this section we discuss modular objects that play a role in the moonshine relation between mock modular forms and finite groups that is the main focus of this paper.

In what follows we take τ in the upper half-plane $\mathbb{H}$ and $z \in \mathbb{C}$, and adopt the shorthand notation $e(x) = e^{2\pi i x}$. We also define $q = e(\tau)$ and $y = e(z)$ and write

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \tag{3.1}$$

for the natural action of $SL_2(\mathbb{R})$ on $\mathbb{H}$, and write

$$\gamma(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \tag{3.2}$$

for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H} \times \mathbb{C}$. We set

$$\mathrm{j}(\gamma, \tau) = (c\tau + d)^{-1} \tag{3.3}$$

and choose the principal branch of the logarithm (i.e. $x^s = |x|^s e^{\mathbf{i}\theta s}$ when $x = |x| e^{\mathbf{i}\theta}$ and $-\pi < \theta \leq \pi$) to define non-integer exponentials.

3.1 Mock Modular Forms

We briefly recall modular forms, mock modular forms, and their vector-valued generalisations.

Let Γ be a discrete subgroup of the group $SL_2(\mathbb{R})$ that is commensurable with the modular group $SL_2(\mathbb{Z})$. For $w \in \frac{1}{2}\mathbb{Z}$ say that a non-zero function $\psi \colon \Gamma \to \mathbb{C}$ is a *multiplier system* for Γ with weight w if

$$\psi(\gamma_1)\psi(\gamma_2)\,\mathrm{j}(\gamma_1, \gamma_2\tau)^w\,\mathrm{j}(\gamma_2, \tau)^w = \psi(\gamma_1\gamma_2)\,\mathrm{j}(\gamma_1\gamma_2, \tau)^w \tag{3.4}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Given such a multiplier system ψ for Γ we may define the (ψ, w)-*action* of Γ on the space $\mathcal{O}(\mathbb{H})$ of holomorphic functions on the upper half-plane by setting

$$(f|_{\psi,w}\gamma)(\tau) = f(\gamma\tau)\psi(\gamma)\,\mathrm{j}(\gamma, \tau)^w \tag{3.5}$$

for $f \in \mathcal{O}(\mathbb{H})$ and $\gamma \in \Gamma$. We then say that $f \in \mathcal{O}(\mathbb{H})$ is an *(unrestricted) modular form* with multiplier ψ and weight w for Γ in the case that f is invariant for this action; i.e. $f|_{\psi,w}\gamma = f$ for all $\gamma \in \Gamma$. We say that an unrestricted modular form f for Γ with multiplier ψ and weight w is a *weakly holomorphic modular form* in case f has at most exponential growth at the cusps of Γ. We say that f is a *modular form* if $(f|_{\widetilde{\psi},w}\sigma)(\tau)$ remains bounded as $\Im(\tau) \to \infty$ for any $\sigma \in SL_2(\mathbb{Z})$, and we say f is a *cusp form* if $(f|_{\widetilde{\psi},w}\sigma)(\tau) \to 0$ as $\Im(\tau) \to \infty$ for any $\sigma \in SL_2(\mathbb{Z})$.

Suppose that ψ is a multiplier system for Γ with weight w, and g is a modular form for Γ with the conjugate multiplier system $\bar{\psi}\colon \gamma \mapsto \overline{\psi(\gamma)}$ and dual weight $2-w$. Then we may use g to twist the (ψ, w)-action of Γ on $\mathcal{O}(\mathbb{H})$ by setting

$$(f|_{\psi,w,g}\gamma)(\tau) = f(\gamma\tau)\psi(\gamma)\,\mathrm{j}(\gamma,\tau)^w + e(\tfrac{w-1}{4})\int_{-\gamma^{-1}\infty}^{\infty}(\tau'+\tau)^{-w}\overline{g(-\bar{\tau}')}\mathrm{d}\tau'. \tag{3.6}$$

With this definition, we say that $f \in \mathcal{O}(\mathbb{H})$ is an *(unrestricted) mock modular form* with multiplier ψ, weight w and shadow g for Γ if f is invariant for this action; i.e. $f|_{\psi,w,g}\gamma = f$ for all $\gamma \in \Gamma$. We say that an unrestricted mock modular form f for Γ with multiplier ψ, weight w and shadow g is a *weakly holomorphic mock modular form* in case f has at most linear exponential growth at the cusps of Γ. From this point of view a (weakly holomorphic) modular form is a (weakly holomorphic) mock modular form with vanishing shadow. This notion of mock modular form developed from the Maass form theory due to Bruinier–Funke [37], and from Zwegers' work [38] on Ramanujan's mock theta functions.

In this paper we will consider the generalisation of the above definition to *vector-valued (weakly holomorphic) mock modular forms* with n components, where the multiplier $\psi\colon \Gamma \to GL_n(\mathbb{C})$ is a (projective) representation of Γ. From the definition (3.6), it is not hard to see that the multiplier ψ of a (vector-valued) mock modular form is necessarily the inverse of that of its shadow. To avoid clutter, we omit the adjective "weakly holomorphic" in the rest of the paper when there is no room for confusion.

Following Zwegers [38] and Zagier [39] we define a *mock theta function* to be a q-series $h = \sum_n a_n q^n$ such that for some $\lambda \in \mathbb{Q}$ the assignment $\tau \mapsto q^\lambda h|_{q=e(\tau)}$ defines a mock modular form of weight $1/2$ whose shadow is a unary (i.e. attached to a quadratic form in one variable) theta series of weight $3/2$. In §6 we conjecture that specific sets of mock theta functions appear as McKay–Thompson series associated to infinite-dimensional modules for the groups G^X (cf. §2.4), where X is a Niemeier root system.

3.2 Jacobi Forms

We first discuss Jacobi forms following [40]. For every pair of integers k and m, we define the m-action of the group $\mathbb{Z}^2$ and the (k,m)-action of the group $SL_2(\mathbb{Z})$ on the space of holomorphic functions $\phi\colon \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ as

$$(\phi|_m(\lambda,\mu))(\tau,z) = e(m(\lambda^2\tau + 2\lambda z))\,\phi(\tau, z+\lambda\tau+\mu) \tag{3.7}$$

$$(\phi|_{k,m}\gamma)(\tau,z) = e(-m\tfrac{cz^2}{c\tau+d})\;\mathrm{j}(\gamma,\tau)^k\phi(\gamma(\tau,z)) \tag{3.8}$$

where $\gamma \in SL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$. We say a holomorphic function $\phi\colon \mathbb{H}\times\mathbb{C} \to \mathbb{C}$ is an *(unrestricted) Jacobi form* of weight k and index m for the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ if it is invariant under the above actions, $\phi = \phi|_{k,m}\gamma$ and $\phi = \phi|_m(\lambda, \mu)$, for all $\gamma \in SL_2(\mathbb{Z})$ and for all $(\lambda, \mu) \in \mathbb{Z}^2$. In what follows we refer to the transformations (3.7) and (3.8) as the *elliptic* and *modular* transformations, respectively.

The invariance of $\phi(\tau, z)$ under $\tau \to \tau + 1$ and $z \to z + 1$ implies a Fourier expansion

$$\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r)q^n y^r \tag{3.9}$$

and the elliptic transformation can be used to show that $c(n, r)$ depends only on the *discriminant* $D = r^2 - 4mn$ and on r mod $2m$. In other words, we have $c(n, r) = C_{\widetilde{r}}(r^2 - 4mn)$ where $\widetilde{r} \in \mathbb{Z}/2m\mathbb{Z}$ and $r = \widetilde{r}$ mod $2m$, for some appropriate function $D \mapsto C_{\widetilde{r}}(D)$. An unrestricted Jacobi form is called a *weak Jacobi form*, a *(strong) Jacobi form*, or a *Jacobi cusp form* when the Fourier coefficients satisfy $c(n, r) = 0$ whenever $n < 0$, $C_{\widetilde{r}}(D) = 0$ whenever $D > 0$, or $C_{\widetilde{r}}(D) = 0$ whenever $D \geq 0$, respectively. In a slight departure from the notation in [40] we denote the space of weak Jacobi forms of weight k and index m by $J_{k,m}$.

In what follows we will need two further generalisations of the above definitions. The first is relatively straightforward and replaces $SL_2(\mathbb{Z})$ by a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$ in the modular transformation law. (One has to consider Fourier expansions (3.9) for each cusp of Γ.) The second is more subtle and leads to *meromorphic Jacobi forms* which obey the modular and elliptic transformation laws but are such that the functions $z \mapsto \phi(\tau, z)$ are allowed to have poles lying at values of $z \in \mathbb{C}$ corresponding to torsion points of the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. Our treatment of meromorphic Jacobi forms (cf. §3.4) mostly follows [38] and [41]. We will only consider functions with simple poles in z.

The elliptic transformation (3.7) implies (cf. [40]) that a (weak) Jacobi form of weight k and index m admits an expansion

$$\phi(\tau, z) = \sum_{r \,(\mathrm{mod}\ 2m)} \widetilde{h}_{m,r}(\tau)\theta_{m,r}(\tau, z) \tag{3.10}$$

in terms of the *index m theta functions*,

$$\theta_{m,r}(\tau, z) = \sum_{\substack{k\in\mathbb{Z} \\ k=r \ \mathrm{mod}\ 2m}} q^{k^2/4m} y^k. \tag{3.11}$$

Recall that the vector-valued function $\theta_m = (\theta_{m,r})$ satisfies

$$\theta_m(-\frac{1}{\tau}, -\frac{z}{\tau}) = \sqrt{-i\tau}\, e\left(\frac{mz^2}{\tau}\right) \mathbf{S}\, \theta_m(\tau, z), \qquad \theta_m(\tau+1, z) = \mathbf{T}\, \theta_m(\tau, z), \tag{3.12}$$

where $\mathbf{S}$ and $\mathbf{T}$ are the $2m \times 2m$ unitary matrices with entries

$$\mathbf{S}_{rr'} = \frac{1}{\sqrt{2m}} e\left(\frac{rr'}{2m}\right), \qquad \mathbf{T}_{rr'} = e\left(\frac{r^2}{4m}\right) \delta_{r,r'}. \tag{3.13}$$

From this we can see that $\widetilde{h} = (\widetilde{h}_{m,r})$ is a $2m$-component vector transforming as a weight $k - 1/2$ modular form for $SL_2(\mathbb{Z})$, with a multiplier system represented by the matrices $\mathbf{S}^\dagger$ and $\mathbf{T}^\dagger$, satisfying $\mathbf{S}\mathbf{S}^\dagger = \mathbf{T}\mathbf{T}^\dagger = I_{2m}$, and corresponding to the modular transformations S and T, respectively. (See [42].) Moreover, the invariance under the modular transformation (3.8) with $\gamma = -I_2$ implies that $\phi(\tau, -z) = (-1)^k \phi(\tau, z)$. Combining this with the identity $\theta_{m,-r}(\tau, z) = \theta_{m,r}(\tau, -z)$ we see that

$$\widetilde{h}_{m,r} = (-1)^k \widetilde{h}_{m,-r}. \tag{3.14}$$

3.3 The Eichler–Zagier Operators and an ADE Classification

We now turn to a discussion of the Eichler–Zagier operators on Jacobi forms and establish an ADE classification of maps satisfying a certain positivity condition.

Recall from §3.2 that a Jacobi form of weight k and index m admits a decomposition (3.10) into a combination of theta functions $\theta_{m,r}$ and the $2m$ components $\widetilde{h}_{m,r}$ of a vector-valued modular form $\widetilde{h}_m$ of weight $k - 1/2$. On the other hand, one can also consider the following question: for a given vector-valued modular form $\widetilde{h}_m$, is the expression in (3.10) the only combination of $\theta_{m,r}$ and $\widetilde{h}_{m,r}$ that has the right transformation property to be a weight k, index m Jacobi form? In other words, we would like to consider all $2m \times 2m$ matrices Ω such that

$$\widetilde{h}_m^T \cdot \Omega \cdot \theta_m = \sum_{r,r' \pmod{2m}} \widetilde{h}_{m,r}\, \Omega_{r,r'} \theta_{m,r'} \tag{3.15}$$

is again a weight k index m Jacobi form.

From (3.12)-(3.13) as well as the transformation under $\gamma = -I_2$, we see that this condition amounts to considering the commutants Ω of $\mathbf{S}$ and $\mathbf{T}$ satisfying

$$\mathbf{S}^\dagger \Omega \mathbf{S} = \mathbf{T}^\dagger \Omega \mathbf{T} = \Omega. \tag{3.16}$$

In particular, as Ω commutes with $\mathbf{S}^2$ we see that it has the reflection symmetry

$$\Omega_{r,r'} = \Omega_{-r,-r'}. \tag{3.17}$$

Such commutants have been classified in [43]. For each positive integer m, the space of $2m \times 2m$ matrices satisfying (3.16) has dimension given by the number of divisors of m, $\sigma_0(m) = \sum_{d|m} 1$, and is spanned by the set of linearly independent matrices $\{\Omega_m(n_1), \Omega_m(n_2), \ldots, \Omega_m(n_{\sigma_0(m)})\}$ whose entries are given by

$$\Omega_m(n_i)_{r,r'} = \begin{cases} 1 & \text{if } r + r' = 0 \bmod 2n_i \text{ and } r - r' = 0 \bmod 2m/n_i, \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

where $1 = n_1 < n_2 < \cdots < m = n_{\sigma_0(m)}$ are the divisors of m. It is easy to check that these matrices automatically satisfy (3.17).

Note that

$$\widetilde{h}_m^T \cdot \Omega_m(n) \cdot \theta_m = (-1)^k \, \widetilde{h}_m^T \cdot \Omega_m(m/n) \cdot \theta_m, \tag{3.19}$$

as is evident from the definition (3.18) of $\Omega_m(n)$ as well as the reflection property (3.14) of the components $\widetilde{h}_{m,r}$ of $\widetilde{h}_m$.

In fact, as we will now show, for a given vector-valued modular form $\widetilde{h}_m = (\widetilde{h}_{m,r})$ and any given divisor n of m, the new Jacobi form $\widetilde{h}_m^T \cdot \Omega_m(n) \cdot \theta_m$ can be obtained from the original one $\widetilde{h}_m^T \cdot \theta_m$ via a natural operator—the so-called Eichler–Zagier operator [40]—on Jacobi forms.

Given positive integers n, m such that $n|m$, we define an *Eichler–Zagier operator* $\mathcal{W}_m(n)$ acting on a function $f\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ by setting

$$\left(f|\mathcal{W}_m(n)\right)(\tau, z) = \frac{1}{n} \sum_{a,b=0}^{n-1} e\left(m\left(\tfrac{a^2}{n^2}\tau + 2\tfrac{a}{n}z + \tfrac{ab}{n^2}\right)\right) f\left(\tau, z + \tfrac{a}{n}\tau + \tfrac{b}{n}\right). \tag{3.20}$$

It is easy to see that the operator $\mathcal{W}_m(n)$ commutes with the index m elliptic transformation (3.7)

$$f|\mathcal{W}_m(n)\,|_m(\lambda, \mu) = f|_m(\lambda, \mu)\,|\mathcal{W}_m(n) \tag{3.21}$$

for all $\mu, \lambda \in \mathbb{Z}$ and in particular preserves the invariance under elliptic transformations. Moreover, one can easily check that the modular invariance (3.8) is also preserved. As a result $\mathcal{W}_m(n)$ maps an unrestricted Jacobi form of weight k and index m to another unrestricted Jacobi form of the same weight and index. Moreover, when $n\|m$ this operation is an involution on the space of strong Jacobi forms. This involution is sometimes referred to as an *Atkin–Lehner involution* for

Jacobi forms due to its intimate relation to Atkin–Lehner involutions for modular forms [44,45]. We will explain and utilise some aspects of this relation in §4.2.

For later use, we define the more general operator $\mathcal{W} = \sum_{n_i|m} c_i \mathcal{W}_m(n_i)$ by setting

$$f|\mathcal{W} = \sum_{n_i|m} c_i \left(f|\mathcal{W}_m(n_i)\right). \tag{3.22}$$

The relation between the Eichler–Zagier operators $\mathcal{W}$ and the transformation on Jacobi forms

$$\widetilde{h}_m^T \cdot \theta_m \mapsto \widetilde{h}_m^T \cdot \Omega \cdot \theta_m, \tag{3.23}$$

where Ω is a linear combination of the matrices $\Omega_m(n)$ in (3.18), can be seen via the action of the former on the theta functions $\theta_{m,r}$. Notice that

$$\theta_{m,r}|\mathcal{W}_m(n) = \sum_{r' \pmod{2m}} \Omega_m(n)_{r,r'} \theta_{m,r'}. \tag{3.24}$$

In terms of the $2m$-component vector $\theta_m = (\theta_{m,r})$, we have

$$\theta_m|\mathcal{W}_m(n) = \Omega_m(n) \cdot \theta_m \ , \tag{3.25}$$

which immediately leads to

$$\widetilde{h}_m^T \cdot \theta_m|\mathcal{W}_m(n) = \widetilde{h}_m^T \cdot \Omega_m(n) \cdot \theta_m. \tag{3.26}$$

In other words, the Jacobi forms we discussed above in terms of the matrices $\Omega_m(n)$ are simply the images of the original Jacobi forms under the corresponding Eichler–Zagier operators. This property makes it obvious that $\widetilde{h}_m^T \cdot \Omega_m(n) \cdot \theta_m$ is also a Jacobi form since $\mathcal{W}_m(n)$ preserves the transformation under the Jacobi group. This relation will be important in the discussion in §4.

Apart from the modularity (Jacobi form) condition, it is also natural to impose a certain positivity condition. As we will see, this additional condition leads to an ADE classification of the matrices Ω. To explain this positivity condition, first recall that all the entries of the matrices $\Omega_m(n)$ for any divisor n of m are non-negative integers and it might seem that any positivity condition would be redundant. However, we have also seen that the description of the theta-coefficients $\widetilde{h}_{m,r}$ of a weight k, index m Jacobi form as a vector with $2m$ components has some redundancy since different components are related to each other by $\widetilde{h}_{m,r} = (-1)^k \widetilde{h}_{m,-r}$ (cf. (3.14)). For the purpose of the present paper we will from now on consider only the case

X	$m(X)$	π^X	Ω^X
A_{m-1}	m	$\frac{m}{1}$	$\Omega_m(1)$
$D_{m/2+1}$	m	$\frac{2.m}{1.(m/2)}$	$\Omega_m(1)+\Omega_m(m/2)$
E_6	12	$\frac{2.3.12}{1.4.6}$	$\Omega_{12}(1)+\Omega_{12}(4)+\Omega_{12}(6)$
E_7	18	$\frac{2.3.18}{1.6.9}$	$\Omega_{18}(1)+\Omega_{18}(6)+\Omega_{18}(9)$
E_8	30	$\frac{2.3.5.30}{1.6.10.15}$	$\Omega_{30}(1)+\Omega_{30}(6)+\Omega_{30}(10)+\Omega_{30}(15)$

Table 5: The ADE classification of matrices Ω producing Jacobi forms $\widetilde{h}_m^T\cdot\Omega\cdot\theta_m$

of odd k, where there are at most $m-1$ independent components in $(\widetilde{h}_{m,r})$. In this case, using the property (3.17) we can rewrite the Jacobi form $\widetilde{h}_m^T\cdot\Omega\cdot\theta_m$ as

$$\widetilde{h}_m^T\cdot\Omega\cdot\theta_m=\sum_{r,r'=1}^{m-1}\widetilde{h}_{m,r}\left(\Omega_{r,r'}-\Omega_{r,-r'}\right)\left(\theta_{m,r'}-\theta_{m,-r'}\right). \tag{3.27}$$

As a result, it is natural to consider the $2m\times 2m$ matrices $\Omega=\sum_{i=1}^{\sigma_0(m)}c_i\Omega_m(n_i)$ where $n_1,n_2,\ldots$ are the (distinct) divisors of m that satisfy the corresponding positivity and integrality condition

$$\Omega_{r,r'}-\Omega_{r,-r'}\in\mathbb{Z}_{\geq 0}\text{ for all }r,r'=1,\ldots,m-1, \tag{3.28}$$

with a natural normalisation

$$\Omega_{1,1}-\Omega_{1,-1}=1. \tag{3.29}$$

Evidently, they are in one-to-one correspondence with the non-negative integer combinations of $(\widetilde{h}_{m,r})$ and $(\theta_{m,r'}-\theta_{m,-r'})$, with $r,r'=1,2,\ldots,m-1$, with the coefficient of the term $\widetilde{h}_{m,1}\theta_{m,1}$ equal to 1. From a conformal field theory point of view this is precisely the requirement of having a unique ground state in the theory.

It turns out that this problem has been studied by Cappelli–Itzykson–Zuber [24]. They found a beautiful ADE classification of such $2m\times 2m$ matrices Ω (see Proposition 2 of [24]) and we present these matrices in Table 5, denoting by Ω^X the matrix corresponding to the irreducible simply-laced root system X. The motivation of [24] was very different from ours: these authors were interested in classifying the modular invariant combinations of chiral and anti-chiral characters of the affine Lie algebra $\widehat{A}_1$ (the $SU(2)$ current algebra). However, as the modular transformation of the $\widehat{A}_1$ characters at level $m-2$ is very closely related to that of the index m theta functions $\theta_{m,r}$, the relevant matrices are also the commutants of the same $\mathbf{S}$ and $\mathbf{T}$ matrices satisfying (3.16).

The relation between the Eichler–Zagier operators and the $\Omega_m(n)$ matrices discussed earlier makes it straightforward to extend the above ADE classification to an ADE classification of Eichler–Zagier operators. Combining the results of the above discussion, we arrive at the following theorem.

Theorem 3.1. *For any integer m and any odd integer k, and any vector-valued modular form $\widetilde{h}_m = (\widetilde{h}_{m,r})$ such that $\widetilde{h}_m^T \cdot \theta_m$ is an (unrestricted) weight k, index m Jacobi form, suppose Ω coincides with a matrix Ω^X corresponding to an irreducible simply-laced root system X with Coxeter number m via Table 5. Then the combination $\widetilde{h}_m^T \cdot \Omega \cdot \theta_m$ is also a weight k, index m (unrestricted) Jacobi form which moreover satisfies the positivity condition*

$$\widetilde{h}_m^T \cdot \Omega \cdot \theta_m = \sum_{r,r'=1}^{m-1} c_{r,r'} \widetilde{h}_{m,r} (\theta_{m,r'} - \theta_{m,-r'}), \;\; c_{r,r'} \in \mathbb{Z}_{\geq 0}, \;\; c_{1,1} = 1. \tag{3.30}$$

Conversely, any $2m \times 2m$ matrix Ω for which the above statement is true necessarily coincides with a matrix Ω^X corresponding to an irreducible simply-laced root system X with Coxeter number m. Moreover, the resulting (unrestricted) Jacobi form is the image of the original Jacobi form $\widetilde{h}_m^T \cdot \theta_m$ under the Eichler–Zagier operator $\mathcal{W}^X$ defined by replacing $\Omega_m(n)$ in Ω^X with $\mathcal{W}_m(n)$ (cf. (3.32)).

Proof. First, the Jacobi form condition on $\widetilde{h}_m^T \cdot \Omega \cdot \theta_m$ requires that Ω satisfies the commutant condition (3.16). It was shown in [43] (cf. Proposition 1 of [24]) that the space of such $2m \times 2m$ matrices are spanned by $\{\Omega_m(n_1), \Omega_m(n_2), \ldots, \Omega_m(n_{\sigma_0(m)})\}$ given in (3.18).

Next, the positivity and integrality conditions on $c_{r,r'}$ are equivalent to those on the entries $\Omega_{r,r'} - \Omega_{r,-r'}$ given in (3.28-3.29). The linear combinations of $\Omega_m(n_i)$ satisfying (3.28-3.29) were shown in [24] to correspond to ADE root systems via Table 5. Finally, the equality (3.32) follows from the equality (3.25). □

The relation between Ω^X and the ADE root system X lies in the following two facts. First, Ω^X is a $2m \times 2m$ matrix where m is the Coxeter number of X. Moreover, $\Omega^X_{r,r} - \Omega^X_{r,-r} = \alpha^X_r$ for $r = 1, \ldots, m-1$ coincides with the multiplicity of r as a Coxeter exponent of X (cf. Table 1). Note the striking similarity between the expression for Ω^X and the denominator of the Coxeter Frame shape π^X (cf. §2.1). For instance, $\Omega^X = \Omega_m(1)$ for $X = A_{m-1}$ is nothing but the $2m \times 2m$ identity matrix.

More generally, for a union $X = \bigcup_i X_i$ of simply-laced root systems with the same Coxeter number, we let

$$\Omega^X = \sum_i \Omega^{X_i}, \text{ and similarly } \; \mathcal{W}^X = \sum_i \mathcal{W}^{X_i}. \tag{3.31}$$

Then we have the relation

$$(\widetilde{h}_m^T \cdot \theta_m)|\mathcal{W}^X = \widetilde{h}_m^T \cdot \Omega^X \cdot \theta_m \tag{3.32}$$

among different (odd) weight k and index m Jacobi forms corresponding to the same vector-valued modular form $\widetilde{h}_m = (\widetilde{h}_{m,r})$. Note that the operator $\mathcal{W}^X$ is in general no longer an involution and often not even invertible. The Eichler–Zagier operators $\mathcal{W}^X$ corresponding to Niemeier root systems (cf. §2.2) will play a central role in §5.

The $\widehat{A}_1$ characters, which have led to the ADE classification of Cappelli–Itzykson–Zuber, are rather ubiquitous in two-dimensional conformal field theory. They can be viewed as the building blocks of, for instance, the characters of $N = 4$ super-conformal algebra and the partition functions of $N = 2$ minimal models (cf. e.g. [9, 43]). Moreover, the positivity and the integrality condition (3.28-3.29) that are necessary to obtain the ADE classification are completely natural from the point of view of the conformal field theories. This might be seen as suggesting a relationship between umbral moonshine and two-dimensional conformal field theories. The concrete realisation of such a relationship is beyond the scope of the present paper.

3.4 From Meromorphic Jacobi Forms to Mock Modular Forms

In §3.1 we have seen the definition of mock modular form and its vector-valued generalisation. One of the natural places where such vector-valued mock modular forms occur is in the theta expansion of meromorphic Jacobi forms. To be more precise, following [38] and [41] we will establish a uniform way to separate a meromorphic Jacobi form ψ into its *polar* and *finite* parts

$$\psi(\tau, z) = \psi^P(\tau, z) + \psi^F(\tau, z). \tag{3.33}$$

The finite part will turn out to be a mock Jacobi form, admitting a theta expansion as in (3.10), whose theta-coefficients are the components of a vector-valued mock modular form. Up to this point, all the mock modular forms playing a role in umbral moonshine, as well as other interesting examples including many of Ramunanjan's mock theta functions, can be obtained in this way as theta-coefficients of finite parts of meromorphic Jacobi forms.

For the purpose of this paper we will focus on the case of weight 1 Jacobi forms with simple poles as a function of z. Consider such a Jacobi form ψ with a pole at $z = z_s$, where z_s is a point inside the fundamental cell $\alpha\tau + \beta$, $\alpha, \beta \in (-1, 0]$. The elliptic transformation (3.7) then forces ψ to have poles at all $z \in z_s + \mathbb{Z} + \tau\mathbb{Z}$. With this property in mind, in the rest of the paper we will only write down the location of the poles inside the fundamental cell. To capture

this translation property of the poles, following [41] we define an averaging operator

$$\mathrm{Av}_m\left[F(y)\right] = \sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} F(q^k y) \tag{3.34}$$

which takes a function of $y = e(z)$ with polynomial growth and returns a function of z which is invariant under the index m elliptic transformations (3.7). For a given pole $z = z_s$ of a weight 1 index m meromorphic Jacobi form ψ, we will consider the image $\psi^P_{z_s}$ under Av_m of a suitably chosen meromorphic function $F_{z_s}(y)$ that has a pole at $z = z_s$, such that $\psi - \psi^P_{z_s}$ is regular at all $z \in z_s + \mathbb{Z} + \tau\mathbb{Z}$.

In the remaining part of this subsection we will first review (following [38, 41]) this construction of mock modular forms in more detail, and then extend the discussion of the Eichler–Zagier operators to meromorphic Jacobi forms and study how they act on the polar and the finite part separately. This will allow us to establish an ADE classification of mock Jacobi forms of a specific type in the next section and constitutes a crucial element in the construction of the umbral mock modular forms H^X.

A Simple Pole at $z = 0$

To start with, consider a meromorphic Jacobi form $\psi(\tau, z)$ of weight 1 and index m, with a simple pole at $z = 0$ and no other poles. Define the polar part of ψ to be

$$\psi^P(\tau, z) = \chi(\tau)\mathrm{Av}_m\left[\frac{y+1}{y-1}\right] \tag{3.35}$$

where $\chi(\tau)/\pi i$ is the residue of $\psi(\tau, z)$ at $z = 0$. For the applications in the present paper we need only consider the case that $\chi(\tau) = \chi$ is a constant. With this definition, one can easily check that $\psi^F = \psi - \psi^P$ is indeed a holomorphic function with no poles in z.

Note that

$$\mu_{m,0}(\tau, z) = \mathrm{Av}_m\left[\frac{y+1}{y-1}\right], \tag{3.36}$$

where we define the generalised Appell–Lerch sum

$$\mu_{m,j}(\tau, z) = (-1)^{1+2j} \sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{(yq^k)^{-2j} + (yq^k)^{-2j+1} + \cdots + (yq^k)^{1+2j}}{1 - yq^k} \tag{3.37}$$

for $j \in \frac{1}{2}\mathbb{Z}$. The function $\mu_{m,0}$ enjoys the following relation to the modular group $SL_2(\mathbb{Z})$. Define

the *completion* of $\mu_{m,0}(\tau,z)$ by setting

$$\widehat{\mu}_m(\tau,\bar{\tau},z) = \mu_{m,0}(\tau,z) - e(-\tfrac{1}{8})\frac{1}{\sqrt{2m}}\sum_{r\,(\mathrm{mod}\,2m)}\theta_{m,r}(\tau,z)\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau', \quad (3.38)$$

where $S_{m,r}(\tau)$ denotes the unary theta series

$$S_{m,r}(\tau) = -S_{m,-r}(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_{m,r}(\tau,z)\big|_{z=0}, \quad (3.39)$$

then $\widehat{\mu}_m$ transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z})$ but is clearly no longer holomorphic whenever $m>1$. From the above definition and the transformation (3.13) of the theta functions, we see that $S_m = (S_{m,r})$ is a (vector-valued) weight 3/2 cusp form for $SL_2(\mathbb{Z})$.

Returning to our weight 1 index m Jacobi form ψ, assumed to have a simple pole at $z=0$, it is now straightforward to see that the finite part $\psi^F = \psi - \psi^P$, a holomorphic function on $\mathbb{H}\times\mathbb{C}$, has a completion given by

$$\widehat{\psi}^F = \psi^F + \chi\, e(-\tfrac{1}{8})\frac{1}{\sqrt{2m}}\sum_{r\,(\mathrm{mod}\,2m)}\theta_{m,r}(\tau,z)\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau' \quad (3.40)$$

that transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z})$. As such, ψ^F is an example of a *mock Jacobi form* (cf. [41, §7.2]). Since both ψ and ψ^P are invariant under the index m elliptic transformation, so is the finite part ψ^F. This fact guarantees a theta expansion of ψ^F analogous to that of a (weak) Jacobi form (3.10)

$$\psi^F(\tau,z) = \sum_{r\,(\mathrm{mod}\,2\mathrm{m})} h_r(\tau)\,\theta_{m,r}(\tau,z), \quad (3.41)$$

where $h=(h_r)$ is a weight 1/2 vector-valued holomorphic function on $\mathbb{H}$ whose completion

$$\widehat{h}_r(\tau) = h_r(\tau) + \chi\, e(-\tfrac{1}{8})\frac{1}{\sqrt{2m}}\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau' \quad (3.42)$$

transforms as a weight 1/2 vector-valued modular form with $2m$ components. As such, we conclude that $h=(h_r)$ is a vector-valued mock modular form for the modular group $SL_2(\mathbb{Z})$ with shadow $\chi\, S_m = (\chi\, S_{m,r})$.

Note that S_1 vanishes identically. This is a reflection of the fact that $\mu_{1,0}$, in contrast to the $\mu_{m,0}$ for $m>1$, coincides with its completion, and is thus (already) a meromorphic Jacobi

form, of weight 1 and index 1. By construction it has simple poles at $z \in \mathbb{Z}\tau + \mathbb{Z}$ and nowhere else, and we also have the explicit formula

$$\mu_{1,0}(\tau, z) = -i \frac{\theta_1(\tau, 2z)\, \eta(\tau)^3}{\theta_1(\tau, z)^2} = \frac{y+1}{y-1} - (y^2 - y^{-2})\, q + \cdots . \tag{3.43}$$

(See §A.2 for $\theta_1(\tau, z)$.) The function $\mu_{1,0}$ is further distinguished by being a meromorphic Jacobi form with vanishing finite part; a "Cheshire cat" in the language of [41, §8.5]. It will play a distinguished role in §4, where it will serve as a device for producing meromorphic Jacobi forms of weight 1 from (weak, holomorphic) Jacobi forms of weight 0.

Simple Poles at n-Torsion Points

Next we would like to consider the more general situation in which we have a weight 1 index m meromorphic Jacobi form ψ with simple poles at more general torsion points $z \in \mathbb{Q}\tau + \mathbb{Q}$. We introduce the row vector with two elements $s = (\alpha\ \beta)$ to label the pole at $z_s = \alpha\tau + \beta$ and write $y_s = e(z_s)$. For the purpose of this paper we will restrict our attention to the n-torsion points satisfying $nz \in \mathbb{Z}\tau + \mathbb{Z}$, where n is a divisor of the index m. Focus on a pole located at say $z_s = \alpha\tau + \beta$ with $\alpha, \beta \in \frac{1}{n}\mathbb{Z}$. Again following [41], we require the corresponding polar term to be given by the formula

$$\psi^P_{z_s}(\tau, z) = \pi i \operatorname{Res}_{z=z_s}(\psi(\tau, z)) \operatorname{Av}_m \left[\left(\frac{y}{y_s} \right)^{-2m\alpha} \frac{y/y_s + 1}{y/y_s - 1} \right], \tag{3.44}$$

generalising (3.35). One can easily check that $\psi - \psi^P_{z_s}$ has no pole at $z \in z_s + \mathbb{Z}\tau + \mathbb{Z}$.

As before, the above polar part is invariant under the elliptic transformation by construction. To discuss its variance under the modular group, first notice that the transformation $(\tau, z) \mapsto \gamma(\tau, z)$ maps the pole at z_s to a different pole according to $s \mapsto s\gamma$. As a result, to obtain a mock Jacobi form for $SL_2(\mathbb{Z})$ from a meromorphic Jacobi form with poles at n-torsion points z_s (where n is the smallest integer such that $z_s \in \frac{\tau}{n}\mathbb{Z} + \frac{1}{n}\mathbb{Z}$), we should consider meromorphic Jacobi forms that have poles at all the n-torsion points. Moreover, the modular transformation of ψ dictates that the residues of the poles satisfy $D_s(\gamma\tau) = D_{s\gamma}(\tau)$, where we have defined, after [41],

$$D_s(\tau) = e(m\alpha z_s) \operatorname{Res}_{z=z_s}(\psi(\tau, z)). \tag{3.45}$$

More specifically, we would like to consider the situation where ψ satisfies

$$\operatorname{Res}_{z=-\frac{a}{n}\tau - \frac{b}{n}} \psi(\tau, z) = \chi e(-ma(a\tau + b)/n^2)/n\pi i, \text{ for } a, b = 0, 1, \ldots, n-1, \tag{3.46}$$

corresponding to the simplest case where the function $D_s(\tau)$ is just a constant. Without loss of generality we will also assume for the moment that ψ has no other poles, as the more general situation can be obtained by taking linear combinations. In this case, using (3.44) it is not hard to see that the polar parts contributed by the poles at these n-torsion points are given by the images under the Eichler–Zagier operator $\mathcal{W}_m(n)$ (cf. (3.20)) of the polar term contributed by the simple pole at the origin, so that

$$\psi^P = \sum_{a,b=0}^{n-1} \psi^P_{-\frac{a}{n}\tau-\frac{b}{n}} = \chi\,\mu_{m,0}\big|\mathcal{W}_m(n). \tag{3.47}$$

From (3.38) and the fact that the Eichler–Zagier operators preserve the Jacobi transformations, we immediately see how considering Jacobi forms with simple poles at torsion points leads us to vector-valued mock modular forms with more general shadows. In this case, from (3.25), (3.38) and (3.47) it is straightforward to see that the completion of the polar part

$$\widehat{\psi}^P = \psi^P - \chi\, e(-\tfrac{1}{8})\frac{1}{\sqrt{2m}} \sum_{r,r' \,(\mathrm{mod}\, 2m)} \theta_{m,r}(\tau,z)\,\Omega_m(n)_{r,r'} \int_{-\bar{\tau}}^{i\infty} (\tau'+\tau)^{-1/2}\overline{S_{m,r'}(-\bar{\tau}')}\,\mathrm{d}\tau' \tag{3.48}$$

again transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z})$.

Following the same argument as before, we conclude that the theta-coefficients of the finite part

$$\psi^F = \psi - \psi^P = \sum_{r\,(\mathrm{mod}\,2\mathrm{m})} h_r(\tau)\,\theta_{m,r}(\tau,z), \tag{3.49}$$

define a vector-valued mock modular form $h = (h_r)$, whose completion is given by

$$\widehat{h}_r(\tau) = h_r(\tau) + \chi\, e(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}} \sum_{r'\,(\mathrm{mod}\,2\mathrm{m})} \Omega_m(n)_{r,r'} \int_{-\bar{\tau}}^{i\infty} (\tau'+\tau)^{-1/2}\overline{S_{m,r'}(-\bar{\tau}')}\,\mathrm{d}\tau', \tag{3.50}$$

and whose shadow is hence given by a vector of unary theta series whose r-th component equals

$$\sum_{r'\,(\mathrm{mod}\,2\mathrm{m})} \Omega_m(n)_{r,r'}\, S_{m,r'} \tag{3.51}$$

where $r \in \mathbb{Z}/2m\mathbb{Z}$. In particular, this means that the vector-valued mock modular forms arising from meromorphic Jacobi forms in this way are closely related to mock theta functions, as their shadows are always given by unary theta series.

Finally, we also note that the Eichler–Zagier operators and the operations of extracting polar and finite parts are commutative in the following sense.

Proposition 3.2. *Suppose ψ is a weight 1 index m meromorphic Jacobi form with simple poles at $\widetilde{n}$-torsion points with $\widetilde{n}|m$ and with no poles elsewhere. Then for any positive integer n such that $n|m$ and $(n,\widetilde{n})=1$ we have*

$$(\psi|\mathcal{W}_m(n))^P = \psi^P|\mathcal{W}_m(n). \tag{3.52}$$

Proof. Denote the set of poles of ψ in the unit cell by S, and focus on the pole of ψ at $z_* = -\widetilde{a}/\widetilde{n}\tau - \widetilde{b}/\widetilde{n} \in S$. From the action of $\mathcal{W}_m(n)$ we see that $\psi|\mathcal{W}_m(n)$ has poles at all $z \in z_* + \frac{1}{n}\mathbb{Z} + \frac{\tau}{n}\mathbb{Z}$. Focussing on the pole at $z = z_s = z_* - (a\tau/n + b/n)$, from (3.20) we get

$$\mathrm{Res}_{z=z_s}(\psi|\mathcal{W}_m(n))(\tau,z) = \frac{1}{n}\, e(\tfrac{2maz_s}{n})e(\tfrac{ma}{n^2}(a\tau+b))\mathrm{Res}_{z=z_*}(\psi(\tau,z)), \tag{3.53}$$

which leads to

$$\begin{aligned}(\psi|\mathcal{W}_m(n))^P(\tau,z) = \frac{\pi i}{n} &\sum_{z_*=-\frac{\widetilde{a}}{\widetilde{n}}\tau-\frac{\widetilde{b}}{\widetilde{n}}\in S} \mathrm{Res}_{z=z_*}(\psi(\tau,z)) \\ &\times \sum_{a,b=0}^{n-1} e(\tfrac{2maz_s}{n})e(\tfrac{ma}{n^2}(a\tau+b)) \sum_{k\in\mathbb{Z}} q^{mk^2}y^{2mk}(q^k y/y_s)^{2m(\frac{a}{n}+\frac{\widetilde{a}}{\widetilde{n}})}\frac{q^k y/y_s+1}{q^k y/y_s-1}\end{aligned}$$

where y_s denotes $y_s = e(z_s) = e(-(\widetilde{a}/\widetilde{n} + a/n)\tau - (\widetilde{b}/\widetilde{n} + b/n))$ in the second line.

By direct comparison using (3.20) and (3.44), this is exactly $\psi^P|\mathcal{W}_m(n)$ and this finishes the proof. □

Since all the operations involved are linear, we also have the following corollary.

Corollary 3.3. *Consider ψ as defined as in Proposition 3.2 and let $\mathcal{W} = \sum_i c_i\mathcal{W}_m(n_i)$ where the n_i are divisors of m satisfying $(n_i,\widetilde{n})=1$. Then*

$$(\psi|\mathcal{W})^P = \psi^P|\mathcal{W} \tag{3.54}$$

and

$$(\psi|\mathcal{W})^F = \psi^F|\mathcal{W}. \tag{3.55}$$

Moreover, if we denote the theta-coefficients of ψ^F by $h_\psi = ((h_\psi)_r)$ and its shadow by $S_\psi = ((S_\psi)_r)$ with $r \in \mathbb{Z}/2m\mathbb{Z}$, then the theta-coefficients of $(\psi|\mathcal{W})^F$ form a vector-valued mock modular form $h_{\psi|\mathcal{W}}$ satisfying

$$h_{\psi|\mathcal{W}} = \Omega\, h_\psi, \tag{3.56}$$

with shadow given by

$$S_{\psi|\mathcal{W}} = \Omega\, S_\psi \tag{3.57}$$

where $\Omega = \sum_i c_i \Omega_m(n_i)$ *(cf. (3.18)).*

As a result, the relations (3.32) between different Jacobi forms of the same index can be applied separately to the polar and the finite part. In the present paper we will mostly be concerned with the application of the above to the case that $\widetilde{n} = 1$. For later use it will be useful to note the following property.

Lemma 3.4. *For* ψ *and* n *as defined as in Proposition 3.2, we have*

$$\psi^F|\mathcal{W}(n) = -\psi^F|\mathcal{W}(m/n), \quad \widehat{\psi}^F|\mathcal{W}(n) = -\widehat{\psi}^F|\mathcal{W}(m/n). \tag{3.58}$$

Proof. From the property (3.47) of ψ^P and the elliptic transformation of $\mu_{m,0}$, it follows that $\psi^P(\tau, z) = -\psi^P(\tau, -z)$ and therefore $\psi^F(\tau, z) = -\psi^F(\tau, -z)$. As such, the vector-valued mock modular form $h = (h_r)$ arising from the theta expansion (3.41) of ψ^F satisfies $h_r = -h_{-r}$. Together with $S_{m,r} = -S_{m,-r}$, the lemma follows from the action of $\mathcal{W}_m(n)$ (3.25) on $\theta_m = (\theta_{m,r})$ and the definition (3.18) of the matrix $\Omega_m(n)$. □

4 The Umbral Mock Modular Forms

Following the general discussion of the relevant automorphic objects in the previous section, in this section we will start specifying concretely the vector-valued mock modular forms which encode, according to our conjecture, the graded dimensions of certain infinite-dimensional modules for the umbral groups defined in §2. We will specify the shadows of these functions—the *umbral mock modular forms*—in §4.1. Subsequently, in §4.2 we will show how these shadows distinguish the Niemeier root systems through a relation to genus zero groups and their principal moduli. Afterwards we will provide explicit expressions for the umbral forms of the A-type Niemeier root systems by specifying a set of weight 0 weak Jacobi forms. The umbral forms of D- and E-type Niemeier root systems will be specified in the next section.

4.1 The Umbral Shadows

In §3.4 we have seen how the theta expansion of the finite part of a meromorphic Jacobi form gives rise to a vector-valued mock modular form, and how different configurations of poles lead to different shadows. The shadows of the mock modular forms obtained in this way are always given by unary theta series. In this subsection we will see how the ADE classification

discussed in §3.3 leads to particular cases of the above construction. Moreover, by combining the ADE classification and the construction of mock modular forms from meromorphic Jacobi forms discussed in §3.4, we will associate a specific shadow S^X, or equivalently a pole structure of the corresponding meromorphic Jacobi form ψ^X, to each of the Niemeier root systems X.

Consider a meromorphic Jacobi form ψ with weight 1 and index m. Recall from (3.47) that the contribution to its polar part ψ^P from the simple poles at the n-torsion points with residues satisfying (3.46), is given by

$$\sum_{a,b=0}^{n-1} \psi^P_{-\frac{a}{n}\tau-\frac{b}{n}} = \chi\, \mu_{m,0}\big|\mathcal{W}_m(n). \tag{4.1}$$

Clearly, one may consider a linear combination of expressions as in (4.1). Consider a weight 1 index m meromorphic Jacobi form ψ with poles at n_1-,...,n_κ-torsion points where $n_j|m$, and where each n_j contributes $c_j e(-ma(a\tau+b)/n_j^2)/n_j\pi i$ to the residue of the pole located at $-\frac{a}{n_j}\tau-\frac{b}{n_j}$. From the above discussion it follows that its polar part is given by

$$\psi^P = \mu_{m,0}\big|\mathcal{W}, \text{ where } \mathcal{W} = \sum_{i=1}^{\kappa} c_i \mathcal{W}_m(n_i). \tag{4.2}$$

By taking a linear combination of expressions as in (3.48) we see that its completion, given by

$$\widehat{\psi}^P = \psi^P - e(-\tfrac{1}{8})\frac{1}{\sqrt{2m}} \sum_{r,r' \,(\mathrm{mod}\, 2m)} \theta_{m,r}(\tau,z)\,\Omega_{r,r'} \int_{-\bar{\tau}}^{i\infty} (\tau'+\tau)^{-1/2}\overline{S_{m,r'}(-\bar{\tau}')}\,\mathrm{d}\tau', \tag{4.3}$$

where $\Omega = \sum_{i=1}^{\kappa} c_i \Omega_m(n_i)$, transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z})$. Immediately we conclude that the theta-coefficients of the finite part $\psi^F = \psi - \psi^P$ constitute a vector-valued mock modular form with (the r-th component of) the shadow given by

$$\sum_{r' \,(\mathrm{mod}\, 2m)} \Omega_{r,r'} S_{m,r'}. \tag{4.4}$$

Now, recall that in §3.3 we used the reflection property $\widetilde{h}_{m,r} = -\widetilde{h}_{m,-r}$ to impose a positivity condition which then led to an ADE classification (cf. Theorem 3.1). Analogously, in the context of meromorphic Jacobi forms we also have a natural positivity condition that we want to impose. Using the reflection property $S_{m,r} = -S_{m,-r}$, for $r \in \mathbb{Z}/2m\mathbb{Z}$ (cf. (3.39)), we may

instead consider a $(m-1)$-component vector with the r-th component given by

$$\sum_{r'=1}^{m-1} (\Omega_{r,r'} - \Omega_{r,-r'}) S_{m,r'}, \quad r = 1, \ldots, m-1. \tag{4.5}$$

Requiring that each component of this $(m-1)$-component vector is a non-negative linear combination of the unary theta series $S_{m,r}$, for $r = 1, 2, \ldots, m-1$, with the normalisation $\Omega_{1,1} - \Omega_{1,-1} = 1$, from Theorem 3.1 we immediately see that such Ω (and, equivalently, $\mathcal{W}$) are classified by ADE root systems. More precisely, to each irreducible simply-laced root system X with Coxeter number m we associate a $2m$-vector-valued cusp form S^X, of weight $3/2$ for $SL_2(\mathbb{Z})$, with r-th component given by

$$S_r^X = \sum_{r' \bmod 2m} \Omega_{r,r'}^X S_{m,r'}, \tag{4.6}$$

where m denotes the Coxeter number of X and the matrix Ω^X is defined as in Table 5. For instance, we have

$$S_r^{A_{m-1}} = S_{m,r}. \tag{4.7}$$

For the D-series root systems with even rank we have

$$S_r^{D_{2n}} = \begin{cases} S_{4n-2,r} + S_{4n-2,4n-2-r} & \text{if } r \text{ is odd,} \\ 0 & \text{if } r \text{ is even,} \end{cases} \tag{4.8}$$

and in the case that n is odd we have

$$S_r^{D_n} = \begin{cases} S_{2n-2,r} & \text{if } r \text{ is odd,} \\ S_{2n-2,2n-2-r} & \text{if } r \text{ is even.} \end{cases} \tag{4.9}$$

For E_6, E_7 and E_8 we have

$$S_r^{E_6} = \begin{cases} S_{12,1} + S_{12,7} & \text{if } r = 1 \text{ or } r = 7, \\ S_{12,4} + S_{12,8} & \text{if } r = 4 \text{ or } r = 8, \\ S_{12,5} + S_{12,11} & \text{if } r = 5 \text{ or } r = 11, \\ 0 & \text{otherwise;} \end{cases} \tag{4.10}$$

$$S_r^{E_7} = \begin{cases} S_{18,1} + S_{18,17} & \text{if } r = 1 \text{ or } r = 17, \\ S_{18,9} & \text{if } r = 3 \text{ or } r = 15, \\ S_{18,5} + S_{18,13} & \text{if } r = 5 \text{ or } r = 13, \\ S_{18,7} + S_{18,11} & \text{if } r = 7 \text{ or } r = 11, \\ S_{18,3} + S_{18,9} + S_{18,15} & \text{if } r = 9, \\ 0 & \text{otherwise;} \end{cases} \tag{4.11}$$

$$S_r^{E_8} = \begin{cases} S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29} & \text{if } r \in \{1, 11, 19, 29\}, \\ S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23} & \text{if } r \in \{7, 13, 17, 23\}, \\ 0 & \text{otherwise;} \end{cases} \tag{4.12}$$

where for simplicity we have only specified $S_r^{E_n}$ for $r \in \{1, \cdots, m-1\} \subset \mathbb{Z}/2m\mathbb{Z}$. The remaining components are determined by the rule $S_{-r}^{E_n} = -S_r^{E_n}$.

Following (3.31), more generally for a union $X = \bigcup_i X_i$ of simply-laced root systems with the same Coxeter number we have $S^X = \sum_i S^{X_i}$. With this definition, S^X is given by the matrix Ω^X as

$$S^X = (S_r^X) \quad , \quad S_r^X = \sum_{r' \bmod 2m} \Omega_{r,r'}^X S_{m,r'}. \tag{4.13}$$

From the above discussion we see that the cusp form S^X arises naturally as the shadow of a vector-valued mock modular form obtained from the theta expansion of a meromorphic Jacobi form ψ^X with simple poles at $z \in \frac{\tau}{n_i}\mathbb{Z} + \frac{1}{n_i}\mathbb{Z}$ for all $n_i | m$ such that $c_i > 0$ in $\Omega^X = \sum_i c_i \Omega_m(n_i)$, whose polar part is given by

$$(\psi^X)^P = \mu_{m,0}(\tau, z) | \mathcal{W}^X. \tag{4.14}$$

It is not hard to see that such meromorphic Jacobi forms exist for any simply-laced root system X where all the irreducible components have the same Coxeter number. Choose an arbitrary weight 0, index $m-1$ weak Jacobi form ϕ with $\phi(\tau, 0) \neq 0$ and assume, without loss of generality, that $\phi(\tau, 0) = 1$. Recall from the definition of weak Jacobi forms in §3.2 that, if $\phi(\tau, z)$ is a weight 0 weak Jacobi form then $\phi(\tau, 0)$ is a weight 0 modular form, which is necessarily a constant. As such, by multiplying with the meromorphic Jacobi form $\mu_{1,0}$ of weight 1 and index 1 (cf. (3.43)) we obtain a weight 1 index m meromorphic Jacobi form with simple poles at $z \in \mathbb{Z}\tau + \mathbb{Z}$ and nowhere else, and with the polar part given by $\mu_{m,0}$. (See [1, §2.3] for a more detailed discussion, but note that $\mu_{1,0}$ is denoted there also by $\Psi_{1,1}$.) The Corollary

3.3 then shows that its image $-\mu_{1,0}\phi|\mathcal{W}^X$ under the corresponding Eichler–Zagier operator is a weight 1 index m meromorphic Jacobi with polar part coinciding with (4.14).

Besides specifying the poles of the meromorphic Jacobi form, in what follows we will also require an *optimal growth* condition (equivalent to the optimal growth condition formulated in [41], with its name derived from the fact that it guarantees the slowest possible growth of the coefficients of the corresponding mock Jacobi form). It turns out that there does not always exist a meromorphic Jacobi form with polar part given by (4.14) that moreover satisfies this optimal growth condition for an arbitrary simply-laced root system X, but when it does exist it is unique.

Theorem 4.1. *Let X be a simply-laced root system with all irreducible components having the same Coxeter number m. There exists at most one weight 1 index m meromorphic Jacobi form ψ^X satisfying the following two conditions. First, its polar part $(\psi^X)^P$ is given by (4.14). Second, its finite part*

$$(\psi^X)^F = \psi^X - (\psi^X)^P = \sum_r h_r^X \theta_{m,r} \tag{4.15}$$

satisfies the optimal growth condition

$$q^{1/4m} h_r^X(\tau) = O(1) \tag{4.16}$$

as $\tau \to i\infty$ for all $r \in \mathbb{Z}/2m\mathbb{Z}$.

Proof. If there are two distinct meromorphic Jacobi forms satisfying the above conditions, then their difference is necessarily a weight 1 index m weak Jacobi form of optimal growth in the sense of [41]. But Theorem 9.7 of [41] is exactly the statement that no such weak Jacobi form exists. □

Corollary 4.2. *Let X be a simply-laced root system with all irreducible components having the same Coxeter number m. Then there exists at most one vector-valued mock modular form h^X for $SL_2(\mathbb{Z})$ with shadow S^X that satisfies the optimal growth condition (4.16).*

Proof. Let h^X be a vector-valued mock modular form satisfying the above conditions. Consider $\psi' = (\psi^X)^P + \sum_r h_r^X \theta_{m,r}$ with $(\psi^X)^P$ given by (4.14). From the fact that the multiplier of a mock modular form is the inverse of that of its shadow and from the discussion in §3.4, ψ' is a weight 1 index m meromorphic Jacobi form satisfying the conditions of Theorem 4.1. It then follows from Theorem 4.1 that such h^X is unique if it exists. □

So far our discussion has been very general, applicable to any simply-laced root system with all irreducible components having the same Coxeter number. In the next subsection we will see how the cusp forms S^X with X given by one of the 23 Niemeier root systems play a distinguished role. The S^X for X a Niemeier root system are called the *umbral shadows*,

$$S^X_r = \sum_{r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega^X_{r,r'} S_{m,r'}. \tag{4.17}$$

We will demonstrate the existence of meromorphic Jacobi forms ψ^X satisfying the conditions of Theorem 4.1 for X an A-type Niemeier root system in Proposition 4.6, and for D- and E-type Niemeier root systems in Proposition 5.2. The resulting vector-valued functions defined by the theta-coefficients of the finite parts of these ψ^X are the umbral mock modular forms, to be denoted $H^X = (H^X_r)$. The first few dozen Fourier coefficients of the umbral mock modular forms are tabulated in Appendix C. For some of the Niemeier root systems X, the meromorphic Jacobi forms ψ^X are closely related to some of the meromorphic Jacobi forms analysed in [41, §9.5]; for other Niemeier root systems X, the corresponding shadows S^X fall outside the range of analysis in [41] and this is why we find mock Jacobi forms of optimal growth for values of m other than those appearing in the work of Dabholkar–Murthy–Zagier.

4.2 From Niemeier Lattices to Principal Moduli

In the last subsection we have seen how ADE root systems have an intimate relation to meromorphic Jacobi forms. More precisely, to a simply-laced root system with all irreducible components having the same Coxeter number m, we associate a pole structure for meromorphic Jacobi forms of weight 1 and index m. Equivalently, we associate a weight 3/2 vector-valued cusp form to every such root system, which plays the role of the shadow of the mock modular form arising from the meromorphic Jacobi form via the relation discussed in §3.4. In this subsection we see how the shadows S^X attached to Niemeier root systems are distinguished, and in particular, how they are related to the genus zero groups Γ^X of §2.3.

Recall [45] that a *skew-holomorphic Jacobi form* of weight 2 and index m is a smooth function $\widetilde{\phi}(\tau, z)$ on $\mathbb{H} \times \mathbb{C}$ which is periodic in both τ and z with period 1, transforms under the S-transformation as

$$\widetilde{\phi}(-\tfrac{1}{\tau}, \tfrac{z}{\tau})\, e(-m\tfrac{z^2}{\tau}) = \bar{\tau}|\tau|\widetilde{\phi}(\tau, z) \tag{4.18}$$

and has a Fourier expansion

$$\widetilde{\phi}(\tau,z) = \sum_{\substack{\Delta,r\in\mathbb{Z}\\ \Delta=r^2 \bmod 4m}} C_{\widetilde{\phi}}(\Delta,r)\ \mathrm{e}\left(\frac{r^2-\Delta}{4m}\Re(\tau)+\frac{r^2+|\Delta|}{4m}i\Im(\tau)+rz\right) \tag{4.19}$$

where $C_{\widetilde{\phi}}(\Delta,r)=0$ for $\Delta<0$. We denote the space of such functions by $J^+_{2,m}$. Recall that an integer Δ is called a fundamental discriminant if $\Delta=1$ or Δ is the discriminant of a quadratic number field. Following Skoruppa [45] (see also [46]), given a pair (Δ_0,r_0) where Δ_0 is a positive fundamental discriminant that is a square modulo $4m$ and $\Delta_0=r_0^2 \bmod 4m$, we may associate a weight 2 modular form

$$\mathscr{S}_{\Delta_0,r_0}(\widetilde{\phi}) = c_{\widetilde{\phi}}(\Delta_0,r_0)+\sum_{n\geq 1} q^n \sum_{a|n}\left(\frac{\Delta_0}{a}\right) C_{\widetilde{\phi}}\left(\Delta_0\tfrac{n^2}{a^2},r_0\tfrac{n}{a}\right) \tag{4.20}$$

for $\Gamma_0(m)$ to each $\widetilde{\phi}\in J^+_{2,m}$, where $\left(\frac{\Delta_0}{a}\right)$ denotes the Jacobi symbol and $c_{\widetilde{\phi}}(\Delta_0,r_0)$ denotes a suitably chosen constant term.

From the discussion in §3.3, it is not difficult to see that given a simply-laced root system X with each of its irreducible components having the same Coxeter number m, we may consider the following skew-holomorphic form

$$\sigma^X = (\overline{S^X})^T\cdot\theta_m = \sum_{r\,(\mathrm{mod}\,2m)} \overline{S^X_r(\tau)}\theta_{m,r}(\tau,z) = \sum_{r,r'\,(\mathrm{mod}\,2m)} \overline{S_{m,r}(\tau)}\,\Omega^X_{r,r'}\theta_{m,r'}(\tau,z) \tag{4.21}$$

of weight 2 and index m.

Applying $\mathscr{S}_{\Delta_0,r_0}$ with the simplest choice $(\Delta_0,r_0)=(1,1)$ to the skew-holomorphic Jacobi form σ^X, and using the fact that the Jacobi symbol $\left(\frac{1}{a}\right)=1$ for all positive integers a, we arrive at a weight 2 form on $\Gamma_0(m)$

$$f^X(\tau)=\mathscr{S}_{1,1}(\sigma^X)(\tau)=\frac{\mathsf{r}}{24}+\sum_{r=1}^{m-1}\alpha^X_r\sum_{\varepsilon=\pm1}\sum_{k=-\lfloor\frac{\varepsilon r}{2m}\rfloor}^{\infty}(2km+\varepsilon r)\frac{q^{2km+\varepsilon r}}{1-q^{2km+\varepsilon r}} \tag{4.22}$$

where r is the rank of the root system X and α^X_r is the multiplicity of the multiplicity of r as a Coxeter exponent of X, which also coincides with the "diagonal" coefficient $\Omega^X_{r,r}-\Omega^X_{r,-r}$ of $S_{m,r}$ in the r-th component of the vector-valued cusp form $S^X=(S^X_r)$ (cf. Table 1 and §3.3).

For $X=A_{m-1}$, one can easily see from the fact that $\alpha^X_r=1$ for all $r\in\{1,2,\ldots,m-1\}$ that the associated weight–two form is nothing but the following Eisenstein form at level m (cf.

X	$m(X)$	π^X	f^X
A_{m-1}	m	$\frac{m}{1}$	λ_m
$D_{m/2+1}$	m	$\frac{2.m}{1.(m/2)}$	$\lambda_2+\lambda_m-\lambda_{m/2}$
E_6	12	$\frac{2.3.12}{1.4.6}$	$\lambda_2+\lambda_3+\lambda_{12}-\lambda_4-\lambda_6$
E_7	18	$\frac{2.3.18}{1.6.9}$	$\lambda_2+\lambda_3+\lambda_{18}-\lambda_6-\lambda_9$
E_8	30	$\frac{2.3.5.30}{1.6.10.15}$	$\lambda_2+\lambda_3+\lambda_5+\lambda_{30}-\lambda_6-\lambda_{10}-\lambda_{15}$

Table 6: The weight 2 modular forms f^X

(A.9))

$$f^{A_{m-1}}(\tau)=\lambda_m(\tau)=q\partial_q\log\left(\frac{\eta(m\tau)}{\eta(\tau)}\right)=\frac{m-1}{24}+\sum_{k>0}\sigma(k)(q^k-mq^{mk}). \tag{4.23}$$

One can compute the function f^X in a similar way for the D- and E-series and arrive at the result in Table 6. From this table one observes that the weight two form f^X has a close relation to the Coxeter Frame shape π^X, and hence also to the matrix Ω^X according to Table 5. We now discuss this further.

From Ω^X where X has Coxeter number m, we can obtain a map on the space spanned by the weight two modular forms $\{\lambda_n(\tau), n|m\}$ of level m by replacing each $\Omega_m(m_1)$ in Ω^X with the operator $w_m(m_1)$ which acts on Eisenstein forms according to

$$\lambda_{m_2}|w_m(m_1)=\lambda_{m_1*m_2}-\lambda_{m_1}, \tag{4.24}$$

where m_1 and m_2 are assumed to divide m, and $m_1*m_2=m_1m_2/(m_1,m_2)^2$ (cf. §2.3). Then the weight two form corresponding to a simply-laced root system X with Coxeter number m is nothing but

$$f^X=\lambda_m|w^X=\lambda_{\pi^X} \tag{4.25}$$

where π^X is the Coxeter Frame shape of X (cf. §2.1) and λ_π, for π an arbitrary Frame shape, is defined in §2.3.

The reader will notice that the map $w_m(e)$ acts in the same way as the Atkin–Lehner involution $W_m(e)$ (2.18) on modular forms for $\Gamma_0(m)$ in the cases where $e\|m$. (Notice that the set $W_m(e)$ is empty if e is not an exact divisor of m.) Indeed, from the definition of the Eisenstein form (A.9) one can compute that

$$(\lambda_f|W_m(e))(\tau)=(\lambda_f|_{\psi=1,w=2}\gamma)(\tau)=e(cm\tau+de)^{-2}\lambda_f(\tfrac{ae\tau+b}{cm\tau+de})=\lambda_{e*f}(\tau)-\lambda_e(\tau), \tag{4.26}$$

where $\gamma = \frac{1}{\sqrt{e}} \left(\begin{smallmatrix} ae & b \\ cm & de \end{smallmatrix} \right) \in SL_2(\mathbb{R})$ (cf. (3.5)). On the other hand, from (3.25) we see that the skew-holomorphic form σ^X can be obtained as the image of the Eichler–Zagier operator

$$\sigma^X = \sigma^{A_{m-1}} | \mathcal{W}^X. \tag{4.27}$$

Taken together, at a given m and for a given union X of simply-laced root systems with Coxeter number m we have the equality

$$f^X = \mathscr{S}_{1,1}\left(\sigma^{A_{m-1}} | \mathcal{W}^X\right) = \sum_i \mathscr{S}_{1,1}\left(\sigma^{A_{m-1}} | \mathcal{W}_m(e_i)\right), \tag{4.28}$$

where we have written $\mathcal{W}^X = \sum_i \mathcal{W}_m(e_i)$ explicitly in terms of its different components. At the same time, for the cases that all $e_i || m$, we also have

$$f^X = \sum_i f^{A_{m-1}} | W_m(e_i). \tag{4.29}$$

For these cases, we note that the equality $f^{A_{m-1}} | W_m(e_i) = \mathscr{S}_{1,1}\left(\sigma^{A_{m-1}} | \mathcal{W}_m(e_i)\right)$ can be viewed as a consequence of the relation between the Eichler–Zagier operators and the Atkin–Lehner involutions observed in [44, 45]. Due to this relation, the Eichler–Zagier operators that define involutions are sometimes referred to as Atkin–Lehner involutions on Jacobi forms in the literature.

We conclude this section by observing that if X is a Niemeier root system then

$$\mathscr{S}_{1,1}(\sigma^X) = f^X = -q\partial_q \log T^X \tag{4.30}$$

where σ^X is the skew-holomorphic Jacobi form defined by S^X in (4.21) and T^X is the principal modulus for Γ^X defined in §2.3. In this way we obtain a direct connection between the umbral shadows S^X and the genus zero groups Γ^X attached to Niemeier root systems in §2.3.

4.3 From Weight Zero Jacobi Forms to Umbral Mock Modular Forms

The goal of this subsection is to construct the umbral mock modular form H^X which (conjecturally) encodes the graded dimension of the umbral module K^X (cf. §6.1) for every A-type Niemeier root system X. In §4.1 we have seen how to associate an umbral shadow S^X to a Niemeier root system X. Equivalently, we can associate a pole structure, which together with the optimal growth condition (4.16) determines a(t most one) meromorphic weight 1 index m Jacobi form ψ^X according to Theorem 4.1. In this subsection we will explicitly construct mero-

morphic Jacobi forms ψ^X satisfying the conditions of Theorem 4.1 for X an A-type root system via certain weight 0 Jacobi forms ϕ^X (cf. Proposition 4.6). After obtaining these ψ^X, the procedure discussed in §3.4 then immediately leads to the umbral forms H^X. It is also possible to specify the D- and E-type Niemeier root systems in a similar way. However, the discussion would become somewhat less illuminating and we will instead specify these in an arguably more elegant way in §5.

Our strategy in the present subsection is the following. As mentioned in §4.1, if we take a weight 0 index $m-1$ weak Jacobi form ϕ with $\phi(\tau,0)=1$, then $-\mu_{1,0}\phi|\mathcal{W}^X$ is a weight 1 index m meromorphic Jacobi whose pole structure is automatically of the desired form (cf. (3.43)). Namely, it always leads to a vector-valued mock modular form whose shadow is given by S^X. In this subsection we will see how to select (uniquely) a weight 0 form ϕ^X such that the resulting weight 1 Jacobi form

$$\psi^X = -\mu_{1,0}\phi^X|\mathcal{W}^X \tag{4.31}$$

satisfies the optimal growth condition (4.16).

For the simplest cases this optimal growth condition can be rephrased in terms of weight 0 Jacobi forms using the language of characters of the $\mathcal{N}=4$ superconformal algebra. Recall from [7, 8] that this algebra contains subalgebras isomorphic to the affine Lie algebra $\widehat{A}_1$ as well as the Virasoro algebra, and in a unitary representation the former of these acts with level $m-1$, for some integer $m>1$, and the latter with central charge $c=6(m-1)$. The unitary irreducible highest weight representations $V^{(m)}_{h,j}$ are labelled by the two "quantum numbers" h and j which are the eigenvalues of L_0 and $\frac{1}{2}J^3_0$, respectively, when acting on the highest weight state. (We adopt a normalisation of the $SU(2)$ current J^3 such that the zero mode J^3_0 has integer eigenvalues.) In the Ramond sector of the superconformal algebra there are two types of highest weight representations: the *short* (or *BPS*, *supersymmetric*) ones with $h=\frac{m-1}{4}$ and $j\in\{0,\frac{1}{2},\cdots,\frac{m-1}{2}\}$, and the *long* (or *non-BPS*, *non-supersymmetric*) ones with $h>\frac{m-1}{4}$ and $j\in\{\frac{1}{2},1,\cdots,\frac{m-1}{2}\}$. Their *(Ramond) characters*, defined as

$$\mathrm{ch}^{(m)}_{h,j}(\tau,z)=\mathrm{tr}_{V^{(m)}_{h,j}}\left((-1)^{J^3_0}y^{J^3_0}q^{L_0-c/24}\right), \tag{4.32}$$

are given by

$$\mathrm{ch}^{(m)}_{\frac{m-1}{4},j}(\tau,z)=\frac{\mu_{m,j}(\tau,z)}{\mu_{1,0}(\tau,z)} \tag{4.33}$$

and

$$\mathrm{ch}^{(m)}_{h,j}(\tau,z)=(-1)^{2j}q^{h-\frac{m-1}{4}-\frac{j^2}{m}}\,\frac{\theta_{m,2j}(\tau,z)-\theta_{m,-2j}(\tau,z)}{\mu_{1,0}(\tau,z)} \tag{4.34}$$

in the short and long cases, respectively, [8], where the function $\mu_{m,j}(\tau,z)$ is defined as in (3.37).

Lambencies $2,3,4,5,7,13$

It turns out that for the pure A-type Niemeier root systems given as the union of $24/(\ell-1)$ copies of $A_{\ell-1}$ for $(\ell-1)|12$, the relevant criterion for $\phi^{(\ell)}$ is that of an *extremal Jacobi form* [1, §2.5]. The idea of an extremal Jacobi form can be viewed as a generalisation of the concept of an extremal Virasoro character, a notion that was introduced in [47] and discussed in [48] in the context of pure AdS_3 gravity.

With the above definitions, following [1], for m a positive integer and ϕ a weak Jacobi form with weight 0 and index $m-1$ we say ϕ is *extremal* if it admits an expression

$$\phi = a_{\frac{m-1}{4},0}\mathrm{ch}^{(m)}_{\frac{m-1}{4},0} + a_{\frac{m-1}{4},\frac{1}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{1}{2}} + \sum_{0<r<m}\sum_{\substack{n>0\\ r^2-4mn<0}} a_{\frac{m-1}{4}+n,\frac{r}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4}+n,\frac{r}{2}} \tag{4.35}$$

for some $a_{h,j}\in\mathbb{C}$. Note the restriction on n in the last summation in (4.35). Write $J^{\mathrm{ext}}_{0,m-1}$ for the subspace of $J_{0,m-1}$ consisting of extremal weak Jacobi forms.

We recall here that the extremal condition has a very natural interpretation in terms of the mock modular forms $h=(h_r)$ of weight $1/2$ via the relation

$$\psi = -\mu_{1,0}\phi, \qquad \psi^F = \sum_{r \bmod 2m} h_r\theta_{m,r}, \tag{4.36}$$

discussed in §4.1. More precisely, the extremal condition is equivalent to the condition that

$$h_r = r\,\delta_{r^2,1}\,a_{\frac{m-1}{4},\frac{1}{2}}q^{-\frac{1}{4m}} + O\left(q^{\frac{1}{4m}}\right) \tag{4.37}$$

as $\tau\to i\infty$, which clearly implies the optimal growth condition $q^{1/4m}h_r(\tau)=O(1)$ of Theorem 4.1.

In [1] we proved that $\dim J^{\mathrm{ext}}_{0,m-1}=1$ in case $m-1$ divides 12, and vanishes otherwise, at least when $m\leq 25$. (Cf. [1, §2.5].) Explicitly, if $m-1$ divides 12 then the one-dimensional space $J^{\mathrm{ext}}_{0,m-1}$ of extremal Jacobi forms with index $m-1$ is spanned by $\varphi_1^{(m)}$, using the basis defined in §A.4. These weight 0 forms $\varphi_1^{(m)}$ arising from the extremal condition (4.35) will determine the unique weight 1 meromorphic Jacobi forms ψ^X satisfying the conditions of Theorem 4.1 for $X\in\{A_1^{24},A_2^{12},A_3^8,A_4^6,A_6^4,A_{12}^2\}$ according to (4.31) where m is the Coxeter number of X.

Lambencies $9,25$

In order to include the other two pure A-type Niemeier root systems ($X = A_8^3$ and $X = A_{24}$), it is sufficient to weaken the extremal condition slightly and consider weight 0 and index $m-1$ Jacobi forms admitting an expression

$$\phi = a_{\frac{m-1}{4},0}\mathrm{ch}^{(m)}_{\frac{m-1}{4},0} + a_{\frac{m-1}{4},\frac{1}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{1}{2}} + \sum_{0<r<m}\sum_{\substack{n>0\\ r^2-4mn\leq 0}} a_{\frac{m-1}{4}+n,\frac{r}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4}+n,\frac{r}{2}} \tag{4.38}$$

for some $a_{h,j} \in \mathbb{C}$. Notice that we have weakened the bound from $r^2-4mn<0$ to $r^2-4mn\leq 0$ in the last summand. This is directly related to the fact that for $m = 9, 25$ there exists $0 < r < m$ such that $r^2 = 0 \bmod 4m$.

There exists at most one solution to (4.38) up to rescaling (look ahead to Lemma 4.3), and inspection reveals that at $m = 9, 25$ we have the non-zero solutions $\varphi_1^{(9)}$ and $\varphi_1^{(25)}$, in the notation of §A.4. These Jacobi forms will determine, by way of (4.31), the unique weight 1 meromorphic Jacobi forms ψ^X satisfying the conditions of Theorem 4.1 for $X \in \{A_8^3, A_{24}\}$.

Lambencies $6, 10$

In order to capture $X = A_5^4D_4$ and $X = A_9^2D_6$ at $m = 6$ and $m = 10$ (the cases with m given by the product of two distinct primes), we will relax the extremal condition further and consider weight 0 and index $m-1$ Jacobi forms admitting an expression

$$\phi = a_{\frac{m-1}{4},0}\mathrm{ch}^{(m)}_{\frac{m-1}{4},0} + a_{\frac{m-1}{4},\frac{1}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{1}{2}} + \sum_{0<r<m}\sum_{\substack{n>0\\ r^2-4mn\leq 1}} a_{\frac{m-1}{4}+n,\frac{r}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4}+n,\frac{r}{2}} \tag{4.39}$$

for some $a_{h,j} \in \mathbb{C}$. For this more general condition we also have a uniqueness property.

Proposition 4.3. *For a given index $m-1$ the dimension of the space of Jacobi forms satisfying (4.39) for some $a_{h,j} \in \mathbb{C}$ is at most 1.*

Proof. If ϕ_1 and ϕ_2 are two weight 0 Jacobi forms that can be written as in (4.39), there exists a linear combination ϕ of ϕ_1 and ϕ_2 satisfying

$$\phi = aq^{-\frac{1}{4m}}\frac{\theta_{m,1}-\theta_{m,-1}}{\mu_{1,0}} + \sum_{0<r<m}\sum_{\substack{n>0\\ r^2-4mn\leq 1}} a_{\frac{m-1}{4}+n,\frac{r}{2}}\mathrm{ch}^{(m)}_{\frac{m-1}{4}+n,\frac{r}{2}} \tag{4.40}$$

for some a, $a_{h,j} \in \mathbb{C}$, where we have used that

$$2\,\mathrm{ch}^{(m)}_{\frac{m-1}{4},0} + \mathrm{ch}^{(m)}_{\frac{m-1}{4},\frac{1}{2}} + q^{-\frac{1}{4m}}\frac{\theta_{m,1}-\theta_{m,-1}}{\mu_{1,0}} = 0. \tag{4.41}$$

Equivalently, $\psi = \mu_{1,0}\phi$ is a weight 1 index m weak Jacobi form with Fourier expansion $\psi(\tau, z) = \sum_{n,\ell} c(n,\ell) q^n y^\ell$, where $c(n,\ell) = 0$ for $\ell^2 - 4mn > 1$. But such a weight 1 index m weak Jacobi form does not exist according to Theorem 9.7 of [41], based on the fact that there is no (strong) Jacobi form of weight 1 and index m for any positive integer m, as shown earlier by Skoruppa [49]. We therefore conclude that $\phi = 0$, and ϕ_1 and ϕ_2 are linearly dependent. □

This more general condition (4.39) singles out the weight 0 Jacobi forms $\varphi_1^{(6)}$ and $\varphi_1^{(10)}$ (cf. §A.4) at $m \in \{6, 10\}$ in addition to those already mentioned, for which m is prime or the square of a prime. As above, these weight 0 forms will determine, by way of (4.31), the unique weight 1 meromorphic Jacobi forms ψ^X satisfying the conditions of Theorem 4.1 for $X \in \{A_5^4 D_6, A_9^2 D_6\}$.

Remark 4.4. We expect that the space of solutions to (4.39) is zero-dimensional for all but finitely many m. The discussion of weight 1 Jacobi forms in [41, §9] suggests that Jacobi forms satisfying (4.39) might only exist for these values $m \in \{2, 3, 4, 5, 6, 7, 9, 10, 13, 25\}$ which are among those of relevance to umbral moonshine.

Lambencies $8, 12, 16, 18$

We are left with the A-type Niemeier root systems with Coxeter numbers that are not square-free and not squares of primes: they are $X = A_7^2D_5^2, A_{11}D_7E_6, A_{15}D_9, A_{17}E_7$ with $m = 8, 12, 16, 18$ respectively. To discuss these cases, let us first point out a subtlety in our procedure for determining the weight 1 meromorphic Jacobi form ψ^X from a weight 0 meromorphic Jacobi form ϕ^X using (4.31). Although the resulting weight 1 form ψ^X is unique following Theorem 4.1, in general the corresponding weight 0 form ϕ^X is not. In other words, there could be more than one ϕ^X satisfying (4.31) for a given ψ^X. For the A-type cases with $m \in \{2, 3, 4, 5, 6, 7, 9, 10, 13, 25\}$ discussed above, there is no such ambiguity since the matrix Ω^X corresponding to the Eichler–Zagier operator $\mathcal{W}^X$ is invertible. On the other hand, the matrix Ω^X is not invertible for $X \in \{A_7^2D_5^2, A_{11}D_7E_6, A_{15}D_9, A_{17}E_7\}$, corresponding to $m \in \{8, 12, 16, 18\}$. At the same time, in these cases there exists a unique $d > 1$ such that d^2 is a proper divisor of m and correspondingly, there is an interesting feature in the space of Jacobi forms. This is due to the fact (cf. [41, §4.4]) that if $\psi_{k,m/d^2}(\tau, z)$ transforms as a weight k index m/d^2 Jacobi form then $\psi_{k,m/d^2}(\tau, dz)$ transforms as a weight k index m Jacobi form. It turns out that the umbral forms discussed above, in particular those with lambency $\ell = 2, 3, 4$, help to determine the weight 0 forms ϕ^X at lambency $\ell = 8, 12, 16, 18$ by requiring the "square relation"

$$-\mu_{1,0}\phi^{(\ell)}|\mathcal{W}_\ell(d)(\tau, z) = \frac{\ell/d^2 - 1}{24}\,\psi^{(\ell/d^2)}(\tau, dz), \tag{4.42}$$

where d is the unique integer such that d^2 is a proper divisor of ℓ different from 1, and $\psi^{(\ell/d^2)}$ is the weight 1 meromorphic Jacobi form with index ℓ/d^2 that we have constructed above via (4.31). This extra condition (4.42) eliminates the kernel of the Eichler–Zagier operator $\mathcal{W}^X$ and renders our choice for ϕ^X unique. Notice that, following Table 3 we have used the lambency ℓ to denote the Niemeier root system X, and the former simply coincides with the Coxeter number for the A-type cases discussed in this subsection.

To specify this particular choice of ϕ^X, let us impose the following condition. For a non-square-free m which is not a square of a prime, we consider the weight 0 index $m - 1$ Jacobi forms ϕ, such that the finite part of the weight 1 index m Jacobi form $-\mu_{1,0}\phi|(\mathbf{1} + \mathcal{W}_m(m/d))$ is a mock Jacobi form with expansion $\sum_{n,\ell} c(n, \ell)q^n y^\ell$ and $c(n, r) = 0$ whenever $r^2 - 4mn > 1$, i.e.

$$\left(-\mu_{1,0}\phi|(\mathbf{1} + \mathcal{W}_m(m/d))\right)^F(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z},n\geq 0\\ r^2-4mn\leq 1}} c(n, r)q^n y^r. \tag{4.43}$$

From the above discussion we arrive at the following uniqueness property for such Jacobi

forms.

Proposition 4.5. *Consider integers $m, \widetilde{m}, d$ satisfying $m = \widetilde{m}d^2$ and $\widetilde{m}, d > 1$. Given any meromorphic weight 1 index n Jacobi form $\psi^{(\widetilde{m})}(\tau, z)$, there exists at most one weight 0 index $m-1$ weak Jacobi form ϕ satisfying (4.43) and the square relation*

$$-\mu_{1,0}\phi|\mathcal{W}_m(d)(\tau, z) = \psi^{(\widetilde{m})}(\tau, dz). \tag{4.44}$$

Proof. Assume that there are two weight 0 index $m-1$ weak Jacobi forms ϕ_1 and ϕ_2 satisfying

$$\mu_{1,0}\phi_1|\mathcal{W}_m(d)(\tau, z) = \mu_{1,0}\phi_2|\mathcal{W}_m(d)(\tau, z) = \psi^{(\widetilde{m})}(\tau, dz) \tag{4.45}$$

and therefore $(\mu_{1,0}\phi_1|\mathcal{W}_m(d))^F = (\mu_{1,0}\phi_2|\mathcal{W}_m(d))^F$. It then follows from Lemma 3.4 that

$$(\mu_{1,0}\phi_1|\mathcal{W}_m(m/d))^F = (\mu_{1,0}\phi_2|\mathcal{W}_m(m/d))^F. \tag{4.46}$$

Note that we also have

$$\begin{aligned} \phi_1(\tau, 0)\mu_{m,0}|\mathcal{W}_m(d) &= (\mu_{1,0}\phi_1|\mathcal{W}_m(d))^P \\ &= (\mu_{1,0}\phi_2|\mathcal{W}_m(d))^P \\ &= \phi_2(\tau, 0)\mu_{m,0}|\mathcal{W}_m(d), \end{aligned} \tag{4.47}$$

which leads to the equality between the two constants $\phi_1(\tau, 0) = \phi_2(\tau, 0)$, and hence

$$(\mu_{1,0}\phi_1|(\mathbf{1} + \mathcal{W}_m(m/d)))^P = (\mu_{1,0}\phi_2|(\mathbf{1} + \mathcal{W}_m(m/d)))^P. \tag{4.48}$$

Next, assume that ϕ_1 and ϕ_2 both satisfy (4.43). Then $\mu_{1,0}(\phi_1 - \phi_2)|(\mathbf{1} + \mathcal{W}_m(m/d))$ is a weight 1 index m weak Jacobi form with expansion $\sum_{n,\ell} c(n, \ell)q^n y^\ell$, where $c(n, \ell) = 0$ for $\ell^2 - 4mn > 1$, which can only be identically zero (cf. [41], Theorem 9.7) and we arrive at

$$\mu_{1,0}\phi_1|(\mathbf{1} + \mathcal{W}_m(m/d)) = \mu_{1,0}\phi_2|(\mathbf{1} + \mathcal{W}_m(m/d)). \tag{4.49}$$

Combining with (4.46) and using Corollary 3.3 we obtain $(\mu_{1,0}\phi_1)^F = (\mu_{1,0}\phi_2)^F$. Again, the polar part $(\mu_{1,0}\phi)^P$ of the meromorphic Jacobi form $\mu_{1,0}\phi$ for any weight 0 index $m-1$ weak Jacobi form ϕ is given by $\phi(\tau, 0)\mu_{m,0}$. This proves that $\mu_{1,0}\phi_1 = \mu_{1,0}\phi_2$ and hence $\phi_1 = \phi_2$. □

At $m \in \{8, 12, 16, 18\}$, applying the above Proposition and choosing

$$\psi^{(\widetilde{m})} = -\mu_{1,0}\phi^{(\widetilde{m})}, \quad \widetilde{m} = 2, 3, 4, \tag{4.50}$$

with $\phi^{(\widetilde{m})}$ as in Table 7 gives the weight 0 forms we need in order to specify the remaining umbral mock modular forms H^X for X of A-type.

The explicit expressions for ϕ^X for all A-type Niemeier root systems X are given in Table 7, where the basis we use for weight 0 weak Jacobi forms is summarised in §A.4 and $\varphi_0^{(\ell)}$ denotes the constant $\varphi_0^{(\ell)} = \varphi_1^{(\ell)}(\tau, 0)$.

Table 7: Weight Zero Jacobi Forms

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	A_8^3	$A_9^2D_6$	A_{12}^2	A_{24}
ℓ	2	3	4	5	6	7	9	10	13	25
$\varphi_0^{(\ell)}\phi^{(\ell)}$	$\varphi_1^{(2)}$	$\varphi_1^{(3)}$	$\varphi_1^{(4)}$	$\varphi_1^{(5)}$	$\varphi_1^{(6)}$	$\varphi_1^{(7)}$	$\varphi_1^{(9)}$	$\varphi_1^{(10)}$	$\varphi_1^{(13)}$	$\varphi_1^{(25)}$

X	$A_7^2D_5^2$	$A_{11}D_7E_6$	$A_{15}D_9$
ℓ	8	12	16
$\varphi_0^{(\ell)}\phi^{(\ell)}$	$\varphi_1^{(8)} + \frac{1}{2}\varphi_2^{(8)}$	$\varphi_1^{(12)} + \varphi_2^{(12)}$	$\varphi_1^{(16)} + \frac{1}{2}\varphi_2^{(16)}$

X	$A_{17}E_7$
ℓ	18
$\varphi_0^{(\ell)}\phi^{(\ell)}$	$\varphi_1^{(18)} + \frac{1}{3}\varphi_3^{(18)} + 4\xi(\varphi_1^{(12)} + 2\varphi_2^{(12)} + \frac{1}{3}\varphi_3^{(12)})$

Given ϕ^X and using the Eichler–Zagier operator $\mathcal{W}^X$ defined in §3.3, the formula (4.31) gives the weight 1 meromorphic Jacobi form ψ^X. From there, using the method described in §3.4 we can separate it into the polar and the finite part $\psi^X = (\psi^X)^P + (\psi^X)^F$ in a canonical way. As can be verified by inspection, the choices of ϕ^X specified in this subsection determine solutions to the hypotheses of Theorem 4.1.

Proposition 4.6. *Let X be one of the 14 Niemeier root systems with an A-type component (cf. (2.8)) and let ϕ^X be as specified in Table 7. Then the meromorphic Jacobi form ψ^X determined by (4.31) is the unique such function satisfying the conditions of Theorem 4.1. Write*

$$(\psi^X)^F = \sum_{r \,(\mathrm{mod}\, 2m)} H_r^X \theta_{m,r}, \tag{4.51}$$

then $H^X = (H^X_r)$ is the unique vector-valued mock modular form with shadow S^X satisfying the optimal growth condition (cf. Corollary 4.2).

As a result, the weight 0 weak Jacobi forms constructed in this subsection define the umbral mock modular forms H^X for each of the A-type Niemeier root systems (cf. (2.8)). The first few dozen coefficients of the components H^X_r are given in Appendix C. It is a reflection of the close relationship between the notions of optimal growth formulated here and in [41] that the umbral forms H^X attached to A-type Niemeier root systems are closely related to the mock modular forms of weight 1 that appear in §A.2 of [41].

5 The Umbral McKay–Thompson Series

The purpose of this section is to discuss the *umbral McKay–Thompson series* H^X_g conjecturally defining the graded character of the group G^X attached to the umbral module K^X (cf. §6.1). In §5.1 we specify their mock modular properties. In §5.2 we discuss the McKay–Thompson series H^X_g attached to the five A-type Niemeier root systems X with prime Coxeter numbers. Subsequently, in §5.3 we discuss the *multiplicative relations* relating the McKay–Thompson series H^X_g and $H^{X'}_{g'}$ attached to different Niemeier root systems X and X' with the Coxeter number of one of the root systems being an integer multiple of the other. In §5.4 we collect the relations between certain McKay–Thompson series and known mock theta series. As we discuss in §5.5 in detail, these relations, together with the constructions presented in §4.3 and §5.2, are sufficient to determine most of the umbral McKay–Thompson series completely. More specifically, we determine all of the umbral McKay–Thompson series H^X_g attached to all conjugacy classes $[g]$ of the umbral group G^X corresponding to all the A-type Niemeier root systems, except for $X = A^4_6, A^2_{12}$, corresponding to lambencies $7, 13$ (cf. (2.3)), for which we provide partial specifications. We also determine all of the umbral McKay–Thompson series H^X_g for all conjugacy classes $[g]$ of the umbral group G^X corresponding to all the D-type Niemeier root systems, except for the lambencies $10 + 5$ and $22 + 11$. For these D-type Niemeier root systems $10 + 5$ and $22 + 11$ we specify all the McKay–Thompson series H^X_g except for $[g] = 4A \subset G^{(10+5)}$ and $[g] = 2A \subset G^{(22+11)}$. For $X = E^4_6$ we specify H^X_g except for $[g] \in \{6A, 8AB\}$, while for $X = E^3_8$ we omit the cases $[g] \in \{2A, 3A\}$. We also provide the first few dozen coefficients of all the umbral McKay–Thompson series in Appendix C. The conjugacy class names are defined in §B.1.

5.1 Shadows

In §2.4 we described the umbral groups G^X and attached twisted Euler characters $\bar{\chi}^{X_A}$, χ^{X_A}, $\bar{\chi}^{X_D}$, χ^{X_D}, &c., to the A-, D-, and E-components of each Niemeier root system. In this section we will explain how to use these characters to define a function S_g^X, for each $g \in G^X$, which turns out to be the shadow of the vector-valued mock modular form H_g^X.

Let X be a Niemeier root system and suppose that m is the Coxeter number of X. Then given $g \in G^X$ we define $2m \times 2m$ matrices $\Omega_g^{X_A}$, $\Omega_g^{X_D}$, and $\Omega_g^{X_E}$, with entries indexed by $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$, as follows. We define $\Omega_g^{X_A}$ by setting

$$\Omega_g^{X_A} = \chi_g^{X_A} P_m^0 + \bar{\chi}_g^{X_A} P_m^1 \tag{5.1}$$

where P_m^s is the diagonal matrix (of size $2m \times 2m$, with entries indexed by $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$) with r-th diagonal equal to 1 or 0 according as $r = s \bmod 2$ or not,

$$(P_m^s)_{r,r'} = \delta_{r,s(2)}\delta_{r,r'(2m)}. \tag{5.2}$$

In (5.2) we write $\delta_{i,j(n)}$ for the function that is 1 when $i = j \pmod n$ and 0 otherwise. Note that $P_m^0 + P_m^1 = \Omega_m(1)$ is the $2m \times 2m$ identity matrix. According to the convention that $\chi_g^{X_A} = \bar{\chi}^{X_A} = 0$ if $X_A = \emptyset$ we have $\Omega_g^{X_A} = 0$ for all $g \in G^X$ in case X_A is empty; i.e. in case there are no type A components in the Niemeier root system X.

If $X_D \neq \emptyset$ then m is even. For $m > 6$ we define $\Omega_g^{X_D}$ by setting

$$\Omega_g^{X_D} = \chi_g^{X_D} P_m^0 + \bar{\chi}_g^{X_D} P_m^1 + \chi^{X_D} \Omega_m(m/2), \tag{5.3}$$

whilst for $m = 6$—an exceptional case due to triality for D_4—we define $\Omega = \Omega_g^{X_D}$ so that

$$\Omega_{r,r'} = \begin{cases} \check{\chi}_g^{X_D}\delta_{r,r'(12)}, & \text{if } r = 0, 3 \bmod 6, \\ \bar{\chi}_g^{X_D}\delta_{r,r'(12)} + \chi_g^{X_D}\delta_{r,-r'(6)}\delta_{r,r'(4)}, & \text{if } r = 1, 5 \bmod 6, \\ \chi_g^{X_D}\delta_{r,r'(12)} + \chi_g^{X_D}\delta_{r,-r'(6)}\delta_{r,r'(4)}, & \text{if } r = 2, 4 \bmod 6. \end{cases} \tag{5.4}$$

The matrices $\Omega_g^{X_E}$ are defined by setting

$$\Omega_g^{X_E} = \begin{cases} (\chi_g^{X_E} P_m^0 + \bar{\chi}_g^{X_E} P_m^1)(\Omega_m(1) + \Omega_m(4)) + \chi_g^{X_E} \Omega_m(6) & \text{if } m = 12, \\ \bar{\chi}_g^{X_E}(\Omega_m(1) + \Omega_m(6) + \Omega_m(9)), & \text{if } m = 18, \\ \bar{\chi}_g^{X_E}(\Omega_m(1) + \Omega_m(6) + \Omega_m(10) + \Omega_m(15)), & \text{if } m = 30. \end{cases} \tag{5.5}$$

Now for X a Niemeier root system we set $\Omega_g^X = \Omega_g^{X_A} + \Omega_g^{X_D} + \Omega_g^{X_E}$, and we define S_g^X by setting

$$S_g^X = \Omega_g^X \cdot S_m. \tag{5.6}$$

This generalises the construction (4.13). We conjecture (cf. §6.2) that the vector-valued function S_g^X is the shadow of the mock modular form H_g^X attached to $g \in G^X$. We will specify (most of) the H_g^X explicitly in the remainder of §5.

Remark 5.1. The matrices Ω^X, corresponding to the case where $[g]$ is the identity class, admit an ADE classification as explained in §3.3. It is natural to ask what the criteria are that characterise these matrices Ω_g^X, attached as above to elements $g \in G^X$ via the twisted Euler characters defined in §2.4.

5.2 Prime Lambencies

In this subsection we review the mock modular forms $H_g^X = (H_{g,r}^X)$ conjecturally encoding the graded characters of the umbral module K^X (cf. Conjecture 6.1) of the umbral group G^X for the five Niemeier root systems X with prime Coxeter numbers. Explicitly, these are the root systems $X = A_1^{24}, A_2^{12}, A_4^6, A_6^4, A_{12}^2$ with $\ell = 2, 3, 5, 7, 13$, and the corresponding McKay–Thompson series are denoted $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$ with $r = 1, 2, \ldots, \ell - 1$, using the notation given in Table 3. In the next subsection we will see that they determine many of the umbral McKay–Thompson series attached to the other Niemeier root systems with non-prime Coxeter numbers. The discussion of this subsection follows that of [1]. The McKay–Thompson series for $\ell = 2$, $X = A_1^{24}$ were first computed in [12–15].

In order to give explicit formulas for the mock modular forms $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$, we consider a slightly different function

$$\widehat{H}_{g,r}^{(\ell)} = H_{g,r}^{(\ell)} - \frac{\chi_r^{(\ell)}(g)}{\chi^{(\ell)}} H_r^{(\ell)}, \tag{5.7}$$

where $H^{(\ell)} = (H_r^{(\ell)})$ is the umbral form specified in §4.3 corresponding to the identity class $1A$

and whose Fourier coefficients are given in Appendix C with $H_r^{(\ell)} = H_{1A,r}^{(\ell)}$. We also let

$$\chi_r^{(\ell)}(g) = \chi_g^{X_A} \text{ for } r = 0 \bmod 2, \quad \chi_r^{(\ell)}(g) = \bar{\chi}_g^{X_A} \text{ for } r = 1 \bmod 2, \tag{5.8}$$

and

$$\chi^{(\ell)} = \chi_{1A}^{X_A} = \bar{\chi}_{1A}^{X_A} = \frac{24}{\ell - 1} \tag{5.9}$$

for

$$X_A = X = A_{\ell-1}^{24/(\ell-1)} \tag{5.10}$$

where the characters $\chi_g^{X_A}$ and $\bar{\chi}_g^{X_A}$ are defined in §2.4 and given explicitly in §B.2. Following the discussion in §5.1 we note that the combination (5.7) of $H_{g,r}^{(\ell)}$ and $H_r^{(\ell)}$ has the property of being a modular form rather than a mock modular form, as the shadows of the two contributions to $\widehat{H}_{g,r}^{(\ell)}$ cancel.

Subsequently, we define weight two modular forms

$$F_g^{(\ell)} = \sum_{r=1}^{\ell-1} \widehat{H}_{g,r}^{(\ell)} S_{\ell,r} \tag{5.11}$$

where $S_{\ell,r}$ is again the unary theta series given in (3.39). For $\ell > 3$ we also specify further weight two modular forms by setting

$$F_g^{(\ell),2} = \sum_{r=1}^{\ell-1} \widehat{H}_{g,r}^{(\ell)} S_{\ell,\ell-r}. \tag{5.12}$$

As explained in detail in [1], specifying $F_g^{(\ell)}$ and $F_g^{(\ell),2}$ is sufficient to determine $H_g^{(\ell)}$ uniquely for $\ell = 2, 3, 5$. In the case $\ell = 2$ there is only one term in the sum (5.11) and it is straightforward to obtain $\widehat{H}_{g,r}^{(\ell)}$ from the weight 2 form $F_g^{(\ell)}$. In the case $\ell = 3, 5$, we also utilise the following fact obeyed by the conjugacy classes of $G^{(\ell)}$.

For any umbral group $G^{(\ell)}$ corresponding to an A-type Niemeier root lattice with Coxeter number $\ell > 2$, for a given conjugacy class $[g]$ with $\chi_1^{(\ell)}(g) > 0$, there exists a (not necessarily different) conjugacy class $[g']$ with the property

$$\chi_1^{(\ell)}(g) = \chi_1^{(\ell)}(g'), \quad \chi_2^{(\ell)}(g) = -\chi_2^{(\ell)}(g') \tag{5.13}$$

and the order of g and g' are either the same or related by a factor of 2 or 1/2. For such paired classes we have

$$H_{g,r}^{(\ell)} + (-1)^r H_{g',r}^{(\ell)} = 0. \tag{5.14}$$

In particular, $H^{(\ell)}_{g,even} = 0$ for the self-paired classes. For $\ell = 7, 13$, this serves to constrain the function $H^{(\ell)}_g$ and supports the claims regarding their modular properties discussed in §6.2. We refer the readers to §4 of [1] for explicit expressions for the weight two forms $F^{(\ell)}_g$ and $F^{(\ell),2}_g$.

Note that, from the discussion in §5.1, the relation (5.13) implies that the shadows attached to such paired classes satisfy

$$S^{(\ell)}_{g,r} + (-1)^r S^{(\ell)}_{g',r} = 0. \tag{5.15}$$

Therefore, the paired relation (5.14) can be viewed as a consequence of (the validity of) Conjecture 6.5.

5.3 Multiplicative Relations

In this subsection we will describe a web of relations, the multiplicative relations, among the mock modular forms attached to different Niemeier lattices. The nomenclature comes from the fact that these relations occur only among mock modular forms attached to Niemeier root systems with one Coxeter number being an integer multiple of the other. To simplify the discussion we will distinguish the following two types. The *horizontal relations* relate umbral McKay–Thompson series H^X_g and $H^{X'}_g$ attached to different Niemeier root systems with the same Coxeter number, *i.e.* $m(X) = m(X')$. The *vertical relations* connect umbral McKay–Thompson series H^X_g and $H^{X'}_g$ attached to different Niemeier root systems X and X' with $m(X)|m(X')$ and $m(X) \neq m(X')$.

First we will discuss the horizontal relations, summarised in Table 8. Note that there are five pairs of Niemeier root systems that share the same Coxeter number $m \in \{6, 10, 12, 18, 30\}$. Let us choose such a pair (X', X). From the definition of the umbral shadows (4.13) and the generalisation in §5.1 to the non-identity conjugacy classes, we see that it can happen that the r'-th component $S^{X'}_{g',r'}$ of the shadow attached to the conjugacy class $[g']$ of the umbral group $G^{X'}$ is expressible as a linear combination of the components $S^X_{g,r}$ of the shadow attached to the conjugacy class $[g]$ of the umbral group G^X. It turns out that for all five equal-Coxeter-number pairs of Niemeier root systems, this indeed happens for various pairs $([g'], [g])$ of conjugacy classes.

Note that the relation between the shadows

$$S^{X'}_{g',r'} = \sum_r c_{r',r}\, S^X_{g,r} \tag{5.16}$$

X'	X	$([g'],[g])$	Relations
6+3	6	$(1A,1A)$ $(2A,2B)$ $(3B,3A)$ $(4A,8AB)$	$H^{X'}_{g',r} = H^X_{g,r} + H^X_{g,6-r}$ r odd
		$(2C,4A)$ $(4B,8AB)$ $(6B,6A)$	$H^{X'}_{g',r} = H^X_{g,r} - H^X_{g,6-r}$ r odd
10+5	10	$(1A,1A)$ $(2A,4AB)$	$H^{X'}_{g',r} = H^X_{g,r} + H^X_{g,10-r}$
		$(2B,4AB)$	$H^{X'}_{g',r} = H^X_{g,r} - H^X_{g,10-r}$
12+4	12	$(1A,1A)$ $(2A,2A)$	$H^{X'}_{g',r} = H^X_{g,r} + H^X_{g,6+r}$, $r = 1,5$ $H^{X'}_{g',4} = H^X_{g,4} + H^X_{g,8}$
18+9	18	(1A,1A)	$H^{X'}_{g',r} = H^X_{g,r} + H^X_{g,18-r}$
		(2A,1A)	$H^{X'}_{g',r} = H^X_{g,r} - H^X_{g,18-r}$
30+6,10,15	30+15	(1A,1A)	$H^{X'}_{g',1} = H^X_{g,1} + H^X_{g,11}$ $H^{X'}_{g',7} = H^X_{g,7} + H^X_{g,13}$

Table 8: Horizontal Relations.

is a necessary but insufficient condition for the linear relation

$$H^{X'}_{g',r'} = \sum_r c_{r',r} H^X_{g,r} \tag{5.17}$$

between the corresponding McKay–Thompson series to hold, since the coincidence of the shadow guarantees the coincidence of the corresponding mock modular form only up to the addition of a modular form. Nevertheless, it turns out that in umbral moonshine the relation between the McKay–Thompson series (5.17) holds whenever the relation between the shadow (5.16) holds non-trivially with $S^{X'}_{g',r'}, S^X_{g,r} \neq 0$. This fact, together with the more general multiplicative relations (5.21), can again be viewed as the consequence of the conjectured uniqueness of such mock modular forms (Conjecture 6.5). See Table 8 for the list of such horizontal relations. In particular, for the identity element a relation $H^{X'}_{r'} = \sum_r c_{r',r} H^X_r$ holds for some $c_{r',r} \in \mathbb{Z}$ for all the five pairs (X', X) of Niemeier root systems with the same Coxeter numbers.

Note that, together with the discussion in §4.3 and Proposition 3.2, this implies more specifically that the umbral mock modular forms $H^{X'}_r$ can be obtained as the theta-coefficients of the

finite part of the meromorphic weight 1 Jacobi form

$$\psi^{X'} = -\mu_{1,0}\phi^{X'}|\mathcal{W}^{X'}\ ,\ \ \phi^{X'} = \phi^{X} \tag{5.18}$$

for the four pairs $(X', X) \in \{(6, 6+3), (10, 10+5), (12, 12+4), (18, 18+9)\}$ with A-type root systems X and with the weight 0 Jacobi forms given in Table 7.

In fact, a linear relation between the shadows attached to different Niemeier root systems can happen more generally and not just among those with the same Coxeter numbers. The first indication that non-trivial relations might exist across different Coxeter numbers is the following property of the building blocks of the umbral shadow. As one can easily check, the unary theta function $S_m = (S_{m,r})$ defined in (3.39) at a given index m can be re-expressed in terms of those at a higher index as

$$S_{m,r}(\tau) = \sum_{\ell=0}^{\lceil \frac{n}{2} \rceil - 1} S_{nm,r+2m\ell}(n\tau) - \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} S_{nm,2m\ell-r}(n\tau), \tag{5.19}$$

for any positive integer n. The above equality makes it possible to have the relation

$$\sum_{r'} c_{r,r'} S^{X'}_{g',r'}(k\tau) = S^{X}_{g,r}(\tau) \tag{5.20}$$

for some $k \in \mathbb{Z}_{>0}$. We will see that this relation between the umbral shadows does occur for many pairs of Niemeier root systems (X', X) with $m(X') = km(X)$. Moreover, whenever this relation holds non-trivially with $S^{X'}_{g',r'}, S^{X}_{g,r} \neq 0$, the corresponding relation among the McKay–Thompson series

$$\sum_{r'} c_{r,r'} H^{X'}_{g',r'}(\tfrac{m(X')}{m(X)}\tau) = H^{X}_{g,r}(\tau) \tag{5.21}$$

also holds.

We summarise a minimal set of such relations in Table 9. In Tables 8 and 9, when it is not explicitly specified, the relation holds for all values of r such that all $H^{X'}_{g',r_1}$ and H^{X}_{g,r_2} appearing on both sides of the equation have $1 \leq r_1 \leq m(X')$ and $1 \leq r_2 \leq m(X)$. From the relations recorded in these tables as well as the paired relations (5.58) and (5.62), many further relations can be derived. For example, combining the relations between the $(1A, 2BC)$ classes for $(X', X) = (16, 8)$, the $(2BC, 4C)$ classes for $(X', X) = (8, 4)$ and the $(4C, 8A)$ classes at $(X', X) = (4, 2)$, one can deduce that there is a multiplicative relation $\sum_{n=0}^{7}(-1)^n H^{(16)}_{1A,2n+1}(8\tau) = H^{(2)}_{8A,1}(\tau)$. See §5.5 for more detailed information.

These multiplicative relations form an intricate web relating umbral moonshine at different

X'	X	$([g'],[g])$	Relations
4	2	$(1A,2A),(2B,4A)$ $(2C,4B),(3A,6A)$ $(4A,4C),(4C,8A)$ $(6BC,12A),(7AB,14AB)$	$(H^{X'}_{g',1}-H^{X'}_{g',3})(2\tau)=H^{X}_{g,1}(\tau)$
6	2	$(1A,3A),(2B,6A)$ $(8AB,12A)$	$\sum_{n=0}^{2}(-1)^n H^{X'}_{g',1+2n}(3\tau)=H^{X}_{g,1}(\tau)$
	3	$(1A,2B),(2A,2C)$ $(2B,4C),(4A,4B)$ $(3A,6C),(6A,6D)$ $(8AB,8CD)$	$(H^{X'}_{g',r}-H^{X'}_{g',6-r})(2\tau)=H^{X}_{g,r}(\tau)$
$6+3$	2	$(5A,15AB)$	$\sum_{n=0}^{2}(-1)^n H^{X'}_{g',1+2n}(3\tau)=H^{X}_{g,1}(\tau)$
8	4	$(1A,2C),(2BC,4C)$ $(4A,4B)$	$(H^{X'}_{g',r}-H^{X'}_{g',8-r})(2\tau)=H^{X}_{g,r}(\tau)$
9	3	$(1A,3A),(2B,6C)$	$(H^{X'}_{g',r}+H^{X'}_{g',r+6}-H^{X'}_{g',6-r})(3\tau)=H^{X}_{g,r}(\tau)$
10	2	$(1A,5A),(4AB,10A)$	$\sum_{n=0}^{4}(-1)^n H^{X'}_{g',1+2n}(5\tau)=H^{X}_{g,1}(\tau)$
	5	$(1A,2C),(4AB,4CD)$	$(H^{X'}_{g',r}-H^{X'}_{g',10-r})(2\tau)=H^{X}_{g,r}(\tau)$
$10+5$	2	$(3A,15AB)$	$(H^{X'}_{g',1}-H^{X'}_{g',3}+H^{X'}_{g',5}/2)(5\tau)=H^{X}_{g,1}(\tau)$
12	4	$(1A,3A)$	$(H^{X'}_{g',r}+H^{X'}_{g',r+8}-H^{X'}_{g',8-r})(3\tau)=H^{X}_{g,r}(\tau)$
	6	$(1A,2B)$	$(H^{X'}_{g',r}-H^{X'}_{g',12-r})(2\tau)=H^{X}_{g,r}(\tau)$
12+4	6	$(3A,3A)$ $(2B,8AB)$	$(H^{X'}_{g',1}-H^{X'}_{g',5})(2\tau)=(H^{X}_{g,1}-H^{X}_{g,5})(\tau)$ $H^{X'}_{g,4}(2\tau)=(H^{X}_{g,2}+H^{X}_{g,4})(\tau)$
14+7	2	$(1A,7AB),(2A,14AB)$ $(3A,21AB)$	$(-\frac{1}{2}H^{X'}_{g',7}+\sum_{n=0}^{2}(-1)^n H^{X'}_{g',1+2n})(7\tau)=H^{X}_{g,1}(\tau)$
16	8	$(1A,2BC)$	$(H^{X'}_{g',r}-H^{X'}_{g',16-r})(2\tau)=H^{X}_{g,r}(\tau)$
18	6	$(1A,3A)$	$(H^{X'}_{g',r}-H^{X'}_{g',12-r}+H^{X'}_{g',12+r})(3\tau)=H^{X}_{g,r}(\tau)$
	9	$(1A,2B)$	$3(H^{X'}_{g',r}-H^{X'}_{g',18-r})(2\tau)=H^{X}_{g,r}(\tau)$
$22+11$	2	$(1A,11A)$	$(-\frac{1}{2}H^{X'}_{g',11}+\sum_{n=0}^{4}(-1)^n H^{X'}_{g',1+2n})(11\tau)=H^{X}_{g,1}(\tau)$
25	5	$(1A,5A)$	$(\sum_{n=0}^{2}H^{X'}_{g',10n+r}-\sum_{n=1}^{2}H^{X'}_{g',10n-r})(5\tau)=H^{X}_{g,r}(\tau)$
30+15	2	$(1A,15AB)$	$(-\frac{1}{2}H^{X'}_{g',15}+\sum_{n=0}^{6}(-1)^n H^{X'}_{g',1+2n})(15\tau)=H^{X}_{g,1}(\tau)$
	$10+5$	$(1A,3A)$	$(H^{X'}_{g',r}+H^{X'}_{g',10-r}-H^{X'}_{g',10+r})(3\tau)=H^{X}_{g,r}(\tau)$
$46+23$	2	$(1A,23AB)$	$(-\frac{1}{2}H^{X'}_{g',23}+\sum_{n=0}^{10}(-1)^n H^{X'}_{g',1+2n})(23\tau)=H^{X}_{g,1}(\tau)$

Table 9: Vertical Relations.

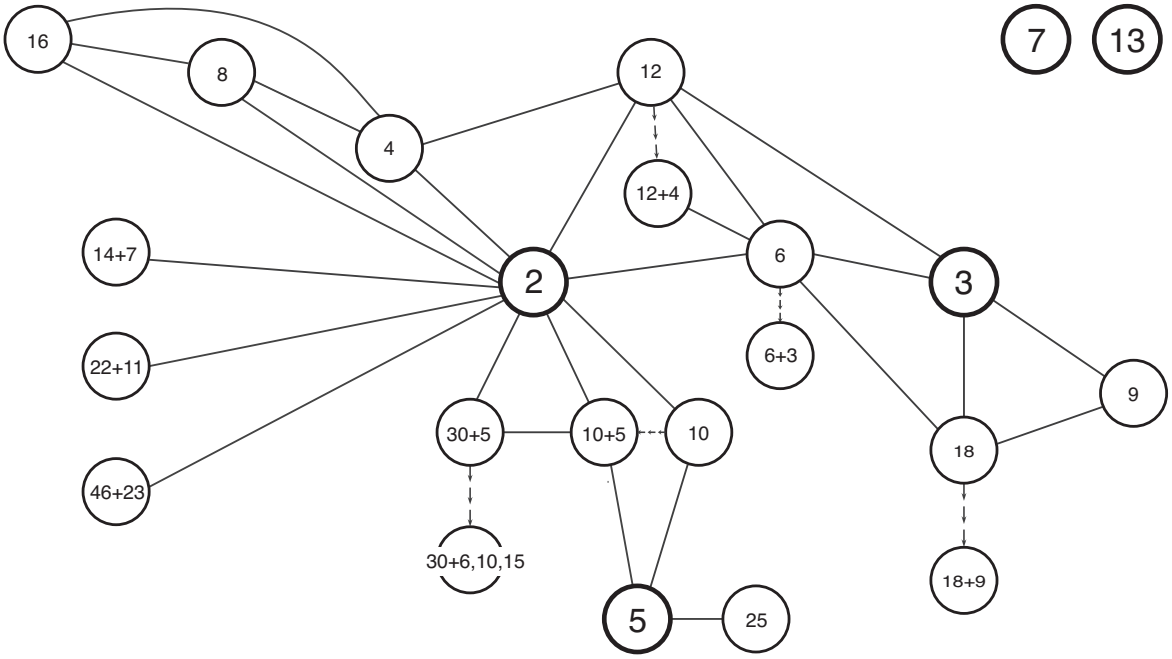


Figure 3: Multiplicative Relations.

lambencies. We summarise this web in Figure 3, where the horizontal relations are indicated by arrowed dashed lines and the vertical relations are indicated by solid lines.

We will finish this section with a discussion of a curious property of the multiplicative relations. Note that these relations occur among the McKay–Thomson series attached to $[g] \subset G^X$ and $[g'] \subset G^{X'}$ with $ord(g')\, m(X')$ coinciding with $ord(g)\, m(X)$ up to a factor of 2. This property can be understood as a consequence of the relation between the level n_g of the automorphy group $\Gamma_0(n_g)$ of the McKay–Thomson series H_g^X and the order $ord(g)$ of the group element (cf. §6.2). The extra factor of 2 can be understood as a consequence of the structure $G^X = 2.\bar{G}^X$ of the associated umbral groups (cf. §2.4).

5.4 Mock Theta Functions

In this subsection we record relations between the McKay–Thompson series of umbral moonshine and known mock theta functions. Many of the mock theta functions arising appear either in Ramanujan's last letter to Hardy or in his lost notebook [50]. In what follows we will give

explicit expressions for the mock theta series using the *q-Pochhammer symbol*

$$(a;q)_n = \prod_{k=0}^{n-1}(1-a\,q^k). \tag{5.22}$$

For lambency 2, two of the functions $H_g^{(2)}(\tau)$ are related to Ramanujan's mock theta functions of orders 2 and 8 through

$$\begin{aligned} H_{4B}^{(2)}(\tau) &= -2\,q^{-1/8}\mu(q) = -2\,q^{-1/8}\sum_{n\geq 0}\frac{(-1)^n\,q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2} \\ H_{8A}^{(2)}(\tau) &= -2\,q^{-1/8}U_0(q) = -2\,q^{-1/8} = \sum_{n\geq 0}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}. \end{aligned} \tag{5.23}$$

For $\ell = 3$ we encounter the following order 3 mock theta functions of Ramanujan:

$$\begin{aligned} &H_{2B,1}^{(3)}(\tau) = H_{2C,1}^{(3)}(\tau) = H_{4C,1}^{(3)}(\tau) = -2q^{-1/12}f(q^2) \\ &H_{6C,1}^{(3)}(\tau) = H_{6D,1}^{(3)}(\tau) = -2q^{-1/12}\chi(q^2) \\ &H_{8C,1}^{(3)}(\tau) = H_{8D,1}^{(3)}(\tau) = -2q^{-1/12}\phi(-q^2) \\ &H_{2B,2}^{(3)}(\tau) = -H_{2C,2}^{(3)}(\tau) = -4q^{2/3}\omega(-q) \\ &H_{6C,2}^{(3)}(\tau) = -H_{6D,2}^{(3)}(\tau) = 2q^{2/3}\rho(-q), \end{aligned} \tag{5.24}$$

where

$$\begin{aligned} f(q) &= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \\ \phi(q) &= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})} \\ \chi(q) &= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})} \\ \omega(q) &= \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2} \\ \rho(q) &= \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})}\ . \end{aligned} \tag{5.25}$$

For $\ell = 4$ and $\ell = 16$ we have the relations

$$\begin{aligned} H^{(4)}_{2C,1}(\tau) &= q^{-\frac{1}{16}}\left(-2S_0(q) + 4T_0(q)\right), \\ H^{(4)}_{2C,3}(\tau) &= q^{\frac{7}{16}}\left(2S_1(q) - 4T_1(q)\right), \\ H^{(4)}_{4C,1}(\tau) &= -2\, q^{-\frac{1}{16}} S_0(q), \\ H^{(4)}_{4C,3}(\tau) &= 2\, q^{\frac{7}{16}}\, S_1(q) \end{aligned} \tag{5.26}$$

and

$$H^{(16)}_{1A,2}(\tau) = H^{(16)}_{1A,14} = 2q^{-1/16}T_0(-q) = \frac{1}{2}\left(H^{(4)}_{2C,1}(\tau) - H^{(4)}_{4C,1}(\tau)\right) \tag{5.27}$$

$$\begin{aligned} H^{(16)}_{1A,4}(\tau) &= H^{(16)}_{1A,12} = 2q^{-1/4}V_1(q) \\ H^{(16)}_{1A,6}(\tau) &= H^{(16)}_{1A,10} = 2q^{7/16}T_1(-q) \end{aligned} \tag{5.28}$$

to the order 8 mock theta functions

$$S_0(\tau) = \sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{5.29}$$

$$S_1(\tau) = \sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{5.30}$$

$$T_0(\tau) = \sum_{n\geq 0} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \tag{5.31}$$

$$T_1(\tau) = \sum_{n\geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \tag{5.32}$$

$$V_1(\tau) = \sum_{n\geq 0} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} \tag{5.33}$$

discussed in [51]. We also have $H^{(16)}_{2A,r} = -H^{(16)}_{1A,r}$ for $r = 2, 4, 6, 10, 12, 14$.

For $\ell = 5$ we encounter four of Ramanujan's order 10 mock theta functions:

$$\begin{aligned} H^{(5)}_{2BC,1}(\tau) &= H^{(5)}_{4CD,1}(\tau) = -2q^{-\frac{1}{20}}\, X(q^2) \\ H^{(5)}_{2BC,3}(\tau) &= H^{(5)}_{4CD,3}(\tau) = -2q^{-\frac{9}{20}}\, \chi_{10}(q^2) \\ H^{(5)}_{2C,2}(\tau) &= -H^{(5)}_{2B,2}(\tau) = 2q^{-\frac{1}{5}}\, \psi_{10}(-q) \\ H^{(5)}_{2C,4}(\tau) &= -H^{(5)}_{2B,4}(\tau) = -2q^{\frac{1}{5}}\, \phi_{10}(-q) \end{aligned} \tag{5.34}$$

given by

$$\phi_{10}(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}} \tag{5.35}$$

$$\psi_{10}(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}} \tag{5.36}$$

$$X(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \tag{5.37}$$

$$\chi_{10}(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} \ . \tag{5.38}$$

At $\ell = 6$ and $\ell = 6+3$ we find

$$H^{(6)}_{2B,3}(\tau) = -2\, q^{-3/8}\psi_6(q) \tag{5.39}$$

$$H^{(6+3)}_{2A,1}(\tau) = -2\, q^{-1/24}\phi_6(q) \tag{5.40}$$

$$H^{(6+3)}_{3B,1}(\tau) = -2\, q^{-1/24}\gamma_6(q) \tag{5.41}$$

$$H^{(6+3)}_{2B,1}(\tau) = -2\, q^{-1/24}f(q) \tag{5.42}$$

$$H^{(6+3)}_{4B,1}(\tau) = -2\, q^{-1/24}\phi(-q) \tag{5.43}$$

$$H^{(6+3)}_{6B,1}(\tau) = -2\, q^{-1/24}\chi(q) \tag{5.44}$$

where $f(q), \phi(q), \chi(q)$ are third order mock theta functions given earlier and

$$\psi_6(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}} \tag{5.45}$$

$$\phi_6(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}} \tag{5.46}$$

$$\gamma_6(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \tag{5.47}$$

are sixth order mock theta functions.

For $\ell = 8$

$$H^{(8)}_{1A,2}(\tau) = H^{(8)}_{1A,6} = 4\, q^{-1/4}A(q) = 4\, q^{-1/4}\sum_{n\geq 0} \frac{q^{(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}} \tag{5.48}$$

$$H^{(8)}_{1A,4}(\tau) = 4\, q^{1/2}B(q) = 4\, q^{1/2}\sum_{n\geq 0} \frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}} \tag{5.49}$$

where $A(q)$, $B(q)$ are both second order mock theta functions discussed in [52].

For $\ell = 12$ and $\ell = 12 + 4$ we have

$$H^{(12)}_{1A,2}(\tau) = H^{(12)}_{1A,10}(\tau) = -2\, q^{-4/48}\sigma(q) \tag{5.50}$$

$$H^{(12)}_{1A,4}(\tau) = H^{(12)}_{1A,8}(\tau) = 2\, q^{2/3}\omega(q) \tag{5.51}$$

$$H^{(12+4)}_{1A,1}(\tau) = -q^{-1/48}\left(f(q^{1/2}) - f(-q^{1/2})\right) \tag{5.52}$$

where $\sigma(q)$ is the order 6 mock theta function

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}} \tag{5.53}$$

and $\omega(q)$ and $f(q)$ are third order mock theta functions given in (5.25).

For $\ell = 30 + 6, 10, 15$ we find four of Ramanujan's mock theta functions of order 5:

$$\begin{aligned}
H^{(30+6,10,15)}_{1A,1}(\tau) &= q^{-1/120}\,(2\,\chi_0(q) - 4)\\
H^{(30+6,10,15)}_{1A,7}(\tau) &= 2\, q^{71/120}\chi_1(q)\\
H^{(30+6,10,15)}_{2A,1}(\tau) &= -2q^{-1/120}\phi_0(-q)\\
H^{(30+6,10,15)}_{2A,7}(\tau) &= 2q^{-49/120}\phi_1(-q)
\end{aligned} \tag{5.54}$$

where

$$\begin{aligned}
\chi_0(\tau) &= \sum_{n\geq 0} \frac{q^n}{(q^{n+1};q)_n}\\
\chi_1(\tau) &= \sum_{n\geq 0} \frac{q^n}{(q^{n+1};q)_{n+1}}\\
\phi_0(\tau) &= \sum_{n\geq 0} q^{n^2}(-q;q^2)_n\\
\phi_1(\tau) &= \sum_{n\geq 0} q^{(n+1)^2}(-q;q^2)_n.
\end{aligned} \tag{5.55}$$

5.5 Specification

In this subsection we will combine the different types of data on the McKay–Thompson series discussed in §4.3 and §5.2-5.4, and explain how they lead to explicit expressions for the umbral McKay–Thompson series.

First we will note one more relation among umbral McKay–Thompson series H^X_g and $H^X_{g'}$

attached to different conjugacy classes of the same umbral group G^X. Notice that for all the A-type Niemeier root systems as well as $X = E_6^4$, the corresponding umbral group takes the form

$$G^X = 2.\bar{G}^X \tag{5.56}$$

and the corresponding McKay–Thompson series display the following paired relation. For the A-type Niemeier root system X, for every conjugacy class $[g] \subset G^X$ with $\bar{\chi}_g^{X_A} > 0$ there is a unique conjugacy class $[g']$ with

$$\chi_g^{X_A} = -\chi_{g'}^{X_A}, \quad \bar{\chi}_g^{X_A} = \bar{\chi}_{g'}^{X_A}, \tag{5.57}$$

that we say to be paired with $[g]$. For such paired classes, the corresponding McKay–Thompson series satisfy the relation

$$H_{g,r}^X + (-1)^r H_{g',r}^X = 0. \tag{5.58}$$

This generalises the paired property for the pure A-series discussed in §5.2. Similarly, for $X = E_6^4$ we have for every conjugacy class $[g] \subset G^X$ with $\chi_g^{X_E} > 0$ a conjugacy class $[g']$ with

$$\chi_g^{X_E} = -\chi_{g'}^{X_E}, \quad \bar{\chi}_g^{X_E} = \bar{\chi}_{g'}^{X_E}, \tag{5.59}$$

that is paired with $[g]$. For such paired classes, the corresponding McKay–Thompson series again satisfy the relation (5.58).

For the lambency $\ell = 6 + 3$, corresponding to $X = D_4^6$, the corresponding umbral group has the form

$$G^X = 3.\bar{G}^X \tag{5.60}$$

and the conjugacy classes with $\bar{\chi}^{X_D}(g) > 0$ form pairs satisfying

$$\chi_g^{X_D} = \chi_{g'}^{X_D}, \quad \check{\chi}_g^{X_D} + 2\check{\chi}_{g'}^{X_D} = 0. \tag{5.61}$$

For these paired conjugacy classes, the McKay–Thompson series have the property

$$H_{g,1}^X - H_{g',1}^X = 0, \quad H_{g,3}^X + 2H_{g',3}^X = 0. \tag{5.62}$$

In particular, we have $H_{g,3}^{(6+3)} = 0$ for all the self-paired classes. From the discussion in §5.1 we see that the relation between the twisted Euler characters and the paired relations is implied by the shadows of the umbral McKay–Thompson series, just as the multiplicative relations

discussed in the §5.3. As a result, one can view the relations (5.58) and (5.62) as a consequence of the apparently general phenomenon that the umbral McKay–Thompson series are determined by their mock modular properties together with the optimal growth condition, cf. Conjecture 6.5.

In the rest of this subsection we will tie these relations together and discuss each lambency separately. First of all, for all the A-type Niemeier root systems

$$\ell \in \{2,3,4,5,6,7,8,9,10,12,13,16,18,25\}, \tag{5.63}$$

the discussion in §4.3 specifies all $H^{(\ell)} = (H^{(\ell)}_{1A,r})$, $r = 1,\ldots,\ell-1$ and it remains only to specify $H^{(\ell)}_g$ with $[g] \subset G^X$ different from the identity class. For $\ell = 2,3,5$, the discussion in §5.2 specifies all the McKay–Thompson series. For $\ell = 7,13$, the discussion in §5.2 gives partial information on all the McKay–Thompson series. For $\ell = 4$, the vertical relations in Table 9 specify all the odd components of $H^{(4)}_{g,r}$ for all conjugacy classes $[g] \in G^{(4)}$ except for $[g] = 4B, 8A$. The odd components for $[g] = 4B, 8A$ can be specified by the identities

$$\begin{aligned}(H^{(4)}_{4B,1} - H^{(4)}_{4B,3})(\tau) &= -2\frac{\eta(\tau/2)\eta(\tau)^4}{\eta(\tau)^2\eta(4\tau)^2},\\ (H^{(4)}_{8A,1} - H^{(4)}_{8A,3})(\tau) &= -2\frac{\eta(\tau)^3}{\eta(\tau/2)\eta(4\tau)}.\end{aligned} \tag{5.64}$$

We are left to determine the second components $H^{(4)}_{g,2}(\tau)$, which are given by

$$\begin{aligned}H^{(4)}_{3A,2}(\tau) &= \frac{1}{4}H^{(4)}_2(\tau) + \frac{1}{2\eta(2\tau)^3}\Big(-3\Lambda_2(\tau) - 4\Lambda_3(\tau) + \Lambda_6(\tau)\Big),\\ H^{(4)}_{7AB,2}(\tau) &= \frac{1}{8}H^{(4)}_2(\tau) + \frac{1}{12\,\eta(2\tau)^3}\Big(-7\Lambda_2(\tau) - 4\Lambda_7(\tau) + \Lambda_{14}(\tau) + 28 f_{14}(\tau)\Big).\end{aligned} \tag{5.65}$$

Together with the paired relation (5.58), Table 9, and

$$H^{(4)}_{g,2}(\tau) = 0 \quad \text{for all } [g] \notin \{1A, 2A, 3A, 6A, 7AB, 14AB\}, \tag{5.66}$$

(5.64)-(5.65) completely specifies all $H^{(4)}_g = (H^{(4)}_{g,r})$ for all $[g] \subset G^{(4)}$.

For $\ell = 6$, the vertical relations in Table 9 to the McKay–Thompson series of lambencies $\ell = 2,3$ suffice to specify all $H^{(6)}_g = (H^{(6)}_{g,r})$ except for the components $r = 1,5$ of the classes $3A$ and $6A$. Subsequently, the relation to $H^{(18)}_{1A,r}$ in Table 9 and the paired relation (5.58) determines $H^{(18)}_{3A}$ and $H^{(18)}_{6A}$.

For $\ell = 8$, the vertical relations to $\ell = 4$ recorded in Table 9, together with the paired

relation (5.58) and $H^{(8)}_{2BC,2n} = H^{(8)}_{4A,2n} = 0$ for all $n = 1, 2, 3$ specify all the $H^{(8)}_g = (H^{(8)}_{g,r})$.

For $\ell = 9$, the vertical relations to $\ell = 3$ recorded in Table 9 together with the paired relation (5.58) determine all $H^{(9)}_g$ except for $[g] = 3A, 6A$. To determine $H^{(9)}_{3A}$ and $H^{(9)}_{6A}$, we have

$$\begin{aligned}
(H^{(9)}_{3A,1} - H^{(9)}_{3A,5} + H^{(9)}_{3A,7})(3\tau) &= -6\,\frac{\eta(\tau)\eta(12\tau)\eta(18\tau)^2}{\eta(6\tau)\eta(9\tau)\eta(36\tau)} \\
(H^{(9)}_{3A,2} - H^{(9)}_{3A,4} + H^{(9)}_{3A,8})(3\tau) &= -3\left(\frac{\eta(\tau)\eta(2\tau)\eta(3\tau)^2}{\eta(4\tau)^2\eta(9\tau)} - \frac{\eta(2\tau)^6\eta(12\tau)\eta(18\tau)^2}{\eta(\tau)\eta(4\tau)^4\eta(6\tau)\eta(9\tau)\eta(36\tau)}\right) \\
H^{(9)}_{3A,3}(\tau) = H^{(9)}_{3A,6}(\tau) &= H^{(9)}_{6A,3}(\tau) = H^{(9)}_{6A,6}(\tau) = 0.
\end{aligned}$$

Together with the paired relation this determines all $H^{(9)}_{3A,r}$ and $H^{(9)}_{6A,r}$ and finishes the specification for $\ell = 9$.

For $\ell = 10$, the vertical relations to $\ell = 2$ and $\ell = 5$ recorded in Table 9 specify all $H^{(10)}_g$. For $\ell = 12, 16, 18, 25$, there is only one conjugacy class $2A$ except for the identity class. The paired relation (5.58) relating the McKay–Thompson series for the $2A$ class to that of the identity class then determines all $H^{(\ell)}_g$.

Next we turn to the D-type Niemeier root systems. For $\ell = 6+3$, the horizontal relations in Table 8 determine all McKay–Thompson series $H^{(6+3)}_g$, except for $[g] \in \{3C, 6C\}$ that are given by

$$\begin{aligned}
H^{(6+3)}_{3C,1}(\tau) = -2\frac{\eta^2(\tau)}{\eta(3\tau)}\,,\ H^{(6+3)}_{6C,1}(\tau) &= -2\frac{\eta(2\tau)\,\eta(3\tau)}{\eta(6\tau)}, \\
H^{(6+3)}_{3C,3} = H^{(6+3)}_{6C,3} &= 0.
\end{aligned} \tag{5.67}$$

For $\ell = 10+5$, the horizontal relations in Table 8 and the vertical relations in Table 9 determine all McKay–Thompson series $H^{(10+5)}_g$, except for $[g] = 4A$. For $\ell = 14+7$, the vertical relations in Table 9 determine all McKay–Thompson series $H^{(14+7)}_g$. For $\ell = 18+9$, the horizontal relations in Table 8 determine all McKay–Thompson series $H^{(18+9)}_g$. For $\ell = 22+11$, the vertical relations in Table 9 determine the umbral form $H^{(22+11)}_{1A}$. For $\ell = 30+15$ and $\ell = 46+23$, the vertical relations in Table 9 determine the only McKay–Thompson series $H^{(\ell)}_{1A}$ attached to the corresponding Niemeier root system.

Finally we discuss the McKay–Thompson series attached to the two E-type Niemeier root systems. For $\ell = 12+4$, the horizontal relations in Table 8 and the vertical relations in Table 9 determine all the McKay–Thompson series $H^{(12+4)}_g$ except for $[g] \in \{6A, 8AB\}$. For $\ell = 30+6, 10, 15$, the horizontal relations in Table 8 and 9 determine the umbral mock modular form $H^{(30+6,10,15)}_{1A}$.

Focusing on the umbral mock modular form $H^X = H^X_{1A}$ corresponding to the identity class, one can explicitly check that the above specification determines the unique vector-valued mock modular form satisfying the conditions of Corollary 4.2.

Proposition 5.2. *Let X be one of the D- or E-type Niemeier root systems (cf. (2.9), (2.10)). The vector-valued mock modular form $H^X = H^X_{1A}$ specified above is the unique vector-valued mock modular form with shadow S^X satisfying the optimal growth condition (cf. Corollary 4.2).*

Together with Proposition 4.6, this proposition establishes our construction of the unique vector-valued mock modular form H^X corresponding to all 23 Niemeier root systems X.

6 Conjectures

In this section we pose the umbral moonshine conjectures connecting the umbral groups G^X and the mock modular forms H^X_g discussed in the previous sections.

6.1 Modules

In this section we formulate a conjecture that relates the umbral McKay–Thompson series H^X_g to an infinite-dimensional G^X-module K^X.

Recall that a *super-space* V is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If $T : V \to V$ is a linear operator preserving the grading then the *super-trace* of T is given by $\mathrm{str}_V\, T = \mathrm{tr}_{V_{\bar{0}}}\, T - \mathrm{tr}_{V_{\bar{1}}}\, T$ where $\mathrm{tr}_W\, T$ denotes the usual trace of T on W. We say that V is purely even (odd) when $V = V_{\bar{0}}$ ($V_{\bar{1}}$). If V is a G-module, with G-action preserving the $\mathbb{Z}/2\mathbb{Z}$-grading, then the function $g \mapsto \mathrm{str}_V\, g$ is called the *super-character* of G determined by V.

Conjecture 6.1. *Let X be a Niemeier root system and let m be the Coxeter number of X.*

There exists a naturally defined $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Q}$-graded super-module

$$K^X = \bigoplus_{r \bmod 2m} K^X_r = \bigoplus_{r \bmod 2m} \bigoplus_{\substack{D \in \mathbb{Z} \\ D = r^2 \bmod 4m}} K^X_{r,-D/4m} \tag{6.1}$$

for G^X such that the graded super-character attached to an element $g \in G^X$ coincides with the vector-valued mock modular form

$$c^X H^X_{g,r}(\tau) = \sum_{\substack{D \in \mathbb{Z} \\ D = r^2 \bmod 4m}} \mathrm{str}_{K^X_{r,-D/4m}}(g)\, q^{-D/4m}, \tag{6.2}$$

where $c^X = 1$ except for $X = A_8^3$, for which $c^X = 3$. Moreover, the homogeneous component $K^X_{r,d}$ of K^X is purely even if $d > 0$.

The reason for the exceptional value $c^X = 3$ for $X = A_8^3$ is the curious fact that there are no integer combinations of irreducible characters of G^X that coincide with the coefficients $q^{-D/36}$, $D = -27\lambda^2$ for some integer λ, of $H^X_{g,r}(\tau)$ (cf. Conjecture 6.11). For example, the minimal positive integer c for which $g \mapsto cH^X_{g,6}$ is a graded virtual super-character of G^X is $c = c^X = 3$.

Combining the above conjecture and the paired relations (5.58) and (5.62) of the McKay–Thompson series we arrive at the following.

Conjecture 6.2. *Let X be a Niemeier root system and set $c = \#G^X/\#\bar{G}^X$. Then the $\mathbb{Q}$-graded G^X-module K^X_r is a faithful representation of G^X when $r \equiv 0 \pmod{c}$ and factors through $\bar{G}^X$ otherwise.*

As discussed in §2, we have $c = 2$ for $\ell \in \{3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25, 12 + 4\}$ and $c = 3$ for $\ell = 6 + 3$, and $c = 1$ in the remaining 8 cases.

As mentioned in §1, for the special case $X = A_1^{24}$, Conjecture 6.1 has been shown to be true by T. Gannon in [16], although the construction of K^X is still absent. In this case, we have $c = 1$ and hence Conjecture 6.2 is automatically true. It should be possible to apply the techniques similar to that in [16] to prove Conjecture 6.1 for other Niemeier root systems X.

6.2 Modularity

We have attached a cusp form S^X_g to each $g \in G^X$ in §5.1 by utilising the naturally defined permutation representations of G^X, and the corresponding twisted Euler characters, that are described in §2.4. We begin this section with an explicit formulation of the conjecture that these cusp forms describe the shadows of the super-characters attached to the conjectural G^X-module K^X.

In preparation for the statement define n_g and h_g for $g \in G^X$ as follows. Take n_g to be the order of the image of g in $\bar{G}^X$ (cf. §2.4), and set $h_g = N_g/n_g$ where N_g denotes the product of the shortest and longest cycle lengths of the permutation which is the image of g under $G^X \to Sym_\Phi$. These values are on display in the tables of §B.2. They may also be read off from the cycle shapes $\widetilde{\Pi}^X_g$ and $\bar{\Pi}^{X_A}_g$, $\bar{\Pi}^{X_D}_g$, &c., attached to the permutation representations constructed in §2.4, for n_g is the maximum of the cycle lengths appearing in $\bar{\Pi}^{X_A}_g$, $\bar{\Pi}^{X_D}_g$ and $\bar{\Pi}^{X_E}_g$, and if $\widetilde{\Pi}^X_g = j_1^{m_1} \cdots j_k^{m_k}$ with $j_1^{m_1} < \cdots < j_1^{m_k}$ and $m_i > 0$ then $h_g = j_1 j_k/n_g$.

Conjecture 6.3. *The graded super-characters (6.2) for fixed X and $g \in G^X$ and varying $r \in \mathbb{Z}/2m\mathbb{Z}$ define the components of a vector-valued mock modular form H^X_g of weight $1/2$ on*

$\Gamma_0(n_g)$ *with shadow function* S_g^X.

Let ν_g^X denote the multiplier system of H_g^X. Since the multiplier system of a mock modular form is the inverse of the multiplier system of its shadow, Conjecture 6.3 completely determines the modular properties of H_g^X—i.e. the matrix-valued function ν_g^X—when S_g^X is non-vanishing. However, it may happen that S_g^X vanishes identically and H_g^X is a(n honest) modular form. The following conjecture puts a strong restriction on ν_g^X even in the case of vanishing shadow.

Conjecture 6.4. *The multiplier system* ν_g^X *for* H_g^X *coincides with the inverse of the multiplier system for* S^X *when restricted to* $\Gamma_0(n_g h_g)$.

6.3 Moonshine

We now formulate a conjecture which may be regarded as the analogue of the principal modulus property (often referred to as the genus zero property) of monstrous moonshine.

The monstrous McKay–Thompson series T_g, for g an element of the monster, are distinguished in that each one is a principal modulus with pole at infinity for a genus zero group Γ_g, meaning that T_g is a Γ_g-invariant holomorphic function on the upper-half plane having a simple pole at the infinite cusp of Γ_g, but having no poles at any other cusps of Γ_g. Equivalently, T_g satisfies the conditions

$$\begin{aligned} &\text{(i)} \quad T_g|_{1,0}\gamma = T_g \text{ for all } \gamma \in \Gamma_g, \\ &\text{(ii)} \quad qT_g(\tau) = O(1) \text{ as } \tau \to i\infty, \\ &\text{(iii)} \quad T_g(\tau) = O(1) \text{ as } \tau \to \alpha \in \mathbb{Q} \text{ whenever } \infty \notin \Gamma_g\alpha, \end{aligned} \tag{6.3}$$

for some group Γ_g, where $T_g|_{1,0}\gamma$ is the function $\tau \mapsto T_g(\gamma\tau)$ by definition (cf. (3.5)). Note that the existence of a non-constant function T_g satisfying the conditions (6.3) implies that the group Γ_g has genus zero, for such a function necessarily induces an isomorphism from X_{Γ_g} (cf. (2.15)) to the Riemann sphere. As such, if we assume that both T_g and T_g' satisfy these conditions and both qT_g and qT_g' have the expansion $1 + O(q)$ near $\tau \to i\infty$, then T_g and T_g' differ by an additive constant; i.e. $T_g' = T_g + C$ for some $C \in \mathbb{C}$. In other words, the space of solutions to (6.3) is 1 or 2 dimensional, according as the genus of X_{Γ_g} is positive or 0.

Observe the similarity between condition (ii) of (6.3) and the optimal growth condition (4.16). Since q^{-1} is the minimal polar term possible for a non-constant Γ_g-invariant function on the upper-half plane, assuming that the stabiliser of infinity in Γ_g is generated by $\pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, the condition (ii) is an optimal growth condition on modular forms of weight 0; the coefficients of a form with higher order poles will grow more quickly. The condition (iii) naturally extends

this to the situation that Γ_g has more than one cusp.

Accordingly, we now formulate an analogue of (6.3)—and an extension of the optimal growth condition (4.16) to vector-valued mock modular forms of higher level—as follows. Suppose that ν is a (matrix-valued) multiplier system on $\Gamma_0(n)$ with weight 1/2, and suppose, for the sake of concreteness, that ν coincides with the inverse of the multiplier system of S^X, for some Niemeier root system X, when restricted to $\Gamma_0(N)$ for some N with $n|N$. Observe that, under these hypotheses, every component H_r has a Fourier expansion in powers of $q^{1/4m}$ where m is the Coxeter number of X, so $q^{-1/4m}$ is the smallest order pole that any component of H may have. Say that a vector-valued function $H = (H_r)$ is a mock modular form of *optimal growth* for $\Gamma_0(n)$ with multiplier ν, weight 1/2 and shadow S if

$$\begin{aligned}
&\text{(i)} \quad H|_{1/2,\nu,S}\gamma = H \text{ for all } \gamma \in \Gamma_0(n),\\
&\text{(ii)} \quad q^{1/4m}H_r(\tau) = O(1) \text{ as } \tau \to i\infty \text{ for all } r,\\
&\text{(iii)} \quad H_r(\tau) = O(1) \text{ for all } r \text{ as } \tau \to \alpha \in \mathbb{Q}, \text{ whenever } \infty \notin \Gamma_g\alpha.
\end{aligned} \tag{6.4}$$

In condition (i) of (6.4) we write $|_{\nu,1/2,S}$ for the weight 1/2 action of $\Gamma_0(n)$ with multiplier ν and twist by S (cf. (3.6)), on holomorphic vector-valued functions on the upper-half plane.

Recall that ν_g^X denotes the multiplier system of H_g^X, and S_g^X is its shadow. Recall also that n_g denotes the order of (the image of) $g \in G^X$ in the quotient group $\bar{G}^X$. We now conjecture that the umbral McKay–Thompson series all have optimal growth in the sense of (6.4), and this serves as a direct analogue of the Conway–Norton conjecture of monstrous moonshine, that all the monstrous McKay–Thompson series are principal moduli for genus zero subgroups of $SL_2(\mathbb{R})$; or equivalently, that they are all functions of optimal growth in the sense of (6.3).

Conjecture 6.5. *Let X be a Niemeier root system and let $g \in G^X$. Then H_g^X is the unique, up to scale, mock modular form of optimal growth for $\Gamma_0(n_g)$ with multiplier ν_g^X, weight 1/2 and shadow S_g^X.*

Conjecture 6.5 should serve as an important step in obtaining a characterisation of the mock modular forms of umbral moonshine. Note that the above conjecture has been proven in §4.1 (cf. Corollary 4.2) for the identity class of G^X for all Niemeier root systems X. Note that in the case of the identity class we have $\Gamma_0(n) = SL_2(\mathbb{Z})$ which has the cusp (representative) at $i\infty$ as the only cusp. As a result, the more general conditions in (6.4) reduce to the condition (4.16) discussed in §4. For $X = A_1^{24}$, this conjecture was proven for all conjugacy classes of $G^X = M_{24}$ in [53]. See also [54] for related results in this case.

6.4 Discriminants

One of the most striking features of umbral moonshine is the apparently intimate relation between the number fields on which the irreducible representations of G^X are defined and the discriminants of the vector-valued mock modular form H^X. In this subsection we will discuss this "discriminant property", extending the discussion in [1].

First, for a Niemeier root system with Coxeter number m we observe that the discriminants of the components H^X_r of the mock modular form $H^X = H^X_{1A}$ determine some important properties of the representations of G^X. Here we say that an integer D is a *discriminant of* H^X if there exists a term $q^d = q^{-\frac{D}{4m}}$ with non-vanishing Fourier coefficient in at least one of the components. The following result can be verified explicitly using the tables in §§B,D.

Proposition 6.6. *Let X be one of the 23 Niemeier root systems. If $n > 1$ is an integer satisfying*

1. *there exists an element of G^X of order n, and*
2. *there exists an integer λ that satisfies at least one of the following conditions and such that $D = -n\lambda^2$ is a discriminant of H^X. First, $(n, \lambda) = 1$, and second, λ^2 is a proper divisor of n,*

then there exists at least one pair of irreducible representations ϱ and ϱ^* of G^X and at least one element $g \in G^X$ such that $\mathrm{tr}_\varrho(g)$ is not rational but

$$\mathrm{tr}_\varrho(g), \mathrm{tr}_{\varrho^*}(g) \in \mathbb{Q}(\sqrt{-n}) \tag{6.5}$$

and n divides $ord(g)$.

The list of integers n satisfying the two conditions of Proposition 6.6 is given in Table 10. We omit from the table lambencies of Niemeier root systems for which there exists no integer n satisfying these conditions.

From now on we say that an irreducible representation ϱ of the umbral group G^X is of *type* n if n is an integer satisfying the two conditions of Proposition 6.6 and the character values of ϱ generate the field $\mathbb{Q}(\sqrt{-n})$. Evidently, irreducible representations of type n come in pairs (ϱ, ϱ^*) with $\mathrm{tr}_{\varrho^*}(g)$ the complex conjugate of $\mathrm{tr}_\varrho(g)$ for all $g \in G^X$. The list of all irreducible representations of type n is also given in Table 10. (See §B.1 for the character tables of the G^X and our notation for irreducible representations.)

Recall that the *Frobenius–Schur indicator* of an irreducible ordinary representation of a finite group is 1, -1 or 0 according as the representation admits an invariant symmetric bilinear form,

an invariant skew-symmetric bilinear form, or no invariant bilinear form, respectively. The representations admitting no invariant bilinear form are precisely those whose character values are not all real. We can now state the next observation.

Proposition 6.7. *For each Niemeier root system X, an irreducible representation ϱ of G^X has Frobenius–Schur indicator* 0 *if and only if it is of type n for some n defined in Proposition 6.6.*

The *Schur index* of an irreducible representation ϱ of a finite group G is the smallest positive integer s such that there exists a degree s extension k of the field generated by the character values $\mathrm{tr}_\varrho(g)$ for $g \in G$ such that ϱ can be realised over k. Inspired by Proposition 6.7 we make the following conjecture.

Conjecture 6.8. *If ϱ is an irreducible representation of G^X of type n then the Schur index of ϱ is equal to* 1.

In other words, we conjecture that the irreducible G^X-representations of type n can be realised over $\mathbb{Q}(\sqrt{-n})$. For $X = A_1^{24}$ this speculation is in fact a theorem, since it is known [55] that the Schur indices for M_{24} are always 1. For $X = A_2^{12}$ it is also known [55] that the Schur indices for $\bar{G}^{(3)} = M_{12}$ are also always 1. Moreover, the representations of $G^{(3)} \simeq 2.\bar{G}^{(3)}$ with characters χ_{16} and χ_{17} in the notation of Table 12 have been constructed explicitly over $\mathbb{Q}(\sqrt{-2})$ in [56]. Finally, Proposition 6.7 constitutes a non-trivial consistency check for Conjecture 6.8 since the Schur index is at least 2 for a representation with Frobenius–Schur indicator equal to -1.

X	n	(ϱ, ϱ^*)
A_1^{24}	$7, 15, 23$	$(\chi_3, \chi_4), (\chi_5, \chi_6), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13}), (\chi_{15}, \chi_{16})$
A_2^{12}	$5, 8, 11, 20$	$(\chi_4, \chi_5), (\chi_{16}, \chi_{17}), (\chi_{20}, \chi_{21}), (\chi_{22}, \chi_{23}), (\chi_{25}, \chi_{26})$
A_3^8	$3, 7$	$(\chi_2, \chi_3), (\chi_{13}, \chi_{14}), (\chi_{15}, \chi_{16})$
A_4^6	4	$(\chi_8, \chi_9), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13})$
$A_5^4 D_4$	8	(χ_6, χ_7)
A_6^4	3	$(\chi_2, \chi_3), (\chi_6, \chi_7)$
$A_9^2 D_6$	4	(χ_3, χ_4)
A_{12}^2	4	(χ_3, χ_4)
D_4^6	15	(χ_{12}, χ_{13})
E_6^4	8	(χ_6, χ_7)

Table 10: The irreducible representations of type n.

Equipped with the preceding discussion we are now ready to state our main observation for

the discriminant property of umbral moonshine.

Proposition 6.9. *Let X be a Niemeier root system with Coxeter number m. Let n be one of the integers in Table 10 and let λ_n be the smallest positive integer such that $D = -n\lambda_n^2$ is a discriminant of H^X. Then $K^X_{r,-D/4m} = \varrho_n \oplus \varrho_n^*$ where ϱ_n and ϱ_n^* are dual irreducible representations of type n. Conversely, if ϱ is an irreducible representation of type n and $-D$ is the smallest positive integer such that $K^X_{r,-D/4m}$ has ϱ as an irreducible constituent then there exists an integer λ such that $D = -n\lambda^2$.*

Extending this we make the following conjecture.

Conjecture 6.10. *Let X be a Niemeier root system with Coxeter number m. If D is a discriminant of H^X which satisfies $D = -n\lambda^2$ for some integer λ then the representation $K^X_{r,-D/4m}$ has at least one dual pair of irreducible representations of type n arising as irreducible constituents.*

Conjecture (6.10) has been verified for the case $X = A_1^{24}$ in [57].

We conclude this section with conjectures arising from the observation (cf. §D) that the conjectural G^X-module $K^X_{r,d}$ is typically isomorphic to several copies of a single representation. We say a G-module V is a *doublet* if it is isomorphic to the direct sum of two copies of a single representation of G, and interpret the term *sextet* similarly.

Conjecture 6.11. *Let X be a Niemeier root system and let m be the Coxeter number of X. Then the representation $K^X_{r,-D/4m}$ is a doublet if and only if $D \neq 0$ and $D \neq -n\lambda^2$ for any integer λ and for any n listed in Table 10 corresponding to X. If $X = A_8^3$ then the representation $K^X_{r,-D/4m}$ is a sextet if and only if $D \neq -27\lambda^2$ for some integer λ.*

In particular, for the nine Niemeier root systems

$$A_7^2D_5^2,\ A_{11}D_7E_6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2,\ D_{12}^2,\ D_{16}E_8,\ D_{24},\ E_8^3, \tag{6.6}$$

that have no irreducible representation with vanishing Forbenius–Schur indicator and have no terms with zero discriminant, we conjecture that all the representations $K^X_{r,-D/4m}$ corresponding to H_g^X are doublets.

7 Conclusions and Discussion

Let us start by recapitulating the main results of this paper. Taking the 23 Niemeier lattices as the starting point, in §2.4 we identify a finite group G^X—the umbral group—for each Niemeier root system X. On the other hand, using the ADE classification discussed in §3.3 and Theorem

4.1, we identify a distinguished vector-valued mock modular form H^X—the umbral form—for each Niemeier root system X. We then conjecture (among other things)

1. (Conjecture 6.1) that the mock modular form H^X encodes the graded super-dimension of a certain infinite-dimensional, $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Q}$-graded module K^X for G^X,
2. (Conjecture 6.3) that the graded super-characters H_g^X arising from the action of G^X on K^X are vector-valued mock modular forms with concretely specified shadows S_g^X, and
3. (Conjecture 6.5) that the umbral McKay–Thompson series H_g^X are uniquely determined by an optimal growth property which is directly analogous to the genus zero property of monstrous moonshine.

To lend evidence in support of these conjectures we explicitly identify (almost all of) the umbral McKay–Thompson series H_g^X.

In this way, from the 23 Niemeier root systems we obtain 23 instances of umbral moonshine, encompassing all the 6 instances previously discussed in [1] and in particular the case with $G^X = M_{24}$ first discussed in the context of the $K3$ elliptic genus [11]. Apart from uncovering 17 new instances, we believe that the relation to Niemeier lattices sheds important light on the underlying structure of umbral moonshine. First, the construction of the umbral group G^X is now completely uniform: G^X is the outer-automorphism group of the corresponding Niemeier lattice (cf. (2.30)). Second, it provides an explanation for why the 6 instances discussed in [1] are naturally labelled by the divisors of 12: they correspond to Niemeier root systems given by evenly many copies (viz., $24/(\ell-1)$) of an A-type root system $A_{\ell-1}$. Third, it also sheds light on the relation between umbral moonshine and meromorphic weight 1 Jacobi forms as well as weight 0 Jacobi forms. For as we have seen in §3.4, the umbral forms H^X can be constructed uniformly by taking theta-coefficients of finite parts of certain weight 1 meromorphic Jacobi forms, but in general the relevant meromorphic Jacobi form has simple poles not only at the origin but also at non-trivial torsion points whenever the corresponding root system has a D- or E-type root system as an irreducible component. As a result, in those cases the relation to weight 0 Jacobi forms is less direct as the Eichler–Zagier operator $\mathcal{W}^X$ of (4.31) is no longer proportional to the identity. In particular, in these cases the umbral mock modular form H^X does not arise in a direct way from the decomposition of a weight 0 (weak) Jacobi form into irreducible characters for the $\mathcal{N}=4$ superconformal algebra.

Recall that the relevant weight 0 Jacobi form in the construction described in §4.3 coincides with the elliptic genus of a $K3$ surface in the case of the Niemeier root system $X = A_1^{24}$ ($\ell = 2$, $G^X = M_{24}$). As the relation to weight 0 forms becomes less straightforward in the more general cases, the relation between umbral moonshine and sigma models, or in fact any kind

of conformal field theory, also becomes more opaque. An interesting question is therefore the following. What, if any, kind of physical theory or geometry should attach to the more general instances of umbral moonshine?

To add to this puzzle, the Borcherds lift of the $K3$ elliptic genus is a Siegel modular form which also plays an important role in type II as well as heterotic string theory compactified on $K3 \times T^2$ [58–60]. As pointed out in [12] and refined in [17], Mathieu moonshine in this context (corresponding here to $X = A_1^{24}$) leads to predictions regarding Siegel modular forms which have been partially proven in [61]. Furthermore, this Siegel modular form also serves as the square of the denominator function of a generalised Kac–Moody algebra developed by Gritsenko–Nikulin in the context of mirror symmetry for $K3$ surfaces [62–66]. As discussed in detail in [1, §5.5], many of these relations to string theory and $K3$ geometry extend to some of the other 5 instances of umbral moonshine discussed in that paper. Since the relation between umbral forms and weight 0 modular forms is modified when D- or E-type root systems are involved, it would be extremely interesting to determine how the above-mentioned relations to string theory, $K3$ surfaces, and generalised Kac–Moody algebras manifest in the more general cases.

Regarding $K3$ surfaces, note that Niemeier lattices have a long history of application to this field, and the study of the symmetries of $K3$ surfaces in particular. See for instance [67]. See [68] for an analysis involving all of the Niemeier lattices. It would be interesting to explore the extent to which recent work [22, 23] applying the Niemeier lattice L^X to the $X = A_1^{24}$ case of umbral moonshine can be extended to other Niemeier root systems in light of [68].

In another direction, the physical context of Mathieu moonshine has been extended recently to $K3$ compactifications of heterotic string theory with 8 supercharges [69]. As the structure of theories with 8 supercharges is much less rigid than those with 16 supercharges, one might speculate that a suitable generalisation of [69] could provide physical realisations of more instances of umbral moonshine.

Apart from posing the umbral moonshine conjecture, in this paper we have also noted various intricate and mysterious properties of this new moonshine story. An important example is the close relation between Niemeier lattices and the genus zero groups discussed in §2.3 and §4.2. Another is the multiplicative relations discussed in §5.3 between McKay–Thompson series attached to different Niemeier root systems that plays an important role in the explicit specification of the umbral McKay–Thompson series in §5.5. While we have a concrete description of these two properties in terms of mock modular forms and related structures, their origin is still unclear. We also observe empirically discriminant relations between number fields underlying irreducible representations of G^X and the discriminants of the vector-valued mock modular form

H^X (§6.4) extending the observations made in [1, §5.4]. It would be extremely interesting to have a representation theoretic explanation of these relations.

Last but not least, the construction of the umbral modules K^X is clearly an important next step in unraveling the mystery of umbral moonshine.

Acknowledgements

We owe particular thanks to George Glauberman for alerting us to the connection between the umbral groups of our previous paper and the stabilisers of deep holes in the Leech lattice. This observation was a crucial catalyst for the present work. We also thank Daniel Allcock, Kathrin Bringmann, Scott Carnahan, Noam Elkies, Igor Frenkel, Terry Gannon, Ching Hung Lam, Geoffrey Mason, Atsushi Matsuo, Sameer Murthy, Ken Ono, Erik Verlinde and Don Zagier for helpful comments and discussions. We thank the referees for numerous helpful suggestions and comments. MC would like to thank the Simons Center for Geometry and Physics, Stanford University and the Haussdorff Center for Mathematics for hospitality. The work of JH was supported by NSF grant 1214409.

A Special Functions

A.1 Dedekind Eta Function

The *Dedekind eta function*, denoted $\eta(\tau)$, is a holomorphic function on the upper half-plane defined by the infinite product

$$\eta(\tau) = q^{1/24} \prod_{n\geq 1} (1 - q^n) \tag{A.1}$$

where $q = \mathrm{e}(\tau) = e^{2\pi \mathbf{i}\tau}$. It is a modular form of weight 1/2 for the modular group $SL_2(\mathbb{Z})$ with multiplier $\epsilon : SL_2(\mathbb{Z}) \to \mathbb{C}^*$, which means that

$$\epsilon(\gamma)\eta(\gamma\tau)\mathrm{jac}(\gamma,\tau)^{1/4} = \eta(\tau) \tag{A.2}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, where $\mathrm{jac}(\gamma,\tau) = (c\tau + d)^{-2}$. The multiplier system ϵ may be described explicitly as

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \mathrm{e}(-b/24), & c = 0,\ d = 1 \\ \mathrm{e}(-(a+d)/24c + s(d,c)/2 + 1/8), & c > 0 \end{cases} \tag{A.3}$$

where $s(d,c) = \sum_{m=1}^{c-1} (d/c)((md/c))$ and $((x))$ is 0 for $x \in \mathbb{Z}$ and $x - \lfloor x \rfloor - 1/2$ otherwise. We can deduce the values $\epsilon(a,b,c,d)$ for $c < 0$, or for $c = 0$ and $d = -1$, by observing that $\epsilon(-\gamma) = \epsilon(\gamma)\,\mathrm{e}(1/4)$ for $\gamma \in SL_2(\mathbb{Z})$.

A.2 Jacobi Theta Functions

We define the *Jacboi theta functions* $\theta_i(\tau, z)$ as follows for $q = e(\tau)$ and $y = e(z)$.

$$\theta_1(\tau,z) = -iq^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}) \tag{A.4}$$

$$\theta_2(\tau,z) = q^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1-q^n)(1+yq^n)(1+y^{-1}q^{n-1}) \tag{A.5}$$

$$\theta_3(\tau,z) = \prod_{n=1}^{\infty} (1-q^n)(1+y\,q^{n-1/2})(1+y^{-1}q^{n-1/2}) \tag{A.6}$$

$$\theta_4(\tau,z) = \prod_{n=1}^{\infty} (1-q^n)(1-y\,q^{n-1/2})(1-y^{-1}q^{n-1/2}) \tag{A.7}$$

Note that there are competing conventions for $\theta_1(\tau, z)$ in the literature and our normalisation may differ from another by a factor of -1 (or possibly $\pm i$).

A.3 Higher Level Modular Forms

The congruence subgroups of the modular group $SL_2(\mathbb{Z})$ that are most relevant for this paper are the *Hecke congruence groups*

$$\Gamma_0(N) \quad = \quad \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), c = 0 \bmod N \right\}. \tag{A.8}$$

A modular form for $\Gamma_0(N)$ is said to have *level* N. For N a positive integer a modular form of weight 2 for $\Gamma_0(N)$ is given by

$$\begin{aligned} \lambda_N(\tau) &= q\partial_q \log\left(\frac{\eta(N\tau)}{\eta(\tau)}\right) \\ &= \frac{N-1}{24} + \sum_{k>0} \sigma(k)(q^k - Nq^{Nk}) \end{aligned} \tag{A.9}$$

where $\sigma(k)$ is the divisor function $\sigma(k) = \sum_{d|k} d$. The function λ_N is, of course, only non-zero when $N > 1$.

Observe that a modular form on $\Gamma_0(N)$ is a modular form on $\Gamma_0(M)$ whenever $N|M$, and for some small N the space of forms of weight 2 is spanned by the $\lambda_d(\tau)$ for d a divisor of N.

A discussion of the ring of weak Jacobi forms of higher level can be found in [70].

A.4 Weight Zero Jacobi Forms

According to [71] the graded ring $J_{0,*} = \bigoplus_{m\geq 1} J_{0,m-1}$, of weak Jacobi forms with weight 0 and integral index (cf. §3.2), is finitely generated, by $\varphi_1^{(2)}$, $\varphi_1^{(3)}$ and $\varphi_1^{(4)}$, where

$$\begin{aligned} \varphi_1^{(2)} &= 4\left(f_2^2 + f_3^2 + f_4^2\right), \\ \varphi_1^{(3)} &= 2\left(f_2^2 f_3^2 + f_3^2 f_4^2 + f_4^2 f_2^2\right), \\ \varphi_1^{(4)} &= 4 f_2^2 f_3^2 f_4^2, \end{aligned} \tag{A.10}$$

and $f_i(\tau, z) = \theta_i(\tau, z)/\theta_i(\tau, 0)$ for $i \in \{2, 3, 4\}$ (cf. §A.2). If we work over $\mathbb{Z}$ then we must include

$$\varphi_1^{(5)} = \frac{1}{4}\left(\varphi_1^{(4)}\varphi_1^{(2)} - (\varphi_1^{(3)})^2\right) \tag{A.11}$$

as a generator also, so that $J^{\mathbb{Z}}_{0,*} = \mathbb{Z}[\varphi_1^{(2)}, \varphi_1^{(3)}, \varphi_1^{(4)}, \varphi_1^{(5)}]$. The ring $J_{0,*}$ has an ideal

$$J_{0,*}(q) = \bigoplus_{m>1} J_{0,m-1}(q) = \left\{ \phi \in J_{0,*} \mid \phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ n>0}} c(n,r) q^n y^r \right\} \tag{A.12}$$

consisting of Jacobi forms that vanish in the limit as $\tau \to i\infty$ (i.e. have vanishing coefficient of $q^0 y^r$, for all r, in their Fourier expansion). This ideal is principal and generated by a weak Jacobi form of weight 0 and index 6 given by

$$\zeta(\tau, z) = \frac{\theta_1(\tau, z)^{12}}{\eta(\tau)^{12}} \tag{A.13}$$

(cf. §A.2 for θ_1 and η). Gritsenko shows [71] that for any positive integer m the quotient $J_{0,m-1}/J_{0,m-1}(q)$ is a vector space of dimension $m-1$ admitting a basis consisting of weight 0 index $m-1$ weak Jacobi forms $\varphi_n^{(m)}$ (denoted $\psi_{0,m-1}^{(n)}$ in [71]) for $1 \le n \le m-1$ such that the coefficient of $q^0 y^k$ in $\varphi_n^{(m)}$ vanishes for $|k| > n$ but does not vanish for $|k| = n$. In fact Gritsenko works in the subring $J^{\mathbb{Z}}_{0,*}$ of Jacobi forms having integer Fourier coefficients and his $\varphi_n^{(m)}$ furnish a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $J^{\mathbb{Z}}_{0,*}/J^{\mathbb{Z}}_{0,*}(q)$. In what follows we record explicit formulas for some of the $\varphi_n^{(m)}$.

Following [71] we define

$$\begin{aligned} \varphi_1^{(7)} &= \varphi_1^{(3)}\varphi_1^{(5)} - (\varphi_1^{(4)})^2, \\ \varphi_1^{(9)} &= \varphi_1^{(3)}\varphi_1^{(7)} - (\varphi_1^{(5)})^2, \\ \varphi_1^{(13)} &= \varphi_1^{(5)}\varphi_1^{(9)} - 2(\varphi_1^{(7)})^2, \end{aligned} \tag{A.14}$$

and define $\varphi_1^{(m)}$ for the remaining positive integers m according to the following recursive procedure. For $(12, m-1) = 1$ and $m > 5$ we set

$$\varphi_1^{(m)} = (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}. \tag{A.15}$$

For $(12, m-1) = 2$ and $m > 10$ we set

$$\varphi_1^{(m)} = \frac{1}{2}\big((12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}\big). \tag{A.16}$$

For $(12, m-1) = 3$ and $m > 9$ we set

$$\varphi_1^{(m)} = \frac{2}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{3}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}. \tag{A.17}$$

For $(12, m-1) = 4$ and $m > 16$ we set

$$\varphi_1^{(m)} = \frac{1}{4}\big((12, m-13)\varphi_1^{(m-12)}\varphi_1^{(13)} + (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m-9)\varphi_1^{(m-8)}\varphi_1^{(9)}\big). \quad \text{(A.18)}$$

For $(12, m-1) = 6$ and $m > 18$ we set

$$\varphi_1^{(m)} = \frac{1}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{6}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - \frac{1}{2}(12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}. \quad \text{(A.19)}$$

Finally, for $(12, m-1) = 12$ and $m > 24$ we set

$$\varphi_1^{(m)} = \frac{1}{6}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} - \frac{1}{4}(12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + \frac{1}{12}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)}, \quad \text{(A.20)}$$

where we have set

$$\varphi_1^{(25)} = \frac{1}{2}\varphi_1^{(21)}\varphi_1^{(5)} - \varphi_1^{(19)}\varphi_1^{(7)} + \frac{1}{2}(\varphi_1^{(13)})^2. \quad \text{(A.21)}$$

The $\varphi_2^{(m)}$ are defined by setting

$$\begin{aligned} \varphi_2^{(3)} &= (\varphi_1^{(2)})^2 - 24\,\varphi_1^{(3)}, \\ \varphi_2^{(4)} &= \varphi_1^{(2)}\varphi_1^{(3)} - 18\,\varphi_1^{(4)}, \\ \varphi_2^{(5)} &= \varphi_1^{(2)}\varphi_1^{(4)} - 16\,\varphi_1^{(5)}, \end{aligned} \quad \text{(A.22)}$$

and

$$\varphi_2^{(m)} = (12, m-4)\,\varphi_1^{(m-3)}\varphi_1^{(4)} - (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m-1)\varphi_1^{(m)} \quad \text{(A.23)}$$

for $m > 5$, and the remaining $\varphi_n^{(m)}$ for $2 \leq m \leq 25$ are given by

$$\begin{aligned} \varphi_n^{(m)} &= \varphi_{n-1}^{(m-3)}\varphi_1^{(4)}, \\ \varphi_{m-2}^{(m)} &= (\varphi_1^{(2)})^{m-3}\varphi_1^{(3)}, \\ \varphi_{m-1}^{(m)} &= (\varphi_1^{(2)})^{m-1}, \end{aligned} \quad \text{(A.24)}$$

where the first equation of (A.24) holds for $3 \leq n \leq m-3$.

B Characters

In §B.1 we give character tables (with power maps and Frobenius–Schur indicators) for each group G^X for X a Niemeier root system. These were computed with the aid of the computer algebra package GAP4 [72]. We use the abbreviations $a_n = \sqrt{-n}$ and $b_n = (-1 + \sqrt{-n})/2$ in these tables.

The tables in §B.2 furnish cycle shapes and character values—the twisted Euler characters—attached to the representations of the groups G^X described in §2.4. Using this data we can obtain explicit expressions for the shadows S_g^X of the vector-valued mock modular forms H_g^X according to the prescription of §5.1.

B.1 Irreducible Characters

Table 11: Character table of $G^X \simeq M_{24}$, $X = A_1^{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
χ_3	∘	45	-3	5	0	3	-3	1	1	0	0	-1	b_7	$\overline{b_7}$	-1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	-1	-1
χ_4	∘	45	-3	5	0	3	-3	1	1	0	0	-1	$\overline{b_7}$	b_7	-1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	-1	-1
χ_5	∘	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	∘	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	+	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
χ_9	+	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	∘	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	∘	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	∘	990	-18	-10	0	3	6	2	-2	0	0	-1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	∘	990	-18	-10	0	3	6	2	-2	0	0	-1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	+	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
χ_{15}	∘	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2b_7$	$2\overline{b_7}$	-1	0	1	0	-1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	∘	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\overline{b_7}$	$2b_7$	-1	0	1	0	-1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	+	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0
χ_{18}	+	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0
χ_{19}	+	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	+	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
χ_{21}	+	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0
χ_{22}	+	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1
χ_{23}	+	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	+	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
χ_{25}	+	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0
χ_{26}	+	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

Table 12: Character table of $G^X \simeq 2.M_{12}$, $X = A_2^{12}$

$[g]$	FS	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^2]$		1A	1A	2A	1A	1A	3A	3A	3B	3B	2B	2B	5A	5A	6B	3A	3A	4B	4B	4C	4C	10A	10A	11B	11B	11A	11A
$[g^3]$		1A	2A	4A	2B	2C	1A	2A	1A	2A	4B	4C	5A	10A	4A	2B	2C	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^5]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	1A	2A	12A	6C	6D	8B	8A	8D	8C	4A	4A	11A	22A	11B	22B
$[g^{11}]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20B	20A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	11	11	-1	3	3	2	2	-1	-1	-1	3	1	1	-1	0	0	-1	-1	1	1	-1	-1	0	0	0	0
χ_3	$+$	11	11	-1	3	3	2	2	-1	-1	3	-1	1	1	-1	0	0	1	1	-1	-1	-1	-1	0	0	0	0
χ_4	$\circ$	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	b_{11}	b_{11}	$\overline{b_{11}}$	$\overline{b_{11}}$
χ_5	$\circ$	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	$\overline{b_{11}}$	$\overline{b_{11}}$	b_{11}	b_{11}
χ_6	$+$	45	45	5	-3	-3	0	0	3	3	1	1	0	0	-1	0	0	-1	-1	-1	-1	0	0	1	1	1	1
χ_7	$+$	54	54	6	6	6	0	0	0	0	2	2	-1	-1	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
χ_8	$+$	55	55	-5	7	7	1	1	1	1	-1	-1	0	0	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
χ_9	$+$	55	55	-5	-1	-1	1	1	1	1	3	-1	0	0	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	$+$	55	55	-5	-1	-1	1	1	1	1	-1	3	0	0	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
χ_{11}	$+$	66	66	6	2	2	3	3	0	0	-2	-2	1	1	0	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{12}	$+$	99	99	-1	3	3	0	0	3	3	-1	-1	-1	-1	-1	0	0	1	1	1	1	-1	-1	0	0	0	0
χ_{13}	$+$	120	120	0	-8	-8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	-1	-1	-1
χ_{14}	$+$	144	144	4	0	0	0	0	-3	-3	0	0	-1	-1	1	0	0	0	0	0	0	-1	-1	1	1	1	1
χ_{15}	$+$	176	176	-4	0	0	-4	-4	-1	-1	0	0	1	1	-1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	$\circ$	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	a_2	$\overline{a_2}$	a_2	$\overline{a_2}$	0	0	-1	1	-1	1
χ_{17}	$\circ$	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	$\overline{a_2}$	a_2	$\overline{a_2}$	a_2	0	0	-1	1	-1	1
χ_{18}	$+$	12	-12	0	4	-4	3	-3	0	0	0	0	2	-2	0	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{19}	$-$	32	-32	0	0	0	-4	4	2	-2	0	0	2	-2	0	0	0	0	0	0	0	0	0	-1	1	-1	1
χ_{20}	$\circ$	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	a_5	$\overline{a_5}$	0	0	0	0
χ_{21}	$\circ$	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	$\overline{a_5}$	a_5	0	0	0	0
χ_{22}	$\circ$	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	a_2	$\overline{a_2}$	$\overline{a_2}$	a_2	0	0	0	0	0	0
χ_{23}	$\circ$	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	$\overline{a_2}$	a_2	a_2	$\overline{a_2}$	0	0	0	0	0	0
χ_{24}	$+$	120	-120	0	8	-8	3	-3	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	-1	1	-1	1
χ_{25}	$\circ$	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-b_{11}$	b_{11}	$-\overline{b_{11}}$	$\overline{b_{11}}$
χ_{26}	$\circ$	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\overline{b_{11}}$	$\overline{b_{11}}$	$-b_{11}$	b_{11}

Table 13: Character table of $G^X \simeq 2.AGL_3(2)$, $X = A_3^8$

$[g]$	FS	1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	7A	14A	7B	14B
$[g^2]$		1A	1A	1A	2A	2B	1A	3A	3A	3A	3A	4A	2C	7A	7A	7B	7B
$[g^3]$		1A	2A	2B	4A	4B	2C	1A	2A	2B	2B	8A	4C	7B	14B	7A	14A
$[g^7]$		1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	b_7	b_7	$\overline{b_7}$	$\overline{b_7}$
χ_3	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	$\overline{b_7}$	$\overline{b_7}$	b_7	b_7
χ_4	+	6	6	6	2	2	2	0	0	0	0	0	0	−1	−1	−1	−1
χ_5	+	7	7	7	−1	−1	−1	1	1	1	1	−1	−1	0	0	0	0
χ_6	+	8	8	8	0	0	0	−1	−1	−1	−1	0	0	1	1	1	1
χ_7	+	7	7	−1	3	−1	−1	1	1	−1	−1	1	−1	0	0	0	0
χ_8	+	7	7	−1	−1	−1	3	1	1	−1	−1	−1	1	0	0	0	0
χ_9	+	14	14	−2	2	−2	2	−1	−1	1	1	0	0	0	0	0	0
χ_{10}	+	21	21	−3	1	1	−3	0	0	0	0	−1	1	0	0	0	0
χ_{11}	+	21	21	−3	−3	1	1	0	0	0	0	1	−1	0	0	0	0
χ_{12}	+	8	−8	0	0	0	0	2	−2	0	0	0	0	1	−1	1	−1
χ_{13}	∘	8	−8	0	0	0	0	−1	1	a_3	$\overline{a_3}$	0	0	1	−1	1	−1
χ_{14}	∘	8	−8	0	0	0	0	−1	1	$\overline{a_3}$	a_3	0	0	1	−1	1	−1
χ_{15}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	$\overline{b_7}$	$-\overline{b_7}$	b_7	$-b_7$
χ_{16}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	b_7	$-b_7$	$\overline{b_7}$	$-\overline{b_7}$

Table 14: Character table of $G^X \simeq GL_2(5)/2$, $X = A_4^6$

$[g]$	FS	1A	2A	2B	2C	3A	6A	5A	10A	4A	4B	4C	4D	12A	12B
$[g^2]$		1A	1A	1A	1A	3A	3A	5A	5A	2A	2A	2C	2C	6A	6A
$[g^3]$		1A	2A	2B	2C	1A	2A	5A	10A	4B	4A	4D	4C	4B	4A
$[g^5]$		1A	2A	2B	2C	3A	6A	1A	2A	4A	4B	4C	4D	12A	12B
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_3	+	4	4	0	0	1	1	−1	−1	2	2	0	0	−1	−1
χ_4	+	4	4	0	0	1	1	−1	−1	−2	−2	0	0	1	1
χ_5	+	5	5	1	1	−1	−1	0	0	1	1	−1	−1	1	1
χ_6	+	5	5	1	1	−1	−1	0	0	−1	−1	1	1	−1	−1
χ_7	+	6	6	−2	−2	0	0	1	1	0	0	0	0	0	0
χ_8	∘	1	−1	1	−1	1	−1	1	−1	a_1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$
χ_9	∘	1	−1	1	−1	1	−1	1	−1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$	a_1
χ_{10}	∘	4	−4	0	0	1	−1	−1	1	$2a_1$	$-2a_1$	0	0	$-a_1$	a_1
χ_{11}	∘	4	−4	0	0	1	−1	−1	1	$-2a_1$	$2a_1$	0	0	a_1	$-a_1$
χ_{12}	∘	5	−5	1	−1	−1	1	0	0	a_1	$-a_1$	$-a_1$	a_1	a_1	$-a_1$
χ_{13}	∘	5	−5	1	−1	−1	1	0	0	$-a_1$	a_1	a_1	$-a_1$	$-a_1$	a_1
χ_{14}	+	6	−6	−2	2	0	0	1	−1	0	0	0	0	0	0

Table 15: Character table of $G^X \simeq GL_2(3)$, $X \in \{A_5^4 D_4, E_6^4\}$

$[g]$	FS	1A	2A	2B	4A	3A	6A	8A	8B
$[g^2]$		1A	1A	1A	2A	3A	3A	4A	4A
$[g^3]$		1A	2A	2B	4A	1A	2A	8A	8B
χ_1	$+$	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	-1	1	1	1	-1	-1
χ_3	$+$	2	2	0	2	-1	-1	0	0
χ_4	$+$	3	3	-1	-1	0	0	1	1
χ_5	$+$	3	3	1	-1	0	0	-1	-1
χ_6	$\circ$	2	-2	0	0	-1	1	a_2	$\overline{a_2}$
χ_7	$\circ$	2	-2	0	0	-1	1	$\overline{a_2}$	a_2
χ_8	$+$	4	-4	0	0	1	-1	0	0

Table 16: Character table of $G^X \simeq 3.Sym_6$, $X = D_4^6$

$[g]$	FS	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	15B	2B	2C	4B	6B	6C
$[g^2]$		1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	15B	1A	1A	2A	3B	3C
$[g^3]$		1A	1A	2A	2A	1A	1A	4A	4A	5A	5A	5A	2B	2C	4B	2B	2C
$[g^5]$		1A	3A	2A	6A	3B	3C	4A	12A	1A	3A	3A	2B	2C	4B	6B	6C
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_3	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	3	-1	1	0	-1
χ_4	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	-3	1	-1	0	1
χ_5	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	-1	3	1	-1	0
χ_6	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	1	-3	-1	1	0
χ_7	$+$	16	16	0	0	-2	-2	0	0	1	1	1	0	0	0	0	0
χ_8	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	3	3	-1	0	0
χ_9	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	-3	-3	1	0	0
χ_{10}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	2	-2	0	-1	1
χ_{11}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	-2	2	0	1	-1
χ_{12}	$\circ$	6	-3	-2	1	0	0	2	-1	1	b_{15}	$\overline{b_{15}}$	0	0	0	0	0
χ_{13}	$\circ$	6	-3	-2	1	0	0	2	-1	1	$\overline{b_{15}}$	b_{15}	0	0	0	0	0
χ_{14}	$+$	12	-6	4	-2	0	0	0	0	2	-1	-1	0	0	0	0	0
χ_{15}	$+$	18	-9	2	-1	0	0	2	-1	-2	1	1	0	0	0	0	0
χ_{16}	$+$	30	-15	-2	1	0	0	-2	1	0	0	0	0	0	0	0	0

Table 17: Character table of $G^X \simeq SL_2(3)$, $X = A_6^4$

$[g]$	FS	1A	2A	4A	3A	6A	3B	6B
$[g^2]$		1A	1A	2A	3B	3A	3A	3B
$[g^3]$		1A	2A	4A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1
χ_2	$\circ$	1	1	1	b_3	$\overline{b_3}$	$\overline{b_3}$	b_3
χ_3	$\circ$	1	1	1	$\overline{b_3}$	b_3	b_3	$\overline{b_3}$
χ_4	$+$	3	3	-1	0	0	0	0
χ_5	$-$	2	-2	0	-1	1	-1	1
χ_6	$\circ$	2	-2	0	$-\overline{b_3}$	b_3	$-b_3$	$\overline{b_3}$
χ_7	$\circ$	2	-2	0	$-b_3$	$\overline{b_3}$	$-\overline{b_3}$	b_3

Table 18: Character table of $G^X \simeq Dih_4$, $X = A_7^2 D_5^2$

$[g]$	FS	1A	2A	2B	2C	4A
$[g^2]$		1A	1A	1A	1A	2A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1
χ_3	$+$	1	1	-1	1	-1
χ_4	$+$	1	1	1	-1	-1
χ_5	$+$	2	-2	0	0	0

Table 19: Character table of $G^X \simeq Dih_6$, $X = A_8^3$

$[g]$	FS	1A	2A	2B	2C	3A	6A
$[g^2]$		1A	1A	1A	1A	3A	3A
$[g^3]$		1A	2A	2B	2C	1A	2A
χ_1	$+$	1	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1	1
χ_3	$+$	2	2	0	0	-1	-1
χ_4	$+$	1	-1	-1	1	1	-1
χ_5	$+$	1	-1	1	-1	1	-1
χ_6	$+$	2	-2	0	0	-1	1

Table 20: Character table of $G^X \simeq 4$, for $X \in \{A_9^2 D_6, A_{12}^2\}$

$[g]$	FS	1A	2A	4A	4B
$[g^2]$		1A	1A	2A	2A
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	$\circ$	1	-1	a_1	$\overline{a_1}$
χ_4	$\circ$	1	-1	$\overline{a_1}$	a_1

Table 21: Character table of $G^X \simeq PGL_2(3) \simeq Sym_4$, $X = D_6^4$

$[g]$	FS	1A	2A	3A	2B	4A
$[g^2]$		1A	1A	3A	1A	2A
$[g^3]$		1A	2A	1A	2B	4A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	1	-1	-1
χ_3	$+$	2	2	-1	0	0
χ_4	$+$	3	-1	0	1	-1
χ_5	$+$	3	-1	0	-1	1

Table 22: Character table of $G^X \simeq 2$, for $X \in \{A_{11} D_7 E_6, A_{15} D_9, A_{17} E_7, A_{24}, D_{10} E_7^2, D_{12}^2\}$

$[g]$	FS	1A	2A
$[g^2]$		1A	1A
χ_1	$+$	1	1
χ_2	$+$	1	-1

Table 23: Character table of $G^X \simeq Sym_3$, $X \in \{D_8^3, E_8^3\}$

$[g]$	FS	1A	2A	3A
$[g^2]$		1A	1A	3A
$[g^3]$		1A	2A	1A
χ_1	$+$	1	1	1
χ_2	$+$	1	-1	1
χ_3	$+$	2	0	-1

B.2 Euler Characters

The tables in this section describe the twisted Euler characters and associated cycle shapes attached to each group G^X in §2.4. According to the prescription of §5.1 the character values $\bar{\chi}_g^{X_A}$, $\chi_g^{X_A}$, &c., can be used to describe the shadows of the vector-valued mock modular forms H_g^X attached to each $g \in G^X$ by umbral moonshine. We also identify symbols $n_g|h_g$ which are used in §6.2 to formulate conjectures about the modularity of H_g^X. By definition n_g is the order of the image of $g \in G^X$ in $\bar{G}^X$ and $h_g = N_g/n_g$ where N_g denotes the product of shortest and longest cycle lengths appearing in the cycle shape $\widetilde{\Pi}_g^X$.

Note that we have $\widetilde{\Pi}_g^X = \widetilde{\Pi}_g^{X_A} = \bar{\Pi}_g^{X_A}$ in case $X = A_1^{24}$. More generally, we will have $\widetilde{\Pi}_g^X = \widetilde{\Pi}_g^{X_A}$ when $X = X_A$ (cf. §2.2), and similarly when $X = X_D$ or $X = X_E$, so we suppress the row (that would otherwise be) labelled $\widetilde{\Pi}_g^X$ in these cases.

Table 24: Twisted Euler characters and Frame shapes at $\ell = 2$, $X = A_1^{24}$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
$n_g\|h_g$	1|1	2|1	2|2	3|1	3|3	4|2	4|1	4|4	5|1	6|1	6|6
$\bar{\chi}_g^{X_A}$	24	8	0	6	0	0	4	0	4	2	0
$\bar{\Pi}_g^{X_A}$	1^{24}	$1^8 2^8$	2^{12}	$1^6 3^6$	3^8	$2^4 4^4$	$1^4 2^2 4^4$	4^6	$1^4 5^4$	$1^2 2^2 3^2 6^2$	6^4

$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
$n_g\|h_g$	7|1	8|1	10|2	11|1	12|2	12|12	14|1	15|1	21|3	23|1
$\bar{\chi}_g^{X_A}$	3	2	0	2	0	0	1	1	0	1
$\bar{\Pi}_g^{X_A}$	$1^3 7^3$	$1^2 2^1 4^1 8^2$	$2^2 10^2$	$1^2 11^2$	$2^1 4^1 6^1 12^1$	12^2	$1^1 2^1 7^1 14^1$	$1^1 3^1 5^1 15^1$	$3^1 21^1$	$1^1 23^1$

Table 25: Twisted Euler characters and Frame shapes at $\ell = 3$, $X = A_2^{12}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
$n_g\|h_g$	1|1	1|4	2|8	2|1	2|2	3|1	3|4	3|3	3|12	4|2	4|1	5|1	5|4	6|24	6|1	6|2	8|4	8|1	10|8	11|1	11|4
$\bar{\chi}_g^{X_A}$	12	12	0	4	4	3	3	0	0	0	4	2	2	0	1	1	0	2	0	1	1
$\chi_g^{X_A}$	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	1	−1
$\bar{\Pi}_g^{X_A}$	1^{12}	1^{12}	2^6	$1^4 2^4$	$1^4 2^4$	$1^3 3^3$	$1^3 3^3$	3^4	3^4	$2^2 4^2$	$1^4 4^2$	$1^2 5^2$	$1^2 5^2$	6^2	$1^1 2^1 3^1 6^1$	$1^1 2^1 3^1 6^1$	$4^1 8^1$	$1^2 2^1 8^1$	$2^1 10^1$	$1^1 11^1$	$1^1 11^1$
$\widetilde{\Pi}_g^{X_A}$	1^{24}	2^{12}	4^6	$1^8 2^8$	2^{12}	$1^6 3^6$	$2^3 6^3$	3^8	6^4	$2^4 4^4$	$1^4 2^2 4^4$	$1^4 5^4$	$2^2 10^2$	12^2	$1^2 2^2 3^2 6^2$	$2^2 6^2$	$4^2 8^2$	$1^2 2^1 4^1 8^2$	$4^1 20^1$	$1^2 11^2$	$2^1 22^1$

Table 26: Twisted Euler characters and Frame shapes at $\ell = 4$, $X = A_3^8$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$n_g\|h_g$	$1\|1$	$1\|2$	$2\|2$	$2\|4$	$4\|4$	$2\|1$	$3\|1$	$3\|2$	$6\|2$	$4\|8$	$4\|1$	$7\|1$	$7\|2$
$\bar{\chi}_g^{X_A}$	8	8	0	0	0	4	2	2	0	0	2	1	1
$\chi_g^{X_A}$	8	-8	0	0	0	0	2	-2	0	0	0	1	-1
$\bar{\Pi}_g^{X_A}$	1^8	1^8	2^4	2^4	4^2	$1^4 2^2$	$1^2 3^2$	$1^2 3^2$	$2^1 6^1$	4^2	$1^2 2^1 4^1$	$1^1 7^1$	$1^1 7^1$
$\widetilde{\Pi}_g^{X_A}$	1^{24}	$1^8 2^8$	2^{12}	$2^4 4^4$	4^6	$1^8 2^8$	$1^6 3^6$	$1^2 2^2 3^2 6^2$	$2^3 6^3$	$4^2 8^2$	$1^4 2^2 4^4$	$1^3 7^3$	$1^1 2^1 7^1 14^1$

Table 27: Twisted Euler characters and Frame shapes at $\ell = 5$, $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$n_g\|h_g$	$1\|1$	$1\|4$	$2\|2$	$2\|1$	$3\|3$	$3\|12$	$5\|1$	$5\|4$	$2\|8$	$4\|1$	$6\|24$
$\bar{\chi}_g^{X_A}$	6	6	2	2	0	0	1	1	0	2	0
$\chi_g^{X_A}$	6	-6	-2	2	0	0	1	-1	0	0	0
$\bar{\Pi}_g^{X_A}$	1^6	1^6	$1^2 2^2$	$1^2 2^2$	3^2	3^2	$1^1 5^1$	$1^1 5^1$	2^3	$1^2 4^1$	6^1
$\widetilde{\Pi}_g^{X_A}$	1^{24}	2^{12}	2^{12}	$1^8 2^8$	3^8	6^4	$1^4 5^4$	$2^2 10^2$	4^6	$1^4 2^2 4^4$	12^2

Table 28: Twisted Euler characters and Frame shapes at $\ell = 6$, $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	$1\|1$	$1\|2$	$2\|1$	$2\|2$	$3\|1$	$3\|2$	$4\|2$
$\bar{\chi}_g^{X_A}$	4	4	2	0	1	1	0
$\chi_g^{X_A}$	4	-4	0	0	1	-1	0
$\bar{\Pi}_g^{X_A}$	1^4	1^4	$1^2 2^1$	2^2	$1^1 3^1$	$1^1 3^1$	4^1
$\widetilde{\Pi}_g^{X_A}$	1^{20}	$1^4 2^8$	$1^6 2^7$	$2^2 4^4$	$1^5 3^5$	$1^1 2^2 3^1 6^2$	$4^1 8^2$
$\bar{\chi}_g^{X_D}$	1	1	1	1	1	1	1
$\chi_g^{X_D}$	1	1	-1	1	1	1	-1
$\check{\chi}_g^{X_D}$	2	2	0	2	-1	-1	0
$\bar{\Pi}_g^{X_D}$	1^1	1^1	1^1	1^1	1^1	1^1	1^1
$\widetilde{\Pi}_g^{X_D}$	1^4	1^4	$1^2 2^1$	1^4	$1^1 3^1$	$1^1 3^1$	$1^2 2^1$
$\widetilde{\Pi}_g^{X}$	1^{24}	$1^8 2^8$	$1^8 2^8$	$1^4 2^2 4^4$	$1^6 3^6$	$1^2 2^2 3^2 6^2$	$1^2 2^1 4^1 8^2$

Table 29: Twisted Euler characters and Frame shapes at $\ell = 6+3$, $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	6C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
$n_g\|h_g$	$1\|1$	$1\|3$	$2\|1$	$2\|3$	$3\|1$	$3\|3$	$4\|2$	$4\|6$	$5\|1$	$5\|3$	$2\|1$	$2\|2$	$4\|1$	$6\|1$	$6\|6$
$\bar{\chi}_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	4	0	2	1	0
$\chi_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	-4	0	-2	-1	0
$\check{\chi}_g^{X_D}$	12	-6	4	-2	0	0	0	0	2	-1	0	0	0	0	0
$\bar{\Pi}_g^{X_D}$	1^6	1^6	$1^2 2^2$	$1^2 2^2$	$1^3 3^1$	3^2	$2^1 4^1$	$2^1 4^1$	$1^1 5^1$	$1^1 5^1$	$1^4 2^1$	2^3	$1^2 4^1$	$1^1 2^1 3^1$	6^1
$\widetilde{\Pi}_g^{X_D}$	1^{24}	$1^6 3^6$	$1^8 2^8$	$1^2 2^2 3^2 6^2$	$1^6 3^6$	3^8	$2^4 4^4$	$2^1 4^1 6^1 12^1$	$1^4 5^4$	$1^1 3^1 5^1 15^1$	$1^2 2^8$	2^{12}	$1^4 2^2 4^4$	$1^2 2^2 3^2 6^2$	6^4

Table 30: Twisted Euler characters and Frame shapes at $\ell = 7$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
$n_g\|h_g$	$1\|1$	$1\|4$	$2\|8$	$3\|1$	$3\|4$
$\bar{\chi}_g^{X_A}$	4	4	0	1	1
$\chi_g^{X_A}$	4	-4	0	1	-1
$\bar{\Pi}_g^{X_A}$	1^4	1^4	2^2	$1^1 3^1$	$1^1 3^1$
$\widetilde{\Pi}_g^{X_A}$	1^{24}	2^{12}	4^6	$1^6 3^6$	$2^3 6^3$

Table 31: Twisted Euler characters and Frame shapes at $\ell = 8$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2B	2C	4A
$n_g\|h_g$	$1\|1$	$1\|2$	$2\|1$	$2\|1$	$2\|4$
$\bar{\chi}_g^{X_A}$	2	2	0	2	0
$\chi_g^{X_A}$	2	-2	0	0	0
$\bar{\Pi}_g^{X_A}$	1^2	1^2	2^1	1^2	2^1
$\widetilde{\Pi}_g^{X_A}$	1^{14}	$1^2 2^6$	2^7	$1^8 2^3$	$2^1 4^3$
$\bar{\chi}_g^{X_D}$	2	2	2	0	0
$\chi_g^{X_D}$	2	-2	0	0	0
$\bar{\Pi}_g^{X_D}$	1^2	1^2	1^2	2^1	2^1
$\widetilde{\Pi}_g^{X_D}$	1^{10}	$1^6 2^2$	$1^8 2^1$	2^5	$2^3 4^1$
$\widetilde{\Pi}_g^{X}$	1^{24}	$1^8 2^8$	$1^8 2^8$	$1^8 2^8$	$2^4 4^4$

Table 32: Twisted Euler characters and Frame shapes at $\ell = 9$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
$n_g \vert h_g$	$1\vert 1$	$1\vert 4$	$2\vert 1$	$2\vert 2$	$3\vert 3$	$3\vert 12$
$\bar{\chi}_g^{X_A}$	3	3	1	1	0	0
$\chi_g^{X_A}$	3	-3	1	-1	0	0
$\bar{\Pi}_g^{X_A}$	1^3	1^3	$1^1 2^1$	$1^1 2^1$	3^1	3^1
$\widetilde{\Pi}_g^{X_A}$	1^{24}	2^{12}	$1^8 2^8$	2^{12}	3^8	6^4

Table 33: Twisted Euler characters and Frame shapes at $\ell = 10$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$	$2\vert 2$
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0
$\bar{\Pi}_g^{X_A}$	1^2	1^2	2^1
$\widetilde{\Pi}_g^{X_A}$	1^{18}	$1^2 2^8$	$2^1 4^4$
$\bar{\chi}_g^{X_D}$	1	1	1
$\chi_g^{X_D}$	1	1	-1
$\bar{\Pi}_g^{X_D}$	1^1	1^1	1^1
$\widetilde{\Pi}_g^{X_D}$	1^6	1^6	$1^4 2^1$
$\widetilde{\Pi}_g^{X}$	1^{24}	$1^8 2^8$	$1^4 2^2 4^4$

Table 34: Twisted Euler characters and Frame shapes at $\ell = 10 + 5$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
$n_g \vert h_g$	$1\vert 1$	$2\vert 2$	$3\vert 1$	$2\vert 1$	$4\vert 4$
$\bar{\chi}_g^{X_D}$	4	0	1	2	0
$\chi_g^{X_D}$	4	0	1	-2	0
$\bar{\Pi}_g^{X_D}$	1^4	2^2	$1^1 3^1$	$1^2 2^1$	4^1
$\widetilde{\Pi}_g^{X_D}$	1^{24}	2^{12}	$1^6 3^6$	$1^8 2^8$	4^6

Table 35: Twisted Euler characters and Frame shapes at $\ell = 12$, $X = A_{11}D_7E_6$

$[g]$	1A	2A
$n_g\|h_g$	$1\|1$	$1\|2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\Pi}_g^{X_A}$	1^1	1^1
$\widetilde{\Pi}_g^{X_A}$	1^{11}	$1^1 2^5$
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\Pi}_g^{X_D}$	1^1	1^1
$\widetilde{\Pi}_g^{X_D}$	1^7	$1^5 2^1$
$\bar{\chi}_g^{X_E}$	1	1
$\chi_g^{X_E}$	1	-1
$\bar{\Pi}_g^{X_E}$	1^1	1^1
$\widetilde{\Pi}_g^{X_E}$	1^6	$1^2 2^2$
$\widetilde{\Pi}_g^{X}$	1^{24}	$1^8 2^8$

Table 36: Twisted Euler characters and Frame shapes at $\ell = 12 + 4$, $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	$1\|1$	$1\|2$	$2\|1$	$2\|4$	$3\|1$	$3\|2$	$4\|8$
$\bar{\chi}_g^{X_E}$	4	4	2	0	1	1	0
$\chi_g^{X_E}$	4	-4	0	0	1	-1	0
$\bar{\Pi}_g^{X_E}$	1^4	1^4	$1^2 2^1$	2^2	$1^1 3^1$	$1^1 3^1$	4^1
$\widetilde{\Pi}_g^{X_E}$	1^{24}	$1^8 2^8$	$1^8 2^8$	$2^4 4^4$	$1^6 3^6$	$1^2 2^2 3^2 6^2$	$4^2 8^2$

Table 37: Twisted Euler characters and Frame shapes at $\ell = 13$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
$n_g \vert h_g$	$1\vert 1$	$1\vert 4$	$2\vert 8$
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0
$\bar{\Pi}_g^{X_A}$	1^2	1^2	2^1
$\widetilde{\Pi}_g^{X_A}$	1^{24}	2^{12}	4^6

Table 38: Twisted Euler characters and Frame shapes at $\ell = 14 + 7$, $X = D_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$	$3\vert 3$
$\bar{\chi}_g^{X_D}$	3	1	0
$\chi_g^{X_D}$	3	1	0
$\bar{\Pi}_g^{X_D}$	1^3	$1^1 2^1$	3^1
$\widetilde{\Pi}_g^{X_D}$	1^{24}	$1^8 2^8$	3^8

Table 39: Twisted Euler characters and Frame shapes at $\ell = 16$, $X = A_{15} D_9$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\Pi}_g^{X_A}$	1^1	1^1
$\widetilde{\Pi}_g^{X_A}$	1^{15}	$1^1 2^7$
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\Pi}_g^{X_D}$	1^1	1^1
$\widetilde{\Pi}_g^{X_D}$	1^9	$1^7 2^1$
$\widetilde{\Pi}_g^{X}$	1^{24}	$1^8 2^8$

Table 40: Twisted Euler characters and Frame shapes at $\ell = 18$, $X = A_{17}E_7$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\Pi}_g^{X_A}$	1^1	1^1
$\widetilde{\Pi}_g^{X_A}$	1^{17}	$1^1 2^8$
$\bar{\chi}_g^{X_E}$	1	1
$\bar{\Pi}_g^{X_E}$	1^1	1^1
$\widetilde{\Pi}_g^{X_E}$	1^7	1^7
$\widetilde{\Pi}_g^{X}$	1^{24}	$1^8 2^8$

Table 41: Twisted Euler characters and Frame shapes at $\ell = 18 + 9$, $X = D_{10}E_7^2$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\Pi}_g^{X_D}$	1^1	1^1
$\widetilde{\Pi}_g^{X_D}$	1^{10}	$1^8 2^1$
$\bar{\chi}_g^{X_E}$	2	0
$\bar{\Pi}_g^{X_E}$	1^2	2^1
$\widetilde{\Pi}_g^{X_E}$	1^{14}	2^7
$\widetilde{\Pi}_g^{X}$	1^{24}	$1^8 2^8$

Table 42: Twisted Euler characters and Frame shapes at $\ell = 22 + 11$, $X = D_{12}^2$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$2\vert 2$
$\bar{\chi}_g^{X_D}$	2	0
$\chi_g^{X_D}$	2	0
$\bar{\Pi}_g^{X_D}$	1^2	2^1
$\widetilde{\Pi}_g^{X_D}$	1^{24}	2^{12}

Table 43: Twisted Euler characters and Frame shapes at $\ell = 25$, $X = A_{24}$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 4$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\Pi}_g^{X_A}$	1^1	1^1
$\widetilde{\Pi}_g^{X_A}$	1^{24}	2^{12}

Table 44: Twisted Euler characters and Frame shapes at $\ell = 30 + 6, 10, 15$, $X = E_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$	$3\vert 3$
$\bar{\chi}_g^{X_E}$	3	1	0
$\bar{\Pi}_g^{X_E}$	1^3	$1^1 2^1$	3^1
$\widetilde{\Pi}_g^{X_E}$	1^{24}	$1^8 2^8$	3^8

C Coefficient Tables

In this section we furnish tables of Fourier coefficients of small degree for the vector-valued mock modular forms H_g^X attached to elements $g \in G^X$. For each Niemeier root system X and each conjugacy class $[g] \in G^X$ we give a table that displays the coefficients of $H_{g,r}^X$ for sufficiently many r that any other component coincides with one of these up to sign. For instance, we always have $H_{g,r}^X = -H_{g,-r}^X$, so it suffices to list the $H_{g,r}^X$ for $0 < r < m$ when the Coxeter number of X is m. When X has no A-type components there are further redundancies, so that $H_{g,2}^X = H_{g,4}^X = 0$ and $H_{g,1}^X = H_{g,5}^X$, for example, when $X = D_4^6$. Recall that H_g^X is conjectured to coincide with the graded trace function of the umbral module K^X for all Niemeier root systems X except for $X = A_3^8$, and the relation has an additional factor of 3 when $X = A_3^8$ (cf. Conjecture 6.1).

For a Niemeier root system X with Coxeter number m, the first row of each table labels the conjugacy classes, and the first column labels exponents of $q^{1/4m}$, so that the entry in the row labelled d and the column labelled nZ in the table captioned $H_{g,r}^X$ is the coefficient of $q^{d/4m}$ in the Fourier expansion of $H_{g,r}^X$ for $[g] = nZ$. Occasionally the functions H_g^X and $H_{g'}^X$ coincide for non-conjugate g and g' and when this happens we condense information into a single column, writing $7AB$ in Table 45, for example, to indicate that the entries in that column are Fourier coefficients for both $H_{7A}^{(2)}$ and $H_{7B}^{(2)}$.

C.1 Lambency 2

Table 45: McKay–Thompson series $H^{(2)}_{g,1} = H^{X}_{g,1}$ for $X = A_1^{24}$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
Γ_g	1	2	2|2	3	3|3	4|2	4	4|4	5	6	6|6	7	8	10|2	11	12|2	12|12	14	15	21|3	23
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
7	90	-6	10	0	6	-6	2	2	0	0	-2	-1	-2	0	2	0	2	1	0	-1	-2
15	462	14	-18	-6	0	-2	-2	6	2	2	0	0	-2	2	0	-2	0	0	-1	0	2
23	1540	-28	20	10	-14	4	-4	-4	0	2	2	0	0	0	0	-2	2	0	0	0	-1
31	4554	42	-38	0	12	-6	2	-6	-6	0	4	4	-2	2	0	0	0	0	0	-2	0
39	11592	-56	72	-18	0	-8	8	0	2	-2	0	0	0	2	-2	-2	0	0	2	0	0
47	27830	86	-90	20	-16	6	-2	6	0	-4	0	-2	2	0	0	0	0	2	0	-2	0
55	61686	-138	118	0	30	6	-10	-2	6	0	-2	2	-2	-2	-2	0	-2	2	0	2	0
63	131100	188	-180	-30	0	-4	4	-12	0	2	0	-3	0	0	2	2	0	-1	0	0	0
71	265650	-238	258	42	-42	-14	10	10	-10	2	6	0	-2	-2	0	-2	-2	0	2	0	0
79	521136	336	-352	0	42	0	-8	16	6	0	2	0	-4	-2	0	0	-2	0	0	0	2
87	988770	-478	450	-60	0	18	-14	-6	0	-4	0	6	2	0	2	0	0	-2	0	0	0
95	1830248	616	-600	62	-70	-8	8	-16	8	-2	-6	0	0	0	2	-2	2	0	2	0	0
103	3303630	-786	830	0	84	-18	22	6	0	0	-4	-6	2	0	0	0	0	-2	0	0	2
111	5844762	1050	-1062	-90	0	10	-6	18	-18	6	0	0	2	-2	0	-2	0	0	0	0	2
119	10139734	-1386	1334	118	-110	22	-26	-10	4	6	2	-4	-2	4	0	-2	2	0	-2	2	0
127	17301060	1764	-1740	0	126	-12	12	-28	0	0	6	0	0	0	-4	0	2	0	0	0	0
135	29051484	-2212	2268	-156	0	-36	28	12	14	-4	0	0	-4	-2	0	0	0	0	-1	0	0
143	48106430	2814	-2850	170	-166	14	-18	38	0	-6	-6	8	-2	0	-2	2	2	0	0	2	-2
151	78599556	-3612	3540	0	210	36	-36	-20	-24	0	-6	0	0	0	2	0	-2	0	0	0	0
159	126894174	4510	-4482	-228	0	-18	14	-42	14	4	0	-6	-2	-2	0	0	0	2	2	0	0
167	202537080	-5544	5640	270	-282	-40	48	16	0	6	6	4	4	0	-2	2	-2	0	0	-2	0
175	319927608	6936	-6968	0	300	24	-16	48	18	0	4	-7	4	2	0	0	0	-1	0	-1	0
183	500376870	-8666	8550	-360	0	54	-58	-18	0	-8	0	0	-2	0	4	0	0	0	0	0	2
191	775492564	10612	-10556	400	-392	-28	28	-60	-36	-8	-8	0	0	4	0	-4	0	0	0	0	0
199	1191453912	-12936	13064	0	462	-72	64	32	12	0	-10	12	-4	4	0	0	2	0	0	0	0
207	1815754710	15862	-15930	-510	0	22	-34	78	0	10	0	0	-6	0	0	-2	0	0	0	0	-1
215	2745870180	-19420	19268	600	-600	84	-76	-36	30	8	8	-10	4	-2	-2	0	0	-2	0	2	0
223	4122417420	23532	-23460	0	660	-36	36	-84	0	0	12	2	0	0	0	0	0	-2	0	2	0
231	6146311620	-28348	28548	-762	0	-92	100	36	-50	-10	0	-6	4	-2	-2	-2	0	2	-2	0	0
239	9104078592	34272	-34352	828	-840	48	-40	96	22	-12	-8	0	4	-2	4	0	0	0	-2	0	0
247	13401053820	-41412	41180	0	966	108	-116	-44	0	0	-10	0	-4	0	0	0	-2	0	0	0	-2

C.2 Lambency 3

Table 46: McKay–Thompson series $H^{(3)}_{g,1} = H^{X}_{g,1}$ for $X = A_2^{12}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
Γ_g	1	1|4	2|8	2	2|2	3	3|4	3|3	3|12	4|2	4	5	5|4	6|24	6	6|2	8|4	8	10|8	11	11|4
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
11	32	32	8	0	0	-4	-4	2	2	0	0	2	2	2	0	0	0	0	-2	-1	-1
23	110	110	-10	-2	-2	2	2	2	2	6	-2	0	0	2	-2	-2	-2	2	0	0	0
35	288	288	8	0	0	0	0	-6	-6	0	0	-2	-2	2	0	0	0	0	-2	2	2
47	660	660	-20	4	4	-6	-6	6	6	-4	4	0	0	-2	-2	-2	4	0	0	0	0
59	1408	1408	32	0	0	4	4	4	4	0	0	-2	-2	-4	0	0	0	0	2	0	0
71	2794	2794	-30	-6	-6	4	4	-8	-8	2	-6	4	4	0	0	0	2	-2	0	0	0
83	5280	5280	40	0	0	-12	-12	6	6	0	0	0	0	-2	0	0	0	0	0	0	0
95	9638	9638	-58	6	6	8	8	2	2	-10	6	-2	-2	2	0	0	-2	-2	2	2	2
107	16960	16960	80	0	0	4	4	-14	-14	0	0	0	0	2	0	0	0	0	0	-2	-2
119	29018	29018	-102	-6	-6	-16	-16	8	8	10	-6	-2	-2	0	0	0	2	2	-2	0	0
131	48576	48576	112	0	0	12	12	6	6	0	0	6	6	-2	0	0	0	0	2	0	0
143	79530	79530	-150	10	10	6	6	-24	-24	-6	10	0	0	0	-2	-2	2	2	0	0	0
155	127776	127776	200	0	0	-24	-24	18	18	0	0	-4	-4	2	0	0	0	0	0	0	0
167	202050	202050	-230	-14	-14	18	18	12	12	10	-14	0	0	4	-2	-2	2	-2	0	2	2
179	314688	314688	272	0	0	12	12	-30	-30	0	0	-2	-2	2	0	0	0	0	2	0	0
191	483516	483516	-348	12	12	-36	-36	24	24	-12	12	6	6	0	0	0	-4	0	2	0	0
203	733920	733920	440	0	0	24	24	12	12	0	0	0	0	-4	0	0	0	0	0	0	0
215	1101364	1101364	-508	-12	-12	16	16	-44	-44	20	-12	-6	-6	-4	0	0	-4	4	2	0	0
227	1635680	1635680	600	0	0	-52	-52	32	32	0	0	0	0	0	0	0	0	0	0	2	2
239	2406116	2406116	-740	20	20	38	38	20	20	-20	20	-4	-4	4	2	2	4	0	0	-2	-2
251	3507680	3507680	888	0	0	20	20	-64	-64	0	0	10	10	0	0	0	0	0	-2	0	0
263	5071000	5071000	-1040	-24	-24	-68	-68	46	46	16	-24	0	0	-2	0	0	0	-4	0	0	0
275	7274464	7274464	1208	0	0	52	52	28	28	0	0	-6	-6	-4	0	0	0	0	-2	-1	-1
287	10359030	10359030	-1450	22	22	30	30	-84	-84	-26	22	0	0	-4	-2	-2	-2	-2	0	0	0
299	14650176	14650176	1744	0	0	-96	-96	60	60	0	0	-4	-4	4	0	0	0	0	4	2	2
311	20585334	20585334	-2018	-26	-26	66	66	36	36	30	-26	14	14	4	-2	-2	-2	2	2	0	0
323	28747840	28747840	2320	0	0	40	40	-116	-116	0	0	0	0	4	0	0	0	0	0	0	0
335	39914402	39914402	-2750	34	34	-130	-130	86	86	-30	34	-8	-8	-2	-2	-2	2	2	0	0	0
347	55114400	55114400	3240	0	0	92	92	50	50	0	0	0	0	-6	0	0	0	0	0	0	0
359	75704904	75704904	-3712	-40	-40	54	54	-156	-156	32	-40	-6	-6	-4	2	2	0	-4	-2	0	0

Table 47: McKay–Thompson series $H^{(3)}_{g,2} = H^{X}_{g,2}$ for $X = A_2^{12}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
Γ_g	1	1|4	2|8	2	2|2	3	3|4	3|3	3|12	4|2	4	5	5|4	6|24	6	6|2	8|4	8	10|8	11	11|4
8	20	-20	0	-4	4	2	-2	-4	4	0	0	0	0	0	2	-2	0	0	0	-2	2
20	88	-88	0	8	-8	-2	2	4	-4	0	0	-2	2	0	2	-2	0	0	0	0	0
32	220	-220	0	-12	12	4	-4	4	-4	0	0	0	0	0	0	0	0	0	0	0	0
44	560	-560	0	16	-16	2	-2	-4	4	0	0	0	0	0	-2	2	0	0	0	-1	1
56	1144	-1144	0	-24	24	-8	8	4	-4	0	0	4	-4	0	0	0	0	0	0	0	0
68	2400	-2400	0	32	-32	6	-6	0	0	0	0	0	0	0	2	-2	0	0	0	2	-2
80	4488	-4488	0	-40	40	6	-6	-12	12	0	0	-2	2	0	2	-2	0	0	0	0	0
92	8360	-8360	0	56	-56	-10	10	8	-8	0	0	0	0	0	2	-2	0	0	0	0	0
104	14696	-14696	0	-72	72	8	-8	8	-8	0	0	-4	4	0	0	0	0	0	0	0	0
116	25544	-25544	0	88	-88	2	-2	-16	16	0	0	4	-4	0	-2	2	0	0	0	2	-2
128	42660	-42660	0	-116	116	-18	18	12	-12	0	0	0	0	0	-2	2	0	0	0	2	-2
140	70576	-70576	0	144	-144	16	-16	4	-4	0	0	-4	4	0	0	0	0	0	0	0	0
152	113520	-113520	0	-176	176	12	-12	-24	24	0	0	0	0	0	4	-4	0	0	0	0	0
164	180640	-180640	0	224	-224	-26	26	16	-16	0	0	0	0	0	2	-2	0	0	0	-2	2
176	281808	-281808	0	-272	272	18	-18	12	-12	0	0	8	-8	0	-2	2	0	0	0	-1	1
188	435160	-435160	0	328	-328	10	-10	-32	32	0	0	0	0	0	-2	2	0	0	0	0	0
200	661476	-661476	0	-404	404	-42	42	24	-24	0	0	-4	4	0	-2	2	0	0	0	2	-2
212	996600	-996600	0	488	-488	30	-30	12	-12	0	0	0	0	0	2	-2	0	0	0	0	0
224	1482536	-1482536	0	-584	584	20	-20	-52	52	0	0	-4	4	0	4	-4	0	0	0	0	0
236	2187328	-2187328	0	704	-704	-50	50	40	-40	0	0	8	-8	0	2	-2	0	0	0	0	0
248	3193960	-3193960	0	-840	840	40	-40	28	-28	0	0	0	0	0	0	0	0	0	0	0	0
260	4629152	-4629152	0	992	-992	20	-20	-64	64	0	0	-8	8	0	-4	4	0	0	0	0	0
272	6650400	-6650400	0	-1184	1184	-78	78	48	-48	0	0	0	0	0	-2	2	0	0	0	-2	2
284	9490536	-9490536	0	1400	-1400	54	-54	24	-24	0	0	-4	4	0	2	-2	0	0	0	0	0
296	13441032	-13441032	0	-1640	1640	36	-36	-96	96	0	0	12	-12	0	4	-4	0	0	0	0	0
308	18920240	-18920240	0	1936	-1936	-100	100	68	-68	0	0	0	0	0	4	-4	0	0	0	-2	2
320	26457464	-26457464	0	-2264	2264	74	-74	44	-44	0	0	-6	6	0	-2	2	0	0	0	0	0
332	36792560	-36792560	0	2640	-2640	38	-38	-124	124	0	0	0	0	0	-6	6	0	0	0	2	-2
344	50865232	-50865232	0	-3088	3088	-140	140	88	-88	0	0	-8	8	0	-4	4	0	0	0	0	0
356	69966336	-69966336	0	3584	-3584	102	-102	48	-48	0	0	16	-16	0	2	-2	0	0	0	0	0

C.3 Lambency 4

Table 48: McKay–Thompson series $H^{(4)}_{g,1} = H^{X}_{g,1}$ for $X = A_3^8$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
Γ_g	1	1|2	2|2	2|4	4|4	2	3	3|2	6|2	4|8	4	7	7|2
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
15	14	14	-2	6	-2	-2	2	2	-2	2	-2	0	0
31	42	42	-6	-6	2	2	0	0	0	2	-2	0	0
47	86	86	6	6	-2	-2	-4	-4	0	-2	2	2	2
63	188	188	-4	-12	-4	4	2	2	2	0	0	-1	-1
79	336	336	0	16	0	-8	0	0	0	0	-4	0	0
95	616	616	-8	-16	0	8	-2	-2	-2	4	0	0	0
111	1050	1050	10	18	2	-6	6	6	-2	-2	2	0	0
127	1764	1764	-12	-28	-4	12	0	0	0	-4	0	0	0
143	2814	2814	14	38	-2	-18	-6	-6	2	2	-2	0	0
159	4510	4510	-18	-42	6	14	4	4	0	2	-2	2	2
175	6936	6936	24	48	0	-16	0	0	0	-4	4	-1	-1
191	10612	10612	-28	-60	-4	28	-8	-8	-4	-4	0	0	0
207	15862	15862	22	78	-2	-34	10	10	-2	2	-6	0	0
223	23532	23532	-36	-84	4	36	0	0	0	4	0	-2	-2
239	34272	34272	48	96	0	-40	-12	-12	0	0	4	0	0
255	49618	49618	-46	-126	-6	50	10	10	2	-6	2	2	2
271	70758	70758	54	150	-2	-66	0	0	0	6	-6	2	2
287	100310	100310	-74	-170	6	70	-10	-10	-2	6	-2	0	0
303	140616	140616	88	192	0	-72	18	18	-2	-4	8	0	0
319	195888	195888	-96	-232	-8	96	0	0	0	-4	0	0	0
335	270296	270296	104	272	0	-120	-22	-22	2	4	-8	-2	-2
351	371070	371070	-130	-306	6	126	18	18	2	6	-2	0	0
367	505260	505260	156	348	4	-140	0	0	0	-4	8	0	0
383	684518	684518	-170	-410	-10	174	-22	-22	-2	-10	2	2	2
399	921142	921142	182	486	-2	-202	28	28	-4	6	-10	-2	-2
415	1233708	1233708	-228	-540	12	220	0	0	0	8	-4	0	0
431	1642592	1642592	272	608	0	-248	-34	-34	2	-8	12	0	0
447	2177684	2177684	-284	-708	-12	292	32	32	4	-8	4	-2	-2
463	2871918	2871918	318	814	-2	-346	0	0	0	6	-14	0	0
479	3772468	3772468	-380	-908	12	380	-38	-38	-2	12	0	0	0
495	4932580	4932580	436	1020	4	-412	46	46	-2	-8	12	2	2
511	6425466	6425466	-486	-1174	-14	490	0	0	0	-14	2	-2	-2
527	8335418	8335418	538	1338	-6	-566	-52	-52	4	10	-14	0	0
543	10776290	10776290	-622	-1494	18	610	50	50	2	14	-6	0	0
559	13879290	13879290	714	1666	2	-678	0	0	0	-10	18	-2	-2
575	17818766	17818766	-786	-1898	-18	790	-58	-58	-6	-14	2	0	0
591	22798188	22798188	860	2148	-4	-900	72	72	-4	8	-20	0	0

Table 49: McKay–Thompson series $H^{(4)}_{g,2} = H^{X}_{g,2}$ for $X = A_3^8$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
Γ_g	1	1|2	2|2	2|4	4|4	2	3	3|2	6|2	4|8	4	7	7|2
12	16	-16	0	0	0	0	-2	2	0	0	0	2	-2
28	48	-48	0	0	0	0	0	0	0	0	0	-1	1
44	112	-112	0	0	0	0	4	-4	0	0	0	0	0
60	224	-224	0	0	0	0	-4	4	0	0	0	0	0
76	432	-432	0	0	0	0	0	0	0	0	0	-2	2
92	784	-784	0	0	0	0	4	-4	0	0	0	0	0
108	1344	-1344	0	0	0	0	-6	6	0	0	0	0	0
124	2256	-2256	0	0	0	0	0	0	0	0	0	2	-2
140	3680	-3680	0	0	0	0	8	-8	0	0	0	-2	2
156	5824	-5824	0	0	0	0	-8	8	0	0	0	0	0
172	9072	-9072	0	0	0	0	0	0	0	0	0	0	0
188	13872	-13872	0	0	0	0	12	-12	0	0	0	-2	2
204	20832	-20832	0	0	0	0	-12	12	0	0	0	0	0
220	30912	-30912	0	0	0	0	0	0	0	0	0	0	0
236	45264	-45264	0	0	0	0	12	-12	0	0	0	2	-2
252	65456	-65456	0	0	0	0	-16	16	0	0	0	-1	1
268	93744	-93744	0	0	0	0	0	0	0	0	0	0	0
284	132944	-132944	0	0	0	0	20	-20	0	0	0	0	0
300	186800	-186800	0	0	0	0	-22	22	0	0	0	-2	2
316	260400	-260400	0	0	0	0	0	0	0	0	0	0	0
332	360208	-360208	0	0	0	0	28	-28	0	0	0	2	-2
348	494624	-494624	0	0	0	0	-28	28	0	0	0	4	-4
364	674784	-674784	0	0	0	0	0	0	0	0	0	-2	2
380	914816	-914816	0	0	0	0	32	-32	0	0	0	0	0
396	1232784	-1232784	0	0	0	0	-36	36	0	0	0	0	0
412	1652208	-1652208	0	0	0	0	0	0	0	0	0	-2	2
428	2202704	-2202704	0	0	0	0	44	-44	0	0	0	0	0
444	2921856	-2921856	0	0	0	0	-48	48	0	0	0	0	0
460	3857760	-3857760	0	0	0	0	0	0	0	0	0	4	-4
476	5070560	-5070560	0	0	0	0	56	-56	0	0	0	-2	2
492	6636000	-6636000	0	0	0	0	-60	60	0	0	0	0	0
508	8649648	-8649648	0	0	0	0	0	0	0	0	0	0	0
524	11230448	-11230448	0	0	0	0	68	-68	0	0	0	-2	2
540	14526848	-14526848	0	0	0	0	-76	76	0	0	0	0	0
556	18724176	-18724176	0	0	0	0	0	0	0	0	0	2	-2
572	24051808	-24051808	0	0	0	0	88	-88	0	0	0	4	-4
588	30793712	-30793712	0	0	0	0	-94	94	0	0	0	-2	2
604	39301584	-39301584	0	0	0	0	0	0	0	0	0	0	0

Table 50: McKay–Thompson series $H^{(4)}_{g,3} = H^{X}_{g,3}$ for $X = A_3^8$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
Γ_g	1	1|2	2|2	2|4	4|4	2	3	3|2	6|2	4|8	4	7	7|2
7	6	6	6	-2	-2	-2	0	0	0	2	2	-1	-1
23	28	28	-4	4	-4	4	-2	-2	2	0	0	0	0
39	56	56	8	0	0	-8	2	2	2	-4	0	0	0
55	138	138	-6	2	2	10	0	0	0	-2	2	-2	-2
71	238	238	14	-10	-2	-10	-2	-2	2	2	2	0	0
87	478	478	-18	6	-2	14	4	4	0	2	-2	2	2
103	786	786	18	-6	2	-22	0	0	0	-2	-2	2	2
119	1386	1386	-22	10	2	26	-6	-6	2	2	2	0	0
135	2212	2212	36	-12	-4	-28	4	4	0	4	4	0	0
151	3612	3612	-36	20	-4	36	0	0	0	0	0	0	0
167	5544	5544	40	-16	0	-48	-6	-6	-2	-4	-4	0	0
183	8666	8666	-54	18	2	58	8	8	0	-2	2	0	0
199	12936	12936	72	-32	0	-64	0	0	0	4	4	0	0
215	19420	19420	-84	36	-4	76	-8	-8	0	0	-4	2	2
231	28348	28348	92	-36	4	-100	10	10	2	-4	-4	-2	-2
247	41412	41412	-108	44	4	116	0	0	0	0	4	0	0
263	59178	59178	138	-62	-6	-126	-12	-12	0	6	6	0	0
279	84530	84530	-158	66	-6	154	14	14	-2	2	-2	-2	-2
295	118692	118692	180	-68	4	-188	0	0	0	-8	-4	0	0
311	166320	166320	-208	88	8	216	-18	-18	2	-4	4	0	0
327	230092	230092	252	-108	-4	-244	16	16	0	8	4	2	2
343	317274	317274	-294	122	-6	282	0	0	0	2	-6	-1	-1
359	432964	432964	324	-132	4	-340	-20	-20	0	-8	-8	0	0
375	588966	588966	-378	150	6	390	24	24	0	-2	6	0	0
391	794178	794178	450	-190	-6	-430	0	0	0	10	10	0	0
407	1067220	1067220	-508	220	-12	500	-30	-30	2	0	-4	0	0
423	1423884	1423884	572	-228	4	-588	30	30	2	-12	-8	0	0
439	1893138	1893138	-654	266	10	666	0	0	0	-2	6	2	2
455	2501434	2501434	762	-326	-6	-742	-32	-32	0	10	10	-2	-2
471	3294256	3294256	-864	360	-8	848	40	40	0	4	-8	0	0
487	4314912	4314912	960	-392	8	-984	0	0	0	-12	-12	0	0
503	5633596	5633596	-1092	452	12	1108	-44	-44	0	0	8	-4	-4
519	7320670	7320670	1262	-522	-10	-1234	46	46	2	18	14	0	0
535	9483336	9483336	-1416	592	-16	1400	0	0	0	4	-8	2	2
551	12233330	12233330	1570	-646	10	-1598	-58	-58	-2	-18	-14	4	4
567	15734606	15734606	-1778	726	14	1798	62	62	-2	-6	10	-1	-1
583	20161302	20161302	2022	-850	-10	-1994	0	0	0	18	14	0	0
599	25761288	25761288	-2264	944	-16	2240	-72	-72	3	4	-12	0	0

C.4 Lambency 5

Table 51: McKay–Thompson series $H^{(5)}_{g,1} = H^X_{g,1}$ for $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
19	8	8	0	0	2	2	-2	-2	4	0	-2
39	18	18	2	2	0	0	-2	-2	-6	2	0
59	40	40	0	0	-2	-2	0	0	4	0	-2
79	70	70	-2	-2	4	4	0	0	-6	-2	0
99	120	120	0	0	0	0	0	0	12	0	0
119	208	208	0	0	-2	-2	-2	-2	-8	0	-2
139	328	328	0	0	4	4	-2	-2	12	0	0
159	510	510	-2	-2	0	0	0	0	-18	-2	0
179	792	792	0	0	-6	-6	2	2	20	0	2
199	1180	1180	4	4	4	4	0	0	-24	4	0
219	1728	1728	0	0	0	0	-2	-2	24	0	0
239	2518	2518	-2	-2	-8	-8	-2	-2	-30	-2	0
259	3600	3600	0	0	6	6	0	0	40	0	-2
279	5082	5082	2	2	0	0	2	2	-42	2	0
299	7120	7120	0	0	-8	-8	0	0	48	0	0
319	9838	9838	-2	-2	10	10	-2	-2	-58	-2	2
339	13488	13488	0	0	0	0	-2	-2	72	0	0
359	18380	18380	4	4	-10	-10	0	0	-80	4	-2
379	24792	24792	0	0	12	12	2	2	84	0	0
399	33210	33210	-6	-6	0	0	0	0	-102	-6	0
419	44248	44248	0	0	-14	-14	-2	-2	116	0	2
439	58538	58538	2	2	14	14	-2	-2	-130	2	2
459	76992	76992	0	0	0	0	2	2	144	0	0
479	100772	100772	-4	-4	-16	-16	2	2	-168	-4	0
499	131160	131160	0	0	18	18	0	0	196	0	-2
519	169896	169896	8	8	0	0	-4	-4	-216	8	0
539	219128	219128	0	0	-22	-22	-2	-2	236	0	2
559	281322	281322	-6	-6	24	24	2	2	-270	-6	0
579	359712	359712	0	0	0	0	2	2	312	0	0
599	458220	458220	4	4	-24	-24	0	0	-336	4	0
619	581416	581416	0	0	28	28	-4	-4	372	0	0
639	735138	735138	-6	-6	0	0	-2	-2	-426	-6	0
659	926472	926472	0	0	-30	-30	2	2	476	0	2
679	1163674	1163674	10	10	34	34	4	4	-526	10	2
699	1457040	1457040	0	0	0	0	0	0	576	0	0
719	1819056	1819056	-8	-8	-42	-42	-4	-4	-644	-8	-2
739	2264376	2264376	0	0	42	42	-4	-4	724	0	-2

Table 52: McKay–Thompson series $H^{(5)}_{g,2} = H^{X}_{g,2}$ for $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
16	10	-10	2	-2	-2	2	0	0	0	0	0
36	30	-30	-2	2	0	0	0	0	0	0	0
56	52	-52	4	-4	4	-4	2	-2	0	0	0
76	108	-108	-4	4	0	0	-2	2	0	0	0
96	180	-180	4	-4	0	0	0	0	0	0	0
116	312	-312	-8	8	0	0	2	-2	0	0	0
136	488	-488	8	-8	-4	4	-2	2	0	0	0
156	792	-792	-8	8	0	0	2	-2	0	0	0
176	1180	-1180	12	-12	4	-4	0	0	0	0	0
196	1810	-1810	-14	14	-2	2	0	0	0	0	0
216	2640	-2640	16	-16	0	0	0	0	0	0	0
236	3868	-3868	-20	20	4	-4	-2	2	0	0	0
256	5502	-5502	22	-22	-6	6	2	-2	0	0	0
276	7848	-7848	-24	24	0	0	-2	2	0	0	0
296	10912	-10912	32	-32	4	-4	2	-2	0	0	0
316	15212	-15212	-36	36	-4	4	2	-2	0	0	0
336	20808	-20808	40	-40	0	0	-2	2	0	0	0
356	28432	-28432	-48	48	4	-4	2	-2	0	0	0
376	38308	-38308	52	-52	-8	8	-2	2	0	0	0
396	51540	-51540	-60	60	0	0	0	0	0	0	0
416	68520	-68520	72	-72	12	-12	0	0	0	0	0
436	90928	-90928	-80	80	-8	8	-2	2	0	0	0
456	119544	-119544	88	-88	0	0	4	-4	0	0	0
476	156728	-156728	-104	104	8	-8	-2	2	0	0	0
496	203940	-203940	116	-116	-12	12	0	0	0	0	0
516	264672	-264672	-128	128	0	0	2	-2	0	0	0
536	341188	-341188	148	-148	16	-16	-2	2	0	0	0
556	438732	-438732	-164	164	-12	12	2	-2	0	0	0
576	560958	-560958	182	-182	0	0	-2	2	0	0	0
596	715312	-715312	-208	208	16	-16	2	-2	0	0	0
616	907720	-907720	232	-232	-20	20	0	0	0	0	0
636	1148928	-1148928	-256	256	0	0	-2	2	0	0	0
656	1447904	-1447904	288	-288	20	-20	4	-4	0	0	0
676	1820226	-1820226	-318	318	-18	18	-4	4	0	0	0
696	2279520	-2279520	352	-352	0	0	0	0	0	0	0
716	2847812	-2847812	-396	396	20	-20	2	-2	0	0	0
736	3545636	-3545636	436	-436	-28	28	-4	4	0	0	0
756	4404384	-4404384	-480	480	0	0	4	-4	0	0	0

Table 53: McKay–Thompson series $H^{(5)}_{g,3} = H^{X}_{g,3}$ for $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
11	8	8	0	0	2	2	-2	-2	-4	0	2
31	22	22	-2	-2	-2	-2	2	2	2	-2	2
51	48	48	0	0	0	0	-2	-2	0	0	0
71	90	90	2	2	0	0	0	0	6	2	0
91	160	160	0	0	-2	-2	0	0	-8	0	-2
111	270	270	-2	-2	0	0	0	0	6	-2	0
131	440	440	0	0	2	2	0	0	-4	0	2
151	700	700	4	4	-2	-2	0	0	8	4	2
171	1080	1080	0	0	0	0	0	0	-12	0	0
191	1620	1620	-4	-4	6	6	0	0	16	-4	-2
211	2408	2408	0	0	-4	-4	-2	-2	-12	0	0
231	3522	3522	2	2	0	0	2	2	18	2	0
251	5048	5048	0	0	2	2	-2	-2	-28	0	2
271	7172	7172	-4	-4	-4	-4	2	2	24	-4	0
291	10080	10080	0	0	0	0	0	0	-24	0	0
311	13998	13998	6	6	6	6	-2	-2	34	6	-2
331	19272	19272	0	0	-6	-6	2	2	-44	0	-2
351	26298	26298	-6	-6	0	0	-2	-2	42	-6	0
371	35600	35600	0	0	8	8	0	0	-48	0	0
391	47862	47862	6	6	-6	-6	2	2	62	6	2
411	63888	63888	0	0	0	0	-2	-2	-72	0	0
431	84722	84722	-6	-6	8	8	2	2	78	-6	0
451	111728	111728	0	0	-10	-10	-2	-2	-80	0	-2
471	146520	146520	8	8	0	0	0	0	96	8	0
491	191080	191080	0	0	10	10	0	0	-124	0	2
511	248008	248008	-8	-8	-14	-14	-2	-2	128	-8	2
531	320424	320424	0	0	0	0	4	4	-132	0	0
551	412088	412088	8	8	14	14	-2	-2	160	8	-2
571	527800	527800	0	0	-14	-14	0	0	-188	0	-2
591	673302	673302	-10	-10	0	0	2	2	198	-10	0
611	855616	855616	0	0	16	16	-4	-4	-216	0	0
631	1083444	1083444	12	12	-18	-18	4	4	248	12	2
651	1367136	1367136	0	0	0	0	-4	-4	-288	0	0
671	1719362	1719362	-14	-14	20	20	2	2	314	-14	-4
691	2155592	2155592	0	0	-22	-22	2	2	-332	0	-2
711	2694276	2694276	12	12	0	0	-4	-4	384	12	0
731	3357664	3357664	0	0	28	28	4	4	-440	0	4
751	4172746	4172746	-14	-14	-26	-26	-4	-4	470	-14	2

Table 54: McKay–Thompson series $H^{(5)}_{g,4} = H^{X}_{g,4}$ for $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
Γ_g	1	1|4	2|2	2	3|3	3|12	5	5|4	2|8	4	6|24
4	2	-2	2	-2	2	-2	2	-2	0	0	0
24	12	-12	-4	4	0	0	2	-2	0	0	0
44	20	-20	4	-4	-4	4	0	0	0	0	0
64	50	-50	-6	6	2	-2	0	0	0	0	0
84	72	-72	8	-8	0	0	2	-2	0	0	0
104	152	-152	-8	8	-4	4	2	-2	0	0	0
124	220	-220	12	-12	4	-4	0	0	0	0	0
144	378	-378	-14	14	0	0	-2	2	0	0	0
164	560	-560	16	-16	-4	4	0	0	0	0	0
184	892	-892	-20	20	4	-4	2	-2	0	0	0
204	1272	-1272	24	-24	0	0	2	-2	0	0	0
224	1940	-1940	-28	28	-4	4	0	0	0	0	0
244	2720	-2720	32	-32	8	-8	0	0	0	0	0
264	3960	-3960	-40	40	0	0	0	0	0	0	0
284	5500	-5500	44	-44	-8	8	0	0	0	0	0
304	7772	-7772	-52	52	8	-8	2	-2	0	0	0
324	10590	-10590	62	-62	0	0	0	0	0	0	0
344	14668	-14668	-68	68	-8	8	-2	2	0	0	0
364	19728	-19728	80	-80	12	-12	-2	2	0	0	0
384	26772	-26772	-92	92	0	0	2	-2	0	0	0
404	35624	-35624	104	-104	-16	16	4	-4	0	0	0
424	47592	-47592	-120	120	12	-12	2	-2	0	0	0
444	62568	-62568	136	-136	0	0	-2	2	0	0	0
464	82568	-82568	-152	152	-16	16	-2	2	0	0	0
484	107502	-107502	174	-174	18	-18	2	-2	0	0	0
504	140172	-140172	-196	196	0	0	2	-2	0	0	0
524	180940	-180940	220	-220	-20	20	0	0	0	0	0
544	233576	-233576	-248	248	20	-20	-4	4	0	0	0
564	298968	-298968	280	-280	0	0	-2	2	0	0	0
584	382632	-382632	-312	312	-24	24	2	-2	0	0	0
604	486124	-486124	348	-348	28	-28	4	-4	0	0	0
624	617112	-617112	-392	392	0	0	2	-2	0	0	0
644	778768	-778768	432	-432	-32	32	-2	2	0	0	0
664	981548	-981548	-484	484	32	-32	-2	2	0	0	0
684	1230732	-1230732	540	-540	0	0	2	-2	0	0	0
704	1541244	-1541244	-596	596	-36	36	4	-4	0	0	0
724	1921240	-1921240	664	-664	40	-40	0	0	0	0	0
744	2391456	-2391456	-736	736	0	0	-4	4	0	0	0

C.5 Lambency 6

Table 55: McKay–Thompson series $H^{(6)}_{g,1} = H^{X}_{g,1}$ for $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|1'$	$2\|1$	$2\|1'$	$3\|1$	$3\|1'$	$4\|2'$
-1	-2	-2	-2	-2	-2	-2	-2
23	4	4	0	4	-2	-2	0
47	12	12	0	-4	0	0	0
71	18	18	-2	2	0	0	-2
95	34	34	2	-6	-2	-2	-2
119	56	56	0	8	2	2	0
143	90	90	2	-6	0	0	2
167	128	128	-4	8	-4	-4	0
191	206	206	2	-10	2	2	-2
215	294	294	-2	14	0	0	2
239	424	424	4	-16	-2	-2	0
263	598	598	-6	14	4	4	-2
287	846	846	6	-18	0	0	-2
311	1152	1152	-4	24	-6	-6	0
335	1598	1598	6	-26	2	2	2
359	2138	2138	-10	26	2	2	-2
383	2886	2886	10	-34	-6	-6	-2
407	3834	3834	-6	42	6	6	2
431	5078	5078	10	-42	2	2	2
455	6622	6622	-18	46	-8	-8	-2
479	8682	8682	14	-54	6	6	-2
503	11214	11214	-14	62	0	0	2
527	14476	14476	20	-68	-8	-8	4
551	18530	18530	-22	74	8	8	-2
575	23684	23684	24	-84	2	2	-4
599	30006	30006	-22	102	-12	-12	2
623	38018	38018	26	-110	8	8	2
647	47792	47792	-36	112	2	2	-4
671	60012	60012	36	-132	-12	-12	-4
695	74942	74942	-34	150	14	14	2
719	93392	93392	44	-160	2	2	4
743	115782	115782	-54	174	-18	-18	-2
767	143388	143388	52	-196	12	12	-4
791	176716	176716	-52	220	4	4	4
815	217416	217416	64	-240	-18	-18	4
839	266496	266496	-76	256	18	18	-4
863	326074	326074	78	-286	4	4	-6
887	397548	397548	-80	324	-24	-24	4
911	483966	483966	90	-346	18	18	6
935	587236	587236	-108	372	4	4	-4
959	711394	711394	114	-414	-26	-26	-6

Table 56: McKay–Thompson series $H^{(6)}_{g,2} = H^{X}_{g,2}$ for $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|1'$	$2\|1$	$2\|1'$	$3\|1$	$3\|1'$	$4\|2'$
20	8	-8	0	0	2	-2	0
44	16	-16	0	0	-2	2	0
68	32	-32	0	0	2	-2	0
92	56	-56	0	0	2	-2	0
116	88	-88	0	0	-2	2	0
140	144	-144	0	0	0	0	0
164	224	-224	0	0	2	-2	0
188	328	-328	0	0	-2	2	0
212	488	-488	0	0	2	-2	0
236	704	-704	0	0	2	-2	0
260	992	-992	0	0	-4	4	0
284	1400	-1400	0	0	2	-2	0
308	1936	-1936	0	0	4	-4	0
332	2640	-2640	0	0	-6	6	0
356	3584	-3584	0	0	2	-2	0
380	4816	-4816	0	0	4	-4	0
404	6408	-6408	0	0	-6	6	0
428	8480	-8480	0	0	2	-2	0
452	11136	-11136	0	0	6	-6	0
476	14512	-14512	0	0	-8	8	0
500	18832	-18832	0	0	4	-4	0
524	24288	-24288	0	0	6	-6	0
548	31136	-31136	0	0	-10	10	0
572	39760	-39760	0	0	4	-4	0
596	50520	-50520	0	0	6	-6	0
620	63888	-63888	0	0	-12	12	0
644	80512	-80512	0	0	4	-4	0
668	101032	-101032	0	0	10	-10	0
692	126280	-126280	0	0	-14	14	0
716	157344	-157344	0	0	6	-6	0
740	195360	-195360	0	0	12	-12	0
764	241752	-241752	0	0	-18	18	0
788	298296	-298296	0	0	6	-6	0
812	366960	-366960	0	0	12	-12	0
836	450112	-450112	0	0	-20	20	0
860	550688	-550688	0	0	8	-8	0
884	671968	-671968	0	0	16	-16	0
908	817840	-817840	0	0	-26	26	0
932	993088	-993088	0	0	10	-10	0
956	1203048	-1203048	0	0	18	-18	0

Table 57: McKay–Thompson series $H^{(6)}_{g,3} = H^{X}_{g,3}$ for $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|1'$	$2\|1$	$2\|1'$	$3\|1$	$3\|1'$	$4\|2'$
15	6	6	-2	-2	0	0	2
39	18	18	2	2	0	0	2
63	30	30	-2	-2	0	0	-2
87	60	60	4	4	0	0	0
111	90	90	-6	-6	0	0	2
135	156	156	4	4	0	0	0
159	228	228	-4	-4	0	0	0
183	360	360	8	8	0	0	0
207	510	510	-10	-10	0	0	2
231	762	762	10	10	0	0	2
255	1062	1062	-10	-10	0	0	-2
279	1518	1518	14	14	0	0	-2
303	2070	2070	-18	-18	0	0	2
327	2880	2880	16	16	0	0	0
351	3870	3870	-18	-18	0	0	-2
375	5256	5256	24	24	0	0	0
399	6948	6948	-28	-28	0	0	4
423	9270	9270	30	30	0	0	2
447	12120	12120	-32	-32	0	0	-4
471	15912	15912	40	40	0	0	0
495	20562	20562	-46	-46	0	0	2
519	26646	26646	46	46	0	0	2
543	34110	34110	-50	-50	0	0	-2
567	43710	43710	62	62	0	0	-2
591	55440	55440	-72	-72	0	0	4
615	70362	70362	74	74	0	0	2
639	88536	88536	-80	-80	0	0	-4
663	111390	111390	94	94	0	0	-2
687	139140	139140	-108	-108	0	0	4
711	173712	173712	112	112	0	0	0
735	215568	215568	-120	-120	0	0	-4
759	267246	267246	142	142	0	0	-2
783	329610	329610	-158	-158	0	0	6
807	406080	406080	168	168	0	0	4
831	498042	498042	-182	-182	0	0	-6
855	610062	610062	206	206	0	0	-2
879	744306	744306	-230	-230	0	0	6
903	906930	906930	242	242	0	0	2
927	1101210	1101210	-262	-262	0	0	-6
951	1335282	1335282	298	298	0	0	-2

Table 58: McKay–Thompson series $H^{(6)}_{g,4} = H^X_{g,4}$ for $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|1'$	$2\|1$	$2\|1'$	$3\|1$	$3\|1'$	$4\|2'$
8	4	-4	0	0	-2	2	0
32	12	-12	0	0	0	0	0
56	24	-24	0	0	0	0	0
80	40	-40	0	0	-2	2	0
104	72	-72	0	0	0	0	0
128	116	-116	0	0	2	-2	0
152	176	-176	0	0	-4	4	0
176	272	-272	0	0	2	-2	0
200	404	-404	0	0	2	-2	0
224	584	-584	0	0	-4	4	0
248	840	-840	0	0	0	0	0
272	1184	-1184	0	0	2	-2	0
296	1640	-1640	0	0	-4	4	0
320	2264	-2264	0	0	2	-2	0
344	3088	-3088	0	0	4	-4	0
368	4152	-4152	0	0	-6	6	0
392	5564	-5564	0	0	2	-2	0
416	7384	-7384	0	0	4	-4	0
440	9712	-9712	0	0	-8	8	0
464	12728	-12728	0	0	2	-2	0
488	16552	-16552	0	0	4	-4	0
512	21388	-21388	0	0	-8	8	0
536	27520	-27520	0	0	4	-4	0
560	35216	-35216	0	0	8	-8	0
584	44832	-44832	0	0	-12	12	0
608	56848	-56848	0	0	4	-4	0
632	71768	-71768	0	0	8	-8	0
656	90208	-90208	0	0	-14	14	0
680	113008	-113008	0	0	4	-4	0
704	141040	-141040	0	0	10	-10	0
728	175376	-175376	0	0	-16	16	0
752	217416	-217416	0	0	6	-6	0
776	268656	-268656	0	0	12	-12	0
800	330940	-330940	0	0	-20	20	0
824	406568	-406568	0	0	8	-8	0
848	498056	-498056	0	0	14	-14	0
872	608472	-608472	0	0	-24	24	0
896	741560	-741560	0	0	8	-8	0
920	901504	-901504	0	0	16	-16	0
944	1093312	-1093312	0	0	-26	26	0

Table 59: McKay–Thompson series $H^{(6)}_{g,5} = H^{X}_{g,5}$ for $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|1'$	$2\|1$	$2\|1'$	$3\|1$	$3\|1'$	$4\|2'$
23	6	6	2	-2	0	0	-2
47	8	8	-4	0	2	2	0
71	24	24	4	0	0	0	0
95	28	28	-4	4	-2	-2	0
119	62	62	6	-2	2	2	-2
143	80	80	-8	0	2	2	0
167	142	142	10	-2	-2	-2	2
191	194	194	-10	2	2	2	-2
215	306	306	10	-6	0	0	-2
239	404	404	-16	4	-4	-4	0
263	622	622	18	-2	4	4	2
287	824	824	-16	8	2	2	0
311	1178	1178	22	-6	-4	-4	-2
335	1564	1564	-28	4	4	4	0
359	2178	2178	30	-6	0	0	2
383	2844	2844	-32	12	-6	-6	0
407	3876	3876	36	-12	6	6	-4
431	5024	5024	-44	8	2	2	0
455	6690	6690	50	-14	-6	-6	2
479	8616	8616	-52	16	6	6	0
503	11286	11286	58	-18	0	0	-2
527	14384	14384	-72	16	-10	-10	0
551	18632	18632	80	-16	8	8	4
575	23578	23578	-82	26	4	4	-2
599	30122	30122	94	-30	-10	-10	-6
623	37882	37882	-110	26	10	10	2
647	47948	47948	120	-28	2	2	4
671	59848	59848	-128	40	-14	-14	0
695	75120	75120	144	-40	12	12	-4
719	93184	93184	-164	40	4	4	0
743	116018	116018	182	-46	-16	-16	6
767	143142	143142	-194	54	12	12	-2
791	176978	176978	210	-62	2	2	-6
815	217108	217108	-244	60	-20	-20	0
839	266838	266838	266	-66	18	18	6
863	325716	325716	-280	84	6	6	0
887	397942	397942	314	-90	-20	-20	-6
911	483524	483524	-352	84	20	20	0
935	587726	587726	382	-98	2	2	6

C.6 Lambency 6+3

Table 60: McKay–Thompson series $H_{g,1}^{(6+3)} = H_{g,1}^{X}$ for $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	3C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
Γ_g	1|1	1|1′	2|1	2|1′	3|1	3|3	4|2	4|2′	5|1	5|1′	2|1′	2|4′	4|1′	6|1′	6|6′
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
23	10	10	2	2	-2	4	-2	-2	0	0	-2	6	2	-2	0
47	20	20	-4	-4	2	2	0	0	0	0	4	-4	0	-2	2
71	42	42	2	2	0	-6	-2	-2	2	2	-6	2	-2	0	2
95	62	62	-2	-2	-4	2	-2	-2	2	2	6	-10	-2	0	2
119	118	118	6	6	4	-2	-2	-2	-2	-2	-6	10	2	0	-2
143	170	170	-6	-6	2	-4	2	2	0	0	10	-6	2	-2	0
167	270	270	6	6	-6	6	2	2	0	0	-14	10	-2	-2	-2
191	400	400	-8	-8	4	4	-4	-4	0	0	12	-12	0	0	0
215	600	600	8	8	0	-6	0	0	0	0	-12	20	4	0	2
239	828	828	-12	-12	-6	6	0	0	-2	-2	20	-20	0	2	-2
263	1220	1220	12	12	8	2	0	0	0	0	-24	16	-4	0	-2
287	1670	1670	-10	-10	2	-10	-2	-2	0	0	22	-26	-2	-2	-2
311	2330	2330	18	18	-10	8	-2	-2	0	0	-26	30	2	-2	0
335	3162	3162	-22	-22	6	0	2	2	2	2	34	-30	2	-2	0
359	4316	4316	20	20	2	-10	0	0	-4	-4	-40	32	-4	2	2
383	5730	5730	-22	-22	-12	6	-2	-2	0	0	42	-46	-2	0	2
407	7710	7710	30	30	12	0	-2	-2	0	0	-42	54	6	0	0
431	10102	10102	-34	-34	4	-14	2	2	2	2	54	-50	2	0	-2
455	13312	13312	32	32	-14	16	0	0	2	2	-68	60	-4	-2	0
479	17298	17298	-38	-38	12	6	-2	-2	-2	-2	66	-70	-2	0	2
503	22500	22500	44	44	0	-18	0	0	0	0	-72	80	4	0	2
527	28860	28860	-52	-52	-18	12	4	4	0	0	92	-84	4	2	0
551	37162	37162	58	58	16	4	2	2	2	2	-102	90	-6	0	0
575	47262	47262	-58	-58	6	-18	-6	-6	2	2	106	-110	-2	-2	-2
599	60128	60128	72	72	-22	20	-4	-4	-2	-2	-116	132	8	-2	0
623	75900	75900	-84	-84	18	6	4	4	0	0	136	-136	0	-2	2
647	95740	95740	84	84	4	-26	0	0	0	0	-156	140	-8	0	2
671	119860	119860	-92	-92	-26	22	-4	-4	0	0	164	-172	-4	2	2
695	150062	150062	110	110	26	2	-2	-2	2	2	-178	190	6	2	-2
719	186576	186576	-120	-120	6	-30	4	4	-4	-4	208	-200	4	-2	-2
743	231800	231800	128	128	-34	26	4	4	0	0	-236	220	-8	-2	-2
767	286530	286530	-142	-142	24	6	-6	-6	0	0	246	-250	-2	0	2
791	353694	353694	158	158	6	-36	-2	-2	4	4	-262	282	10	2	0
815	434524	434524	-180	-180	-38	28	4	4	4	4	308	-300	4	2	0
839	533334	533334	190	190	36	6	2	2	-6	-6	-342	322	-10	0	-2
863	651790	651790	-202	-202	10	-44	-6	-6	0	0	358	-370	-6	-2	-4
887	795490	795490	234	234	-44	40	-2	-2	0	0	-394	414	10	-4	0
911	967490	967490	-262	-262	38	14	6	6	0	0	442	-430	6	-2	2
935	1174962	1174962	274	274	6	-54	2	2	2	2	-490	470	-10	2	2

Table 61: McKay–Thompson series $H_{g,3}^{(6+3)} = H_{g,3}^{X}$ for $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	3C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
Γ_g	1|1	1|1′	2|1	2|1′	3|1	3|3	4|2	4|2′	5|1	5|1′	2|1′	2|4′	4|1′	6|1′	6|6′
15	12	-6	-4	2	0	0	4	-2	2	-1	0	0	0	0	0
39	36	-18	4	-2	0	0	4	-2	-4	2	0	0	0	0	0
63	60	-30	-4	2	0	0	-4	2	0	0	0	0	0	0	0
87	120	-60	8	-4	0	0	0	0	0	0	0	0	0	0	0
111	180	-90	-12	6	0	0	4	-2	0	0	0	0	0	0	0
135	312	-156	8	-4	0	0	0	0	2	-1	0	0	0	0	0
159	456	-228	-8	4	0	0	0	0	-4	2	0	0	0	0	0
183	720	-360	16	-8	0	0	0	0	0	0	0	0	0	0	0
207	1020	-510	-20	10	0	0	4	-2	0	0	0	0	0	0	0
231	1524	-762	20	-10	0	0	4	-2	4	-2	0	0	0	0	0
255	2124	-1062	-20	10	0	0	-4	2	4	-2	0	0	0	0	0
279	3036	-1518	28	-14	0	0	-4	2	-4	2	0	0	0	0	0
303	4140	-2070	-36	18	0	0	4	-2	0	0	0	0	0	0	0
327	5760	-2880	32	-16	0	0	0	0	0	0	0	0	0	0	0
351	7740	-3870	-36	18	0	0	-4	2	0	0	0	0	0	0	0
375	10512	-5256	48	-24	0	0	0	0	2	-1	0	0	0	0	0
399	13896	-6948	-56	28	0	0	8	-4	-4	2	0	0	0	0	0
423	18540	-9270	60	-30	0	0	4	-2	0	0	0	0	0	0	0
447	24240	-12120	-64	32	0	0	-8	4	0	0	0	0	0	0	0
471	31824	-15912	80	-40	0	0	0	0	4	-2	0	0	0	0	0
495	41124	-20562	-92	46	0	0	4	-2	4	-2	0	0	0	0	0
519	53292	-26646	92	-46	0	0	4	-2	-8	4	0	0	0	0	0
543	68220	-34110	-100	50	0	0	-4	2	0	0	0	0	0	0	0
567	87420	-43710	124	-62	0	0	-4	2	0	0	0	0	0	0	0
591	110880	-55440	-144	72	0	0	8	-4	0	0	0	0	0	0	0
615	140724	-70362	148	-74	0	0	4	-2	4	-2	0	0	0	0	0
639	177072	-88536	-160	80	0	0	-8	4	-8	4	0	0	0	0	0
663	222780	-111390	188	-94	0	0	-4	2	0	0	0	0	0	0	0
687	278280	-139140	-216	108	0	0	8	-4	0	0	0	0	0	0	0
711	347424	-173712	224	-112	0	0	0	0	4	-2	0	0	0	0	0
735	431136	-215568	-240	120	0	0	-8	4	6	-3	0	0	0	0	0
759	534492	-267246	284	-142	0	0	-4	2	-8	4	0	0	0	0	0
783	659220	-329610	-316	158	0	0	12	-6	0	0	0	0	0	0	0
807	812160	-406080	336	-168	0	0	8	-4	0	0	0	0	0	0	0
831	996084	-498042	-364	182	0	0	-12	6	4	-2	0	0	0	0	0
855	1220124	-610062	412	-206	0	0	-4	2	4	-2	0	0	0	0	0
879	1488612	-744306	-460	230	0	0	12	-6	-8	4	0	0	0	0	0
903	1813860	-906930	484	-242	0	0	4	-2	0	0	0	0	0	0	0
927	2202420	-1101210	-524	262	0	0	-12	6	0	0	0	0	0	0	0
951	2670564	-1335282	596	-298	0	0	-4	2	4	-2	0	0	0	0	0

C.7 Lambency 7

Table 62: $H^{(7)}_{g,1}$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
-1	-2	-2	-2	-2	-2
27	4	4	4	1	1
55	6	6	-2	0	0
83	10	10	2	-2	-2
111	20	20	-4	2	2
139	30	30	6	0	0
167	42	42	-6	0	0
195	68	68	4	2	2
223	96	96	-8	0	0
251	130	130	10	-2	-2
279	188	188	-12	2	2
307	258	258	10	0	0
335	350	350	-10	-4	-4
363	474	474	18	3	3
391	624	624	-16	0	0
419	826	826	18	-2	-2
447	1090	1090	-22	4	4
475	1410	1410	26	0	0
503	1814	1814	-26	-4	-4
531	2338	2338	26	4	4
559	2982	2982	-34	0	0
587	3774	3774	38	-6	-6
615	4774	4774	-42	4	4
643	5994	5994	42	0	0
671	7494	7494	-50	-6	-6
699	9348	9348	60	6	6
727	11586	11586	-62	0	0
755	14320	14320	64	-8	-8
783	17654	17654	-74	8	8
811	21654	21654	86	0	0
839	26488	26488	-88	-8	-8
867	32334	32334	94	9	9
895	39324	39324	-108	0	0
923	47680	47680	120	-8	-8
951	57688	57688	-128	10	10
979	69600	69600	136	0	0
1007	83760	83760	-152	-12	-12
1035	100596	100596	172	12	12

Table 63: $H^{(7)}_{g,2}$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
24	4	-4	0	-2	2
52	12	-12	0	0	0
80	20	-20	0	2	-2
108	32	-32	0	-1	1
136	48	-48	0	0	0
164	80	-80	0	2	-2
192	108	-108	0	-3	3
220	168	-168	0	0	0
248	232	-232	0	4	-4
276	328	-328	0	-2	2
304	444	-444	0	0	0
332	620	-620	0	2	-2
360	812	-812	0	-4	4
388	1104	-1104	0	0	0
416	1444	-1444	0	4	-4
444	1904	-1904	0	-4	4
472	2460	-2460	0	0	0
500	3208	-3208	0	4	-4
528	4080	-4080	0	-6	6
556	5244	-5244	0	0	0
584	6632	-6632	0	8	-8
612	8400	-8400	0	-6	6
640	10524	-10524	0	0	0
668	13224	-13224	0	6	-6
696	16408	-16408	0	-8	8
724	20436	-20436	0	0	0
752	25216	-25216	0	10	-10
780	31120	-31120	0	-8	8
808	38148	-38148	0	0	0
836	46784	-46784	0	8	-8
864	56976	-56976	0	-12	12
892	69432	-69432	0	0	0
920	84144	-84144	0	12	-12
948	101904	-101904	0	-12	12
976	122868	-122868	0	0	0
1004	148076	-148076	0	14	-14
1032	177656	-177656	0	-16	16
1060	213072	-213072	0	0	0

Table 64: $H^{(7)}_{g,3}$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
19	6	6	-2	0	0
47	12	12	4	0	0
75	22	22	-2	1	1
103	36	36	4	0	0
131	58	58	-6	-2	-2
159	90	90	2	0	0
187	132	132	-4	0	0
215	190	190	6	-2	-2
243	274	274	-6	1	1
271	384	384	8	0	0
299	528	528	-8	0	0
327	722	722	10	2	2
355	972	972	-12	0	0
383	1300	1300	12	-2	-2
411	1724	1724	-12	2	2
439	2256	2256	16	0	0
467	2938	2938	-22	-2	-2
495	3806	3806	22	2	2
523	4890	4890	-22	0	0
551	6244	6244	28	-2	-2
579	7940	7940	-28	2	2
607	10038	10038	30	0	0
635	12620	12620	-36	-4	-4
663	15814	15814	38	4	4
691	19722	19722	-46	0	0
719	24490	24490	50	-2	-2
747	30310	30310	-50	4	4
775	37362	37362	58	0	0
803	45908	45908	-68	-4	-4
831	56236	56236	68	4	4
859	68646	68646	-74	0	0
887	83556	83556	84	-6	-6
915	101436	101436	-92	6	6
943	122790	122790	102	0	0
971	148254	148254	-106	-6	-6
999	178566	178566	118	6	6
1027	214548	214548	-132	0	0
1055	257190	257190	142	-6	-6

Table 65: $H^{(7)}_{g,4}$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
12	4	-4	0	1	-1
40	12	-12	0	0	0
68	16	-16	0	-2	2
96	36	-36	0	0	0
124	48	-48	0	0	0
152	84	-84	0	0	0
180	116	-116	0	2	-2
208	180	-180	0	0	0
236	244	-244	0	-2	2
264	360	-360	0	0	0
292	480	-480	0	0	0
320	676	-676	0	-2	2
348	896	-896	0	2	-2
376	1224	-1224	0	0	0
404	1588	-1588	0	-2	2
432	2128	-2128	0	1	-1
460	2736	-2736	0	0	0
488	3588	-3588	0	0	0
516	4576	-4576	0	4	-4
544	5904	-5904	0	0	0
572	7448	-7448	0	-4	4
600	9500	-9500	0	2	-2
628	11892	-11892	0	0	0
656	14992	-14992	0	-2	2
684	18628	-18628	0	4	-4
712	23256	-23256	0	0	0
740	28688	-28688	0	-4	4
768	35532	-35532	0	3	-3
796	43560	-43560	0	0	0
824	53528	-53528	0	-4	4
852	65256	-65256	0	6	-6
880	79656	-79656	0	0	0
908	96564	-96564	0	-6	6
936	117196	-117196	0	4	-4
964	141360	-141360	0	0	0
992	170600	-170600	0	-4	4
1020	204848	-204848	0	8	-8
1048	245988	-245988	0	0	0

Table 66: $H^{(7)}_{g,5}$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
3	2	2	2	-1	-1
31	6	6	-2	0	0
59	14	14	-2	2	2
87	22	22	-2	-2	-2
115	36	36	4	0	0
143	56	56	0	2	2
171	82	82	2	-2	-2
199	126	126	-2	0	0
227	182	182	6	2	2
255	250	250	-6	-2	-2
283	354	354	2	0	0
311	490	490	-6	4	4
339	656	656	8	-4	-4
367	882	882	-6	0	0
395	1180	1180	4	4	4
423	1550	1550	-10	-4	-4
451	2028	2028	12	0	0
479	2638	2638	-10	4	4
507	3394	3394	10	-5	-5
535	4362	4362	-14	0	0
563	5562	5562	18	6	6
591	7032	7032	-16	-6	-6
619	8886	8886	14	0	0
647	11166	11166	-18	6	6
675	13940	13940	28	-7	-7
703	17358	17358	-26	0	0
731	21536	21536	24	8	8
759	26594	26594	-30	-10	-10
787	32742	32742	38	0	0
815	40180	40180	-36	10	10
843	49124	49124	36	-10	-10
871	59916	59916	-44	0	0
899	72852	72852	52	12	12
927	88296	88296	-56	-12	-12
955	106788	106788	52	0	0
983	128816	128816	-64	14	14
1011	154948	154948	76	-14	-14

Table 67: $H^{(7)}_{g,6}$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
Γ_g	1	1|4	2|8	3	3|4
20	4	-4	0	-2	2
48	4	-4	0	1	-1
76	12	-12	0	0	0
104	12	-12	0	0	0
132	32	-32	0	2	-2
160	36	-36	0	0	0
188	64	-64	0	-2	2
216	80	-80	0	2	-2
244	132	-132	0	0	0
272	160	-160	0	-2	2
300	252	-252	0	3	-3
328	312	-312	0	0	0
356	448	-448	0	-2	2
384	572	-572	0	2	-2
412	792	-792	0	0	0
440	992	-992	0	-4	4
468	1348	-1348	0	4	-4
496	1680	-1680	0	0	0
524	2220	-2220	0	-6	6
552	2776	-2776	0	4	-4
580	3600	-3600	0	0	0
608	4460	-4460	0	-4	4
636	5712	-5712	0	6	-6
664	7044	-7044	0	0	0
692	8892	-8892	0	-6	6
720	10932	-10932	0	6	-6
748	13656	-13656	0	0	0
776	16672	-16672	0	-8	8
804	20672	-20672	0	8	-8
832	25116	-25116	0	0	0
860	30856	-30856	0	-8	8
888	37352	-37352	0	8	-8
916	45564	-45564	0	0	0
944	54884	-54884	0	-10	10
972	66572	-66572	0	11	-11
1000	79848	-79848	0	0	0
1028	96256	-96256	0	-14	14

C.8 Lambency 8

Table 68: $H^{(8)}_{g,1}$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2BC	4A
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$
-1	-2	-2	-2	-2
31	2	2	-2	2
63	4	4	0	-4
95	8	8	0	0
127	12	12	0	-4
159	14	14	-2	6
191	28	28	0	-4
223	36	36	0	4
255	50	50	2	-6
287	70	70	-2	6
319	96	96	0	-8
351	126	126	-2	6
383	174	174	2	-10
415	220	220	-4	12
447	292	292	4	-12
479	380	380	0	12
511	490	490	2	-14
543	610	610	-6	18
575	790	790	2	-18
607	986	986	-2	18
639	1240	1240	4	-24
671	1546	1546	-6	26
703	1926	1926	6	-26
735	2364	2364	-4	28
767	2934	2934	6	-34
799	3574	3574	-10	38
831	4384	4384	8	-40
863	5334	5334	-6	38
895	6482	6482	10	-46
927	7814	7814	-14	54
959	9466	9466	10	-54
991	11362	11362	-10	58
1023	13646	13646	14	-66
1055	16312	16312	-16	72
1087	19498	19498	14	-78
1119	23178	23178	-14	82
1151	27598	27598	18	-90
1183	32662	32662	-22	102
1215	38678	38678	22	-106
1247	45646	45646	-18	110

Table 69: $H^{(8)}_{g,2}$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2BC	4A
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$
28	4	-4	0	0
60	8	-8	0	0
92	12	-12	0	0
124	20	-20	0	0
156	32	-32	0	0
188	44	-44	0	0
220	64	-64	0	0
252	92	-92	0	0
284	124	-124	0	0
316	172	-172	0	0
348	232	-232	0	0
380	304	-304	0	0
412	404	-404	0	0
444	528	-528	0	0
476	680	-680	0	0
508	876	-876	0	0
540	1120	-1120	0	0
572	1416	-1416	0	0
604	1788	-1788	0	0
636	2248	-2248	0	0
668	2796	-2796	0	0
700	3476	-3476	0	0
732	4304	-4304	0	0
764	5292	-5292	0	0
796	6500	-6500	0	0
828	7948	-7948	0	0
860	9672	-9672	0	0
892	11748	-11748	0	0
924	14224	-14224	0	0
956	17156	-17156	0	0
988	20648	-20648	0	0
1020	24784	-24784	0	0
1052	29652	-29652	0	0
1084	35412	-35412	0	0
1116	42188	-42188	0	0
1148	50120	-50120	0	0
1180	59440	-59440	0	0
1212	70344	-70344	0	0
1244	83052	-83052	0	0
1276	97896	-97896	0	0

Table 70: $H^{(8)}_{g,3}$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2BC	4A
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$
23	4	4	0	-4
55	10	10	2	2
87	14	14	-2	-2
119	26	26	2	2
151	36	36	0	-4
183	58	58	2	2
215	76	76	-4	-4
247	116	116	4	4
279	154	154	-2	-6
311	216	216	4	8
343	282	282	-6	-6
375	390	390	6	6
407	500	500	-4	-12
439	666	666	6	10
471	848	848	-8	-8
503	1108	1108	8	12
535	1400	1400	-8	-16
567	1798	1798	10	14
599	2240	2240	-12	-16
631	2842	2842	14	18
663	3524	3524	-12	-20
695	4406	4406	14	22
727	5418	5418	-18	-22
759	6722	6722	18	26
791	8206	8206	-18	-34
823	10082	10082	22	34
855	12238	12238	-26	-34
887	14918	14918	26	38
919	18012	18012	-24	-44
951	21806	21806	30	46
983	26176	26176	-36	-48
1015	31510	31510	38	54
1047	37650	37650	-38	-62
1079	45050	45050	42	66
1111	53582	53582	-50	-66
1143	63816	63816	52	72
1175	75578	75578	-50	-86
1207	89584	89584	56	88
1239	105694	105694	-66	-90

Table 71: $H^{(8)}_{g,4}$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2BC	4A
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$
16	4	-4	0	0
48	8	-8	0	0
80	16	-16	0	0
112	24	-24	0	0
144	36	-36	0	0
176	56	-56	0	0
208	80	-80	0	0
240	112	-112	0	0
272	160	-160	0	0
304	216	-216	0	0
336	288	-288	0	0
368	392	-392	0	0
400	516	-516	0	0
432	672	-672	0	0
464	880	-880	0	0
496	1128	-1128	0	0
528	1440	-1440	0	0
560	1840	-1840	0	0
592	2320	-2320	0	0
624	2912	-2912	0	0
656	3648	-3648	0	0
688	4536	-4536	0	0
720	5616	-5616	0	0
752	6936	-6936	0	0
784	8516	-8516	0	0
816	10416	-10416	0	0
848	12720	-12720	0	0
880	15456	-15456	0	0
912	18720	-18720	0	0
944	22632	-22632	0	0
976	27248	-27248	0	0
1008	32728	-32728	0	0
1040	39232	-39232	0	0
1072	46872	-46872	0	0
1104	55872	-55872	0	0
1136	66472	-66472	0	0
1168	78880	-78880	0	0
1200	93400	-93400	0	0
1232	110400	-110400	0	0

Table 72: $H^{(8)}_{g,5}$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2BC	4A
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$
7	2	2	-2	2
39	8	8	0	0
71	10	10	-2	2
103	22	22	2	-2
135	28	28	-4	4
167	48	48	4	0
199	64	64	-4	0
231	100	100	4	-4
263	126	126	-6	6
295	188	188	4	-4
327	244	244	-4	4
359	340	340	8	-4
391	430	430	-10	6
423	588	588	8	-4
455	742	742	-10	6
487	984	984	12	-8
519	1234	1234	-14	10
551	1598	1598	14	-10
583	1994	1994	-14	10
615	2546	2546	18	-14
647	3142	3142	-22	14
679	3964	3964	20	-12
711	4870	4870	-22	14
743	6064	6064	28	-16
775	7398	7398	-30	22
807	9132	9132	28	-20
839	11076	11076	-32	20
871	13558	13558	38	-26
903	16358	16358	-42	30
935	19868	19868	44	-28
967	23854	23854	-46	30
999	28788	28788	52	-36
1031	34384	34384	-60	40
1063	41256	41256	60	-40
1095	49090	49090	-62	42
1127	58566	58566	74	-50
1159	69382	69382	-82	54
1191	82394	82394	82	-54
1223	97234	97234	-90	58

Table 73: $H^{(8)}_{g,6}$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2BC	4A
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$
28	4	-4	0	0
60	8	-8	0	0
92	12	-12	0	0
124	20	-20	0	0
156	32	-32	0	0
188	44	-44	0	0
220	64	-64	0	0
252	92	-92	0	0
284	124	-124	0	0
316	172	-172	0	0
348	232	-232	0	0
380	304	-304	0	0
412	404	-404	0	0
444	528	-528	0	0
476	680	-680	0	0
508	876	-876	0	0
540	1120	-1120	0	0
572	1416	-1416	0	0
604	1788	-1788	0	0
636	2248	-2248	0	0
668	2796	-2796	0	0
700	3476	-3476	0	0
732	4304	-4304	0	0
764	5292	-5292	0	0
796	6500	-6500	0	0
828	7948	-7948	0	0
860	9672	-9672	0	0
892	11748	-11748	0	0
924	14224	-14224	0	0
956	17156	-17156	0	0
988	20648	-20648	0	0
1020	24784	-24784	0	0
1052	29652	-29652	0	0
1084	35412	-35412	0	0
1116	42188	-42188	0	0
1148	50120	-50120	0	0
1180	59440	-59440	0	0
1212	70344	-70344	0	0
1244	83052	-83052	0	0

Table 74: $H^{(8)}_{g,7}$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2BC	4A
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$
15	2	2	2	2
47	2	2	-2	2
79	8	8	4	0
111	6	6	-2	-2
143	18	18	2	2
175	16	16	-4	0
207	34	34	6	2
239	40	40	-4	0
271	66	66	6	2
303	72	72	-8	0
335	120	120	8	0
367	140	140	-8	-4
399	202	202	10	2
431	248	248	-12	0
463	346	346	14	2
495	412	412	-12	-4
527	566	566	14	6
559	678	678	-18	-2
591	900	900	20	4
623	1092	1092	-20	-4
655	1406	1406	22	6
687	1692	1692	-28	-4
719	2170	2170	30	2
751	2608	2608	-28	-8
783	3266	3266	34	10
815	3930	3930	-38	-6
847	4878	4878	42	6
879	5828	5828	-44	-12
911	7178	7178	46	10
943	8552	8552	-56	-8
975	10420	10420	60	12
1007	12404	12404	-60	-12
1039	14992	14992	68	16
1071	17756	17756	-76	-12
1103	21358	21358	82	14
1135	25212	25212	-84	-20
1167	30108	30108	92	20
1199	35464	35464	-104	-16
1231	42140	42140	112	20

C.9 Lambency 9

Table 75: $H^{(9)}_{g,1}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
-1	-2	-2	-2	-2	-2	-2
35	0	0	0	0	0	0
71	4	4	0	0	-2	-2
107	4	4	0	0	-2	-2
143	6	6	-2	-2	0	0
179	12	12	0	0	0	0
215	16	16	0	0	-2	-2
251	20	20	0	0	2	2
287	30	30	-2	-2	0	0
323	40	40	0	0	-2	-2
359	54	54	2	2	0	0
395	72	72	0	0	0	0
431	92	92	0	0	-4	-4
467	116	116	0	0	2	2
503	156	156	0	0	0	0
539	196	196	0	0	-2	-2
575	242	242	-2	-2	2	2
611	312	312	0	0	0	0
647	388	388	0	0	-2	-2
683	476	476	0	0	2	2
719	594	594	-2	-2	0	0
755	728	728	0	0	-4	-4
791	890	890	2	2	2	2
827	1092	1092	0	0	0	0
863	1322	1322	-2	-2	-4	-4
899	1600	1600	0	0	4	4
935	1938	1938	2	2	0	0
971	2324	2324	0	0	-4	-4
1007	2782	2782	-2	-2	4	4
1043	3336	3336	0	0	0	0
1079	3978	3978	2	2	-6	-6
1115	4720	4720	0	0	4	4
1151	5610	5610	-2	-2	0	0
1187	6636	6636	0	0	-6	-6
1223	7830	7830	2	2	6	6
1259	9236	9236	0	0	2	2
1295	10846	10846	-2	-2	-8	-8
1331	12720	12720	0	0	6	6

Table 76: $H^{(9)}_{g,2}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
32	4	-4	0	0	-2	2
68	6	-6	2	-2	0	0
104	8	-8	0	0	2	-2
140	16	-16	0	0	-2	2
176	18	-18	-2	2	0	0
212	30	-30	2	-2	0	0
248	40	-40	0	0	-2	2
284	54	-54	2	-2	0	0
320	74	-74	-2	2	2	-2
356	102	-102	2	-2	0	0
392	126	-126	-2	2	0	0
428	174	-174	2	-2	0	0
464	218	-218	-2	2	-4	4
500	284	-284	4	-4	2	-2
536	356	-356	-4	4	2	-2
572	460	-460	4	-4	-2	2
608	564	-564	-4	4	0	0
644	716	-716	4	-4	2	-2
680	884	-884	-4	4	-4	4
716	1098	-1098	6	-6	0	0
752	1342	-1342	-6	6	4	-4
788	1658	-1658	6	-6	-4	4
824	2000	-2000	-8	8	2	-2
860	2456	-2456	8	-8	2	-2
896	2960	-2960	-8	8	-4	4
932	3582	-3582	10	-10	0	0
968	4294	-4294	-10	10	4	-4
1004	5174	-5174	10	-10	-4	4
1040	6156	-6156	-12	12	0	0
1076	7378	-7378	14	-14	4	-4
1112	8748	-8748	-12	12	-6	6
1148	10400	-10400	16	-16	2	-2
1184	12280	-12280	-16	16	4	-4
1220	14544	-14544	16	-16	-6	6
1256	17084	-17084	-20	20	2	-2
1292	20140	-20140	20	-20	4	-4
1328	23590	-23590	-22	22	-8	8
1364	27656	-27656	24	-24	2	-2

Table 77: $H^{(9)}_{g,3}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
27	4	4	0	0	0	0
63	6	6	-2	-2	0	0
99	12	12	0	0	0	0
135	18	18	2	2	0	0
171	24	24	0	0	0	0
207	36	36	0	0	0	0
243	52	52	0	0	0	0
279	72	72	0	0	0	0
315	96	96	0	0	0	0
351	126	126	-2	-2	0	0
387	168	168	0	0	0	0
423	222	222	2	2	0	0
459	288	288	0	0	0	0
495	366	366	-2	-2	0	0
531	468	468	0	0	0	0
567	594	594	2	2	0	0
603	744	744	0	0	0	0
639	930	930	-2	-2	0	0
675	1156	1156	0	0	0	0
711	1434	1434	2	2	0	0
747	1764	1764	0	0	0	0
783	2160	2160	0	0	0	0
819	2640	2640	0	0	0	0
855	3210	3210	2	2	0	0
891	3888	3888	0	0	0	0
927	4692	4692	-4	-4	0	0
963	5652	5652	0	0	0	0
999	6786	6786	2	2	0	0
1035	8112	8112	0	0	0	0
1071	9678	9678	-2	-2	0	0
1107	11520	11520	0	0	0	0
1143	13674	13674	2	2	0	0
1179	16188	16188	0	0	0	0
1215	19116	19116	-4	-4	0	0
1251	22536	22536	0	0	0	0
1287	26508	26508	4	4	0	0
1323	31108	31108	0	0	0	0
1359	36438	36438	-2	-2	0	0

Table 78: $H^{(9)}_{g,4}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
20	2	-2	-2	2	2	-2
56	8	-8	0	0	2	-2
92	10	-10	-2	2	-2	2
128	18	-18	2	-2	0	0
164	26	-26	-2	2	2	-2
200	42	-42	2	-2	0	0
236	50	-50	-2	2	2	-2
272	78	-78	2	-2	0	0
308	100	-100	-4	4	-2	2
344	140	-140	4	-4	2	-2
380	180	-180	-4	4	0	0
416	244	-244	4	-4	-2	2
452	302	-302	-6	6	2	-2
488	404	-404	4	-4	2	-2
524	502	-502	-6	6	-2	2
560	648	-648	8	-8	0	0
596	806	-806	-6	6	2	-2
632	1024	-1024	8	-8	-2	2
668	1250	-1250	-10	10	2	-2
704	1574	-1574	10	-10	2	-2
740	1916	-1916	-12	12	-4	4
776	2372	-2372	12	-12	2	-2
812	2876	-2876	-12	12	2	-2
848	3530	-3530	14	-14	-4	4
884	4240	-4240	-16	16	4	-4
920	5168	-5168	16	-16	2	-2
956	6186	-6186	-18	18	-6	6
992	7460	-7460	20	-20	2	-2
1028	8894	-8894	-22	22	2	-2
1064	10664	-10664	24	-24	-4	4
1100	12634	-12634	-26	26	4	-4
1136	15070	-15070	26	-26	4	-4
1172	17790	-17790	-30	30	-6	6
1208	21080	-21080	32	-32	2	-2
1244	24794	-24794	-34	34	2	-2
1280	29250	-29250	38	-38	-6	6
1316	34248	-34248	-40	40	6	-6
1352	40234	-40234	42	-42	4	-4

Table 79: $H^{(9)}_{g,5}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
11	4	4	0	0	-2	-2
47	6	6	2	2	0	0
83	12	12	0	0	0	0
119	16	16	0	0	-2	-2
155	24	24	0	0	0	0
191	36	36	0	0	0	0
227	52	52	0	0	-2	-2
263	68	68	0	0	2	2
299	96	96	0	0	0	0
335	130	130	2	2	-2	-2
371	168	168	0	0	0	0
407	224	224	0	0	2	2
443	292	292	0	0	-2	-2
479	372	372	0	0	0	0
515	480	480	0	0	0	0
551	608	608	0	0	-4	-4
587	764	764	0	0	2	2
623	962	962	2	2	2	2
659	1196	1196	0	0	-4	-4
695	1478	1478	-2	-2	2	2
731	1832	1832	0	0	2	2
767	2248	2248	0	0	-2	-2
803	2744	2744	0	0	2	2
839	3348	3348	0	0	0	0
875	4064	4064	0	0	-4	-4
911	4910	4910	2	2	2	2
947	5924	5924	0	0	2	2
983	7116	7116	0	0	-6	-6
1019	8516	8516	0	0	2	2
1055	10184	10184	0	0	2	2
1091	12132	12132	0	0	-6	-6
1127	14404	14404	0	0	4	4
1163	17084	17084	0	0	2	2
1199	20202	20202	2	2	-6	-6
1235	23824	23824	0	0	4	4
1271	28054	28054	-2	-2	4	4
1307	32956	32956	0	0	-8	-8
1343	38626	38626	2	2	4	4

Table 80: $H^{(9)}_{g,6}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
0	1	-1	-1	1	0	0
36	6	-6	2	-2	0	0
72	6	-6	-2	2	0	0
108	14	-14	2	-2	0	0
144	18	-18	-2	2	0	0
180	30	-30	2	-2	0	0
216	36	-36	-4	4	0	0
252	60	-60	4	-4	0	0
288	72	-72	-4	4	0	0
324	108	-108	4	-4	0	0
360	132	-132	-4	4	0	0
396	186	-186	6	-6	0	0
432	230	-230	-6	6	0	0
468	312	-312	8	-8	0	0
504	384	-384	-8	8	0	0
540	504	-504	8	-8	0	0
576	618	-618	-10	10	0	0
612	798	-798	10	-10	0	0
648	972	-972	-12	12	0	0
684	1236	-1236	12	-12	0	0
720	1494	-1494	-14	14	0	0
756	1872	-1872	16	-16	0	0
792	2256	-2256	-16	16	0	0
828	2790	-2790	18	-18	0	0
864	3348	-3348	-20	20	0	0
900	4098	-4098	22	-22	0	0
936	4896	-4896	-24	24	0	0
972	5942	-5942	26	-26	0	0
1008	7068	-7068	-28	28	0	0
1044	8514	-8514	30	-30	0	0
1080	10080	-10080	-32	32	0	0
1116	12060	-12060	36	-36	0	0
1152	14226	-14226	-38	38	0	0
1188	16920	-16920	40	-40	0	0
1224	19884	-19884	-44	44	0	0
1260	23520	-23520	48	-48	0	0
1296	27540	-27540	-52	52	0	0
1332	32424	-32424	56	-56	0	0

Table 81: $H^{(9)}_{g,7}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
23	2	2	-2	-2	2	2
59	4	4	0	0	-2	-2
95	8	8	0	0	2	2
131	12	12	0	0	0	0
167	18	18	-2	-2	0	0
203	24	24	0	0	0	0
239	38	38	2	2	2	2
275	52	52	0	0	-2	-2
311	66	66	-2	-2	0	0
347	92	92	0	0	2	2
383	124	124	0	0	-2	-2
419	156	156	0	0	0	0
455	206	206	-2	-2	2	2
491	268	268	0	0	-2	-2
527	338	338	2	2	2	2
563	428	428	0	0	2	2
599	538	538	-2	-2	-2	-2
635	672	672	0	0	0	0
671	842	842	2	2	2	2
707	1040	1040	0	0	-4	-4
743	1274	1274	-2	-2	2	2
779	1568	1568	0	0	2	2
815	1922	1922	2	2	-4	-4
851	2328	2328	0	0	0	0
887	2824	2824	-4	-4	4	4
923	3416	3416	0	0	-4	-4
959	4106	4106	2	2	2	2
995	4936	4936	0	0	4	4
1031	5906	5906	-2	-2	-4	-4
1067	7040	7040	0	0	2	2
1103	8392	8392	4	4	4	4
1139	9960	9960	0	0	-6	-6
1175	11784	11784	-4	-4	0	0
1211	13936	13936	0	0	4	4
1247	16434	16434	2	2	-6	-6
1283	19316	19316	0	0	2	2
1319	22680	22680	-4	-4	6	6

Table 82: $H^{(9)}_{g,8}$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
Γ_g	1	1|4	2	2|2	3|3	3|12
8	2	-2	2	-2	2	-2
44	2	-2	-2	2	2	-2
80	6	-6	2	-2	0	0
116	2	-2	-2	2	2	-2
152	12	-12	4	-4	0	0
188	10	-10	-2	2	-2	2
224	20	-20	4	-4	2	-2
260	20	-20	-4	4	2	-2
296	36	-36	4	-4	0	0
332	38	-38	-6	6	2	-2
368	66	-66	6	-6	0	0
404	70	-70	-6	6	-2	2
440	104	-104	8	-8	2	-2
476	120	-120	-8	8	0	0
512	172	-172	8	-8	-2	2
548	194	-194	-10	10	2	-2
584	276	-276	12	-12	0	0
620	316	-316	-12	12	-2	2
656	418	-418	14	-14	4	-4
692	494	-494	-14	14	2	-2
728	640	-640	16	-16	-2	2
764	746	-746	-18	18	2	-2
800	960	-960	20	-20	0	0
836	1124	-1124	-20	20	-4	4
872	1408	-1408	24	-24	4	-4
908	1658	-1658	-26	26	2	-2
944	2054	-2054	26	-26	-4	4
980	2398	-2398	-30	30	4	-4
1016	2952	-2952	32	-32	0	0
1052	3450	-3450	-34	34	-6	6
1088	4186	-4186	38	-38	4	-4
1124	4898	-4898	-42	42	2	-2
1160	5892	-5892	44	-44	-6	6
1196	6864	-6864	-48	48	6	-6
1232	8220	-8220	52	-52	0	0
1268	9558	-9558	-54	54	-6	6
1304	11348	-11348	60	-60	8	-8

C.10 Lambency 10

Table 83: $H^{(10)}_{g,1}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	$1\|1$	$1\|1'$	$2\|2'$
-1	-2	-2	-2
39	2	2	2
79	2	2	-2
119	2	2	2
159	6	6	-2
199	8	8	4
239	10	10	-2
279	14	14	2
319	20	20	-4
359	24	24	4
399	34	34	-6
439	40	40	4
479	54	54	-6
519	72	72	8
559	90	90	-6
599	106	106	6
639	140	140	-8
679	170	170	10
719	210	210	-10
759	258	258	10
799	316	316	-12
839	378	378	14
879	466	466	-14
919	554	554	14
959	672	672	-16
999	804	804	20
1039	960	960	-20
1079	1132	1132	20
1119	1362	1362	-22
1159	1602	1602	26
1199	1894	1894	-26
1239	2226	2226	26
1279	2622	2622	-30
1319	3062	3062	34
1359	3600	3600	-36
1399	4184	4184	36
1439	4884	4884	-40
1479	5678	5678	46
1519	6598	6598	-46
1559	7618	7618	46

Table 84: $H^{(10)}_{g,2}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	$1\|1$	$1\|1'$	$2\|2'$
36	2	-2	0
76	4	-4	0
116	8	-8	0
156	8	-8	0
196	14	-14	0
236	20	-20	0
276	24	-24	0
316	36	-36	0
356	48	-48	0
396	60	-60	0
436	80	-80	0
476	104	-104	0
516	128	-128	0
556	164	-164	0
596	208	-208	0
636	256	-256	0
676	318	-318	0
716	396	-396	0
756	480	-480	0
796	588	-588	0
836	720	-720	0
876	864	-864	0
916	1048	-1048	0
956	1260	-1260	0
996	1504	-1504	0
1036	1800	-1800	0
1076	2144	-2144	0
1116	2540	-2540	0
1156	3010	-3010	0
1196	3560	-3560	0
1236	4184	-4184	0
1276	4920	-4920	0
1316	5776	-5776	0
1356	6744	-6744	0
1396	7880	-7880	0
1436	9188	-9188	0
1476	10672	-10672	0
1516	12396	-12396	0
1556	14368	-14368	0
1596	16608	-16608	0

Table 85: $H^{(10)}_{g,3}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	$1\|1$	$1\|1'$	$2\|2'$
31	2	2	-2
71	6	6	2
111	8	8	0
151	14	14	2
191	16	16	-4
231	26	26	2
271	34	34	-2
311	48	48	4
351	60	60	-4
391	84	84	4
431	104	104	-4
471	140	140	4
511	170	170	-6
551	222	222	6
591	274	274	-6
631	348	348	8
671	422	422	-10
711	532	532	8
751	644	644	-8
791	796	796	12
831	956	956	-12
871	1172	1172	12
911	1400	1400	-12
951	1702	1702	14
991	2018	2018	-18
1031	2430	2430	18
1071	2880	2880	-16
1111	3436	3436	20
1151	4044	4044	-24
1191	4806	4806	22
1231	5636	5636	-24
1271	6652	6652	28
1311	7770	7770	-30
1351	9128	9128	32
1391	10624	10624	-32
1431	12420	12420	36
1471	14404	14404	-40
1511	16764	16764	40
1551	19390	19390	-42
1591	22480	22480	48

Table 86: $H^{(10)}_{g,4}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	1|1	1|1′	2|2′
24	4	-4	0
64	6	-6	0
104	8	-8	0
144	14	-14	0
184	20	-20	0
224	28	-28	0
264	40	-40	0
304	52	-52	0
344	68	-68	0
384	92	-92	0
424	120	-120	0
464	152	-152	0
504	196	-196	0
544	248	-248	0
584	312	-312	0
624	392	-392	0
664	484	-484	0
704	596	-596	0
744	736	-736	0
784	898	-898	0
824	1092	-1092	0
864	1328	-1328	0
904	1600	-1600	0
944	1924	-1924	0
984	2312	-2312	0
1024	2762	-2762	0
1064	3288	-3288	0
1104	3912	-3912	0
1144	4632	-4632	0
1184	5472	-5472	0
1224	6456	-6456	0
1264	7588	-7588	0
1304	8900	-8900	0
1344	10424	-10424	0
1384	12176	-12176	0
1424	14192	-14192	0
1464	16520	-16520	0
1504	19180	-19180	0
1544	22232	-22232	0
1584	25740	-25740	0

Table 87: $H^{(10)}_{g,5}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	1|1	1|1′	2|2′
15	2	2	2
55	6	6	-2
95	8	8	0
135	14	14	-2
175	18	18	2
215	30	30	-2
255	34	34	2
295	54	54	-2
335	68	68	4
375	92	92	-4
415	114	114	2
455	158	158	-2
495	190	190	6
535	252	252	-4
575	308	308	4
615	394	394	-6
655	480	480	8
695	608	608	-8
735	726	726	6
775	912	912	-8
815	1096	1096	8
855	1342	1342	-10
895	1602	1602	10
935	1958	1958	-10
975	2318	2318	14
1015	2802	2802	-14
1055	3316	3316	12
1095	3968	3968	-16
1135	4674	4674	18
1175	5566	5566	-18
1215	6514	6514	18
1255	7716	7716	-20
1295	9016	9016	24
1335	10598	10598	-26
1375	12336	12336	24
1415	14454	14454	-26
1455	16752	16752	32
1495	19536	19536	-32
1535	22592	22592	32
1575	26220	26220	-36

Table 88: $H^{(10)}_{g,6}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	1|1	1|1′	2|2′
4	2	-2	0
44	4	-4	0
84	8	-8	0
124	12	-12	0
164	16	-16	0
204	24	-24	0
244	32	-32	0
284	44	-44	0
324	62	-62	0
364	80	-80	0
404	104	-104	0
444	136	-136	0
484	174	-174	0
524	220	-220	0
564	280	-280	0
604	348	-348	0
644	432	-432	0
684	540	-540	0
724	664	-664	0
764	812	-812	0
804	992	-992	0
844	1204	-1204	0
884	1456	-1456	0
924	1760	-1760	0
964	2112	-2112	0
1004	2524	-2524	0
1044	3016	-3016	0
1084	3588	-3588	0
1124	4256	-4256	0
1164	5040	-5040	0
1204	5944	-5944	0
1244	6996	-6996	0
1284	8224	-8224	0
1324	9636	-9636	0
1364	11264	-11264	0
1404	13152	-13152	0
1444	15314	-15314	0
1484	17800	-17800	0
1524	20664	-20664	0
1564	23928	-23928	0

Table 89: $H^{(10)}_{g,7}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	$1\|1$	$1\|1'$	$2\|2'$
31	4	4	0
71	4	4	0
111	10	10	2
151	10	10	-2
191	20	20	0
231	24	24	0
271	38	38	2
311	42	42	-2
351	66	66	2
391	78	78	-2
431	110	110	2
471	132	132	-4
511	178	178	2
551	214	214	-2
591	284	284	4
631	336	336	-4
671	436	436	4
711	520	520	-4
751	658	658	6
791	778	778	-6
831	972	972	4
871	1156	1156	-4
911	1420	1420	8
951	1678	1678	-10
991	2044	2044	8
1031	2404	2404	-8
1071	2906	2906	10
1111	3404	3404	-12
1151	4078	4078	10
1191	4774	4774	-10
1231	5674	5674	14
1271	6610	6610	-14
1311	7814	7814	14
1351	9082	9082	-14
1391	10674	10674	18
1431	12364	12364	-20
1471	14462	14462	18
1511	16704	16704	-20

Table 90: $H^{(10)}_{g,8}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	$1\|1$	$1\|1'$	$2\|2'$
16	2	-2	0
56	4	-4	0
96	4	-4	0
136	8	-8	0
176	12	-12	0
216	16	-16	0
256	22	-22	0
296	32	-32	0
336	40	-40	0
376	52	-52	0
416	72	-72	0
456	88	-88	0
496	116	-116	0
536	148	-148	0
576	182	-182	0
616	232	-232	0
656	288	-288	0
696	352	-352	0
736	436	-436	0
776	536	-536	0
816	648	-648	0
856	788	-788	0
896	956	-956	0
936	1144	-1144	0
976	1376	-1376	0
1016	1652	-1652	0
1056	1960	-1960	0
1096	2336	-2336	0
1136	2772	-2772	0
1176	3268	-3268	0
1216	3860	-3860	0
1256	4544	-4544	0
1296	5326	-5326	0
1336	6244	-6244	0
1376	7300	-7300	0
1416	8504	-8504	0
1456	9904	-9904	0
1496	11512	-11512	0

Table 91: $H^{(10)}_{g,9}$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
Γ_g	$1\|1$	$1\|1'$	$2\|2'$
39	0	0	0
79	4	4	0
119	2	2	2
159	8	8	0
199	4	4	0
239	12	12	0
279	12	12	0
319	22	22	-2
359	20	20	0
399	40	40	0
439	38	38	2
479	58	58	-2
519	64	64	0
559	96	96	0
599	102	102	2
639	146	146	-2
679	160	160	0
719	218	218	-2
759	250	250	2
799	326	326	-2
839	366	366	2
879	480	480	0
919	544	544	4
959	684	684	-4
999	786	786	2
1039	978	978	-2
1079	1118	1118	6
1119	1380	1380	-4
1159	1578	1578	2
1199	1916	1916	-4
1239	2204	2204	4
1279	2646	2646	-6
1319	3032	3032	4
1359	3632	3632	-4
1399	4156	4156	8
1439	4916	4916	-8
1479	5636	5636	4
1519	6638	6638	-6

C.11 Lambency 10+5

Table 92: $H_{g,1}^{(10+5)}$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
Γ_g	1|1	2|2	3|1	2|1′	4|4′
-1	-2	-2	-2	-2	-2
39	2	2	2	2	2
79	6	-2	0	-2	2
119	4	4	-2	0	0
159	14	-2	2	-2	-2
199	12	4	0	4	0
239	22	-2	-2	-2	2
279	26	2	2	2	-2
319	42	-6	0	-2	-2
359	44	4	-4	4	0
399	74	-6	2	-6	2
439	78	6	0	2	-2
479	112	-8	-2	-4	0
519	136	8	4	8	0
559	186	-6	0	-6	2
599	208	8	-2	4	0
639	286	-10	4	-6	-2
679	330	10	0	10	2
719	428	-12	-4	-8	4
759	508	12	4	8	0
799	642	-14	0	-10	-2
839	744	16	-6	12	0
879	946	-14	4	-14	2
919	1098	18	0	10	-2
959	1356	-20	-6	-12	-4
999	1590	22	6	18	2
1039	1938	-22	0	-18	2
1079	2250	26	-6	14	-2
1119	2742	-26	6	-18	-2
1159	3180	28	0	24	0
1199	3810	-30	-6	-22	2
1239	4430	30	8	22	-2
1279	5268	-36	0	-24	-4
1319	6094	38	-8	30	2
1359	7232	-40	8	-32	4
1399	8340	44	0	28	0
1439	9800	-48	-10	-32	-4
1479	11314	50	10	42	2
1519	13236	-52	0	-40	4
1559	15200	56	-10	36	0

Table 93: $H_{g,3}^{(10+5)}$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
Γ_g	1|1	2|2	3|1	2|1′	4|4′
31	6	-2	0	-2	2
71	10	2	-2	2	-2
111	18	2	0	-2	-2
151	24	0	0	4	0
191	36	-4	0	-4	0
231	50	2	2	2	2
271	72	0	0	-4	0
311	90	2	0	6	2
351	126	-2	0	-6	2
391	162	2	0	6	-2
431	214	-2	-2	-6	-2
471	272	0	2	8	0
511	348	-4	0	-8	0
551	436	4	-2	8	0
591	558	-2	0	-10	-2
631	684	4	0	12	0
671	858	-6	0	-14	2
711	1052	4	2	12	0
751	1302	-2	0	-14	-2
791	1574	6	-4	18	2
831	1928	-8	2	-16	0
871	2328	8	0	16	0
911	2820	-4	0	-20	0
951	3380	4	2	24	0
991	4062	-10	0	-26	2
1031	4834	10	-2	26	-2
1071	5786	-6	2	-26	-2
1111	6840	8	0	32	0
1151	8122	-14	-2	-34	2
1191	9580	12	4	32	0
1231	11310	-10	0	-38	-2
1271	13262	14	-4	42	2
1311	15584	-16	2	-44	4
1351	18210	18	0	46	-2
1391	21298	-14	-2	-50	-2
1431	24784	16	4	56	0
1471	28866	-22	0	-58	2
1511	33468	20	-6	60	0
1551	38846	-18	2	-66	-2
1591	44886	22	0	74	2

Table 94: $H_{g,5}^{(10+5)}$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
Γ_g	$1\|1$	$2\|2$	$3\|1$	$2\|1'$	$4\|4'$
15	4	4	-2	0	0
55	12	-4	0	0	0
95	16	0	4	0	0
135	28	-4	-2	0	0
175	36	4	0	0	0
215	60	-4	0	0	0
255	68	4	-4	0	0
295	108	-4	0	0	0
335	136	8	4	0	0
375	184	-8	-2	0	0
415	228	4	0	0	0
455	316	-4	4	0	0
495	380	12	-4	0	0
535	504	-8	0	0	0
575	616	8	4	0	0
615	788	-12	-4	0	0
655	960	16	0	0	0
695	1216	-16	4	0	0
735	1452	12	-6	0	0
775	1824	-16	0	0	0
815	2192	16	8	0	0
855	2684	-20	-4	0	0
895	3204	20	0	0	0
935	3916	-20	4	0	0
975	4636	28	-8	0	0
1015	5604	-28	0	0	0
1055	6632	24	8	0	0
1095	7936	-32	-8	0	0
1135	9348	36	0	0	0
1175	11132	-36	8	0	0
1215	13028	36	-10	0	0
1255	15432	-40	0	0	0
1295	18032	48	8	0	0
1335	21196	-52	-8	0	0
1375	24672	48	0	0	0
1415	28908	-52	12	0	0
1455	33504	64	-12	0	0
1495	39072	-64	0	0	0
1535	45184	64	16	0	0
1575	52440	-72	-12	0	0

C.12 Lambency 12

Table 95: $H_{1A}^{(12)}$, $X = A_{11}D_7E_6$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11
0	-2										
48	0	2	2	2	2	2	0	0	0	0	0
96	2	2	4	4	4	4	4	2	2	0	0
144	2	4	4	6	6	6	4	4	2	2	0
192	2	6	8	8	10	8	8	6	6	2	2
240	4	6	10	12	10	12	10	8	4	4	0
288	6	10	14	16	18	16	16	12	10	6	4
336	6	14	16	20	22	22	16	16	10	6	2
384	10	16	24	28	30	28	28	20	18	10	6
432	10	22	30	36	36	36	32	28	18	14	4
480	14	28	40	44	50	48	44	36	28	16	10
528	20	34	46	58	58	60	52	44	32	22	6
576	24	44	62	72	80	76	72	58	46	28	18
624	26	56	74	88	94	94	82	72	50	34	14
672	36	66	94	112	120	116	110	88	72	44	22
720	44	82	112	136	144	144	128	112	80	56	22
768	52	102	142	164	182	176	164	136	108	66	36
816	64	120	168	202	210	216	194	164	120	82	34
864	78	148	206	244	266	260	244	202	158	102	54
912	90	178	242	292	314	312	280	244	182	120	52
960	114	210	298	352	382	376	352	292	230	148	76
1008	130	254	348	420	448	450	410	352	262	178	80
1056	158	302	420	496	544	536	498	420	328	210	108
1104	188	354	490	592	632	636	580	496	376	254	110
1152	220	420	588	700	762	752	704	592	464	302	154
1200	254	496	686	820	884	888	810	700	528	354	164
1248	306	578	812	968	1050	1044	974	820	642	420	210
1296	356	680	940	1132	1220	1224	1124	968	736	496	230
1344	414	796	1108	1320	1442	1432	1334	1132	886	578	292
1392	482	922	1282	1544	1658	1668	1540	1320	1010	680	314
1440	564	1074	1502	1792	1954	1944	1814	1544	1204	796	398
1488	646	1248	1726	2076	2250	2258	2078	1792	1374	922	434
1536	758	1438	2014	2408	2620	2616	2442	2076	1626	1074	534

Table 96: $H_{2A}^{(12)}$, $X = A_{11}D_7E_6$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11
0	-2										
48	0	-2	2	-2	2	-2	0	0	0	0	0
96	2	-2	4	-4	4	-4	4	-2	2	0	0
144	2	-4	4	-6	6	-6	4	-4	2	-2	0
192	2	-6	8	-8	10	-8	8	-6	6	-2	2
240	4	-6	10	-12	10	-12	10	-8	4	-4	0
288	6	-10	14	-16	18	-16	16	-12	10	-6	4
336	6	-14	16	-20	22	-22	16	-16	10	-6	2
384	10	-16	24	-28	30	-28	28	-20	18	-10	6
432	10	-22	30	-36	36	-36	32	-28	18	-14	4
480	14	-28	40	-44	50	-48	44	-36	28	-16	10
528	20	-34	46	-58	58	-60	52	-44	32	-22	6
576	24	-44	62	-72	80	-76	72	-58	46	-28	18
624	26	-56	74	-88	94	-94	82	-72	50	-34	14
672	36	-66	94	-112	120	-116	110	-88	72	-44	22
720	44	-82	112	-136	144	-144	128	-112	80	-56	22
768	52	-102	142	-164	182	-176	164	-136	108	-66	36
816	64	-120	168	-202	210	-216	194	-164	120	-82	34
864	78	-148	206	-244	266	-260	244	-202	158	-102	54
912	90	-178	242	-292	314	-312	280	-244	182	-120	52
960	114	-210	298	-352	382	-376	352	-292	230	-148	76
1008	130	-254	348	-420	448	-450	410	-352	262	-178	80
1056	158	-302	420	-496	544	-536	498	-420	328	-210	108
1104	188	-354	490	-592	632	-636	580	-496	376	-254	110
1152	220	-420	588	-700	762	-752	704	-592	464	-302	154
1200	254	-496	686	-820	884	-888	810	-700	528	-354	164
1248	306	-578	812	-968	1050	-1044	974	-820	642	-420	210
1296	356	-680	940	-1132	1220	-1224	1124	-968	736	-496	230
1344	414	-796	1108	-1320	1442	-1432	1334	-1132	886	-578	292
1392	482	-922	1282	-1544	1658	-1668	1540	-1320	1010	-680	314
1440	564	-1074	1502	-1792	1954	-1944	1814	-1544	1204	-796	398
1488	646	-1248	1726	-2076	2250	-2258	2078	-1792	1374	-922	434
1536	758	-1438	2014	-2408	2620	-2616	2442	-2076	1626	-1074	534

C.13 Lambency 12+4

Table 97: McKay–Thompson series $H_{g,1}^{(12+4)} = H_{g,1}^{X}$ for $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$	$3\|1$	$3\|2'$	$4\|8'$
-1	-2	-2	-2	-2	-2	-2	-2
47	4	4	0	4	-2	-2	0
95	6	6	-2	-2	0	0	2
143	10	10	2	2	-2	-2	-2
191	12	12	0	-4	0	0	0
239	20	20	0	4	2	2	0
287	22	22	-2	-2	-2	-2	2
335	34	34	2	2	-2	-2	2
383	42	42	-2	-6	0	0	-2
431	54	54	2	6	0	0	2
479	66	66	-2	-6	0	0	2
527	92	92	4	4	2	2	0
575	106	106	-2	-6	-2	-2	-2
623	136	136	0	8	-2	-2	0
671	164	164	-4	-4	2	2	0
719	208	208	4	8	-2	-2	0
767	246	246	-2	-10	0	0	-2
815	308	308	4	12	2	2	0
863	358	358	-6	-10	-2	-2	2
911	442	442	6	10	-2	-2	-2
959	524	524	-4	-12	2	2	0
1007	628	628	4	12	-2	-2	0
1055	738	738	-6	-14	0	0	2
1103	892	892	8	12	4	4	0
1151	1030	1030	-6	-18	-2	-2	-2
1199	1228	1228	4	20	-2	-2	0
1247	1430	1430	-10	-18	2	2	2
1295	1690	1690	10	18	-2	-2	-2
1343	1954	1954	-6	-22	-2	-2	-2
1391	2296	2296	8	24	4	4	0
1439	2642	2642	-10	-22	-4	-4	2
1487	3088	3088	12	24	-2	-2	0
1535	3556	3556	-12	-28	4	4	-4
1583	4120	4120	12	32	-2	-2	0
1631	4722	4722	-14	-30	0	0	2
1679	5472	5472	16	32	6	6	0
1727	6242	6242	-14	-38	-4	-4	-2
1775	7184	7184	12	40	-4	-4	0
1823	8194	8194	-18	-38	4	4	2
1871	9392	9392	20	40	-4	-4	0
1919	10680	10680	-16	-48	0	0	-4

Table 98: McKay–Thompson series $H_{g,4}^{(12+4)} = H_{g,4}^{X}$ for $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$	$3\|1$	$3\|2'$	$4\|8'$
32	4	-4	0	0	-2	2	0
80	8	-8	0	0	2	-2	0
128	12	-12	0	0	0	0	0
176	16	-16	0	0	-2	2	0
224	24	-24	0	0	0	0	0
272	32	-32	0	0	2	-2	0
320	40	-40	0	0	-2	2	0
368	56	-56	0	0	2	-2	0
416	72	-72	0	0	0	0	0
464	88	-88	0	0	-2	2	0
512	116	-116	0	0	2	-2	0
560	144	-144	0	0	0	0	0
608	176	-176	0	0	-4	4	0
656	224	-224	0	0	2	-2	0
704	272	-272	0	0	2	-2	0
752	328	-328	0	0	-2	2	0
800	404	-404	0	0	2	-2	0
848	488	-488	0	0	2	-2	0
896	584	-584	0	0	-4	4	0
944	704	-704	0	0	2	-2	0
992	840	-840	0	0	0	0	0
1040	992	-992	0	0	-4	4	0
1088	1184	-1184	0	0	2	-2	0
1136	1400	-1400	0	0	2	-2	0
1184	1640	-1640	0	0	-4	4	0
1232	1936	-1936	0	0	4	-4	0
1280	2264	-2264	0	0	2	-2	0
1328	2640	-2640	0	0	-6	6	0
1376	3088	-3088	0	0	4	-4	0
1424	3584	-3584	0	0	2	-2	0
1472	4152	-4152	0	0	-6	6	0
1520	4816	-4816	0	0	4	-4	0
1568	5564	-5564	0	0	2	-2	0
1616	6408	-6408	0	0	-6	6	0
1664	7384	-7384	0	0	4	-4	0
1712	8480	-8480	0	0	2	-2	0
1760	9712	-9712	0	0	-8	8	0
1808	11136	-11136	0	0	6	-6	0
1856	12728	-12728	0	0	2	-2	0
1904	14512	-14512	0	0	-8	8	0

Table 99: McKay–Thompson series $H_{g,5}^{(12+4)} = H_{g,5}^{X}$ for $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
Γ_g	$1\|1$	$1\|2'$	$2\|1$	$2\|4'$	$3\|1$	$3\|2'$	$4\|8'$
23	2	2	-2	2	2	2	-2
71	6	6	2	-2	0	0	-2
119	6	6	-2	-2	0	0	2
167	14	14	2	-2	2	2	2
215	12	12	-4	4	0	0	0
263	24	24	4	0	0	0	0
311	26	26	-2	2	2	2	2
359	40	40	4	0	-2	-2	0
407	42	42	-6	2	0	0	-2
455	68	68	4	-4	2	2	0
503	72	72	-4	0	0	0	0
551	102	102	6	-2	0	0	2
599	116	116	-8	4	2	2	0
647	156	156	8	-4	0	0	0
695	178	178	-6	2	-2	-2	2
743	236	236	8	-4	2	2	0
791	262	262	-10	6	-2	-2	-2
839	342	342	10	-2	0	0	-2
887	394	394	-10	2	4	4	2
935	490	490	10	-6	-2	-2	2
983	558	558	-14	6	0	0	-2
1031	698	698	14	-6	2	2	-2
1079	796	796	-12	4	-2	-2	0
1127	972	972	16	-4	0	0	0
1175	1114	1114	-18	10	4	4	-2
1223	1342	1342	18	-10	-2	-2	-2
1271	1534	1534	-18	6	-2	-2	2
1319	1840	1840	20	-8	4	4	0
1367	2092	2092	-24	12	-2	-2	0
1415	2488	2488	24	-8	-2	-2	0
1463	2842	2842	-22	10	4	4	2
1511	3334	3334	26	-10	-2	-2	2
1559	3796	3796	-32	12	-2	-2	-4
1607	4450	4450	30	-14	4	4	-2
1655	5050	5050	-30	10	-2	-2	2
1703	5874	5874	34	-14	0	0	2
1751	6666	6666	-38	18	6	6	-2
1799	7712	7712	40	-16	-4	-4	0
1847	8734	8734	-38	14	-2	-2	2
1895	10068	10068	44	-20	6	6	0

C.14 Lambency 13

Table 100: $H^{(13)}_{g,1}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
-1	-2	-2	-2
51	2	2	2
103	2	2	-2
155	0	0	0
207	2	2	-2
259	2	2	2
311	4	4	0
363	6	6	2
415	6	6	-2
467	8	8	4
519	12	12	-4
571	14	14	2
623	14	14	-2
675	20	20	4
727	24	24	-4
779	28	28	4
831	36	36	-4
883	42	42	6
935	50	50	-6
987	62	62	6
1039	70	70	-6
1091	84	84	8
1143	102	102	-6
1195	118	118	6
1247	136	136	-8
1299	162	162	10
1351	190	190	-10
1403	216	216	8
1455	254	254	-10
1507	292	292	12
1559	336	336	-12
1611	392	392	12
1663	446	446	-14
1715	510	510	14
1767	592	592	-16
1819	672	672	16
1871	764	764	-16
1923	876	876	20

Table 101: $H^{(13)}_{g,2}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
48	0	0	0
100	2	-2	0
152	4	-4	0
204	4	-4	0
256	6	-6	0
308	8	-8	0
360	8	-8	0
412	12	-12	0
464	16	-16	0
516	20	-20	0
568	24	-24	0
620	32	-32	0
672	36	-36	0
724	48	-48	0
776	56	-56	0
828	68	-68	0
880	80	-80	0
932	100	-100	0
984	112	-112	0
1036	140	-140	0
1088	164	-164	0
1140	192	-192	0
1192	224	-224	0
1244	268	-268	0
1296	306	-306	0
1348	364	-364	0
1400	420	-420	0
1452	488	-488	0
1504	560	-560	0
1556	656	-656	0
1608	744	-744	0
1660	864	-864	0
1712	988	-988	0
1764	1134	-1134	0
1816	1292	-1292	0
1868	1484	-1484	0
1920	1676	-1676	0
1972	1920	-1920	0

Table 102: $H^{(13)}_{g,3}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
43	2	2	-2
95	2	2	2
147	4	4	0
199	6	6	2
251	8	8	-4
303	10	10	2
355	14	14	-2
407	18	18	2
459	22	22	-2
511	26	26	2
563	34	34	-2
615	44	44	4
667	52	52	-4
719	64	64	4
771	78	78	-2
823	96	96	4
875	114	114	-6
927	136	136	4
979	164	164	-4
1031	194	194	6
1083	230	230	-6
1135	270	270	6
1187	318	318	-6
1239	374	374	6
1291	434	434	-10
1343	506	506	10
1395	592	592	-8
1447	686	686	10
1499	792	792	-12
1551	914	914	10
1603	1054	1054	-10
1655	1214	1214	14
1707	1394	1394	-14
1759	1594	1594	14
1811	1822	1822	-14
1863	2084	2084	16
1915	2374	2374	-18
1967	2698	2698	18

Table 103: $H^{(13)}_{g,4}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
36	2	-2	0
88	4	-4	0
140	4	-4	0
192	8	-8	0
244	8	-8	0
296	12	-12	0
348	16	-16	0
400	22	-22	0
452	24	-24	0
504	36	-36	0
556	40	-40	0
608	52	-52	0
660	64	-64	0
712	80	-80	0
764	92	-92	0
816	116	-116	0
868	136	-136	0
920	168	-168	0
972	196	-196	0
1024	238	-238	0
1076	272	-272	0
1128	332	-332	0
1180	384	-384	0
1232	456	-456	0
1284	528	-528	0
1336	620	-620	0
1388	712	-712	0
1440	840	-840	0
1492	960	-960	0
1544	1120	-1120	0
1596	1280	-1280	0
1648	1484	-1484	0
1700	1688	-1688	0
1752	1952	-1952	0
1804	2216	-2216	0
1856	2544	-2544	0
1908	2888	-2888	0
1960	3304	-3304	0

Table 104: $H^{(13)}_{g,5}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
27	2	2	2
79	4	4	0
131	6	6	2
183	6	6	-2
235	10	10	2
287	14	14	-2
339	16	16	0
391	22	22	-2
443	30	30	2
495	36	36	-4
547	46	46	2
599	58	58	-2
651	68	68	4
703	86	86	-2
755	106	106	2
807	124	124	-4
859	152	152	4
911	184	184	-4
963	216	216	4
1015	258	258	-6
1067	308	308	4
1119	362	362	-6
1171	426	426	6
1223	502	502	-6
1275	584	584	8
1327	684	684	-8
1379	798	798	6
1431	920	920	-8
1483	1070	1070	10
1535	1238	1238	-10
1587	1422	1422	10
1639	1638	1638	-10
1691	1884	1884	12
1743	2156	2156	-12
1795	2468	2468	12
1847	2822	2822	-14
1899	3212	3212	16
1951	3660	3660	-16

Table 105: $H^{(13)}_{g,6}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
16	2	-2	0
68	4	-4	0
120	4	-4	0
172	8	-8	0
224	8	-8	0
276	16	-16	0
328	16	-16	0
380	24	-24	0
432	28	-28	0
484	38	-38	0
536	44	-44	0
588	60	-60	0
640	68	-68	0
692	88	-88	0
744	104	-104	0
796	132	-132	0
848	152	-152	0
900	190	-190	0
952	220	-220	0
1004	268	-268	0
1056	312	-312	0
1108	376	-376	0
1160	432	-432	0
1212	520	-520	0
1264	596	-596	0
1316	708	-708	0
1368	812	-812	0
1420	956	-956	0
1472	1092	-1092	0
1524	1280	-1280	0
1576	1460	-1460	0
1628	1696	-1696	0
1680	1932	-1932	0
1732	2236	-2236	0
1784	2536	-2536	0
1836	2924	-2924	0
1888	3308	-3308	0
1940	3792	-3792	0

Table 106: $H^{(13)}_{g,7}$, $X = A^2_{12}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
3	2	2	-2
55	2	2	2
107	4	4	0
159	8	8	0
211	10	10	-2
263	12	12	0
315	16	16	0
367	22	22	2
419	26	26	-2
471	34	34	2
523	44	44	0
575	54	54	2
627	68	68	-4
679	82	82	2
731	102	102	-2
783	124	124	4
835	148	148	-4
887	176	176	4
939	214	214	-2
991	256	256	4
1043	300	300	-4
1095	356	356	4
1147	420	420	-4
1199	494	494	6
1251	580	580	-8
1303	674	674	6
1355	786	786	-6
1407	918	918	6
1459	1060	1060	-8
1511	1226	1226	6
1563	1418	1418	-6
1615	1632	1632	8
1667	1874	1874	-10
1719	2150	2150	10
1771	2464	2464	-8
1823	2816	2816	12
1875	3214	3214	-14

Table 107: $H^{(13)}_{g,8}$, $X = A^2_{12}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
40	4	-4	0
92	4	-4	0
144	6	-6	0
196	6	-6	0
248	12	-12	0
300	12	-12	0
352	20	-20	0
404	24	-24	0
456	32	-32	0
508	36	-36	0
560	52	-52	0
612	56	-56	0
664	76	-76	0
716	88	-88	0
768	112	-112	0
820	128	-128	0
872	164	-164	0
924	184	-184	0
976	232	-232	0
1028	268	-268	0
1080	324	-324	0
1132	372	-372	0
1184	452	-452	0
1236	512	-512	0
1288	616	-616	0
1340	704	-704	0
1392	832	-832	0
1444	950	-950	0
1496	1120	-1120	0
1548	1268	-1268	0
1600	1486	-1486	0
1652	1688	-1688	0
1704	1956	-1956	0
1756	2220	-2220	0
1808	2568	-2568	0
1860	2896	-2896	0
1912	3336	-3336	0

Table 108: $H^{(13)}_{g,9}$, $X = A^2_{12}$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
23	2	2	-2
75	4	4	0
127	4	4	0
179	6	6	2
231	8	8	0
283	12	12	0
335	14	14	-2
387	20	20	0
439	26	26	-2
491	30	30	2
543	40	40	0
595	50	50	2
647	60	60	0
699	74	74	2
751	90	90	-2
803	108	108	4
855	134	134	-2
907	158	158	2
959	188	188	-4
1011	226	226	2
1063	266	266	-2
1115	314	314	2
1167	372	372	-4
1219	436	436	4
1271	508	508	-4
1323	596	596	4
1375	692	692	-4
1427	802	802	6
1479	932	932	-4
1531	1074	1074	6
1583	1238	1238	-6
1635	1430	1430	6
1687	1640	1640	-8
1739	1878	1878	6
1791	2150	2150	-6
1843	2456	2456	8
1895	2800	2800	-8

Table 109: $H^{(13)}_{g,10}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
4	2	-2	0
56	0	0	0
108	4	-4	0
160	4	-4	0
212	8	-8	0
264	8	-8	0
316	12	-12	0
368	12	-12	0
420	20	-20	0
472	20	-20	0
524	32	-32	0
576	34	-34	0
628	48	-48	0
680	52	-52	0
732	72	-72	0
784	78	-78	0
836	104	-104	0
888	116	-116	0
940	148	-148	0
992	164	-164	0
1044	208	-208	0
1096	232	-232	0
1148	288	-288	0
1200	324	-324	0
1252	396	-396	0
1304	444	-444	0
1356	536	-536	0
1408	604	-604	0
1460	720	-720	0
1512	812	-812	0
1564	960	-960	0
1616	1080	-1080	0
1668	1268	-1268	0
1720	1428	-1428	0
1772	1664	-1664	0
1824	1872	-1872	0

Table 110: $H^{(13)}_{g,11}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
35	2	2	2
87	2	2	2
139	2	2	-2
191	4	4	0
243	4	4	0
295	8	8	0
347	10	10	-2
399	10	10	2
451	16	16	0
503	20	20	0
555	22	22	-2
607	28	28	0
659	36	36	0
711	44	44	0
763	54	54	-2
815	64	64	0
867	76	76	0
919	94	94	2
971	114	114	-2
1023	130	130	2
1075	156	156	0
1127	188	188	0
1179	216	216	-4
1231	254	254	2
1283	300	300	0
1335	346	346	2
1387	404	404	-4
1439	470	470	2
1491	542	542	-2
1543	630	630	2
1595	724	724	-4
1647	828	828	4
1699	954	954	-2
1751	1100	1100	4
1803	1250	1250	-6
1855	1428	1428	4

Table 111: $H^{(13)}_{g,12}$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
Γ_g	1	1|4	2|8
12	0	0	0
64	2	-2	0
116	0	0	0
168	4	-4	0
220	0	0	0
272	4	-4	0
324	2	-2	0
376	8	-8	0
428	4	-4	0
480	12	-12	0
532	8	-8	0
584	16	-16	0
636	16	-16	0
688	24	-24	0
740	20	-20	0
792	36	-36	0
844	32	-32	0
896	48	-48	0
948	48	-48	0
1000	68	-68	0
1052	68	-68	0
1104	96	-96	0
1156	98	-98	0
1208	128	-128	0
1260	136	-136	0
1312	176	-176	0
1364	184	-184	0
1416	240	-240	0
1468	252	-252	0
1520	312	-312	0
1572	340	-340	0
1624	416	-416	0
1676	448	-448	0
1728	548	-548	0
1780	592	-592	0
1832	708	-708	0

C.15 Lambency 14+7

Table 112: $H^{(14+7)}_{g,1}$, $X = D_8^3$

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
-1	-2	-2	-2
55	2	2	2
111	0	0	0
167	4	0	-2
223	2	-2	2
279	6	2	0
335	6	-2	0
391	8	0	2
447	6	-2	0
503	16	4	-2
559	14	-2	2
615	18	2	0
671	22	-2	-2
727	32	4	2
783	30	-2	0
839	46	2	-2
895	44	-4	2
951	60	4	0
1007	68	-4	-4
1063	86	2	2
1119	90	-6	0
1175	122	6	-4
1231	128	-4	2
1287	156	4	0
1343	178	-6	-2
1399	220	8	4
1455	234	-6	0
1511	290	6	-4
1567	316	-8	4
1623	378	10	0
1679	424	-8	-2
1735	496	8	4
1791	546	-10	0
1847	656	12	-4
1903	718	-10	4
1959	834	10	0
2015	930	-14	-6
2071	1078	14	4

Table 113: $H^{(14+7)}_{g,3}$, $X = D_8^3$

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
47	2	-2	2
103	6	2	0
159	6	-2	0
215	10	2	-2
271	10	-2	-2
327	18	2	0
383	18	-2	0
439	30	2	0
495	30	-2	0
551	44	4	2
607	48	-4	0
663	66	2	0
719	72	-4	0
775	96	4	0
831	108	-4	0
887	138	6	0
943	154	-6	-2
999	198	6	0
1055	218	-6	2
1111	270	6	0
1167	306	-6	0
1223	372	8	0
1279	420	-8	0
1335	504	8	0
1391	566	-10	2
1447	678	10	0
1503	762	-10	0
1559	902	10	2
1615	1012	-12	-2
1671	1188	12	0
1727	1332	-12	0
1783	1554	14	0
1839	1746	-14	0
1895	2016	16	0
1951	2260	-16	-2
2007	2604	16	0
2063	2912	-20	2
2119	3340	20	-2

Table 114: $H_{g,5}^{(14+7)}$, $X = D_8^3$

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
31	4	0	-2
87	6	-2	0
143	8	0	2
199	12	0	0
255	18	2	0
311	20	0	2
367	28	0	-2
423	36	0	0
479	44	0	2
535	54	-2	0
591	72	0	0
647	84	0	0
703	106	2	-2
759	126	-2	0
815	152	0	2
871	184	0	-2
927	222	2	0
983	258	-2	0
1039	310	2	-2
1095	366	-2	0
1151	430	2	4
1207	502	-2	-2
1263	594	2	0
1319	684	0	0
1375	802	2	-2
1431	924	-4	0
1487	1070	2	2
1543	1234	-2	-2
1599	1428	4	0
1655	1628	-4	2
1711	1874	2	-4
1767	2142	-2	0
1823	2446	2	4
1879	2782	-6	-2
1935	3180	4	0
1991	3598	-2	4
2047	4094	6	-4
2103	4626	-6	0

Table 115: $H_{g,7}^{(14+7)}$, $X = D_8^3$

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
7	2	-2	2
63	6	2	0
119	8	0	-4
175	14	2	2
231	12	-4	0
287	24	0	0
343	26	-2	2
399	36	4	0
455	44	-4	-4
511	60	4	0
567	66	-2	0
623	92	4	-4
679	100	-4	4
735	132	4	0
791	156	-4	0
847	194	6	2
903	216	-8	0
959	276	4	0
1015	316	-4	4
1071	384	8	0
1127	440	-8	-4
1183	530	6	2
1239	600	-8	0
1295	728	8	-4
1351	820	-12	4
1407	972	12	0
1463	1112	-8	-4
1519	1300	12	4
1575	1470	-14	0
1631	1724	12	-4
1687	1948	-12	4
1743	2256	16	0
1799	2552	-16	-4
1855	2944	16	4
1911	3312	-16	0
1967	3812	20	-4
2023	4278	-22	6
2079	4884	20	0

C.16 Lambency 16

Table 116: $H_{1A}^{(16)}$, $X = A_{15}D_9$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-2														
64	0	0	2	2	0	2	2	1	0	0	0	0	0	0	0
128	0	2	2	2	2	2	4	2	2	2	2	0	0	0	0
192	0	2	2	2	4	4	2	4	2	2	2	2	0	0	0
256	2	2	4	4	4	4	6	4	4	4	4	2	2	0	2
320	0	2	4	6	4	6	6	6	4	4	4	2	0	2	0
384	2	4	6	6	8	8	8	8	8	6	6	4	4	2	2
448	4	4	6	8	8	10	10	8	8	8	4	6	2	2	0
512	2	6	8	10	12	12	14	12	12	10	10	6	6	2	2
576	2	8	10	12	14	16	14	16	12	12	10	8	4	4	2
640	4	8	14	16	18	18	20	18	18	16	14	10	8	4	4
704	6	10	14	18	20	22	22	24	20	18	14	12	8	6	0
768	6	14	18	22	28	28	30	28	28	22	22	16	12	8	6
832	8	14	22	28	28	34	34	32	28	28	22	18	12	8	2
896	10	18	26	32	38	40	42	40	38	34	30	22	18	10	6
960	10	22	30	38	44	48	46	48	44	40	32	28	18	14	4
1024	14	24	38	46	52	56	60	56	56	48	42	32	26	14	10
1088	14	30	42	54	60	66	68	66	60	56	46	38	24	18	6
1152	18	36	52	62	74	78	82	80	76	66	60	46	36	22	14
1216	22	40	56	74	82	92	92	92	84	78	62	54	38	24	10
1280	24	48	70	86	100	106	112	108	104	92	82	62	50	30	16
1344	26	56	80	98	114	124	124	128	116	106	90	74	50	36	14
1408	34	64	92	116	134	144	152	144	140	124	108	86	66	40	22
1472	38	74	104	132	152	166	168	168	156	144	120	98	70	48	18
1536	44	86	124	152	180	192	200	196	188	166	148	116	90	56	32
1600	50	96	140	178	198	220	226	224	208	192	160	132	94	64	28
1664	58	112	162	202	234	252	264	258	246	220	194	152	118	74	38
1728	64	130	182	230	266	290	294	296	276	252	214	178	126	86	36
1792	76	144	212	264	306	330	346	336	324	290	252	202	156	96	52
1856	84	166	238	300	342	376	386	384	358	330	280	230	166	112	48
1920	96	190	274	340	398	428	446	440	422	376	330	264	202	130	68
1984	112	212	306	388	442	486	498	496	468	428	362	300	220	144	64
2048	126	244	350	438	510	550	576	564	540	486	426	340	260	166	86

Table 117: $H_{2A}^{(16)}$, $X = A_{15}D_9$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	-2														
64	0	0	2	-2	0	-2	2	-1	0	0	0	0	0	0	0
128	0	-2	2	-2	2	-2	4	-2	2	-2	2	0	0	0	0
192	0	-2	2	-2	4	-4	2	-4	2	-2	2	-2	0	0	0
256	2	-2	4	-4	4	-4	6	-4	4	-4	4	-2	2	0	2
320	0	-2	4	-6	4	-6	6	-6	4	-4	4	-2	0	-2	0
384	2	-4	6	-6	8	-8	8	-8	8	-6	6	-4	4	-2	2
448	4	-4	6	-8	8	-10	10	-8	8	-8	4	-6	2	-2	0
512	2	-6	8	-10	12	-12	14	-12	12	-10	10	-6	6	-2	2
576	2	-8	10	-12	14	-16	14	-16	12	-12	10	-8	4	-4	2
640	4	-8	14	-16	18	-18	20	-18	18	-16	14	-10	8	-4	4
704	6	-10	14	-18	20	-22	22	-24	20	-18	14	-12	8	-6	0
768	6	-14	18	-22	28	-28	30	-28	28	-22	22	-16	12	-8	6
832	8	-14	22	-28	28	-34	34	-32	28	-28	22	-18	12	-8	2
896	10	-18	26	-32	38	-40	42	-40	38	-34	30	-22	18	-10	6
960	10	-22	30	-38	44	-48	46	-48	44	-40	32	-28	18	-14	4
1024	14	-24	38	-46	52	-56	60	-56	56	-48	42	-32	26	-14	10
1088	14	-30	42	-54	60	-66	68	-66	60	-56	46	-38	24	-18	6
1152	18	-36	52	-62	74	-78	82	-80	76	-66	60	-46	36	-22	14
1216	22	-40	56	-74	82	-92	92	-92	84	-78	62	-54	38	-24	10
1280	24	-48	70	-86	100	-106	112	-108	104	-92	82	-62	50	-30	16
1344	26	-56	80	-98	114	-124	124	-128	116	-106	90	-74	50	-36	14
1408	34	-64	92	-116	134	-144	152	-144	140	-124	108	-86	66	-40	22
1472	38	-74	104	-132	152	-166	168	-168	156	-144	120	-98	70	-48	18
1536	44	-86	124	-152	180	-192	200	-196	188	-166	148	-116	90	-56	32
1600	50	-96	140	-178	198	-220	226	-224	208	-192	160	-132	94	-64	28
1664	58	-112	162	-202	234	-252	264	-258	246	-220	194	-152	118	-74	38
1728	64	-130	182	-230	266	-290	294	-296	276	-252	214	-178	126	-86	36
1792	76	-144	212	-264	306	-330	346	-336	324	-290	252	-202	156	-96	52
1856	84	-166	238	-300	342	-376	386	-384	358	-330	280	-230	166	-112	48
1920	96	-190	274	-340	398	-428	446	-440	422	-376	330	-264	202	-130	68
1984	112	-212	306	-388	442	-486	498	-496	468	-428	362	-300	220	-144	64
2048	126	-244	350	-438	510	-550	576	-564	540	-486	426	-340	260	-166	86

C.17 Lambency 18

Table 118: $H_{1A}^{(18)}$, $X = A_{17}E_7$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	-2																
72	0	2	0	0	2	2	0	2	0	0	0	0	0	0	0	0	0
144	0	0	2	2	2	2	2	2	2	2	2	1	0	0	0	0	0
216	0	2	2	2	2	2	2	4	2	2	2	2	0	2	0	0	0
288	0	2	2	2	4	4	4	4	4	2	4	2	2	2	2	0	0
360	2	2	2	4	4	4	4	4	4	4	2	4	2	2	0	0	0
432	2	2	4	4	6	6	6	6	6	6	6	4	4	2	2	2	2
504	0	4	4	4	6	8	6	8	6	6	6	4	2	4	2	0	0
576	2	4	6	8	8	8	10	8	10	8	8	6	6	4	4	2	2
648	2	4	6	8	10	10	10	12	10	10	8	8	6	6	2	2	0
720	2	6	8	10	12	12	14	14	14	12	12	10	8	6	6	2	2
792	4	6	10	12	12	16	16	16	14	14	12	12	8	6	4	4	0
864	4	8	12	14	18	18	20	20	20	18	18	14	14	10	8	4	4
936	4	10	12	16	20	22	20	24	22	20	18	16	12	12	6	4	2
1008	6	10	16	20	24	26	28	26	28	26	24	20	18	12	10	6	4
1080	6	14	18	24	26	30	32	32	30	30	26	24	18	16	10	8	2
1152	8	16	22	26	34	36	38	38	38	34	34	28	24	18	16	8	6
1224	10	16	24	32	36	40	42	44	42	42	34	32	26	22	14	10	2
1296	10	20	30	38	44	48	52	52	52	48	46	38	34	26	20	12	8
1368	10	24	34	42	50	56	56	60	56	54	50	44	34	30	20	12	4
1440	14	26	40	50	58	64	70	68	70	64	60	52	46	34	28	16	10
1512	16	32	44	56	66	74	76	80	76	74	66	60	48	40	26	20	8
1584	18	36	52	64	78	84	90	92	92	84	80	68	60	46	36	22	12
1656	20	40	58	76	84	96	102	104	100	98	86	80	64	52	36	24	8
1728	24	46	68	86	102	110	120	120	120	112	106	90	80	62	48	30	18
1800	26	54	74	96	114	126	130	136	132	126	114	104	84	70	48	32	14
1872	32	58	88	110	130	144	154	154	154	146	136	118	102	78	62	38	20
1944	34	68	98	124	144	162	170	176	170	166	150	134	110	92	64	44	18
2016	38	78	112	140	168	184	196	200	200	186	176	152	132	104	80	48	28
2088	46	86	124	160	184	208	218	224	220	212	190	172	144	116	82	56	22
2160	50	98	142	180	214	234	252	254	254	240	224	196	170	134	102	64	36
2232	54	110	158	200	236	264	276	288	278	268	246	220	180	150	108	72	30
2304	64	122	180	228	268	296	318	320	322	304	284	248	216	168	130	82	42

Table 119: $H_{2A}^{(18)}$, $X = A_{17}E_7$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	-2																
72	0	-2	0	0	2	-2	0	-2	0	0	0	0	0	0	0	0	0
144	0	0	2	-2	2	-2	2	-2	2	-2	2	-1	0	0	0	0	0
216	0	-2	2	-2	2	-2	2	-4	2	-2	2	-2	0	-2	0	0	0
288	0	-2	2	-2	4	-4	4	-4	4	-2	4	-2	2	-2	2	0	0
360	2	-2	2	-4	4	-4	4	-4	4	-4	2	-4	2	-2	0	0	0
432	2	-2	4	-4	6	-6	6	-6	6	-6	6	-4	4	-2	2	-2	2
504	0	-4	4	-4	6	-8	6	-8	6	-6	6	-4	2	-4	2	0	0
576	2	-4	6	-8	8	-8	10	-8	10	-8	8	-6	6	-4	4	-2	2
648	2	-4	6	-8	10	-10	10	-12	10	-10	8	-8	6	-6	2	-2	0
720	2	-6	8	-10	12	-12	14	-14	14	-12	12	-10	8	-6	6	-2	2
792	4	-6	10	-12	12	-16	16	-16	14	-14	12	-12	8	-6	4	-4	0
864	4	-8	12	-14	18	-18	20	-20	20	-18	18	-14	14	-10	8	-4	4
936	4	-10	12	-16	20	-22	20	-24	22	-20	18	-16	12	-12	6	-4	2
1008	6	-10	16	-20	24	-26	28	-26	28	-26	24	-20	18	-12	10	-6	4
1080	6	-14	18	-24	26	-30	32	-32	30	-30	26	-24	18	-16	10	-8	2
1152	8	-16	22	-26	34	-36	38	-38	38	-34	34	-28	24	-18	16	-8	6
1224	10	-16	24	-32	36	-40	42	-44	42	-42	34	-32	26	-22	14	-10	2
1296	10	-20	30	-38	44	-48	52	-52	52	-48	46	-38	34	-26	20	-12	8
1368	10	-24	34	-42	50	-56	56	-60	56	-54	50	-44	34	-30	20	-12	4
1440	14	-26	40	-50	58	-64	70	-68	70	-64	60	-52	46	-34	28	-16	10
1512	16	-32	44	-56	66	-74	76	-80	76	-74	66	-60	48	-40	26	-20	8
1584	18	-36	52	-64	78	-84	90	-92	92	-84	80	-68	60	-46	36	-22	12
1656	20	-40	58	-76	84	-96	102	-104	100	-98	86	-80	64	-52	36	-24	8
1728	24	-46	68	-86	102	-110	120	-120	120	-112	106	-90	80	-62	48	-30	18
1800	26	-54	74	-96	114	-126	130	-136	132	-126	114	-104	84	-70	48	-32	14
1872	32	-58	88	-110	130	-144	154	-154	154	-146	136	-118	102	-78	62	-38	20
1944	34	-68	98	-124	144	-162	170	-176	170	-166	150	-134	110	-92	64	-44	18
2016	38	-78	112	-140	168	-184	196	-200	200	-186	176	-152	132	-104	80	-48	28
2088	46	-86	124	-160	184	-208	218	-224	220	-212	190	-172	144	-116	82	-56	22
2160	50	-98	142	-180	214	-234	252	-254	254	-240	224	-196	170	-134	102	-64	36
2232	54	-110	158	-200	236	-264	276	-288	278	-268	246	-220	180	-150	108	-72	30
2304	64	-122	180	-228	268	-296	318	-320	322	-304	284	-248	216	-168	130	-82	42

C.18 Lambency 18+9

Table 120: $H^{(18+9)}_{g,1}$, $X = D_{10}E_7^2$

$[g]$	1A	2A
Γ_g	1|1	2|1
-1	-2	-2
71	0	0
143	2	-2
215	0	0
287	2	-2
359	2	2
431	4	0
503	0	0
575	6	-2
647	4	0
719	6	-2
791	6	2
863	10	-2
935	6	2
1007	14	-2
1079	10	2
1151	18	-2
1223	18	2
1295	22	-2
1367	18	2
1439	32	-4
1511	30	2
1583	38	-2
1655	38	2
1727	52	-4
1799	48	4
1871	68	-4
1943	64	4
2015	80	-4
2087	86	6
2159	104	-4
2231	104	4
2303	134	-6
2375	136	4
2447	162	-6
2519	170	6
2591	206	-6
2663	210	6
2735	254	-6
2807	262	6

Table 121: $H^{(18+9)}_{g,3}$, $X = D_{10}E_7^2$

$[g]$	1A	2A
Γ_g	1|1	2|1
63	2	-2
135	2	2
207	4	0
279	4	0
351	6	-2
423	6	2
495	10	-2
567	10	2
639	14	-2
711	14	2
783	20	0
855	22	2
927	28	-4
999	30	2
1071	38	-2
1143	42	2
1215	52	-4
1287	56	4
1359	70	-2
1431	76	4
1503	92	-4
1575	100	4
1647	120	-4
1719	132	4
1791	154	-6
1863	170	6
1935	200	-4
2007	220	4
2079	254	-6
2151	278	6
2223	322	-6
2295	354	6
2367	404	-8
2439	444	8
2511	506	-6
2583	556	8
2655	630	-10
2727	690	10
2799	780	-8
2871	854	10

Table 122: $H^{(18+9)}_{g,5}$, $X = D_{10}E_7^2$

$[g]$	1A	2A
Γ_g	1|1	2|1
47	2	2
119	4	0
191	4	0
263	8	0
335	6	2
407	12	0
479	12	0
551	16	0
623	18	2
695	26	-2
767	24	0
839	36	0
911	38	2
983	48	0
1055	52	0
1127	68	0
1199	70	2
1271	90	-2
1343	98	2
1415	118	-2
1487	130	2
1559	158	-2
1631	168	0
1703	204	0
1775	224	4
1847	262	-2
1919	288	0
1991	338	-2
2063	364	4
2135	430	-2
2207	470	2
2279	538	-2
2351	592	4
2423	680	-4
2495	738	2
2567	846	-2
2639	928	4
2711	1048	-4
2783	1150	2
2855	1300	-4

Table 123: $H_{g,7}^{(18+9)}$, $X = D_{10}E_7^2$

$[g]$	1A	2A
Γ_g	1|1	2|1
23	2	-2
95	4	0
167	6	-2
239	6	2
311	10	-2
383	12	0
455	14	-2
527	18	2
599	22	-2
671	26	2
743	34	-2
815	38	2
887	44	-4
959	54	2
1031	66	-2
1103	72	4
1175	88	-4
1247	102	2
1319	116	-4
1391	136	4
1463	156	-4
1535	176	4
1607	208	-4
1679	234	6
1751	266	-6
1823	304	4
1895	346	-6
1967	386	6
2039	442	-6
2111	498	6
2183	560	-8
2255	628	8
2327	708	-8
2399	788	8
2471	892	-8
2543	990	10
2615	1102	-10
2687	1232	8
2759	1374	-10

Table 124: $H_{g,9}^{(18+9)}$, $X = D_{10}E_7^2$

$[g]$	1A	2A
Γ_g	1|1	2|1
63	4	0
135	4	0
207	8	0
279	8	0
351	12	0
423	12	0
495	20	0
567	20	0
639	28	0
711	28	0
783	40	0
855	44	0
927	56	0
999	60	0
1071	76	0
1143	84	0
1215	104	0
1287	112	0
1359	140	0
1431	152	0
1503	184	0
1575	200	0
1647	240	0
1719	264	0
1791	308	0
1863	340	0
1935	400	0
2007	440	0
2079	508	0
2151	556	0
2223	644	0
2295	708	0
2367	808	0
2439	888	0
2511	1012	0
2583	1112	0
2655	1260	0
2727	1380	0
2799	1560	0

C.19 Lambency 22+11

Table 125: $H^{(22+11)}_{g,1}$, $X = D^2_{12}$

$[g]$	1A	2A
Γ_g	1|1	2|2
-1	-2	-2
87	2	2
175	0	0
263	0	0
351	0	0
439	2	2
527	0	0
615	4	0
703	2	-2
791	2	2
879	2	-2
967	4	0
1055	2	-2
1143	6	2
1231	2	-2
1319	6	2
1407	6	-2
1495	10	2
1583	6	-2
1671	12	0
1759	10	-2
1847	12	4
1935	14	-2
2023	18	2
2111	14	-2
2199	24	4
2287	18	-2
2375	26	2
2463	26	-2
2551	32	4
2639	30	-2
2727	42	2
2815	40	-4
2903	48	4
2991	48	-4
3079	60	4
3167	58	-6
3255	76	4
3343	72	-4
3431	84	4

Table 126: $H^{(22+11)}_{g,3}$, $X = D^2_{12}$

$[g]$	1A	2A
Γ_g	1|1	2|2
79	0	0
167	2	2
255	2	-2
343	4	0
431	2	-2
519	4	0
607	4	0
695	6	2
783	4	0
871	10	2
959	8	0
1047	12	0
1135	10	-2
1223	16	0
1311	16	0
1399	20	0
1487	18	-2
1575	26	2
1663	26	-2
1751	34	2
1839	32	0
1927	42	2
2015	42	-2
2103	54	2
2191	54	-2
2279	66	2
2367	70	-2
2455	82	2
2543	84	-4
2631	102	2
2719	106	-2
2807	126	2
2895	130	-2
2983	152	4
3071	160	-4
3159	186	2
3247	192	-4
3335	224	4
3423	238	-2
3511	272	4

Table 127: $H^{(22+11)}_{g,5}$, $X = D^2_{12}$

$[g]$	1A	2A
Γ_g	1|1	2|2
63	2	2
151	2	-2
239	4	0
327	2	-2
415	6	2
503	6	-2
591	6	2
679	8	0
767	12	0
855	10	-2
943	14	2
1031	16	0
1119	18	2
1207	22	-2
1295	26	2
1383	26	-2
1471	34	2
1559	38	-2
1647	42	2
1735	46	-2
1823	58	2
1911	60	-4
1999	70	2
2087	78	-2
2175	88	4
2263	96	-4
2351	112	4
2439	120	-4
2527	140	4
2615	152	-4
2703	168	4
2791	184	-4
2879	212	4
2967	226	-6
3055	256	4
3143	280	-4
3231	310	6
3319	338	-6
3407	378	6
3495	406	-6

Table 128: $H_{g,7}^{(22+11)}$, $X = D_{12}^2$

$[g]$	1A	2A
Γ_g	1|1	2|2
39	2	-2
127	4	0
215	2	2
303	6	2
391	4	0
479	8	0
567	8	0
655	12	0
743	10	-2
831	16	0
919	16	0
1007	20	0
1095	22	-2
1183	28	0
1271	28	0
1359	38	2
1447	38	-2
1535	48	0
1623	52	0
1711	62	2
1799	64	0
1887	80	0
1975	84	0
2063	100	0
2151	106	-2
2239	126	2
2327	132	0
2415	158	2
2503	166	-2
2591	192	0
2679	208	0
2767	238	2
2855	254	-2
2943	294	2
3031	312	0
3119	354	2
3207	382	-2
3295	432	0
3383	460	0

Table 129: $H_{g,9}^{(22+11)}$, $X = D_{12}^2$

$[g]$	1A	2A
Γ_g	1|1	2|2
7	2	2
95	2	-2
183	4	0
271	6	-2
359	6	2
447	8	0
535	8	0
623	10	-2
711	14	2
799	14	-2
887	18	2
975	22	-2
1063	26	2
1151	28	0
1239	34	2
1327	38	-2
1415	44	4
1503	50	-2
1591	58	2
1679	64	-4
1767	74	2
1855	82	-2
1943	94	2
2031	108	-4
2119	120	4
2207	132	-4
2295	150	2
2383	168	-4
2471	188	4
2559	208	-4
2647	232	4
2735	256	-4
2823	286	6
2911	316	-4
2999	348	4
3087	386	-6
3175	426	6
3263	466	-6
3351	518	6

Table 130: $H_{g,11}^{(22+11)}$, $X = D_{12}^2$

$[g]$	1A	2A
Γ_g	1|1	2|2
55	4	0
143	4	0
231	4	0
319	4	0
407	8	0
495	8	0
583	12	0
671	12	0
759	16	0
847	16	0
935	24	0
1023	20	0
1111	28	0
1199	32	0
1287	40	0
1375	40	0
1463	52	0
1551	52	0
1639	68	0
1727	72	0
1815	84	0
1903	92	0
1991	108	0
2079	112	0
2167	136	0
2255	148	0
2343	168	0
2431	180	0
2519	212	0
2607	224	0
2695	260	0
2783	280	0
2871	316	0
2959	344	0
3047	392	0
3135	416	0
3223	472	0
3311	512	0
3399	572	0

C.20 Lambency 25

Table 131: $H_{1A}^{(25)}$, $X = A_{24}$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	-2																							
100	0	0	0	2	0	0	2	0	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
200	0	0	0	2	2	0	0	2	2	2	0	2	2	2	0	0	0	0	0	0	0	0	0	0
300	0	2	0	0	2	0	2	2	2	0	2	2	0	2	2	0	2	0	0	0	0	0	0	0
400	0	0	2	2	0	2	2	2	2	2	2	2	2	2	2	2	0	2	2	1	0	0	0	0
500	0	0	0	2	2	2	2	2	2	2	2	2	2	2	2	0	2	0	2	0	0	0	0	0
600	0	2	2	2	2	2	2	2	4	4	2	4	2	4	2	2	2	2	2	2	0	2	0	2
700	0	0	2	2	2	2	4	2	4	4	2	2	4	2	2	2	2	2	2	0	0	0	0	0
800	0	2	2	2	2	4	4	4	4	4	4	4	4	4	4	4	2	2	2	2	0	2	2	0
900	0	2	2	2	4	4	4	4	4	4	6	4	4	4	4	2	4	2	2	2	2	0	0	0
1000	2	2	2	4	4	4	4	6	6	6	4	6	6	6	4	4	4	4	2	2	2	2	0	2
1100	0	2	2	4	4	4	6	6	6	6	6	6	6	6	4	4	4	2	4	2	2	0	0	0
1200	0	2	4	4	6	6	6	8	8	8	8	8	6	8	6	6	6	6	4	4	2	2	2	2
1300	2	2	4	6	6	6	8	6	8	8	8	8	8	6	8	6	6	4	4	2	2	2	0	0
1400	2	2	4	6	6	8	8	10	10	10	10	10	10	10	8	8	6	6	4	6	2	2	2	2
1500	2	4	4	6	8	8	10	10	12	10	12	10	10	10	10	8	8	6	6	4	4	2	2	0
1600	2	4	6	8	8	10	12	12	12	14	12	14	12	12	12	10	8	8	8	6	4	4	2	2
1700	2	4	6	8	10	10	12	14	14	14	14	14	14	12	12	10	10	8	8	4	4	2	2	0
1800	2	6	6	10	12	14	14	16	16	16	16	18	16	16	14	14	12	12	8	8	6	6	2	2
1900	4	4	8	10	12	14	16	16	18	18	18	18	18	16	16	14	12	10	10	6	6	4	2	0
2000	2	6	10	12	14	16	18	20	20	22	22	22	20	20	18	18	14	14	12	10	6	6	4	2
2100	2	8	10	12	16	18	20	22	24	22	24	22	22	22	20	16	18	14	12	10	8	4	4	2
2200	4	8	12	16	18	20	22	24	26	28	26	26	26	26	22	22	18	18	14	12	8	8	4	4
2300	4	8	12	16	20	22	26	26	28	28	30	28	28	26	26	22	22	16	16	12	10	6	4	0
2400	4	10	14	18	22	26	28	32	32	34	34	34	32	32	30	28	24	22	18	16	12	10	6	4
2500	6	10	16	20	24	28	32	32	36	36	36	36	36	32	32	28	26	22	20	14	12	8	6	2
2600	6	12	16	24	28	32	34	38	40	42	40	42	40	40	34	34	30	26	22	20	14	12	6	4
2700	6	14	20	24	30	34	38	42	44	44	46	44	44	42	40	34	34	28	24	20	16	10	8	2
2800	8	14	22	28	32	40	44	46	48	52	50	52	50	48	44	42	36	34	28	24	18	14	8	6
2900	8	16	24	30	36	42	48	50	54	54	56	54	54	50	48	44	40	34	30	24	20	12	10	2
3000	8	18	26	34	42	48	52	58	60	62	62	64	60	60	54	52	46	42	34	30	22	18	10	8
3100	10	18	28	38	44	50	58	60	66	68	68	66	66	62	60	54	50	42	38	28	24	16	10	4
3200	10	22	32	42	50	58	64	70	72	76	76	76	74	72	66	62	54	50	42	36	26	22	14	8
3300	12	24	34	44	56	62	70	74	80	80	84	82	80	76	74	64	62	52	46	36	30	20	14	6
3400	14	26	38	50	60	70	76	84	88	92	90	92	90	86	80	76	66	62	50	44	34	26	14	10
3500	14	28	42	54	66	74	84	90	96	98	100	98	98	92	88	80	74	62	56	44	36	24	18	6

Table 132: $H_{2A}^{(25)}$, $X = A_{24}$

$4k\ell, r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	-2																							
100	0	0	0	-2	0	0	2	0	2	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
200	0	0	0	-2	2	0	0	-2	2	-2	0	-2	2	-2	0	0	0	0	0	0	0	0	0	0
300	0	-2	0	0	2	0	2	-2	2	0	2	-2	0	-2	2	0	2	0	0	0	0	0	0	0
400	0	0	2	-2	0	-2	2	-2	2	-2	2	-2	2	-2	2	-2	0	-2	2	-1	0	0	0	0
500	0	0	0	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	0	2	0	2	0	0	0	0	0
600	0	-2	2	-2	2	-2	2	-2	4	-4	2	-4	2	-4	2	-2	2	-2	2	-2	0	-2	0	-2
700	0	0	2	-2	2	-2	4	-2	4	-4	2	-2	4	-2	2	-2	2	-2	2	0	0	0	0	0
800	0	-2	2	-2	2	-4	4	-4	4	-4	4	-4	4	-4	4	-4	2	-2	2	-2	0	-2	2	0
900	0	-2	2	-2	4	-4	4	-4	4	-4	6	-4	4	-4	4	-2	4	-2	2	-2	2	0	0	0
1000	2	-2	2	-4	4	-4	4	-6	6	-6	4	-6	6	-6	4	-4	4	-4	2	-2	2	-2	0	-2
1100	0	-2	2	-4	4	-4	6	-6	6	-6	6	-6	6	-6	4	-4	4	-2	4	-2	2	0	0	0
1200	0	-2	4	-4	6	-6	6	-8	8	-8	8	-8	6	-8	6	-6	6	-6	4	-4	2	-2	2	-2
1300	2	-2	4	-6	6	-6	8	-6	8	-8	8	-8	8	-6	8	-6	6	-4	4	-2	2	-2	0	0
1400	2	-2	4	-6	6	-8	8	-10	10	-10	10	-10	10	-10	8	-8	6	-6	4	-6	2	-2	2	-2
1500	2	-4	4	-6	8	-8	10	-10	12	-10	12	-10	10	-10	10	-8	8	-6	6	-4	4	-2	2	0
1600	2	-4	6	-8	8	-10	12	-12	12	-14	12	-14	12	-12	12	-10	8	-8	8	-6	4	-4	2	-2
1700	2	-4	6	-8	10	-10	12	-14	14	-14	14	-14	14	-12	12	-10	10	-8	8	-4	4	-2	2	0
1800	2	-6	6	-10	12	-14	14	-16	16	-16	16	-18	16	-16	14	-14	12	-12	8	-8	6	-6	2	-2
1900	4	-4	8	-10	12	-14	16	-16	18	-18	18	-18	18	-16	16	-14	12	-10	10	-6	6	-4	2	0
2000	2	-6	10	-12	14	-16	18	-20	20	-22	22	-22	20	-20	18	-18	14	-14	12	-10	6	-6	4	-2
2100	2	-8	10	-12	16	-18	20	-22	24	-22	24	-22	22	-22	20	-16	18	-14	12	-10	8	-4	4	-2
2200	4	-8	12	-16	18	-20	22	-24	26	-28	26	-26	26	-26	22	-22	18	-18	14	-12	8	-8	4	-4
2300	4	-8	12	-16	20	-22	26	-26	28	-28	30	-28	28	-26	26	-22	22	-16	16	-12	10	-6	4	0
2400	4	-10	14	-18	22	-26	28	-32	32	-34	34	-34	32	-32	30	-28	24	-22	18	-16	12	-10	6	-4
2500	6	-10	16	-20	24	-28	32	-32	36	-36	36	-36	36	-32	32	-28	26	-22	20	-14	12	-8	6	-2
2600	6	-12	16	-24	28	-32	34	-38	40	-42	40	-42	40	-40	34	-34	30	-26	22	-20	14	-12	6	-4
2700	6	-14	20	-24	30	-34	38	-42	44	-44	46	-44	44	-42	40	-34	34	-28	24	-20	16	-10	8	-2
2800	8	-14	22	-28	32	-40	44	-46	48	-52	50	-52	50	-48	44	-42	36	-34	28	-24	18	-14	8	-6
2900	8	-16	24	-30	36	-42	48	-50	54	-54	56	-54	54	-50	48	-44	40	-34	30	-24	20	-12	10	-2
3000	8	-18	26	-34	42	-48	52	-58	60	-62	62	-64	60	-60	54	-52	46	-42	34	-30	22	-18	10	-8
3100	10	-18	28	-38	44	-50	58	-60	66	-68	68	-66	66	-62	60	-54	50	-42	38	-28	24	-16	10	-4
3200	10	-22	32	-42	50	-58	64	-70	72	-76	76	-76	74	-72	66	-62	54	-50	42	-36	26	-22	14	-8
3300	12	-24	34	-44	56	-62	70	-74	80	-80	84	-82	80	-76	74	-64	62	-52	46	-36	30	-20	14	-6
3400	14	-26	38	-50	60	-70	76	-84	88	-92	90	-92	90	-86	80	-76	66	-62	50	-44	34	-26	14	-10
3500	14	-28	42	-54	66	-74	84	-90	96	-98	100	-98	98	-92	88	-80	74	-62	56	-44	36	-24	18	-6

C.21 Lambency 30+15

Table 133: $H_{1A}^{(30+15)}$, $X = D_{16}E_8$

$4k\ell, r$	1	3	5	7	9	11	13	15
0	-2							
120	0	0	2	0	2			
240	0	2	0	2	2	2	2	2
360	0	0	2	2	2	2	2	2
480	0	2	2	2	2	4	2	4
600	2	0	2	2	4	2	4	2
720	0	2	2	4	4	4	4	4
840	0	2	4	2	4	4	4	4
960	0	2	2	6	4	6	6	6
1080	2	2	4	4	6	6	6	4
1200	0	4	4	6	6	8	6	8
1320	2	2	4	6	8	6	8	8
1440	0	4	6	8	8	10	10	10
1560	2	2	8	6	10	10	10	8
1680	2	6	6	10	10	12	12	12
1800	2	4	8	10	12	12	14	12
1920	0	6	8	12	14	16	14	16
2040	4	6	10	12	16	14	18	14
2160	2	8	10	16	16	20	18	20
2280	2	6	14	14	18	20	20	20
2400	2	10	12	20	20	24	24	24
2520	4	8	16	18	24	24	26	24
2640	2	12	16	22	26	30	28	28
2760	6	10	18	24	28	28	32	30
2880	4	14	20	28	30	36	36	36
3000	6	12	24	26	36	36	38	36
3120	4	16	22	34	38	42	42	44
3240	6	16	28	34	42	44	48	44
3360	4	20	30	40	44	52	50	52
3480	10	18	32	40	50	52	58	52
3600	6	24	34	48	54	62	62	62
3720	8	22	40	48	60	64	66	64
3840	8	28	40	58	64	72	74	76
3960	12	26	46	58	72	76	80	76
4080	8	34	48	66	76	88	86	88
4200	14	32	54	70	84	88	96	92
4320	10	38	58	80	90	102	104	104
4440	14	38	66	78	100	106	112	108
4560	14	44	66	94	106	120	122	122
4680	18	44	76	96	116	126	132	128
4800	14	52	80	108	124	142	142	144

C.22 Lambency 30+6,10,15

Table 134: $H_{g,1}^{(30+6,10,15)}$, $X = E_8^3$

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
-1	-2	-2	-2
119	2	2	2
239	2	-2	2
359	4	0	-2
479	2	-2	2
599	6	2	0
719	4	0	-2
839	6	2	0
959	6	-2	0
1079	10	2	-2
1199	6	-2	0
1319	12	0	0
1439	10	-2	-2
1559	14	2	2
1679	14	-2	2
1799	18	2	0
1919	14	-2	2
2039	24	4	0
2159	22	-2	-2
2279	26	2	2
2399	26	-2	2
2519	34	2	-2
2639	30	-2	0
2759	42	2	0
2879	40	-4	-2
2999	48	4	0
3119	48	-4	0
3239	58	2	-2
3359	56	-4	2
3479	72	4	0
3599	70	-2	-2
3719	80	4	2
3839	84	-4	0
3959	100	4	-2
4079	96	-4	0
4199	116	4	2
4319	116	-4	-4
4439	134	6	2
4559	140	-4	2

Table 135: $H_{g,7}^{(30+6,10,15)}$, $X = E_8^3$

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
71	2	-2	2
191	4	0	-2
311	4	0	-2
431	6	2	0
551	6	-2	0
671	8	0	2
791	8	0	2
911	12	0	0
1031	10	-2	-2
1151	14	2	2
1271	16	0	-2
1391	18	2	0
1511	18	-2	0
1631	24	0	0
1751	24	0	0
1871	30	2	0
1991	30	-2	0
2111	36	0	0
2231	38	-2	2
2351	46	2	-2
2471	46	-2	-2
2591	54	2	0
2711	60	0	0
2831	66	2	0
2951	68	-4	2
3071	82	2	-2
3191	84	0	0
3311	98	2	2
3431	102	-2	0
3551	114	2	0
3671	122	-2	2
3791	138	2	0
3911	144	-4	0
4031	162	2	0
4151	174	-2	0
4271	192	4	0
4391	200	-4	2
4511	226	2	-2
4631	238	-2	-2

C.23 Lambency 46+23

Table 136: $H_{1A}^{(46+23)}$, $X = D_{24}$

$4k\ell, r$	1	3	5	7	9	11	13	15	17	19	21	23
0	-2											
184	2	0	0	0	2	0	2					
368	0	0	0	2	0	2	0	2	2	2		
552	0	0	2	0	0	2	0	2	2	0	2	2
736	0	0	0	2	0	2	2	2	0	2	2	2
920	0	0	2	0	2	0	2	0	2	2	2	0
1104	0	2	0	0	2	2	2	2	2	2	2	2
1288	0	0	0	2	2	0	2	2	2	2	2	0
1472	0	2	0	2	2	2	2	2	2	2	2	4
1656	0	0	2	0	2	2	2	2	4	2	2	2
1840	0	2	0	2	2	2	2	4	2	4	2	4
2024	0	0	2	2	2	2	4	2	2	2	2	4
2208	0	0	2	2	2	4	2	4	4	4	4	4
2392	2	0	2	2	2	2	4	2	4	4	4	2
2576	0	2	2	2	2	4	4	4	4	4	4	4
2760	0	0	2	2	4	4	4	4	4	4	4	4
2944	0	2	2	4	2	4	4	6	4	6	6	4
3128	2	0	2	2	4	4	4	4	6	4	6	4
3312	0	2	2	4	4	6	4	6	6	6	6	6
3496	2	2	2	4	4	4	6	6	6	6	6	4
3680	0	2	2	4	4	6	6	8	6	8	6	8
3864	0	2	4	2	6	6	6	6	8	6	8	8
4048	0	2	2	6	6	6	6	8	8	8	8	8
4232	2	2	4	4	6	6	8	8	8	8	8	8
4416	0	2	4	6	6	8	8	10	10	10	10	10
4600	2	2	4	4	8	6	10	8	10	10	10	8
4784	0	4	4	6	6	10	10	10	10	12	10	12
4968	0	2	6	6	8	8	10	10	12	10	12	12
5152	0	4	4	8	8	10	12	12	12	14	12	12
5336	2	2	6	6	10	10	12	12	14	12	14	12
5520	0	4	6	8	10	12	12	14	14	16	14	16
5704	2	4	6	8	10	10	14	14	14	16	16	14
5888	0	4	6	10	10	14	14	16	16	18	16	16
6072	2	4	8	8	12	14	16	16	18	16	18	16
6256	2	6	6	12	12	16	16	18	18	20	18	20
6440	2	4	8	10	14	14	18	18	20	20	20	20
6624	0	6	8	12	14	18	18	22	22	22	22	24
6808	4	4	10	12	16	16	20	20	22	22	24	20
6992	2	6	8	14	16	20	20	24	24	26	24	24
7176	2	6	12	12	18	20	22	24	26	24	26	26
7360	2	8	10	16	18	22	24	28	26	30	28	28

D Decompositions

As explained in §5 (see also §6.1) our conjectural proposals for the umbral McKay–Thompson series $H^X_{g,r}(\tau) = \sum_d c^X_{g,r}(d)q^d$ (cf. §§5,C) determine the G^X-modules $K^X_{r,d}$ up to isomorphism for $d > 0$, at least for those values of d for which we can identify all the Fourier coefficients $c^X_{g,r}(d)$. In this section we furnish tables of explicit decompositions of $K^X_{r,d}$ into irreducible representations for G^X, for the first few values of d. The coefficient $c^X_r(d)$ of $H^X_{g,r}$ is non-zero only when $d = n - r^2/4m$ for some integer $n \geq 0$, where m denotes the Coxeter number of X. For each of the tables in this section the rows are labelled by the values $4md$, so that the entry in row k and column χ_i indicates the multiplicity of the irreducible representation of G^X with character χ_i (in the notation of §B.1) appearing in the G^X-module $K^X_{r,k/4m}$. One can observe that these tables support Conjectures 6.1, 6.10 and 6.11, as well as Conjecture 6.2.

As we have seen in §2.4, for some Niemeier root systems, especially those with higher Coxeter numbers, the umbral groups G^X are very small and the decompositions of the G^X-module into irreducibles are extremely simple and do not need to be tabulated. For those cases we give instead the following expression for the module in terms of the umbral McKay–Thompson series H^X_g whose low order coefficients are tabulated in §C.

For the Niemeier root system $X = A_9^2 D_6$ corresponding to $\ell = 10$, we have $G^X \cong 4$ and the decomposition is simply given by

$$\sum_{\substack{d>0 \\ 40d=-r^2 \bmod 40}} q^d K^{(10)}_{r,d} = \frac{\chi_3 + \chi_4}{2} H^{(10)}_{1A,r}, \quad r \text{ even}, \tag{D.1}$$

$$\sum_{\substack{d>0 \\ 40d=-r^2 \bmod 40}} q^d K^{(10)}_{r,d} = \frac{\chi_1 + \chi_2}{2} H^{(10)}_{1A,r} + \frac{\chi_1 - \chi_2}{2} H^{(10)}_{4AB,r}, \ r \text{ odd}. \tag{D.2}$$

For the six Niemeier root systems

$$A_{11}D_7E_6,\ A_{15}D_9,\ A_{17}E_7,\ A_{24},\ D_{10}E_7^2,\ D_{12}^2, \tag{D.3}$$

corresponding to $\ell \in \{12, 16, 18, 25, 18+9, 22+11\}$ the associated umbral group is $G^X \cong 2$ and the decomposition of the representations is simply given by

$$\sum_{\substack{d>0 \\ 4md=-r^2 \bmod 4m}} q^d K^X_{r,d} = \frac{\chi_1 + \chi_2}{2} H^X_{1A,r} + \frac{\chi_1 - \chi_2}{2} H^X_{2A,r}, \tag{D.4}$$

where m again denotes the Coxeter number of X. In particular, for the A-type cases above,

begin $A_{11}D_7E_6$, $A_{15}D_9$, $A_{17}E_7$, and A_{24}, this reduces to

$$\sum_{\substack{d>0 \\ 4md=-r^2 \bmod 4m}} q^d K^X_{r,d} = \chi_1 H^X_{1A,r}, \quad r \text{ odd}, \tag{D.5}$$

$$\sum_{\substack{d>0 \\ 4md=-r^2 \bmod 4m}} q^d K^X_{r,d} = \chi_2 H^X_{1A,r}, \quad r \text{ even}, \tag{D.6}$$

where m again denotes the Coxeter number of X.

For the two Niemeier lattices $D_{16}E_8$ and D_{24} the umbral group is $G^X \cong 1$ and the decomposition is completely trivial.

D.1 Lambency 2

Table 137: Decomposition of $K_1^{(2)} = K_1^X$ for $X = A_1^{24}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
-1	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	2	2	2	2
63	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	2	0	0	2	2	2	4	2	2	6
71	0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	2	0	2	2	2	4	4	4	8	8	10
79	0	0	0	0	2	2	0	2	2	0	0	2	2	2	2	2	4	4	4	6	6	8	12	10	10	24
87	0	0	0	0	0	0	0	0	0	4	4	4	4	6	4	4	2	8	10	8	14	12	22	24	26	40
95	0	2	0	0	2	2	2	4	4	6	6	8	8	4	8	8	12	12	12	18	26	30	40	38	40	80
103	0	0	2	2	2	2	4	2	6	10	10	14	14	18	14	14	16	26	30	28	44	44	70	80	84	136
111	0	0	0	0	8	8	4	6	14	16	16	24	24	22	24	24	34	38	46	58	80	86	128	126	132	254
119	0	0	2	2	8	8	12	8	18	38	38	40	40	46	44	44	46	78	86	88	138	144	218	238	246	424
127	0	2	2	2	18	18	18	22	36	50	50	72	72	68	72	72	100	122	140	170	232	252	378	382	400	742
135	0	2	8	8	25	25	30	26	54	94	94	116	116	130	124	124	140	212	246	262	392	410	630	670	704	1222
143	0	6	6	6	50	50	50	58	100	148	148	194	194	192	202	202	256	342	388	454	654	704	1044	1074	1120	2058
151	0	4	18	18	68	68	80	72	150	252	252	318	318	346	332	332	394	582	664	722	1062	1116	1702	1800	1880	3320
159	0	14	20	20	126	126	128	138	254	390	390	516	516	520	536	536	676	904	1036	1196	1716	1836	2764	2846	2980	5408
167	2	20	40	40	182	182	214	200	396	652	652	814	814	872	860	860	1020	1476	1684	1862	2742	2902	4384	4622	4828	8572
175	2	32	55	55	314	314	328	346	640	988	988	1298	1298	1336	1348	1348	1686	2302	2630	3000	4324	4616	6950	7204	7532	13620
183	2	40	98	98	460	460	512	496	972	1590	1590	2020	2020	2144	2118	2118	2546	3638	4162	4624	6768	7166	10856	11376	11898	21204

D.2 Lambency 3

Table 138: $K_1^{(3)} = K_1^X$ for $X = A_2^{12}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	2	0	2	0	0	2
59	0	0	0	0	0	2	2	0	0	0	2	2	2	2	2
71	0	0	2	0	0	0	0	2	2	2	2	2	4	4	6
83	0	0	0	2	2	4	4	2	2	2	4	6	6	8	10
95	0	0	2	0	0	4	4	8	8	6	6	10	12	14	18
107	0	2	2	4	4	8	12	8	8	8	14	16	22	28	30

Table 139: $K_2^{(3)} = K_2^X$ for $X = A_2^{12}$

	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
8	1	1	0	0	0	0	0	0	0	0	0
20	0	0	0	0	1	1	0	0	0	0	0
32	0	0	0	0	0	0	1	1	0	0	0
44	0	0	0	0	0	0	0	0	2	1	1
56	0	0	0	2	0	0	2	2	0	2	2
68	0	0	2	0	2	2	2	2	4	4	4
80	2	2	0	0	1	1	6	6	4	8	8
92	0	0	2	4	6	6	8	8	12	14	14
104	2	2	0	4	6	6	20	20	16	24	24
116	2	2	6	8	12	12	26	26	36	44	44

D.3 Lambency 4

Table 140: $K_1^{(4)} = K_1^X$ for $X = A_3^8$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
-1	-2	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	2	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	2
47	0	0	0	0	0	2	0	0	2	2	0
63	0	1	1	0	2	0	0	2	2	2	4
79	0	0	0	2	2	2	4	0	4	6	4
95	0	2	2	2	2	4	2	4	6	8	12
111	2	2	2	6	6	6	8	6	10	18	14
127	0	4	4	6	10	10	6	12	18	26	30
143	2	6	6	14	14	18	18	10	32	46	40
159	4	10	10	18	24	26	20	26	44	68	76

Table 141: $K_3^{(4)} = K_3^X$ for $X = A_3^8$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
7	0	1	1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	2	0	0
39	0	0	0	0	2	0	0	0	0	2	0
55	0	0	0	2	0	0	0	2	2	2	2
71	0	2	2	0	2	2	0	0	2	4	4
87	2	0	0	2	2	2	4	4	6	6	8
103	0	2	2	2	6	6	4	2	6	14	12
119	2	2	2	8	4	8	6	8	18	20	22
135	2	8	8	8	14	14	12	10	20	36	34
151	4	6	6	18	16	20	20	22	42	54	56

Table 142: $K_2^{(4)} = K_2^X$ for $X = A_3^8$

	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
12	0	1	1	0	0
28	0	0	0	1	1
44	2	0	0	2	2
60	0	2	2	4	4
76	2	2	2	8	8
92	6	4	4	14	14
108	6	9	9	24	24
124	14	14	14	40	40
140	24	20	20	66	66
156	32	36	36	104	104

D.4 Lambency 5

Table 143: $K_1^{(5)}$, $X = A_4^6$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
-1	-2	0	0	0	0	0	0
19	0	0	2	0	0	0	0
39	0	0	0	2	0	2	0
59	0	0	2	0	2	2	2
79	0	2	2	4	2	2	4
99	2	0	6	2	6	4	6
119	0	2	6	8	8	10	10
139	4	2	14	10	14	12	16
159	2	6	14	20	20	22	26
179	8	4	28	22	36	32	40

Table 144: $K_3^{(5)}$, $X = A_4^6$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
11	0	0	0	2	0	0	0
31	0	0	0	0	2	0	2
51	0	0	2	2	2	2	2
71	2	0	4	2	4	4	4
91	0	2	4	6	6	8	8
111	2	2	10	8	12	10	14
131	4	4	14	16	18	18	22
151	8	4	24	22	30	30	34
171	8	10	34	38	44	46	54
191	14	14	58	52	68	64	82

Table 145: $K_2^{(5)}$, $X = A_4^6$

	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}
16	0	0	0	0	1	1	0
36	0	0	1	1	1	1	2
56	2	2	2	2	2	2	2
76	0	0	4	4	4	4	6
96	2	2	6	6	8	8	8
116	2	2	10	10	12	12	18
136	4	4	16	16	22	22	22
156	6	6	26	26	32	32	42
176	12	12	40	40	50	50	56
196	13	13	60	60	74	74	94

Table 146: $K_4^{(5)}$, $X = A_4^6$

	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}
4	1	1	0	0	0	0	0
24	0	0	0	0	0	0	2
44	0	0	0	0	2	2	0
64	0	0	2	2	1	1	4
84	2	2	2	2	4	4	2
104	0	0	4	4	6	6	10
124	4	4	8	8	10	10	8
144	1	1	13	13	14	14	22
164	6	6	18	18	26	26	24
184	6	6	30	30	34	34	50

D.5 Lambency 6

Table 147: $K_1^{(6)}$, $X = A_5^4 D_4$

	χ_1	χ_2	χ_3	χ_4	χ_5
-1	-2	0	0	0	0
23	0	0	2	0	0
47	0	0	0	2	2
71	0	2	2	2	2
95	0	0	2	4	6
119	4	4	6	6	6
143	4	2	6	12	12
167	4	6	14	16	14
191	8	8	14	26	28
215	14	14	28	36	34
239	16	14	32	54	56

Table 148: $K_3^{(6)}$, $X = A_5^4 D_4$

	χ_1	χ_2	χ_3	χ_4	χ_5
15	0	0	0	2	0
39	2	0	2	2	2
63	0	2	2	4	4
87	4	2	6	6	8
111	2	4	6	14	10
135	8	6	14	18	20
159	8	10	18	30	28
183	18	14	32	42	46
207	18	22	40	68	62
231	36	30	66	92	96

Table 149: $K_5^{(6)}$, $X = A_5^4 D_4$

	χ_1	χ_2	χ_3	χ_4	χ_5
23	0	0	0	0	2
47	0	2	0	2	0
71	2	0	2	2	4
95	0	2	4	4	2
119	4	2	4	6	10
143	2	6	6	12	8
167	8	2	12	16	20
191	6	12	16	26	22
215	14	10	24	36	42
239	12	20	36	54	46
263	32	22	50	74	82

Table 150: $K_2^{(6)}$, $X = A_5^4 D_4$

	χ_6	χ_7	χ_8
20	0	0	2
44	2	2	2
68	2	2	6
92	4	4	10
116	8	8	14
140	12	12	24
164	18	18	38
188	28	28	54
212	40	40	82
236	58	58	118
260	84	84	164

Table 151: $K_4^{(6)}$, $X = A_5^4 D_4$

	χ_6	χ_7	χ_8
8	1	1	0
32	1	1	2
56	2	2	4
80	4	4	6
104	6	6	12
128	9	9	20
152	16	16	28
176	22	22	46
200	33	33	68
224	50	50	96
248	70	70	140
272	98	98	198

D.6 Lambency 6+3

Table 152: $K_1^{(6+3)}$, $X = D_4^6$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
-1	-2	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	2	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	2	0
71	0	0	0	2	0	0	2	0	0	0	0
95	0	0	0	0	0	2	2	0	0	2	0
119	0	0	2	2	2	0	2	2	2	0	2
143	0	0	2	0	0	0	4	2	2	4	2
167	0	2	0	2	4	2	6	4	4	2	4
191	0	0	4	2	2	4	8	4	4	8	6
215	2	0	4	6	6	2	14	8	8	6	8
239	0	0	6	2	4	8	18	10	10	14	12
263	2	4	8	12	10	8	26	16	16	14	18
287	0	2	14	10	8	12	38	20	20	26	22
311	4	4	14	18	22	16	52	30	30	28	32
335	4	4	24	18	18	22	70	38	38	50	44
359	6	6	28	36	32	28	96	56	56	54	60
383	6	6	40	34	36	44	128	70	70	86	78

Table 153: $K_3^{(6+3)}$, $X = D_4^6$

	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
15	1	1	0	0	0
39	0	0	0	2	0
63	0	0	0	0	2
87	0	0	2	2	2
111	2	2	0	2	4
135	1	1	4	4	6
159	2	2	2	6	10
183	2	2	8	10	14
207	6	6	6	12	22
231	6	6	16	20	30
255	10	10	16	24	46
279	10	10	28	40	62
303	20	20	30	50	88
327	22	22	52	74	118
351	34	34	60	94	164
375	41	41	94	134	216
399	62	62	108	172	292

D.7 Lambency 7

Table 154: $K_1^{(7)}$, $X = A_6^4$

	χ_1	χ_2	χ_3	χ_4
-1	−2	0	0	0
27	2	1	1	0
55	0	0	0	2
83	0	2	2	2
111	2	0	0	6
139	4	4	4	6
167	2	2	2	12
195	8	6	6	16
223	6	6	6	26
251	12	14	14	30

Table 155: $K_3^{(7)}$, $X = A_6^4$

	χ_1	χ_2	χ_3	χ_4
19	0	0	0	2
47	2	2	2	2
75	2	1	1	6
103	4	4	4	8
131	2	4	4	16
159	8	8	8	22
187	10	10	10	34
215	16	18	18	46
243	22	21	21	70
271	34	34	34	94

Table 156: $K_5^{(7)}$, $X = A_6^4$

	χ_1	χ_2	χ_3	χ_4
3	0	1	1	0
31	0	0	0	2
59	2	0	0	4
87	0	2	2	6
115	4	4	4	8
143	6	4	4	14
171	6	8	8	20
199	10	10	10	32
227	18	16	16	44
255	18	20	20	64

Table 157: $K_2^{(7)}$, $X = A_6^4$

	χ_5	χ_6	χ_7
24	2	0	0
52	2	2	2
80	2	4	4
108	6	5	5
136	8	8	8
164	12	14	14
192	20	17	17
220	28	28	28
248	36	40	40
276	56	54	54

Table 158: $K_4^{(7)}$, $X = A_6^4$

	χ_5	χ_6	χ_7
12	0	1	1
40	2	2	2
68	4	2	2
96	6	6	6
124	8	8	8
152	14	14	14
180	18	20	20
208	30	30	30
236	42	40	40
264	60	60	60

Table 159: $K_6^{(7)}$, $X = A_6^4$

	χ_5	χ_6	χ_7
20	2	0	0
48	0	1	1
76	2	2	2
104	2	2	2
132	4	6	6
160	6	6	6
188	12	10	10
216	12	14	14
244	22	22	22
272	28	26	26

D.8 Lambency 8

Consider the case $\ell = 8$, $X = A_7^2 D_5^2$. For the components with $r = 0 \bmod 2$ we have the module structure

$$\sum_{\substack{d>0 \\ 32d=-r^2 \bmod 32}} q^d K_{r,d}^{(8)} = \frac{1}{2}\chi_5 H_r^{(8)}. \tag{D.7}$$

Table 160: $K_1^{(8)}$, $X = A_7^2 D_5^2$

	χ_1	χ_2	$\chi_3 + \chi_4$
-1	-2	0	0
31	0	2	0
63	0	0	2
95	2	2	2
127	2	2	4
159	4	6	2
191	6	6	8
223	10	10	8
255	12	10	14
287	18	20	16

Table 161: $K_5^{(8)}$, $X = A_7^2 D_5^2$

	χ_1	χ_2	$\chi_3 + \chi_4$
7	0	2	0
39	2	2	2
71	2	4	2
103	6	4	6
135	6	10	6
167	14	10	12
199	14	18	16
231	26	22	26
263	30	36	30
295	48	44	48

Table 162: $K_3^{(8)}$, $X = A_7^2 D_5^2$

	χ_1	χ_2	$\chi_3 + \chi_4$
23	0	0	2
55	4	2	2
87	2	4	4
119	8	6	6
151	8	8	10
183	16	14	14
215	16	20	20
247	32	28	28
279	36	38	40
311	58	54	52

Table 163: $K_7^{(8)}$, $X = A_7^2 D_5^2$

	χ_1	χ_2	$\chi_3 + \chi_4$
15	2	0	0
47	0	2	0
79	4	0	2
111	0	2	2
143	6	4	4
175	2	6	4
207	12	6	8
239	8	12	10
271	20	14	16
303	14	22	18

D.9 Lambency 9

Table 164: $K_1^{(9)}$, $X = A_8^3$

	χ_1	χ_2	χ_3
-1	−6	0	0
35	0	0	0
71	0	0	6
107	0	0	6
143	0	6	6
179	6	6	12
215	6	6	18
251	12	12	18
287	12	18	30
323	18	18	42

Table 165: $K_5^{(9)}$, $X = A_8^3$

	χ_1	χ_2	χ_3
11	0	0	6
47	6	0	6
83	6	6	12
119	6	6	18
155	12	12	24
191	18	18	36
227	24	24	54
263	36	36	66
299	48	48	96
335	66	60	132

Table 166: $K_7^{(9)}$, $X = A_8^3$

	χ_1	χ_2	χ_3
23	0	6	0
59	0	0	6
95	6	6	6
131	6	6	12
167	6	12	18
203	12	12	24
239	24	18	36
275	24	24	54
311	30	36	66
347	48	48	90

Table 167: $K_2^{(9)}$, $X = A_8^3$

	χ_5	χ_6	χ_7
32	0	0	6
68	0	6	6
104	6	6	6
140	6	6	18
176	12	6	18
212	12	18	30
248	18	18	42
284	24	30	54
320	42	36	72
356	48	54	102

Table 168: $K_4^{(9)}$, $X = A_8^3$

	χ_5	χ_6	χ_7
20	6	0	0
56	6	6	6
92	6	0	12
128	6	12	18
164	18	12	24
200	18	24	42
236	30	24	48
272	36	42	78
308	54	42	102
344	66	78	138

Table 169: $K_8^{(9)}$, $X = A_8^3$

	χ_5	χ_6	χ_7
8	0	6	0
44	6	0	0
80	0	6	6
116	6	0	0
152	0	12	12
188	6	0	12
224	6	18	18
260	18	6	18
296	12	24	36
332	30	12	36

Table 170: $K_3^{(9)}$, $X = A_8^3$

	χ_1	χ_2	χ_3
27	2	2	4
63	0	6	6
99	6	6	12
135	12	6	18
171	12	12	24
207	18	18	36
243	26	26	52
279	36	36	72
315	48	48	96
351	60	66	126
387	84	84	168
423	114	108	222
459	144	144	288
495	180	186	366
531	234	234	468
567	300	294	594
603	372	372	744
639	462	468	930
675	578	578	1156

Table 171: $K_6^{(9)}$, $X = A_8^3$

	χ_4	χ_5	χ_6
0	2	-1	1
36	0	6	6
72	6	0	6
108	4	10	14
144	12	6	18
180	12	18	30
216	24	12	36
252	24	36	60
288	42	30	72
324	48	60	108
360	72	60	132
396	84	102	186
432	124	106	230
468	144	168	312
504	204	180	384
540	240	264	504
576	324	294	618
612	384	414	798
648	504	468	972

D.10 Lambency 10+5

Table 172: $K_1^{(10+5)}$, $X = D_6^4$

	χ_1	χ_2	χ_3	χ_4	χ_5
-1	-2	0	0	0	0
39	2	0	0	0	0
79	0	0	0	0	2
119	0	0	2	0	0
159	0	2	0	2	2
199	2	0	2	2	0
239	0	0	2	2	4
279	2	2	2	4	2
319	0	2	2	6	6
359	2	0	6	6	4
399	2	4	4	8	12
439	4	4	8	10	8
479	2	4	8	14	16
519	10	6	12	18	14
559	6	8	14	22	26

Table 173: $K_3^{(10+5)}$, $X = D_6^4$

	χ_1	χ_2	χ_3	χ_4	χ_5
31	0	0	0	0	2
71	0	0	2	2	0
111	0	2	2	2	2
151	2	0	2	4	2
191	0	2	2	4	6
231	4	2	4	6	6
271	2	4	6	8	10
311	6	2	8	12	10
351	4	6	10	14	18
391	8	6	14	22	18
431	6	10	18	26	28
471	14	10	22	36	32
511	12	16	28	42	46
551	20	16	38	56	52
591	20	26	46	68	72

Table 174: $K_5^{(10+5)}$, $X = D_6^4$

	χ_1	χ_2	χ_3	χ_4	χ_5
15	0	0	2	0	0
55	0	0	0	2	2
95	2	2	0	2	2
135	0	0	2	4	4
175	2	2	4	4	4
215	2	2	4	8	8
255	2	2	8	8	8
295	4	4	8	14	14
335	8	8	12	16	16
375	6	6	14	24	24
415	10	10	20	28	28
455	14	14	24	40	40
495	16	16	36	46	46
535	20	20	40	64	64
575	28	28	52	76	76

D.11 Lambency 12+4

Table 175: $K_1^{(12+4)}$, $X = E_6^4$

	χ_1	χ_2	χ_3	χ_4	χ_5
-1	-2	0	0	0	0
47	0	0	2	0	0
95	0	0	0	2	0
143	0	0	2	0	2
191	0	0	0	2	2
239	2	2	2	2	2
287	0	0	2	4	2
335	2	0	4	4	4
383	0	2	2	6	6
431	4	2	6	6	6

Table 176: $K_5^{(12+4)}$, $X = E_6^4$

	χ_1	χ_2	χ_3	χ_4	χ_5
23	0	2	0	0	0
71	0	0	0	0	2
119	0	0	0	2	0
167	2	0	0	2	2
215	0	2	2	2	0
263	2	0	2	2	4
311	2	2	2	4	2
359	2	0	4	4	6
407	0	4	4	6	4
455	4	2	4	8	10

Table 177: $K_4^{(12+4)}$, $X = E_6^4$

	χ_6	χ_7	χ_8
32	1	1	0
80	0	0	2
128	1	1	2
176	2	2	2
224	2	2	4
272	2	2	6
320	4	4	6
368	4	4	10
416	6	6	12
464	8	8	14

D.12 Lambency 13

Table 178: $K_1^{(13)}$, $X = A_{12}^2$

	χ_1	χ_2
-1	-2	0
51	2	0
103	0	2
155	0	0
207	0	2
259	2	0
311	2	2
363	4	2
415	2	4
467	6	2

Table 179: $K_3^{(13)}$, $X = A_{12}^2$

	χ_1	χ_2
43	0	2
95	2	0
147	2	2
199	4	2
251	2	6
303	6	4
355	6	8
407	10	8
459	10	12
511	14	12

Table 180: $K_5^{(13)}$, $X = A_{12}^2$

	χ_1	χ_2
27	2	0
79	2	2
131	4	2
183	2	4
235	6	4
287	6	8
339	8	8
391	10	12
443	16	14
495	16	20

Table 181: $K_2^{(13)}$, $X = A_{12}^2$

	χ_3	χ_4
48	0	0
100	1	1
152	2	2
204	2	2
256	3	3
308	4	4
360	4	4
412	6	6
464	8	8
516	10	10

Table 182: $K_4^{(13)}$, $X = A_{12}^2$

	χ_3	χ_4
36	1	1
88	2	2
140	2	2
192	4	4
244	4	4
296	6	6
348	8	8
400	11	11
452	12	12
504	18	18

Table 183: $K_6^{(13)}$, $X = A_{12}^2$

	χ_3	χ_4
16	1	1
68	2	2
120	2	2
172	4	4
224	4	4
276	8	8
328	8	8
380	12	12
432	14	14
484	19	19

Table 184: $K_7^{(13)}$, $X = A_{12}^2$

	χ_1	χ_2
3	0	2
55	2	0
107	2	2
159	4	4
211	4	6
263	6	6
315	8	8
367	12	10
419	12	14
471	18	16

Table 185: $K_9^{(13)}$, $X = A_{12}^2$

	χ_1	χ_2
23	0	2
75	2	2
127	2	2
179	4	2
231	4	4
283	6	6
335	6	8
387	10	10
439	12	14
491	16	14

Table 186: $K_{11}^{(13)}$, $X = A_{12}^2$

	χ_1	χ_2
35	2	0
87	2	0
139	0	2
191	2	2
243	2	2
295	4	4
347	4	6
399	6	4
451	8	8
503	10	10

Table 187: $K_8^{(13)}$, $X = A_{12}^2$

	χ_3	χ_4
40	2	2
92	2	2
144	3	3
196	3	3
248	6	6
300	6	6
352	10	10
404	12	12
456	16	16
508	18	18

Table 188: $K_{10}^{(13)}$, $X = A_{12}^2$

	χ_3	χ_4
4	1	1
56	0	0
108	2	2
160	2	2
212	4	4
264	4	4
316	6	6
368	6	6
420	10	10
472	10	10

Table 189: $K_{12}^{(13)}$, $X = A_{12}^2$

	χ_3	χ_4
12	0	0
64	1	1
116	0	0
168	2	2
220	0	0
272	2	2
324	1	1
376	4	4
428	2	2
480	6	6

D.13 Lambency 14+7

Table 190: $K_1^{(14+7)}$, $X = D_8^3$

	χ_1	χ_2	χ_3
-1	-2	0	0
55	2	0	0
111	0	0	0
167	0	0	2
223	0	2	0
279	2	0	2
335	0	2	2
391	2	2	2
447	0	2	2
503	4	0	6

Table 191: $K_3^{(14+7)}$, $X = D_8^3$

	χ_1	χ_2	χ_3
47	0	2	0
103	2	0	2
159	0	2	2
215	2	0	4
271	0	2	4
327	4	2	6
383	2	4	6
439	6	4	10
495	4	6	10
551	10	6	14

Table 192: $K_5^{(14+7)}$, $X = D_8^3$

	χ_1	χ_2	χ_3
31	0	0	2
87	0	2	2
143	2	2	2
199	2	2	4
255	4	2	6
311	4	4	6
367	4	4	10
423	6	6	12
479	8	8	14
535	8	10	18

Table 193: $K_7^{(14+7)}$, $X = D_8^3$

	χ_1	χ_2	χ_3
7	0	2	0
63	2	0	2
119	0	0	4
175	4	2	4
231	0	4	4
287	4	4	8
343	4	6	8
399	8	4	12
455	4	8	16
511	12	8	20

D.14 Lambency 30+6,10,15

Table 194: $K_1^{(30+6,10,15)}$, $X = E_8^3$

	χ_1	χ_2	χ_3
-1	-2	0	0
119	2	0	0
239	0	2	0
359	0	0	2
479	0	2	0
599	2	0	2
719	0	0	2
839	2	0	2
959	0	2	2
1079	2	0	4

Table 195: $K_7^{(30+6,10,15)}$, $X = E_8^3$

	χ_1	χ_2	χ_3
71	0	2	0
191	0	0	2
311	0	0	2
431	2	0	2
551	0	2	2
671	2	2	2
791	2	2	2
911	2	2	4
1031	0	2	4
1151	4	2	4

References

[1] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine," arXiv:1204.2779 [math.RT].

[2] H.-V. Niemeier, "Definite quadratische Formen der Dimension 24 und Diskriminante 1," *J. Number Theory* **5** (1973) 142–178.

[3] J. Leech, "Notes on Sphere Packings," *Canad. J. Math.* **19** (1967) 251–267.

[4] J. G. Thompson, "Finite groups and modular functions," *Bull. London Math. Soc.* **11** (1979) no. 3, 347–351.

[5] J. H. Conway and S. P. Norton, "Monstrous Moonshine," *Bull. London Math. Soc.* **11** (1979) 308 339.

[6] T. Gannon, *Moonshine beyond the monster. The bridge connecting algebra, modular forms and physics.* Cambridge University Press, 2006.

[7] T. Eguchi and A. Taormina, "Unitary representations of the $N = 4$ superconformal algebra," *Phys. Lett. B* **196** (1987) no. 1, 75–81. http://dx.doi.org/10.1016/0370-2693(87)91679-0.

[8] T. Eguchi and A. Taormina, "Character formulas for the $N = 4$ superconformal algebra," *Phys. Lett. B* **200** (1988) no. 3, 315–322. http://dx.doi.org/10.1016/0370-2693(88)90778-2.

[9] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, "Superconformal Algebras and String Compactification on Manifolds with $SU(N)$ Holonomy," *Nucl. Phys.* **B315** (1989) 193.

[10] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions," *J.Phys.A* **42:304010,2009** (Dec., 2008) , 0812.1151 [math-ph].

[11] T. Eguchi, H. Ooguri, and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," *Exper.Math.* **20** (2011) 91–96, arXiv:1004.0956 [hep-th].

[12] M. C. N. Cheng, "$K3$ Surfaces, $N = 4$ Dyons, and the Mathieu Group M_{24}," 1005.5415 [hep-th].

[13] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu twining characters for $K3$," *JHEP* **1009** (2010) 058, arXiv:1006.0221 [hep-th].

[14] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," *JHEP* **1010** (2010) 062, arXiv:1008.3778 [hep-th].

[15] T. Eguchi and K. Hikami, "Note on Twisted Elliptic Genus of K3 Surface," *Phys.Lett.* **B694** (2011) 446–455, arXiv:1008.4924 [hep-th].

[16] T. Gannon, "Much ado about Mathieu," arXiv:1211.5531 [math.RT].

[17] M. C. N. Cheng and J. F. R. Duncan, "The Largest Mathieu Group and (Mock) Automorphic Forms," 1201.4140 [math.RT].

[18] M. R. Gaberdiel and R. Volpato, "Mathieu Moonshine and Orbifold $K3$'s," arXiv:1206.5143 [hep-th].

[19] T. Creutzig, G. Hoehn, and T. Miezaki, "The McKay-Thompson series of Mathieu Moonshine modulo two," arXiv:1211.3703 [math.NT].

[20] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch, and R. Volpato, "Generalised Mathieu Moonshine," arXiv:1211.7074 [hep-th].

[21] M. R. Gaberdiel, D. Persson, and R. Volpato, "Generalised Moonshine and Holomorphic Orbifolds," arXiv:1302.5425 [hep-th].

[22] A. Taormina and K. Wendland, "Symmetry-surfing the Moduli Space of Kummer $K3$'s," arXiv:1303.2931 [hep-th].

[23] A. Taormina and K. Wendland, "A Twist in the M_{24} Moonshine Story," arXiv:1303.3221 [hep-th].

[24] A. Cappelli, C. Itzykson, and J. Zuber, "The ADE Classification of Minimal and $A_1^{(1)}$ Conformal Invariant Theories," *Commun.Math.Phys.* **113** (1987) 1.

[25] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[26] J. E. Humphreys, *Introduction to Lie algebras and representation theory.* Springer-Verlag, New York, 1972. Graduate Texts in Mathematics, Vol. 9.

[27] J. H. Conway, "A characterisation of Leech's lattice," *Invent. Math.* **7** (1969) 137–142.

[28] J. Leech, "Some Sphere Packings in Higher Space," *Canad. J. Math.* **16** (1964) 657–682.

[29] J. H. Conway, R. A. Parker, and N. J. A. Sloane, "The covering radius of the Leech lattice," *Proc. Roy. Soc. London Ser. A* **380** (1982) no. 1779, 261–290. http://dx.doi.org/10.1098/rspa.1982.0042.

[30] J. H. Conway and N. J. A. Sloane, "Twenty-three constructions for the Leech lattice," *Proc. Roy. Soc. London Ser. A* **381** (1982) no. 1781, 275–283. http://dx.doi.org/10.1098/rspa.1982.0071.

[31] G. Shimura, *Introduction to the arithmetic theory of automorphic functions.* Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971. Kanô Memorial Lectures, No. 1.

[32] C. R. Ferenbaugh, "The genus-zero problem for $n|h$-type groups," *Duke Math. J.* **72** (1993) no. 1, 31–63.

[33] J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.* Oxford: Clarendon Press, 1985.

[34] J. McKay, "Graphs, singularities, and finite groups," in *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, vol. 37 of *Proc. Sympos. Pure Math.*, pp. 183–186. Amer. Math. Soc., Providence, R.I., 1980.

[35] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, vol. 815 of *Lecture Notes in Mathematics.* Springer, Berlin, 1980.

[36] G. Gonzalez-Sprinberg and J.-L. Verdier, "Construction géométrique de la correspondance de McKay," *Ann. Sci. École Norm. Sup. (4)* **16** (1983) no. 3, 409–449 (1984). http://www.numdam.org/item?id=ASENS_1983_4_16_3_409_0.

[37] J. H. Bruinier and J. Funke, "On two geometric theta lifts," *Duke Math. J.* **125** (2004) no. 1, 45–90. http://dx.doi.org/10.1215/S0012-7094-04-12513-8.

[38] S. Zwegers, *Mock Theta Functions.* PhD thesis, Utrecht University, 2002.

[39] D. Zagier, "Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)," *Astérisque* (2009) no. 326, Exp. No. 986, vii–viii, 143–164 (2010). Séminaire Bourbaki. Vol. 2007/2008.

[40] M. Eichler and D. Zagier, *The theory of Jacobi forms.* Birkhäuser, 1985.

[41] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," arXiv:1208.4074 [hep-th].

[42] G. Shimura, "On modular forms of half integral weight," *Ann. of Math. (2)* **97** (1973) 440–481.

[43] D. Gepner and Z.-a. Qiu, "Modular Invariant Partition Functions for Parafermionic Field Theories," *Nucl.Phys.* **B285** (1987) 423.

[44] N.-P. Skoruppa and D. Zagier, "Jacobi forms and a certain space of modular forms," *Invent. Math.* **94** (1988) no. 1, 113–146. http://dx.doi.org/10.1007/BF01394347.

[45] N.-P. Skoruppa, "Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms," *Invent. Math.* **102** (1990) no. 3, 501–520. http://dx.doi.org/10.1007/BF01233438.

[46] N.-P. Skoruppa, "Heegner cycles, modular forms and Jacobi forms," *Sém. Théor. Nombres Bordeaux (2)* **3** (1991) no. 1, 93–116. http://jtnb.cedram.org/item?id=JTNB_1991__3_1_93_0.

[47] G. Hoehn, "Selbstduale Vertexoperatorsuperalgebren und das Babymonster (Self-dual Vertex Operator Super Algebras and the Baby Monster)," *Bonner Mathematische Schriften, Vol.* **286,** (June, 2007) 1–85,Bonn1996, 0706.0236 [math.QA].

[48] E. Witten, "Three-Dimensional Gravity Revisited," 0706.3359 [hep-th].

[49] N.-P. Skoruppa, *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts.* Bonner Mathematische Schriften [Bonn Mathematical Publications], 159. Universität Bonn Mathematisches Institut, Bonn, 1985. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1984.

[50] S. Ramanujan, *The lost notebook and other unpublished papers.* Springer-Verlag, Berlin, 1988. With an introduction by George E. Andrews.

[51] B. Gordon and R. J. McIntosh, "Some Eighth Order Mock Theta Functions," *J. London Math. Soc. (2)* **62** (2000) no. 2, 321–335. http://dx.doi.org/10.1112/S0024610700008735.

[52] R. J. McIntosh, "Second Order Mock Theta Functions," *Canad. Math. Bull.* **50** (2007) no. 2, 284–290. http://dx.doi.org/10.4153/CMB-2007-028-9.

[53] M. C. N. Cheng and J. F. R. Duncan, "On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine," 1110.3859 [math.RT].

[54] M. C. N. Cheng and J. F. R. Duncan, "On the Discrete Groups of Mathieu Moonshine," *ArXiv e-prints* (Dec., 2012) , arXiv:1212.0906 [math.NT].

[55] M. Benard, "Schur indexes of sporadic simple groups," *J. Algebra* **58** (1979) no. 2, 508–522. http://dx.doi.org/10.1016/0021-8693(79)90177-7.

[56] R. S. Margolin, "Representations of M_{12}," *J. Algebra* **156** (1993) no. 2, 362–369. http://dx.doi.org/10.1006/jabr.1993.1078.

[57] T. Creutzig, G. Hoehn, and T. Miezaki, "The McKay-Thompson series of Mathieu Moonshine modulo two," *ArXiv e-prints* (Nov., 2012) , arXiv:1211.3703 [math.NT].

[58] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "Counting Dyons in N=4 String Theory," *Nucl.Phys.B* **484:543-561,1997** (Nucl.Phys.B484:543-561,1997) , hep-th/9607026.

[59] T. Kawai, "$N = 2$ Heterotic String Threshold Correction, $K3$ Surface and Generalized Kac-Moody Superalgebra," *Phys.Lett.* **B372** (1996) 59–64, arXiv:hep-th/9512046 [hep-th].

[60] G. Lopes Cardoso, G. Curio, and D. Lust, "Perturbative couplings and modular forms in N=2 string models with a Wilson line," *Nucl.Phys.* **B491** (1997) 147–183, arXiv:hep-th/9608154 [hep-th].

[61] M. Raum, "M_{24}-twisted Product Expansions are Siegel Modular Forms," arXiv:1208.3453 [math.NT].

[62] V. A. Gritsenko and V. V. Nikulin, "$K3$ surfaces, Lorentzian Kac-Moody algebras and mirror symmetry," *Math. Res. Lett.* **3** (1996) no. 2, 211–229.

[63] V. A. Gritsenko and V. V. Nikulin, "The Arithmetic Mirror Symmetry and Calabi-Yau manifolds," *Comm. Math. Phys.* **210** (2000) no. 1, 1–11. http://dx.doi.org/10.1007/s002200050769.

[64] V. A. Gritsenko and V. V. Nikulin, "Automorphic forms and Lorentzian Kac-Moody algebras. I," *Internat. J. Math.* **9** (1998) no. 2, 153–199. http://dx.doi.org/10.1142/S0129167X98000105.

[65] V. A. Gritsenko and V. V. Nikulin, "Automorphic forms and Lorentzian Kac-Moody algebras. II," *Internat. J. Math.* **9** (1998) no. 2, 201–275. http://dx.doi.org/10.1142/S0129167X98000117.

[66] V. A. Gritsenko and V. V. Nikulin, "Siegel Automorphic Form Corrections of Some Lorentzian Kac-Moody Lie algebras," *Amer. J. Math.* **119** (1997) no. 1, 181–224. http://muse.jhu.edu/journals/american_journal_of_mathematics/v119/119.1gritsenko.pdf.

[67] S. Kondo, "Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces. With an appendix by Shigeru Mukai.," *Duke Math. J* **92, No.3** (1998) 593–603.

[68] V. V. Nikulin, "Kählerian $K3$ Surfaces and Niemeier Lattices, I," arXiv:1109.2879 [math.AG].

[69] M. C. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, *et al.*, "Mathieu Moonshine and N=2 String Compactifications," arXiv:1306.4981 [hep-th].

[70] H. Aoki and T. Ibukiyama, "Simple graded rings of Siegel modular forms, differential operators and Borcherds products," *Int. J. Math.* **16, No. 3** (2005) 249–279.

[71] V. Gritsenko, "Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms," *Algebra i Analiz* **11** (1999) no. 5, 100–125.

[72] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2005. (`http://www.gap-system.org`).

arXiv:1406.0619v4 [hep-th] 18 Jul 2017

Umbral Moonshine and $K3$ Surfaces

Miranda C. N. Cheng*[1] and Sarah Harrison†[2]

[1]Institute of Physics and Korteweg-de Vries Institute for Mathematics,
University of Amsterdam, Amsterdam, the Netherlands‡

[2]Stanford Institute for Theoretical Physics, Department of Physics, and Theory Group,
SLAC, Stanford University, Stanford, CA 94305, USA

Abstract

Recently, 23 cases of umbral moonshine, relating mock modular forms and finite groups, have been discovered in the context of the 23 even unimodular Niemeier lattices. One of the 23 cases in fact coincides with the so-called Mathieu moonshine, discovered in the context of $K3$ non-linear sigma models. In this paper we establish a uniform relation between all 23 cases of umbral moonshine and $K3$ sigma models, and thereby take a first step in placing umbral moonshine into a geometric and physical context. This is achieved by relating the ADE root systems of the Niemeier lattices to the ADE du Val singularities that a $K3$ surface can develop, and the configuration of smooth rational curves in their resolutions. A geometric interpretation of our results is given in terms of the marking of $K3$ surfaces by Niemeier lattices.

*mcheng@uva.nl
†sarharr@stanford.edu
‡On leave from CNRS, Paris.

Contents

1 Introduction and Summary 3

2 The Elliptic Genus of Du Val Singularities 8

3 Umbral Moonshine and Niemeier Lattices 14

4 Umbral Moonshine and the (Twined) $K3$ Elliptic Genus 20

5 Geometric Interpretation 27

6 Discussion 30

A Special Functions 32

B Calculations and Proofs 34

C The Twining Functions 41

References 49

1 Introduction and Summary

Mock modular forms are interesting functions playing an increasingly important role in various areas of mathematics and theoretical physics. The "Mathieu moonshine" phenomenon relating certain mock modular forms and the sporadic group M_{24} was surprising, and its apparent relation to non-linear sigma models of $K3$ surfaces even more so. The fundamental role played by two-dimensional supersymmetric conformal field theories and $K3$ compactifications makes this moonshine relation interesting not just for mathematicians but also for string theorists. In 2013 it was realised that this Mathieu moonshine is but just one case out of 23 such relations, called "umbral moonshine". The 23 cases admit a uniform construction from the 23 even unimodular positive-definite lattices of rank 24 labeled by their non-trivial root systems. While the discovery of these 23 cases of moonshine perhaps adds to the beauty of the Mathieu moonshine relation, it also adds more mystery. In particular, it was previously entirely unclear what the physical or geometrical context for these other 22 instances of umbral moonshine could be. In this paper we establish a relation between $K3$ sigma models and all 23 cases of umbral moonshine, and thereby take a first step in incorporating umbral moonshine into the realm of geometry and theoretical physics.

Background

In mathematics, the term "moonshine" is used to refer to a particular type of relation between modular objects and finite groups. It was first introduced to describe the remarkable "monstrous moonshine" phenomenon [1] relating modular functions such as the J-function discussed below and the "Fischer–Griess monster group" $\mathbb{M}$, the largest of the 26 sporadic groups in the classification of finite simple groups. The study of this mysterious phenomenon was initiated by the observation by J. McKay that the second coefficient in the Fourier expansion of the modular function

$$\begin{aligned} J(\tau) &= J(\tau+1) = J(-1/\tau) \\ &= \sum_{m\geq -1} a(m)\, q^m = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots \end{aligned} \tag{1.1}$$

with $q = e^{2\pi i\tau}$ satisfies $196884 = 196883+1$, and 196883 is precisely the dimension of the smallest non-trivial representation of $\mathbb{M}$. Note that the J-function has the mathematical significance as the unique holomorphic function on the upper-half plane $\mathbb{H}$ invariant under the natural action of $PSL_2(\mathbb{Z})$ generated by $\tau \to \tau+1$ and $\tau \to -1/\tau$, that moreover has the behaviour $J(\tau) = q^{-1} + O(q)$ near the cusp $\tau \to i\infty$. Why and how the specific modular functions and the

monster group, usually thought of as belonging to two very different branches of mathematics, are related to each other, remained a puzzle until about a decade after its discovery.

The key structure that unifies the two turns out to be that of a (chiral) 2d conformal field theory (CFT), or vertex operator algebra in more mathematical terms [2,3]. The two sides of moonshine – the modularity and the finite group symmetry – can naturally be viewed as the manifestation of two kinds of symmetries – the world-sheet and the space-time symmetries– the CFT possesses. The mathematical proof of monstrous moonshine is achieved by constructing a generalised Kac–Moody algebra based on the above chiral CFT and utilising the no-ghost theorem of string theory, which roughly corresponds to considering the full 26 dimensions including the 2 light-cone directions of the bosonic string theory [4]. We refer to, for instance, [5] for an introduction on the theory of modular forms and to [6] or the introduction of [7] for a summary of monstrous moonshine.

In 2010, an entirely unexpected new observation, pointing towards a new type of moonshine relating "mock modular forms" and finite groups, was made in the context of the elliptic genus of $K3$ surfaces. Mock modular forms embody a novel variation of the concept of modular forms and are interesting due to their significance in number theory as well as a wide range of applications (cf. (3.3)). See, for instance, [8,9] for an expository account on mock modular forms. From a physical point of view, as demonstrated in a series of recent works, the "mockness" of mock modular forms is often related to the non-compactness of relevant spaces in the theory. See, for instance, [10–13].

As we will discuss in more detail in §4, the elliptic genus $\mathbf{EG}(K3)$ of $K3$ surfaces enumerates the BPS states of a $K3$ non-linear sigma model, and by taking the $\mathcal{N}=4$ superconformal symmetry of this theory into account, one arrives at a weight $1/2$ mock modular form with Fourier expansion [14–16]

$$H_1^{X=A_1^{24}}(\tau) = 2q^{-1/8}(-1+45\,q+231\,q^2+770\,q^3+O(q^4)). \tag{1.2}$$

The observation by Eguchi–Ooguri–Tachikawa then states that the numbers 45, 231, and 770 are all dimensions of certain irreducible representations of the sporadic Mathieu group M_{24} [17]. This connection has since been studied, refined, extended, and finally established in [18–28]. From a mathematical point of view, the prospect of a novel type of moonshine for mock modular forms is extremely exciting. From a physical point of view, the ubiquity of $K3$ surfaces and the importance of BPS spectra in the study of string theory makes this "Mathieu moonshine" potentially much more relevant than the previous monstrous moonshine. See [29] for a review and [30–34], [35–40] for some of the explorations in string theory and $K3$ conformal field theories

inspired by this connection.

In 2013, the above relation was realised to be just the tip of the iceberg, or less metaphorically just one case out of a series of such relations, called "umbral moonshine" [7, 41]. As will be reviewed in more detail in §3, to each one of the 23 Niemeier lattices L^X – the 23 even unimodular positive-definite lattices of rank 24 labeled by their non-trivial root systems X – one can attach on the one hand a finite group G^X and on the other hand a vector-valued mock modular form H^X, such that the Fourier coefficients of H^X are again suggestive of a relation to certain representations of G^X, analogous to the observation on the functions $J(\tau)$ and $H_1^{X=A_1^{24}}(\tau)$ in (1.1) and (1.2). Further evidence for this relation was provided by relating characters of the same G^X-representations to the Fourier coefficients of other mock modular forms H_g^X, for each conjugacy class $[g]$ of G^X. More precisely, it was conjectured that an infinite-dimensional G^X-module K^X reproduces the mock modular forms H_g^X as its graded g-characters. The finite group G^X is defined by considering the symmetries of the Niemeier lattice L^X, while the mock modular form is determined by its root system X. The important role played by the rank 24 root systems X suggests the importance of the corresponding 24-dimensional representation of G^X. For instance, for the Niemeier lattice with the simplest root system $X = A_1^{24}$, the mock modular form $H^{A_1^{24}}$ is simply given by the function (1.2) above, and the finite group is $G^X \cong M_{24}$. In this case the umbral moonshine *is* the Mathieu moonshine first observed in the context of the $K3$ elliptic genus that we described above. Given the uniform construction of the 23 instances of umbral moonshine from the Niemeier lattices L^X, one is naturally led to the following questions: *What about the other 22 cases of umbral moonshine with $X \neq A_1^{24}$? What, if any, is the physical and geometrical relevance of umbral moonshine? Are they also related to string or conformal field theories on $K3$? What is the relation between $K3$ and the Niemeier lattices L^X? And the group G^X? The mock modular form H^X and the underlying G^X–module K^X?*

<u>*Summary*</u>

In the present paper we propose a first step in answering the above questions. To discuss the relation between the mock modular form H^X and the $K3$ elliptic genus, we first take a closer look at the construction of H^X from the root system X. For any of the 23 Niemeier lattices, the root system is a union of simply-laced root systems with an ADE classification with the same Coxeter number m. As is well-known, a wide variety of elegant structures in mathematics and physics admit an ADE classification. Apart from the simply-laced root systems, another such structure that will be important for us is that of modular invariant combinations of characters of the $A_1^{(1)}$ Kac–Moody algebra at level $m-2$ [42]. As will be reviewed in more detail in §2,

this classification leads to the introduction of the so-called Cappelli–Itzykson–Zuber matrices for every ADE root system, and these matrices in turn determine the relevant mock modular properties, which uniquely determine H^X when combined with a certain analyticity condition. Hence, the Cappelli–Itzykson–Zuber matrices Ω^Φ constitute a key element in the construction of the 23 instances of umbral moonshine.

By itself, the question of the classification of certain modular invariants seems remote from any physics or geometry. However, the parafermionic description of the $\mathcal{N}=2$ minimal models relates this classification to that of the $\mathcal{N}=2$ minimal superconformal field theories [43–46]. Moreover, their seemingly mysterious ADE classification can be related to the ADE classification of du Val (or Kleinian, or rational) surface singularities [45,46], whose minimal resolution gives rise to smooth rational (genus 0) curves with intersection given by the corresponding ADE Dynkin diagram. A third way to think about the ADE classification is the fact that these du Val singularities are isomorphic to the quotient singularity $\mathbb{C}^2/G$, with G being the finite subgroup of $SU_2(\mathbb{C})$ with the corresponding ADE classification [47]. Therefore, a perhaps simple-minded but logical step towards understanding the physical and geometrical context of umbral moonshine would be to take the ADE origin of the mock modular form H^X seriously. In particular we would like to explore if the ADE-ology in umbral moonshine can be related to that of the du Val singularities.

Recall that the du Val singularities are precisely the singularities a $K3$ surface can develop. After computing the elliptic genus of du Val singularities (see §2), one realises that the $K3$ elliptic genus can naturally be split into two parts: one is the contribution from the configuration of the singularities given by X and the other is the contribution from the mock modular form H^X. Equipped with the mock modular form H^X for the other 22 Niemeier root systems X constructed in umbral moonshine, one finds that the same splitting holds uniformly for all 23 instances of umbral moonshine (cf. (4.9)). Note that this splitting makes no reference to the $\mathcal{N}=4$ characters, although for the special case $X=A_1^{24}$ the two considerations render the same result.

While the above fact might be surprising and suggestive, one should be careful not to claim a strong connection between umbral moonshine and $K3$ string theory too quickly: it's logically possible that the above relation is just a consequence of the fact that the space of the relevant modular objects, the Jacobi forms of weight 0 and index 1 to be more precise, is very constrained and in fact only one-dimensional. See Appendix B for more details.

To gather more evidence that the umbral moonshine – a conjecture on the existence of a G^X–module K^X which (re)produces the mock modular forms H_g^X, $[g]\subset G^X$ as its graded characters – and the $K3$ sigma model, one should compare the way G^X acts on K^X with the

way the BPS spectrum of the $K3$ CFT transforms under its finite group symmetry G, when such a non-trivial G exists. Let us first focus on the geometric symmetries of $K3$ surfaces (as opposed to "stringy" CFT symmetries without direct geometric origins). As we will review in more detail in §5, thanks to the global Torelli theorem for $K3$, we know that a finite group G is the group of hyper-Kähler-preserving symmetries of a certain $K3$ surface M if and only if it acts on the 24-dimensional $K3$ cohomology lattice $H^*(M,\mathbb{Z})$ in a certain way. Relating this 24-dimensional representation of G to the natural 24-dimensional representation of G^X induced from its action on the root system X, this translates into a criterion for a conjugacy class $[g] \subset G^X$ to arise as a $K3$ symmetries for each of the 23 G^X.

On the one hand, umbral moonshine suggests a "twined" function Z_g^X for each $[g] \subset G^X$, where $Z_g^X = \mathbf{EG}(K3)$ for the special case that $[g]$ is the identity class (cf. (4.12)). In particular, from this consideration we arrive at a conjecture for the elliptic genus of the du Val singularity twined by its symmetries given by the automorphism of the corresponding Dynkin diagram. On the other hand, whenever the CFT admits a non-trivial finite automorphism group G, one can compute the elliptic genus "twined" by any $g \in G$. These twined elliptic genera $\mathbf{EG}_g(K3)$ provide information about the Hilbert space as a representation of G. As a result, for a conjugacy class $[g] \subset G^X$ arising from $K3$ symmetries, we have two ways to attach a twined function – Z_g^X and $\mathbf{EG}_g(K3)$ – to such a "geometric" conjugacy class of $[g] \subset G^X$. It turns out that they coincide for all the geometric conjugacy classes $[g]$ of any one of the 23 G^X. This identity clearly provides non-trivial evidence that all 23 instances of umbral moonshine are related to $K3$ non-linear sigma models.

Recall that in arriving at the above relation we have interpreted the ADE root systems X as the configuration of rational curves given by the ADE singularities. The above result hence suggests that it might be fruitful to study the symmetries of different $K3$ surfaces with distinct configurations of rational curves in a different framework corresponding to the 23 cases of umbral moonshine. In fact, this has been implemented in a recent analysis of the relation between the $K3$ Picard lattice, $K3$ symplectic automorphisms, and the Niemeier lattices, through a "marking" of a $K3$ surface M by one of the L^X such that the Dynkin diagram obtained from the smooth rational curves of M is a sub-diagram of X [48,49]. As will be discussed in more detail in §5, through this marking by the Niemeier lattice L^X, the root system X obtains the interpretation as the "enveloping configuration of smooth rational curves" while the finite group G^X is naturally interpreted as the "enveloping symmetry group" of the $K3$ surfaces that can be marked by the given L^X. On the one hand, this provides a geometric interpretation of our results. On the other hand, one can view our results as a moonshine manifestation and extension of the geometric analysis in [48].

The organisation of the paper is as follows. In §2 we compute the elliptic genus of the ADE du Val singularities that $K3$ surfaces can develop. In §3 we review the umbral moonshine construction from 23 Niemeier lattices and introduce the necessary ingredients for later calculations. Utilising the results of §2, in §4 we establish the relation between the (twined) elliptic genus and the mock modular forms of umbral moonshine. In §5 we provide a geometric interpretation of this result. In §6 we close this paper by discussing some open questions and point to some possible future directions. In Appendix A we collect useful definitions. In Appendix B we present the calculations and proofs, and present our conjectures for the twined (or equivariant) elliptic genus for the du Val singularities. The explicit results for the twining functions are recorded in the Appendix C.

2 The Elliptic Genus of Du Val Singularities

The rational singularities in two (complex) dimensions famously admit an ADE classification. See, for instance, [50]. They are also called the du Val or Kleinian singularities and are isomorphic to the quotient singularity $\mathbb{C}^2/G$, with G being the finite subgroup of $SU_2(\mathbb{C})$ with the corresponding ADE classification [47]. Any such singularity has a unique minimal resolution. The so-called resolution graph, the graph of the intersections of the smooth rational (genus 0) curves of the minimal resolution, gives precisely the corresponding ADE Dynkin diagram. We will denote by Φ the corresponding simply-laced irreducible root system. In terms of hypersurfaces, it is given by $W^0_\Phi = 0$ with

$$W^0_{A_{m-1}} = x_1^2 + x_2^2 + x_3^m \tag{2.1}$$

$$W^0_{D_{m/2+1}} = x_1^2 + x_2^2 x_3 + x_3^{m/2} \tag{2.2}$$

$$W^0_{E_6} = x_1^2 + x_2^3 + x_3^4 \tag{2.3}$$

$$W^0_{E_7} = x_1^2 + x_2^3 + x_2 x_3^3 \tag{2.4}$$

$$W^0_{E_8} = x_1^2 + x_2^3 + x_3^5. \tag{2.5}$$

These singularities show up naturally as singularities of $K3$ surfaces and play an important role in various physical setups, such as in heterotic–type II dualities and in geometric engineering, in string theory compactifications. See, for instance, [51,52] and [53].

The 2d conformal field theory description of these (isolated) singularities was proposed in [54] to be the product of a non-compact super-coset model $\frac{SL(2,\mathbb{R})}{U(1)}$ (the Kazama–Suzuki model [55]) and an $\mathcal{N}=2$ minimal model, followed by an orbifoldisation by the discrete group $\mathbb{Z}/m\mathbb{Z}$, where

m is the Coxeter number of the corresponding simply-laced root system (cf. Table 1). In other words, we consider the super-string background that is schematically given by

$$\text{Minkowski space-time } \mathbb{R}^{5,1} \otimes \left(\mathcal{N}=2 \text{ minimal } \otimes \mathcal{N}=2 \ \frac{SL(2,\mathbb{R})}{U(1)} \text{ coset} \right) / (\mathbb{Z}/m\mathbb{Z}). \tag{2.6}$$

Recall that, when the minimal model is chosen to be the "diagonal" A_{m-1} theory, the above theory also describes the near-horizon geometry of m NS five-branes [56]. Note that this point of view plays an important role in the work of [37, 57], also in the context of discussing the possible physical context of umbral moonshine.

To resolve the singularity let us consider $W^0_\Phi = \mu$. In [54] it was proposed that the sigma model with the non-compact target space $W^0_\Phi = \mu$ has an alternative description as the Landau–Ginsburg model with superpotential

$$\tilde{W}_\Phi = -\mu x_0^{-m} + W^0_\Phi,$$

where x_0 is an additional chiral superfield and m is again given by the Coxeter number of Φ.

The purpose of the rest of the section is to compute the elliptic genus of (the supersymmetric sigma model with the target space being) the du Val singularities. First let us focus on the minimal model part. The $\mathcal{N}=2$ minimal models are known to have an ADE classification [43–46] [1], based on an ADE classification of the modular invariant combinations of chiral (holomorphic) and anti-chiral (anti-holomorphic) characters of the $A_1^{(1)}$ Kac–Moody algebra [42]. In this language, the ADE classification can be thought of as a classification of the possible ways to consistently combine left- and right-movers. To be more precise, in [42] it was found that a physically acceptable and modular invariant combination of characters of the $A_1^{(1)}$ Kac–Moody algebra at level $m-2$ is necessarily given by a $2m \times 2m$ matrix Ω^Φ corresponding to an ADE root system Φ, where we say that a modular invariant is physically acceptable if it satisfies certain integrality, positivity and normalisation conditions. See [42] for more details. The list of these matrices is given in Table 3. The relation between Ω^Φ and the ADE root system Φ lies in the following two facts. First, Ω^Φ is a $2m \times 2m$ matrix where m is the Coxeter number of Φ. Moreover, $\Omega^\Phi_{r,r} - \Omega^\Phi_{r,-r} = \alpha^\Phi_r$ for $r = 1, \ldots, m-1$ coincides with the multiplicity of r as a Coxeter exponent of Φ (cf. Table 1). Recall that a Coxeter element $\prod_{i=1}^r r_i$ of the Weyl group of a rank-r root system is the product of reflections with respect to all simple roots (the order in which the product is taken does not change the conjugacy class of the element), and

[1] Strictly speaking, this classification applies when one requires the presence of a spectral flow symmetry. See for instance [58, 59] for a discussion on related subtleties.

the Coxeter number is the order of such a Coxeter element.

	A_{m-1}	$D_{1+m/2}$	E_6	E_7	E_8
Coxeter number	m	m	12	18	30
Coxeter exponents	$1, 2, 3, \ldots, m-1$	$1, 3, 5, \ldots, m-1,$ $m/2$	1,4,5, 7, 8, 11	1,5,7,9, 11,13,17	1,7,11,13, 17,19,23,29

Table 1: Simply-laced root systems, Coxeter numbers and Coxeter exponents

A quantity that played an important role in the the CFT/LG correspondence [60] as well as in the recent developments of mock modular form moonshine is the elliptic genus. From a physical point of view, the elliptic genus for a 2d $\mathcal{N} = (2, 2)$ superconformal field theory $\mathcal{T}$ is defined as [61]

$$Z_{\mathcal{T}}(\tau, z) = \mathrm{tr}_{\mathcal{H}_{\mathcal{T},\mathrm{RR}}} \left((-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R} \right) \tag{2.7}$$

where $F_{R,L}$ denotes the right- and left-moving fermion number respectively. Moreover, the left- (right-) moving Hamiltonian is given by $H_L = L_0 - c_L/24$ ($H_R = \bar{L}_0 - c_R/24$), where $J_0, L_0, \bar{J}_0, \bar{L}_0$ are the zero modes of the left- and right-moving copies of the $U(1)$ R-current and Virasoro parts of the $\mathcal{N} = 2$ superconformal algebra, respectively. $\mathcal{H}_{\mathcal{T},RR}$ denotes the space of quantum states of theory $\mathcal{T}$ in the Ramond–Ramond sector, and c_L and c_R denote the left- and right-moving central charge of the SCFT. In the above formula, τ takes values in the upper-half plane $\mathbb{H}$ while z takes values in the complex plane $\mathbb{C}$, and we have written $q = \mathrm{e}(\tau)$ and $y = \mathrm{e}(z)$. Throughout the paper we use $\mathrm{e}(x) := e^{2\pi i x}$. Because of the insertion $(-1)^{F_R}$, the elliptic genus only receives contributions from left-moving states that are paired with a right-moving Ramond ground state and is therefore holomorphic, at least when the spectrum of the theory is discrete. As such, it is rigid in the sense of being invariant under any continuous deformation of the theory.

The elliptic genus of the $\mathcal{N} = 2$ minimal model can be computed in various ways. First, from the relation to the parafermion theory, we obtain that the building block of the elliptic genus is the function $\tilde{\chi}^r_s(\tau, z)$, where $|s| \leq r - 1 < m$ [44,62]. See Appendix B for the definition of $\tilde{\chi}^r_s$. From the known spectrum of the minimal model given in terms of the matrix Ω^Φ and the identity $\tilde{\chi}^r_s(\tau, 0) = \delta_{r,s} - \delta_{r,-s}$ it is straightforward to see that the elliptic genus of the minimal model corresponding to the ADE root system Φ is given by [63,64]

$$Z^\Phi_{\mathrm{minimal}}(\tau, z) = \sum_{r,r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega^\Phi_{r,r'} \tilde{\chi}^r_{r'}(\tau, z) = \mathrm{Tr}(\Omega^\Phi \cdot \tilde{\chi}). \tag{2.8}$$

We again refer to Appendix B for more details.

On the other hand, the Landau-Ginzburg description facilitates a free-field computation for the elliptic genus and one obtains an infinite-product expression for $Z^{\Phi}_{\text{minimal}}(\tau, z)$ [61]. In terms of the Jacobi theta function (A.1), the results are [61,64]

$$Z^{\Phi}_{\text{minimal}} = \frac{\theta_1\left(\tau, \frac{m-1}{m}z\right)}{\theta_1\left(\tau, \frac{z}{m}\right)} \quad \text{for} \quad \Phi = A_{m-1} \tag{2.9}$$

for the A-series where $m \geq 2$,

$$Z^{\Phi}_{\text{minimal}} = \frac{\theta_1\left(\tau, \frac{m-2}{m}z\right)\theta_1\left(\tau, \frac{m+2}{2m}z\right)}{\theta_1\left(\tau, \frac{2z}{m}\right)\theta_1\left(\tau, \frac{m-2}{2m}z\right)} \quad \text{for} \quad \Phi = D_{\frac{m}{2}+1} \tag{2.10}$$

for the D series where $m \geq 6$ and even, and finally

$$Z^{E_6}_{\text{minimal}} = \frac{\theta_1\left(\tau, \frac{3}{4}z\right)\theta_1\left(\tau, \frac{2}{3}z\right)}{\theta_1\left(\tau, \frac{z}{4}\right)\theta_1\left(\tau, \frac{z}{3}\right)} \tag{2.11}$$

$$Z^{E_7}_{\text{minimal}} = \frac{\theta_1\left(\tau, \frac{7}{9}z\right)\theta_1\left(\tau, \frac{2}{3}z\right)}{\theta_1\left(\tau, \frac{2}{9}z\right)\theta_1\left(\tau, \frac{z}{3}\right)} \tag{2.12}$$

$$Z^{E_8}_{\text{minimal}} = \frac{\theta_1\left(\tau, \frac{4}{5}z\right)\theta_1\left(\tau, \frac{2}{3}z\right)}{\theta_1\left(\tau, \frac{z}{5}\right)\theta_1\left(\tau, \frac{z}{3}\right)} \tag{2.13}$$

for the E-type cases. The central charge of these minimal models are given by the Coxeter number m of the corresponding simply-laced root system by

$$\hat{c} = c/3 = 1 - \frac{2}{m}. \tag{2.14}$$

In order to obtain the elliptic genus of the isolated ADE singularities, another ingredient we need is the elliptic genus of the $\frac{SL(2,\mathbb{R})}{U(1)}$ super-coset model. The $\frac{SL(2,\mathbb{R})}{U(1)}$ super-coset model is known to describe the geometry of a semi-infinite cigar (a 2d Euclidean black hole) [65] and is mirror to the $\mathcal{N} = 2$ super Liouville theory [56,66]. The level of the super-coset model is related to the mass of the corresponding 2d black hole, and the central charge of the super Liouville theory. Here, we will consider $SL(2,\mathbb{R})$ super-current algebra of (super) level m. The central charge of the corresponding super-coset theory is

$$\hat{c} = 1 + \frac{2}{m}.$$

Due to the presence of the adjoint fermions, there is a shift between the level of the super Kac–Moody algebra $[\hat{g}]_k$ and the level of its bosonic sub-algebra $\hat{g}_{\bar{k}}$ given by the corresponding

quadratic invariant as

$$\bar{k} = k - c_2(g) \quad ,$$

which is given explicitly in terms of structure constants by $c_2(g)\delta_{ab} = f_a{}^{cd} f_{bcd}$.

The spectrum of the super-coset model and the corresponding torus conformal blocks has been discussed in [67,68], following the earlier work [69,70]. Since the model is non-compact, the spectrum not surprisingly contains both discrete and continuous states. In the geometric picture, the discrete states are those localised at the tip of the cigar while the continuous ones are those states whose wave-functions spread into the infinitely long half-cylinder and are only present above a "mass gap" $\frac{1}{4m}$ on the conformal weight [71]. The fact that the torus conformal blocks of the super-coset theory coincide with the characters of the corresponding highest weight representations of the $\mathcal{N} = 2$ superconformal algebra constitutes non-trivial evidence for its equivalence to the $\mathcal{N} = 2$ super Liouville theory. Moreover, the continuous states correspond to massive (or long) $\mathcal{N} = 2$ highest weight representations while the discrete states correspond to massless (or short) ones. As such, it is easy to see from the Hilbert space (Hamiltonian) definition (2.7) of the elliptic genus that it only receives contribution from the discrete part of the spectrum. Accepting the above argument, the building block of the elliptic genus is the Ramond character graded by $(-1)^F$

$$\mathrm{Ch}^{(\tilde{R})}_{\mathrm{massless}}(\tau, z; s) = \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \sum_{k\in\mathbb{Z}} y^{2k} q^{mk^2} \frac{(yq^{mk})^{\frac{s-1}{m}}}{1 - yq^{mk}}$$

where $\eta(\tau) = q^{1/24} \prod_{n\geq 1}(1 - q^n)$ is the Dedekind eta function and $s/2$ is the $U(1)$ charge of the highest weight. The above formula can also be identified as $\mathcal{N} = 2$ characters extended by spectral flow. Putting them together, from the spectrum of the super-coset model it is straightforward to work out the elliptic genus of the theory

$$Z_{L_m}(\tau, z) = \frac{1}{2} \sum_{s=1}^{m} \mathrm{Ch}^{(\tilde{R})}_{\mathrm{massless}}(\tau, z; m+2-s) + \mathrm{Ch}^{(\tilde{R})}_{\mathrm{massless}}(\tau, z; s) = \frac{1}{2}\mu_{m,0}\big(\tau, \frac{z}{m}\big) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}, \quad (2.15)$$

where we have used the (specialised) Appell–Lerch sum

$$\mu_{m,0}(\tau, z) = -\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2km} \frac{1 + yq^k}{1 - yq^k}. \quad (2.16)$$

The above partition function has also been calculated in [11] using an alternative free-field representation of the theory. See also [72,73].

From this we can derive the elliptic genus of the super coset theory coupled to the rational

theory

$$\left(\mathcal{N}=2 \text{ minimal } \otimes \mathcal{N}=2 \ \frac{SL(2,\mathbb{R})}{U(1)} \text{ coset}\right)/(\mathbb{Z}/m\mathbb{Z}),$$

describing the corresponding du Val surface singularities of type Φ, by using the orbifoldisation formula [63]

$$Z^{\Phi,S}(\tau,z) = \frac{1}{m}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2}y^{2a}\, Z^{\Phi}_{\text{minimal}}(\tau,z+a\tau+b) Z_{L_m}(\tau,z+a\tau+b) \tag{2.17}$$

$$= \frac{1}{2m}\frac{i\theta_1(\tau,z)}{\eta^3(\tau)}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} (-1)^{a+b} q^{a^2/2}y^a\, Z^{\Phi}_{\text{minimal}}(\tau,z+a\tau+b)\ \mu_{m,0}(\tau,\frac{z+a\tau+b}{m}). \tag{2.18}$$

Note that the above elliptic genus is not modular, as opposed to the familiar situation with elliptic genera of a supersymmetric conformal field theory. In fact, it is mock modular in the following sense [74]. Let the "completion" of $\mu_{m,0}(\tau,z)$ be

$$\hat{\mu}_{m,0}(\tau,z) = \mu_{m,0}(\tau,z) - \mathrm{e}(-\tfrac{1}{8})\frac{1}{\sqrt{2m}}\sum_{r\in\mathbb{Z}/2m\mathbb{Z}}\theta_{m,r}(\tau,z)\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau', \tag{2.19}$$

then $\hat{\mu}_{m,0}$ transforms like a Jacobi form of weight 1 and index m under the Jacobi group $SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$ but is not holomorphic. (See Appendix A for the definition of Jacobi forms.) In the above formula, $S_m=(S_{m,r})$ denotes the vector-valued cusp form for $SL_2(\mathbb{Z})$ whose components are given by the unary theta function (cf. (A.3))

$$S_{m,r}(\tau) = \sum_{k=r \pmod{2m}} k\, q^{k^2/4m} = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_{m,r}(\tau,z)|_{z=0}. \tag{2.20}$$

In Appendix B.2, we will also conjecture the answer for the elliptic genera of these ADE-singularities twined by automorphisms of the corresponding Dynkin diagram, which can be thought of as permuting the smooth rational curves in the minimal resolution.

This absence of the usual modularity can be attributed to the fact that the target space of the theory is non-compact and hence the spectrum contains a continuous part [11]. This is however seemingly in contradiction with the expectation that a path integral formulation of the elliptic genus should render a function transforming nicely under $SL_2(\mathbb{Z})$, corresponding to the $SL_2(\mathbb{Z})$ mapping class group of the world-sheet torus underlying the path integral formulation. This issue has been recently addressed in [11], and further refined in [75, 76], for the cigar theory. These authors found that a path integral computation indeed renders an answer that is modular but non-holomorphic, and the breakdown of holomorphicity is attributed to the

imperfect cancellation between contributions of the bosonic and fermonic states to the elliptic genus (2.7) in the continuous part of the spectrum. Analogously, we expect the path integral formulation of the elliptic genus of the ADE singularities will render as the answer the real Jacobi form

$$\hat{Z}^{\Phi,S}(\tau,z)=\frac{1}{2m}\frac{i\theta_1(\tau,z)}{\eta^3(\tau)}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}}(-1)^{a+b}q^{a^2/2}y^a\,Z^{\Phi}_{\text{minimal}}(\tau,z+a\tau+b)\;\hat{\mu}_{m,0}(\tau,\frac{z+a\tau+b}{m}). \tag{2.21}$$

Finally, we note that there is a different definition of elliptic genus that is purely geometric. For a compact complex manifold M with $\dim_{\mathbb{C}}M=d_0$, the elliptic genus is defined as the character-valued Euler characteristic of the formal vector bundle [61, 77–80]

$$\mathbf{E}_{q,y}=y^{d/2}\bigwedge\nolimits_{-y^{-1}}T^*_M\bigotimes\nolimits_{n\geq 1}\bigwedge\nolimits_{-y^{-1}q^n}T^*_M\bigotimes\nolimits_{n\geq 1}\bigwedge\nolimits_{-yq^n}T_M\bigotimes\nolimits_{n\geq 0}S_{q^n}(T_M\oplus T^*_M),$$

where T_M and T^*_M are the holomorphic tangent bundle and its dual, and we adopt the notation

$$\bigwedge\nolimits_q V=1+qV+q^2\bigwedge\nolimits^2 V+\ldots,\quad S_qV=1+qV+q^2S^2V+\cdots\cdots,$$

with S^kV denoting the k-th symmetric power of V. In other words, we have

$$\mathbf{EG}(\tau,z;M)=\int_M ch(\mathbf{E}_{q,y})\mathrm{Td}(M) \tag{2.22}$$

where $\mathrm{Td}(M)$ is the Todd class of T_M. For M a (compact) Calabi–Yau manifold, the above geometric definition and the conformal field theory definition, when the CFT is taken to be the 2d non-linear sigma model of M, are believed to give the same function [78, 81]. The fact that the CFT elliptic genus is rigid corresponds to the geometric fact that $\mathbf{EG}(\tau,z;M)$ is a topological invariant. Note that the above definition is manifestly holomorphic. We expect that a suitable generalisation of the above definition which handles non-compact geometries will lead to the geometric elliptic genus $\mathbf{EG}(\tau,z;\Phi)=Z^{\Phi,S}(\tau,z)$ of the du Val singularity. In this paper we will simply refer to $Z^{\Phi,S}(\tau,z)$ as the elliptic genus of the ADE singularity of type Φ.

3 Umbral Moonshine and Niemeier Lattices

In this section we will briefly review the umbral moonshine conjecture and its construction from the 23 Niemeier lattices [41]. The readers are referred to [41] for more details. Let us start by recalling what the Niemeier lattices are. Consider positive-definite lattices of rank 24, we would

like to know which of them are even and unimodular. In string theory, one is often interested in even, unimodular lattices due to the modular invariance of their theta functions. In the classification of positive-definite even unimodular lattices, a special role will be played by the root system of the lattice L, given by $\Delta(L) = \{v \in L | \langle v, v\rangle = 2\}$.

The even unimodular positive-definite lattices of rank 24 were classified by Niemeier [82]. There are 24 of them (up to isomorphisms). The Leech lattice is the unique even, unimodular, positive-definite lattice of rank 24 with no roots [83], discovered shortly before the classification of Niemeier [84,85]. Apart from the Leech lattice, there are 23 other inequivalent even unimodular lattices of rank 24. They are uniquely determined by their root systems $\Delta(L)$, that are all unions of the simply-laced root systems. Moreover, the 23 root systems of the 23 Niemeier lattices are precisely the 23 unions of ADE root systems satisfying the following two simple conditions: first, all of the irreducible components have the same Coxeter numbers; second, the total rank is 24. They are listed in Table 2, where n denotes $\mathbb{Z}/n\mathbb{Z}$. Here and in the rest of the paper we will adopt the shorthand notation $A_{m-1}^{d_A} D_{m/2+1}^{d_D} (E^{(m)})^{d_E}$ for the direct sum of d_A copies of A_{m-1}, d_D copies of $D_{m/2+1}$ and d_E copies of

$$E^{(m)} = \begin{cases} E_6, E_7, E_8 & \text{for } m = 12, 18, 30 \\ \emptyset & \text{otherwise} \end{cases} . \tag{3.1}$$

Let X be one of the 23 root systems listed above, and denote by L^X the unique (up to isomorphism) Niemeier lattice with root system X. For each of these 23 L^X we will have an instance of umbral moonshine as we will explain now. First, we need to define the finite group relevant for this new type of moonshine. Let us consider the automorphism group $\mathrm{Aut}(L^X)$ of the lattice L^X. Clearly, any element of the Weyl group $\mathrm{Weyl}(X)$ generated by reflections with respect to any root vector leaves the lattice invariant. In fact, $\mathrm{Weyl}(X)$ is a normal subgroup of $\mathrm{Aut}(L^X)$ and we define the "umbral group" G^X to be the corresponding quotient

$$G^X = \mathrm{Aut}(L^X)/\mathrm{Weyl}(X). \tag{3.2}$$

The list of the 23 G^X is given in Table 2.

After defining the relevant finite group G^X, we will now define the relevant (vector-valued) mock modular forms H_g^X, $[g] \subset G^X$, for the umbral moonshine. As explained in §2, the ADE classification of the modular invariant combinations of $\hat{A}_1^{(1)}$ characters is given by a symmetric matrix Ω^Φ of size $2m$, where m denotes the Coxeter number of Φ, for every simply-laced root system Φ. As we have seen, the Cappelli–Itzykson–Zuber matrix Ω^Φ also controls the spectrum

Table 2: Umbral Groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

and hence the elliptic genus (2.9) of the 2d minimal model of type Φ. Now consider any one of the 23 Niemeier root systems X listed above. Since they are unions $X = \cup_i \Phi_i$ of simply-laced root systems Φ_i with the same Coxeter number, we can extend the definition of the Ω-matrix to $\Omega^X = \sum_i \Omega^{\Phi_i}$. Using these Ω-matrices we can then define for each Niemeier lattice L^X the vector-valued weight 3/2 cusp form

$$S^X = \Omega^X S_m = (S_r^X), \quad r \in \mathbb{Z}/2m\mathbb{Z}$$

with the r-th component given by

$$S_r^X = \sum_{r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega^X_{r,r'} S_{m,r'}$$

in terms of the unary theta function (2.20). From $(\Omega^X)_{r,r'} = (\Omega^X)_{-r,-r'}$ and $S_{m,r} = -S_{m,-r}$ it is easy to see that $S_r^X = -S_{-r}^X$.

Given the cusp form S^X, we can now specify the mock modular form H^X by the following two conditions. First we specify its mock modular property: we require H^X to be a weight 1/2

vector-valued mock modular form whose shadow is given by S^X. More precisely, let

$$\hat{H}_r^X(\tau) = H_r^X(\tau) + \mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}}\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-\frac{1}{2}}\,\overline{S_r^X(-\bar{\tau}')}\,\mathrm{d}\tau',$$

then

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \hat{H}_r^X(\tau)\,\theta_{m,r}(\tau,z)$$

transforms as a Jacobi form of weight 1 and index m under the Jacobi group $SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$. Recall that the shadow $s(\tau)$ of a mock modular form $f(\tau)$ of weight w is the function, a modular form of weight $2-w$ itself for the same $\Gamma < SL_2(\mathbb{R})$, whose integral gives the non-holomorphic completion

$$\hat{f}(\tau) = f(\tau) + \mathrm{e}(\frac{w-1}{4})\int_{-\bar{\tau}}^{i\infty}(\tau+\tau')^{-w}\,\overline{s(-\bar{\tau}')}\,d\tau' \tag{3.3}$$

of f which transforms as a weight w modular form. This definition has a straightforward generalisation to the vector-valued case which we have employed above.

After specifying the mock modularity, we impose the following analyticity condition : we require its growth near the cusp to be

$$q^{1/4m}H_r^X(\tau) = O(1) \quad \text{as } \tau\to i\infty \tag{3.4}$$

for every element $r\in\mathbb{Z}/2m\mathbb{Z}$. The above two conditions turn out to be sufficient to determine H^X uniquely (up to a rescaling), as shown in [12, 41, 86]. We also fix the scaling by requiring $q^{1/4m}H_1^X(\tau) = -2 + O(q)$.

For instance, when considering the Niemeier lattice with the simplest root system, $X = A_1^{24}$, the unique mock modular form determined by the above condition reads

$$H_1^{X=A_1^{24}}(\tau) = -H_{-1}^{X=A_1^{24}}(\tau) = \frac{-2E_2(\tau)+48F_2^{(2)}(\tau)}{\eta(\tau)^3} \tag{3.5}$$

$$= 2q^{-1/8}(-1+45\,q+231\,q^2+770\,q^3+O(q^4)). \tag{3.6}$$

where $E_2(\tau)$ stands for the weight 2 Eisenstein series and

$$F_2^{(2)}(\tau) = \sum_{\substack{r>s>0\\ r-s=1 \bmod 2}} (-1)^r\,s\,q^{rs/2} = q+q^2-q^3+q^4+\dots\,.$$

As mentioned in §1, the first observation that led to the recent development in the moonshine phenomenon for mock modular forms is the fact that the above numbers 45, 231, 770 coincide

with the dimensions of certain irreducible representations of the corresponding umbral group $G^X \cong M_{24}$ for $X = A_1^{24}$.

Note that without the non-holomorphic completion, the function

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} H_r^X\, \theta_{m,r}$$

does not transform nicely under the modular group; it is a mock Jacobi form according to the definition given in [12]. In [41], following [12], this mock Jacobi form is interpreted as the finite part of a meromorphic (as a function of z) Jacobi form with simple poles at m-torsion points. For later convenience, we will define another mock Jacobi form

$$\phi^X(\tau,z) = \frac{i\theta_1(\tau,mz)\theta_1(\tau,(m-1)z)}{\eta^3(\tau)\theta_1(\tau,z)} \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} H_r^X(\tau)\,\theta_{m,r}(\tau,z) \tag{3.7}$$

which contains exactly the same information as the vector-valued mock modular form H^X.

In order to relate such functions to representations of the finite group G^X that we have constructed, we need as many vector-valued functions similar to H^X as the number of conjugacy classes of G^X to encode the characters of the underlying representation. Hence, for every Niemeier lattice X, and for every conjugacy class $[g] \subset G^X$ we would like to define a vector-valued mock modular form H_g^X. As before, first we need to specify their mock modular properties. The relevant congruence subgroup $\Gamma_0(n_g) \subseteq SL_2(\mathbb{Z})$ (see (A.6)), is determined by n_g, the order group element g. This is similar to the situation both in monstrous moonshine [1] and, not unrelatedly, 2-dimensional CFT.

We need two more pieces of data to completely specify the mock modularity of H_g^X. The first one is the shadow. By studying the action of $\langle g\rangle$, the cyclic group generated by g, we can analogously define a $2m\times 2m$ matrix Ω_g^X and the corresponding cusp form $S_g^X = \Omega_g^X S_m = S_{g,r}^X$. See §5.1 of [41] for the list of Ω_g^X. The second piece of data we need is the multiplier system system on $\Gamma_0(n_g)$, namely a projective representation $\nu_g : \Gamma_0(n_g) \to GL_{2m}(\mathbb{C})$ of the congruence subgroup $\Gamma_0(n_g)$. In the case where the specified shadow S_g^X does not vanish, the definition of the shadow stipulates the multiplier of the mock modular form to be the inverse of the shadow. As a result, this second piece of data is implied by the first. If however $S_g^X = 0$, namely when the mock modular form H_g^X is in fact modular, one needs to specify the multiplier system independently. It turns out that ν_g is identical to the inverse of the multiplier of S^X on a group

$\Gamma_0(n_g h_g) < \Gamma_0(n_g)$ for certain integral $h_g > 1$. See [41] for more details. In particular, let

$$\hat{H}^X_{g,r}(\tau) = H^X_{g,r}(\tau) + \mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}}\int_{-\bar{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\overline{S^X_{g,r}(-\bar{\tau}')}\,\mathrm{d}\tau',$$

then

$$\sum_{r\in\mathbb{Z}/2m\mathbb{Z}}\hat{H}^X_{g,r}(\tau)\,\theta_{m,r}(\tau,z)$$

transforms like a Jacobi form of weight 1 and index m under the group $\Gamma_0(n_g h_g)\ltimes\mathbb{Z}^2$. By the same token, the function $\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} H^X_{g,r}\,\theta_{m,r}$ is a mock Jacobi form of weight 1 and index m under $\Gamma_0(n_g h_g)\ltimes\mathbb{Z}^2$.

As before, after specifying the mock modular property we also need to fix the analyticity property of H^X_g. For $\Gamma_0(n_g)$ with $n_g > 1$, there is more than one cusp (representative), namely more than one $\Gamma_0(n_g)$-orbit among $\mathbb{Q}\cup i\infty$. For the cusp (representative) located at $\tau\to i\infty$ we require the same growth condition

$$q^{1/4m}H^X_{g,r}(\tau) = O(1)\quad\text{as } \tau\to i\infty. \tag{3.8}$$

for every $r\in\mathbb{Z}/2m\mathbb{Z}$. Moreover we require the function to be bounded

$$H^X_{g,r}(\tau) = O(1)\quad\text{as } \tau\to\alpha\in\mathbb{Q},\quad \alpha\notin\Gamma_0(n_g)\infty. \tag{3.9}$$

at all other cusps.

After specifying the shadow S^X_g, the multiplier system ν_g and the behaviour at the cusps, a vector-valued mock modular form H^X_g of weight 1/2 for $\Gamma_0(n_g)$ was then given in [41] for every $[g]\subset G^X$ and for all 23 Niemeier lattices L^X. See [41] for explicit Fourier coefficients of the q-expansions of $H^X_{g,r}$. Finally, it was conjectured in [41] that H^X_g is the unique (up to rescaling) vector-valued mock modular form with the above mock modularity and poles. For later convenience, we will also define

$$\phi^X_g(\tau,z) = \frac{i\theta_1(\tau,mz)\theta_1(\tau,(m-1)z)}{\eta^3(\tau)\theta_1(\tau,z)}\sum_{r\in\mathbb{Z}/2m\mathbb{Z}}H^X_{g,r}(\tau)\,\theta_{m,r}(\tau,z) \tag{3.10}$$

Note that we recover H^X and ϕ^X by putting $[g]$ to be the identity class in the above discussions on H^X_g and ϕ^X_g.

After constructing the finite group G^X and the set of vector-valued mock modular forms $H^X_g = (H^X_{g,r})$ for each Niemeier lattice L^X, we can now formulate the umbral moonshine con-

jecture [41]. This conjecture states that for every Niemeier lattice X, for every $1 \leq r \leq m-1$ we have an infinite-dimensional $\mathbb{Z}$-graded module $K_r^X = \oplus_D K_{r,D}^X$ for G^X such that $H_{g,r}^X$ is essentially given by the graded characters $\sum_{D=-r^2 \pmod{4m}, D>0} q^{D/4m} \mathrm{Tr}_{K_{r,D}^X} g$, up to the possible inclusion of a polar term $-2q^{-1/4m}$ and a constant term (as well as an additional factor of 3 in the case $X = A_8^3$). See §6.1 of [41] for the precise statement of the conjecture. In summary, umbral moonshine conjectures for each of the 23 Niemeier lattices the existence of a special module K^X of the finite group G^X, which underlies the special mock Jacobi forms ϕ_g^X. This conjecture has so far been proven for the case $X = A_1^{24}$ [28], and explicitly verified till the first hundred terms in the q-expansion for the other 22 cases. In the following section we will demonstrate the relation between the mock Jacobi forms ϕ_g^X and the elliptic genus of $K3$ surfaces. Subsequently we will explore the relation between the Niemeier lattices L^X, the finite group G^X, the (conjectural) G^X-module K^X, and the (stringy) symmetry of $K3$ surfaces.

4 Umbral Moonshine and the (Twined) $K3$ Elliptic Genus

In §2 we have computed the elliptic genus of Du Val singularities a $K3$ surface can develop. In §3 we have briefly reviewed the umbral moonshine conjecture relating a finite group G^X and a set of mock Jacobi forms ϕ_g^X for every Niemeier lattice L^X via an underlying G^X-module K^X. In this section we will see how these two separate topics meet in the framework of (twined) elliptic genera for $K3$ surfaces.

Let's start by briefly reviewing the relation between the elliptic genus of $K3$ surfaces and the Mathieu group M_{24}, which is also the umbral group G^X for the Niemeier lattice with root system $X = A_1^{24}$. The 2d non-linear sigma model of a $K3$ surface is a CFT with central charge $c = 6$ and with a (small) $\mathcal{N} = 4$ superconformal symmetry. As explained in §2, the elliptic genus (2.7) is the same for different $K3$ sigma models and coincides with the geometric elliptic genus of $K3$. It is computed to be (cf. (A.1)) [16]

$$\mathbf{EG}(\tau, z; K3) = 8 \sum_{i=2,3,4} \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)} \right)^2 . \tag{4.1}$$

The $\mathcal{N} = 4$ superconformal symmetry of the theory implies that the spectrum is composed of irreducible representations ("multiplets") of the $\mathcal{N} = 4$ superconformal algebra, and the elliptic genus permits a decomposition into their characters.

Recall that the $\mathcal{N} = 4$ superconformal algebra contains subalgebras isomorphic to the affine Lie algebra $\hat{\mathfrak{sl}}_2$ and the Virasoro algebra, and in a unitary representation the former of these

acts with level $m-1$ and the latter with central charge $c=6(m-1)$ for some integer $m>1$. The unitary irreducible highest weight representations $v_{m;h,j}$ are labelled by the two quantum numbers h and j which are the eigenvalues of L_0 and $\frac{1}{2}J_0^3$ of the highest weight state, respectively, when acting on the highest weight state [15,87]. (We adopt a normalisation of the $SU(2)$ current J^3 such that the zero mode J_0^3 has integer eigenvalues. The shift by -1 in the central charge and the level of the current algebra is due to the -1 difference between the level and the index of the theta functions underlying the characters, as we will see below.) The algebra has two types of highest weight representations: the *short* (or *BPS*, *supersymmetric*) ones and the *long* (or *non-BPS*, *non-supersymmetric*) ones. In the Ramond sector, the former has $h=\frac{c}{24}=\frac{m-1}{4}$ and $j\in\{0,\frac{1}{2},\cdots,\frac{m-1}{2}\}$, while the latter has $h>\frac{m-1}{4}$ and $j\in\{\frac{1}{2},1,\cdots,\frac{m-1}{2}\}$. Their (Ramond) graded characters, defined as

$$\mathrm{ch}_{m;h,j}(\tau,z)=\mathrm{tr}_{v_{m;h,j}}\left((-1)^{J_0^3}y^{J_0^3}q^{L_0-c/24}\right), \tag{4.2}$$

are given by

$$\mathrm{ch}_{m;h,j}(\tau,z)=\frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\mu_{m,j}(\tau,z) \tag{4.3}$$

and

$$\mathrm{ch}_{m;h,j}(\tau,z)=\frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\,q^{h-\frac{c}{24}-\frac{j^2}{m}}\left(\theta_{m,2j}(\tau,z)-\theta_{m,-2j}(\tau,z)\right) \tag{4.4}$$

in the short and long cases, respectively [15]. In the above formulas, $\mu_{m,j}$ is given by $\mu_{m,0}$ (2.16) and the identity

$$\mu_{m,\frac{r}{2}}=(-1)^r(r+1)\mu_{m,0}+(-1)^{r-n+1}\sum_{n=1}^{r}n\,q^{-\frac{(r-n+1)^2}{4m}}(\theta_{m,r-n+1}-\theta_{m,-(r-n+1)}).$$

When applying the above formula to the $K3$ sigma models which have $c=6\,(m=2)$, we obtain the following rewriting of the function in (4.1):

$$\begin{aligned}\mathbf{EG}(\tau,z;K3)&=20\,\mathrm{ch}_{2;\frac{1}{4},0}-2\,\mathrm{ch}_{2;\frac{1}{4},\frac{1}{2}}+\left(90\,\mathrm{ch}_{2;\frac{5}{4},\frac{1}{2}}+462\,\mathrm{ch}_{2;\frac{9}{4},\frac{1}{2}}+1540\,\mathrm{ch}_{2;\frac{13}{4},\frac{1}{2}}+\dots\right) &(4.5)\\ &=\frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\Big\{24\,\mu_{2,0}(\tau,z)+(\theta_{2,-1}(\tau,z)-\theta_{2,1}(\tau,z))\\ &\quad\times(-2q^{-1/8}+90q^{7/8}+462q^{15/8}+1540q^{23/8}+\dots)\Big\} &(4.6)\end{aligned}$$

where $\dots$ corresponds to terms in $\mathbf{EG}(\tau,z;K3)$ of the form $\frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}q^\alpha y^\beta$ with $\alpha-\beta^2/8>3$. Note that the q-series in the last line is nothing but the umbral mock modular form (3.5) coresponding to the Niemeier lattice with root system $X=A_1^{24}$ that we introduced in the

previous section. As mentioned in §1, it was precisely in this context of decomposing the $K3$ elliptic genus into $\mathcal{N}=4$ characters that the first case of moonshine for mock modular forms was observed [17].

From the above discussion, we see that the two contributions to $\mathbf{EG}(\tau,z;K3)$, given by

$$24\,\mu_{2,0}(\tau,z)$$

and

$$-\sum_{r\in\mathbb{Z}/4\mathbb{Z}} H_r^{X=A_1^{24}}(\tau)\theta_{2,r}(\tau,z),$$

in the $\{\}$ bracket, can roughly be thought of as the contributions from the BPS and non-BPS $\mathcal{N}=4$ multiplets respectively[2].

However, there is a possible alternative interpretation, thanks to the identity between the short $\mathcal{N}=4$ characters and the elliptic genus of an $\Phi=A_1$ singularity:

$$Z^{A_1,S}(\tau,z)=\mathrm{ch}_{2;\frac{1}{4},0}(\tau,z), \tag{4.7}$$

which follows from the identity

$$\frac{1}{2}\sum_{a,b=0}^{1} q^{a^2}y^{2a}\theta_1(\tau,z+a\tau+b)\,\mu_{2,0}(\tau,\frac{z+a\tau+b}{2})=\frac{\theta_1(\tau,z)^2}{i\theta_1(\tau,2z)}\,\mu_{2,0}(\tau,z).$$

In other words, we can re-express the elliptic genus of $K3$ as

$$\mathbf{EG}(\tau,z;K3)=24Z^{A_1,S}(\tau,z)-\frac{i\,\theta_1(\tau,z)^2}{\eta^3(\tau)\theta_1(\tau,2z)}\sum_{r\in\mathbb{Z}/4\mathbb{Z}} H_r^{A_1^{24}}\theta_{2,r}(\tau,z). \tag{4.8}$$

Using the identity

$$\theta_{2,1}(\tau,z)-\theta_{2,-1}(\tau,z)=-i\theta_1(\tau,2z),$$

and

$$-q^{1/2}y\,\theta_1(\tau,z+\tau)=\theta_1(\tau,z)$$

[2]Strictly speaking, the polar term "$-2q^{-1/8}$" of $H_1^{X=A_1^{24}}$ also corresponds to the contributions from BPS multiplets, while all the infinitely many other terms are contributions from non-BPS multiplets.

we can rewrite the above expression as

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a}\, \phi^X\big(\tau, \frac{z + a\tau + b}{m}\big) \tag{4.9}$$

for $X = A_1^{24}$, where ϕ^X is the function defined in (3.7) that encodes the umbral moonshine mock modular form H^X. In the above, for a root system X that is the union of simply-laced root systems with the same Coxeter number m (cf. (3.1))

$$X = A_{m-1}^{d_A} D_{m/2+1}^{d_D} (E^{(m)})^{d_E},$$

we write

$$Z^{X,S} = d_A Z^{A_{m-1}} + d_D Z^{D_{m/2+1}} + d_E Z^{E^{(m)}}, \tag{4.10}$$

corresponding to a collection of non-interacting ADE theories with the total Hilbert space given by the direct sum of the Hilbert spaces of the component theories.

In other words, instead of interpreting the two contributions to the $K3$ elliptic genus as that of the BPS and that of the non-BPS $\mathcal{N} = 4$ multiplets, one might interpret them as the contribution from the 24 copies of A_1-type surface singularities and the "umbral moonshine" contribution given by the umbral moonshine mock modular forms H^X with $X = A_1^{24}$.

The first surprise we encounter is that such an interpretation actually holds for all 23 cases of umbral moonshine. In particular, the equality (4.9) is valid not only for the case $X = A_1^{24}$ but also for all other 22 cases corresponding to all the 23 Niemeier lattices L^X. The detailed proof will be supplied in Appendix B. Put differently, corresponding to the 23 Niemeier lattices L^X we have 23 different ways of separating $\mathbf{EG}(K3)$ into two parts. On the one hand, by replacing the Niemeier root system X with the corresponding configuration of singularities, we obtain a contribution to the $K3$ elliptic genus by the singularities. On the other hand, the umbral moonshine construction attaches a mock Jacobi form ϕ^X to every L^X, which gives the rest of $\mathbf{EG}(K3)$ after a summation procedure reminiscent of the "orbifoldisation" formula for the elliptic genus of orbifold SCFTs [63].

Recall that in umbral moonshine for a given Niemeier lattice L^X, the mock Jacobi form ϕ^X is conjectured to encode the graded dimension of an infinite-dimensional module K^X of the umbral finite group G^X. The existence of such a module is supported by the construction of the other mock Jacobi forms ϕ_g^X for the other (non-identity) conjugacy classes $[g]$ of the umbral group G^X (cf. (3.2)), that are conjectured to encode the graded characters of K^X. Given the above relation between the $K3$ elliptic genus and the mock modular form $H^X = H_g^X$ for $[g]$

being the identity class, a natural question is whether a $K3$ interpretation also exists for other mock modular forms H_g^X corresponding to other conjugacy classes of the group G^X.

To discuss the relation between the graded characters in umbral moonshine and the elliptic genus of $K3$, let us first discuss how the equality (4.9) might be "twined" in the presence of a non-trivial group element. On the left-hand side (the $K3$ side) of the equation is the elliptic genus, defined in terms of the Ramond-Ramond Hilbert space $\mathcal{H}_{\mathcal{T},RR}$ of the underlying supersymmetric sigma model $\mathcal{T}$ as in (2.7). In the event that every Hilbert subspace $\mathcal{H}_{h,j;\mathcal{T},RR} \subset \mathcal{H}_{\mathcal{T},RR}$, consisting of states with the same L_0, J_0 eigenvalues h and j, is a representation of the cyclic group generated by g, or that g acts on the theory and commutes with the superconformal algebra in other words, we can define the so-called "twisted elliptic genus" as the graded character

$$\mathbf{EG}_g(\tau, z; K3) = \mathrm{tr}_{\mathcal{H}_{\mathcal{T},\mathrm{RR}}} \Big(g\,(-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R} \Big). \tag{4.11}$$

Let us now turn to the right-hand side (the umbral moonshine side) of the equation. Assuming the (linear) relevance of the umbral moonshine module K^X for the calculation of $\mathbf{EG}(K3)$, the unique way to twine the second term

$$\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a}\ \phi^X\big(\tau, \frac{z+a\tau+b}{m}\big)$$

is to replace it with

$$\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a}\ \phi_g^X\big(\tau, \frac{z+a\tau+b}{m}\big)$$

where ϕ_g^X is defined in (3.10). This is equivalent to replacing the graded dimension of the module K^X with its graded character. What remains to be twined is the first term in (4.9), the contribution from the configuration of singularities stipulated by the root system X of the Niemeier lattice L^X. For an element g of the umbral group G^X, consider its action on the rank 24 root system X. In the case that g simply permutes the different irreducible components of its root system, it is easy to write down the twining of the singularity part $Z^{X,S}$ of $\mathbf{EG}(K3)$: $Z_g^{X,S}$ is simply given by the contribution from the irreducible components of X that are left invariant by the action of g. For instance, for $X = A_1^{24}$, consider the order 2 element g of the umbral group $G^X = M_{24}$ whose action on L^X is to exchange 8 pairs of A_1 root systems and leave the other 8 copies of A_1 invariant when restricted to the root vectors of L^X. In this case the twined singularity part of the elliptic genus is simply $Z_g^{X,S} = 8Z^{A_1,S}$. It can also happen that g also involves a non-trivial automorphism of the individual irreducible components of the root system, such as the $\mathbb{Z}/2\mathbb{Z}$ symmetry of the A_n, $n > 1$ Dynkin diagram and the $\mathbb{Z}/3\mathbb{Z}$ symmetry

of the D_4 Dynkin diagram. In this case the computation for $Z_g^{X,S}$ is more involved and will be discussed in Appendix B.2. Combining the two parts, we can now define the twining for the right-hand side (the umbral moonshine side) (4.9) which we denote by

$$Z_g^X(\tau,z) = Z_g^{X,S}(\tau,z) + \frac{1}{2m}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2}y^{2a}\,\phi_g^X\big(\tau,\frac{z+a\tau+b}{m}\big). \tag{4.12}$$

The second surprise is that these twining functions given by umbral moonshine precisely reproduce the elliptic genus twined by a geometric symmetry of the underlying $K3$ surface whenever the latter interpretation is available, a fact we will now explain. The symmetries of a $K3$ surface M that are of interest for the purpose of studying the elliptic genus are the so-called finite symplectic automorphisms of M, as we need to require the symmetry to preserve the hyper-Kähler structure in order for it to commute with the $\mathcal{N}=4$ superconformal algebra. As we will discuss in §5, a necessary condition for a subgroup $G\subseteq G^X$ to admit such an interpretation as the group of finite symplectic automorphisms of a certain $K3$ surface is that it has at least 5 orbits and 1 fixed point on the 24-dimensional representation of G^X. See [88] for a proof by S. Kondō utilising the previous results by V. Nikulin [89,90], and [48] for a more refined analysis.

For convenience, above and in the rest of the paper we will simply refer to the 24-dimensional representation that encodes the action of G^X on X as "the 24-dimensional representation" of G^X. As above and in §5, this representation is also the relevant one when describing the action of various subgroups of G^X on the $K3$ cohomology lattice, via the embedding of its sub-lattice into L^X. The action of an element $g\in G^X$ on the 24-dimensional representation is encoded in the 24 eigenvalues, or equivalently its "24-dimensional cycle shape"

$$\Pi_g^X = \prod_i^k \ell_i^{m_i},\ \text{where } m_i\in\mathbb{Z}_{>0},\ 0<\ell_1<\cdots<\ell_k \text{ and } \sum_i^k m_i\ell_i = 24, \tag{4.13}$$

where the relation between the cycle shape and the eigenvalues $\lambda_1,\ldots,\lambda_{24}$ is given by

$$\prod_i^k (x^{\ell_i}-1)^{m_i} = (x-\lambda_1)\cdots(x-\lambda_{24}). \tag{4.14}$$

We will say that an element $g\in G^X$ satisfies the "geometric condition" if it satisfies the criterium of Mukai, namely when it has at least 5 orbits ($\sum_i^k m_i\geq 5$) and one fixed point ($\ell_1=1$) on the 24-dimensional representation.

Moreover, this implies that G must be (isomorphic to) a subgroup of one of the 11 maxi-

mal subgroups of M_{23} listed in [91] and provides an alternative proof of Mukai's theorem [91]. Conversely, given any G^X among the 23 umbral groups and for any element $g \in G^X$ satisfying the geometric condition, there exists a $K3$ surface M whose finite group of symplectic automorphisms has a subgroup isomorphic to $\langle g \rangle$. This can be shown using the global Torelli theorem and in fact holds not just for the Abelian groups but also for all 11 maximal subgroups of M_{23}. See the Appendix by S. Mukai in [88].

As a result, for any of the 23 G^X for any element $g \in G^X$ satisfying the geometric condition, one can compute $\mathbf{EG}_g(K3)$ geometrically by considering the supersymmetric sigma model on a $K3$ surface with $\langle g \rangle$ symmetry. Note that the latter is well-defined because of the uniqueness of the $\langle g \rangle$ action. To be more precise, it was shown in [92] that if $G_i \cong \langle g \rangle$ acts on a $K3$ surface M_i faithfully and symplectically ($i = 1, 2$), then there exists a lattice isomorphism $\alpha : H^2(M_1, \mathbb{Z}) \to H^2(M_2, \mathbb{Z})$ preserving the intersection forms such that $\alpha \cdot G_1 \cdot \alpha^{-1} = G_2$ in $H^2(M_2, \mathbb{Z})$ (see [93] for a generalisation of this result to many non-Abelian groups). Together with the global Torelli theorem, which states that any lattice isomorphism $\varphi^* : H^2(M, \mathbb{Z}) \to H^2(M', \mathbb{Z})$ between the second cohomology groups of two $K3$ surfaces that preserves the Hodge structure and the effectiveness of the cycles is induced by a unique isomorphism $\varphi : M \to M'$, this shows the uniqueness of the symplectic action of $\langle g \rangle$ on $K3$ and thereby that of $\mathbf{EG}_g(K3)$.

On the other hand, using the prescription of umbral moonshine (4.12) one can compute Z_g^X. The first non-trivial fact is that, whenever $g_1 \in G^{X_1}$ and $g_2 \in G^{X_2}$ both satisfy the geometric condition and moreover have the same 24-dimensional cycle shape $\Pi_{g_1}^{X_1} = \Pi_{g_2}^{X_2}$, we obtain

$$Z_{g_1}^{X_1} = Z_{g_2}^{X_2} \tag{4.15}$$

despite the fact that they are defined in a very different way and each consists of two very different contributions (cf. (4.12)). Second, the result also coincides with the geometrically twined elliptic genus for a $K3$ admitting $\langle g \rangle$-symmetry

$$Z_g^X = \mathbf{EG}_g(K3) \tag{4.16}$$

whose induced action on 24-dimensional representation is isomorphic to that of $g \in G^X$.

For the conjugacy classes $g \in G^X$ that do not satisfy the geometric condition, the interpretation of the function Z_g^X is much less clear, similar to the situation in the M_{24}-moonshine. Just like the more familiar case when $X = A_1^{24}$ [32], some of them correspond to SCA-preserving symmetries of certain SCFT $\mathcal{T}$ in the same moduli space as that of $K3$ sigma model, while some of them don't. We will discuss their interpretation in §6. The explicit formulas for the Z_g^X for

all the conjugacy classes $[g] \subset G^X$ for all 23 X can be found in Appendix C.

5 Geometric Interpretation

The result of the previous section suggests that it can be fruitful to study the symmetries of (the non-linear sigma models on) different $K3$ surfaces with different configurations of rational curves in a different framework corresponding to the 23 different cases of umbral moonshine. In this section we will see how, on the geometric side, this has in fact been implemented in a recent analysis of the relation between the $K3$ Picard lattice, $K3$ symplectic automorphisms, and the Niemeier lattices [48, 49]. On the one hand, this provides a geometric interpretation of the results in this paper. On the other hand, one can view our results as a moonshine manifestation and extension of the geometric analysis in [48].

To discuss this interpretation, let us first briefly review the result in [48], in which Nikulin advocates a more refined study of the geometric and arithmetic properties of $K3$ surfaces by introducing an additional marking using Niemeier lattices. (Note that the idea of using the specific Niemeier lattice that has root lattice A_1^{24} to mark the K3 lattice was first proposed and concretely realised in [31].) Usually, to specify a "marking" of a $K3$ surface M is to specify an isomorphism between the rank 22 lattice $H^2(M,\mathbb{Z})$ and the unique (up to isomorphism) even unimodular lattice $\Gamma_{3,19} \cong 2E_8(-1) \oplus 3U$ of signature (3,19), where U is the hyperbolic lattice $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ [3]. To introduce an additional marking by Niemeier lattices, on top of the marking described above, an important ingredient is the Picard lattice

$$\mathrm{Pic}(M) = H^2(M,\mathbb{Z}) \cap H^{1,1}(M)$$

of M. The real space $H^{1,1}(M,\mathbb{R})$ has signature $(1,19)$ and the Picard lattice is either: a. negative definite with $0 \leq \mathrm{rk}\,(\mathrm{Pic}(M)) \leq 19$; b. hyperbolic of signature $(1, \mathrm{rk}\,(\mathrm{Pic}(M)) - 1)$ and with $1 \leq \mathrm{rk}\,(\mathrm{Pic}(M)) \leq 20$; c. semi-negative definite with a null direction and with $1 \leq \mathrm{rk}\,(\mathrm{Pic}(M)) \leq 19$. The condition b. holds if and only if M is algebraic. On the other hand, a generic non-algebraic $K3$ suface satisfies the first condition. Unless differently stated, we will focus on these two, the "generic" (a.) and the "algebraic" (b.), cases.

To obtain an additional marking of M by a Niemeier lattice, consider the maximal negative definite sublattice of the Picard lattice, denoted by $S_M(-1) \subseteq \mathrm{Pic}(M)$. To be more explicit,

[3] The "(-1)" means that we multiply the lattice bilinear form by a factor of -1. This (-1) comes from the fact that the signature of the $K3$ cohomology lattice is mostly negative while the usual convention for the signature of the simply-laced root system and hence the Niemeier lattices is positive definite. The same goes for the (-1) factor in the definition of S_M below.

in the generic case we have simply $S_M(-1) = \mathrm{Pic}(M)$, while in the algebraic case $S_M(-1) = h^{\perp}_{\mathrm{Pic}(M)}$ is the orthogonal complement in the Picard lattice of the one-dimensional sublattice generated by the primitive $h \in \mathrm{Pic}(M)$ with $h^2 > 0$ corresponding to a nef divisor on M. Using the properties of the Torelli period map, one can show that a lattice S_M may arise in the above way from a $K3$ surface M if and only if $S_M(-1)$ admits a primitive embedding into $\Gamma_{3,19}$, a condition that can be further translated into more concrete terms using the lattice embedding results in [89].

We say $\iota_{M,X}$ is a marking of the $K3$ surface M by the Niemeier lattice L^X if $\iota_{M,X} : S_M \to L^X$ is a primitive embedding of S_M into L^X. The first result of [48] states that every $K3$ surface admits a marking by (at least) one of the 23 Niemeier lattices. This can be shown using the fact that $S_M(-1)$ admits a primitive embedding into $\Gamma_{3,19}$ and the embedding theorem in [89][4]. We will denote by $\tilde{S}_M$ the image of S_M, and $(\tilde{S}_M)^{\perp}_{L^X}$ by its orthonormal complement in L^X.

The second result, demonstrating the importance of *all* 23 Niemeier lattices for the study $K3$ surfaces, proves that for every L^X with the exception of $X = A_{24}$ and $X = A_{12}^2$, there exists a $K3$ surface that can only be marked using L^X and not by any other Niemeier lattice. It was also conjectured in [48] that the same statement also holds for $X = A_{24}$ and $X = A_{12}^2$. In particular, from this point of view the case $X = A_1^{24}$ is not more special than any other of the 22 cases. The third result on the additional Niemeier marking states that, for any L^X, any primitive sublattice of L^X which can be primitively embedded into $\Gamma_{3,19}(-1)$ arises from the Picard lattice $\mathrm{Pic}(M)$ in the way described above for a certain $K3$ surface M.

The above three results show that the additional marking of $K3$ lattices is general and universally applicable. Now we will see that such an extra marking is also useful. In [48], two applications of the Niemeier marking are discussed. As we will see, both are crucial for the geometric interpretation of our results. The first application is to use the Niemeier marking to constrain the configuration of smooth rational curves in a $K3$ surface: for the generic cases, a $K3$ surface M that can be marked by L^X has the configuration of all smooth rational curves given by $X \cap S_M$; for the algebraic cases, this holds modulo multiples of the primitive nef element. In particular, if one thinks of the rational curves as arising from the minimal resolutions of the du Val singularities, then the singularities have to be given by a sub-diagram of the Dynkin diagram corresponding to X. The second application involves studying the symmetries of $K3$. If M is a $K3$ surface of the generic or the algebraic type and M admits a marking by L^X, then the finite symplectic automorphism group G_M of M is a subgroup of G^X. More precisely, we

[4]The trick of considering $S_M \oplus A_1$, also used in [88] to prove Mukai's theorem, is employed here to exclude the Leech lattice.

have

$$G_M = \{g \in G^X | gv = v \text{ for all } v \in (\tilde{S}_M)^{\perp}_{L^X}\}.$$

In the other direction, $G \subset G^X$ is the finite symplectic automorphism group of some $K3$ surface if the orthonormal complement $(L^X)_G \subset L^X$ of the fixed point lattice $\{v \in L^X | gv = v \text{ for all } g \in G\}$ can be primitively embedded into $\Gamma_{3,19}(-1)$. Such $G \subset G^X$ that arise from $K3$ symmetries have been computed in [48] for all 23 L^X. In particular, it is easy to see that they indeed satisfy the geometric condition mentioned in §4: they must have at least 5 orbits on the 24-dimensional representation and at least 1 fixed point.

From the above two applications, we see that the marking by Niemeier lattices facilitates a more refined study of $K3$ geometry by labelling a $K3$ surface by one of the Niemeier lattices L^X via marking. This labelling is, as explained above, sometimes unique and sometimes not. It tends to be unique when the $K3$ surface has very large symmetry – the type of $K3$ surfaces especially of interest to us. In the above two applications, the two most important pieces of data associated to the Niemeier lattice L^X for the construction of umbral moonshine – the root system X and the umbral group G^X – acquire the meaning of the "enveloping smooth rational curve configuration" and the "enveloping symmetry group" respectively, for all the $K3$ surfaces that can be labelled by L^X. Employing this obvious interpretation for X and G^X, the contribution from the ADE singularities to the (twined) $K3$ elliptic genus (cf. (4.9) and (4.12)) acquires the interpretation of the contribution from the "enveloping smooth rational curve configuration" of the (class of) $K3$ surface, while the twining given by umbral moonshine is to be interpreted as encoding the action of the "enveloping symmetry group" on the non-linear sigma model.

Before closing the section, let us give a few examples to illustrate the above discussion. Consider a $K3$ surface M with 16 smooth rational curves giving the root system A_1^{16}, generating a primitive sublattice Π_K of $\mathrm{Pic}(M)$. It is known that such a $K3$ surface is a Kummer surface, *i.e.* a resolution of $T^4/\mathbb{Z}_2$ by replacing the 16 A_1 du Val singularities with 16 rational curves [94]. Note that the $K3$ is not necessarily algebraic since the T^4 can be non-algebraic. From the above discussion we see that M can only be marked by the Niemeier lattice L^X with $X = A_1^{24}$ and hence its finite symplectic automorphism group is a subgroup of M_{24}. More precisely, it is a subgroup of $\{g \in M_{24} | g(\Pi_K) = \Pi_K\}$. Similarly, let's consider as the second example a $K3$ surface M with 18 smooth rational curves giving the root system A_2^9. It can arise in the Kummer-type construction, where we consider the minimal resolution of the nine A_2 type singularities of $T^4/\mathbb{Z}_3$ (for a certain type of T^4 and a certain $\mathbb{Z}_3$). Similarly, M can only marked by the Niemeier lattice L^X with $X = A_2^{12}$ and hence its finite symplectic automorphism group is a subgroup of $G^X \cong 2.M_{12}$. For a certain $T^4/\mathbb{Z}_6$ model, by resolving the singularities of type $A_5 \oplus A_2^4 \oplus A_1^5$

we obtain a $K3$ surface that can be marked by L^X with $X = A_7^2 D_5^2$. See [95,96] for the detailed description of these $K3$ at the orbifold limit. From the above analysis the symmetry of this $K3$ lies in $G^X \cong Dih_4$.

6 Discussion

In this paper we established a relation between umbral moonshine and the $K3$ elliptic genus, thereby taking a first step in placing umbral moonshine into a geometric and physical context. However, many questions remain unanswered and much work still needs to be done before one can solve the mystery of umbral moonshine. In this section we discuss some of the open questions and future directions.

- In §5 we have provided an interpretation of the umbral group G^X as the "enveloping symmetry group" of the (sigma model of) $K3$ surfaces that can be marked by the given Niemeier lattice L^X. It would be interesting to investigate to what extent this general idea of "enveloping symmetry group" can be made precise and can be confirmed by combining geometric symmetries at different points in the moduli space, similar to the idea explored in [35]. Abstractly, it seems rather clear that varying the moduli induces a varying primitive embedding of S_M into L^X and can generate a subgroup of G^X that doesn't necessarily admit an interpretation as a group of geometric symmetries of any specific $K3$ surface. As a concrete example, one family of $K3$ surfaces that that might be amenable to an explicit analysis is the torus orbifold $T^4/\mathbb{Z}_3$, where one can easily vary the moduli of the T^4. As discussed in §5, the umbral group relevant for this family is $G^X \cong 2.M_{12}$ with $X = A_2^{12}$, analogous to the M_{24} case for the torus orbifold $T^4/\mathbb{Z}_2$ studied in [35].
- Another obvious possible interpretation for the conjugacy classes $[g]$ that do not admit a geometric interpretation in the present context is as stringy symmetries of certain $K3$ sigma models preserving the $\mathcal{N} = (4,4)$ superconformal symmetries that have no counterpart in classical geometry. Note that they must have at least 4 orbits in the 24-dimensional representation in order for this interpretation to be possible [32,97]. As a result, it is clear that not all conjugacy classes of all of the 23 G^X admit such a possible interpretation. When a conjugacy class $[g]$ does have at least 4 orbits, often the resulting umbral moonshine twining Z_g^X is observed to coincide with a known elliptic genus $\mathbf{EG}_{g'}(K3)$ twined by a certain symmetry g' of the non-linear sigma model whose induced action on the 24-dimensional representation is isomorphic to that of g, *i.e.* they have the same cycle shape. However, we have not been able to match all Z_g^X with some known CFT twining results

for all $[g] \subset G^X$ with at least 4 orbits. Moreover, for non-geometric classes g the twining Z_g^X is not uniquely determined by the cycle shape Π_g^X and it can occur that $Z_g^X \neq Z_{g'}^{X'}$ even when $\Pi_g^X = \Pi_{g'}^{X'}$. See the following point for a closely-related discussion.

Curiously, various twining functions Z_g^X coincide with those obtained in the work of [40]. It will be interesting to understand better the relation of the two analysis.

- It seems possible and natural to generalise the analysis in §5 beyond the realm of geometric symmetries to include the CFT symmetries. To do so, one should consider the "quantum Picard lattice" $\mathrm{Pic}(M) \oplus U$ instead of $\mathrm{Pic}(M)$ and consider its embedding into $\Gamma_{4,20} = \Gamma_{3,19} \oplus U$ instead of $\Gamma_{3,19}$. The relevant symmetry groups are again subgroups of G^X, now with at least 4 orbits on the 24-dimensional representation. The analysis should amount to a combination of that in [48] and in [32]. However, a lack of a Torelli type theorem means some of the very strong results in [48] will not necessarily hold for the CFT generalisation. Finally, given a fixed Niemeier marking one may also generalise the "symmetry surfing" analysis (see above) into the realm of CFT symmetries.
- It would be illuminating to provide the CFT underpinning of the separation of $\mathbf{EG}(K3)$ into the contribution from the singularities and the rest (4.9), by for instance analysing the twisted and untwisted fields in the orbifold $K3$ models.
- It would be interesting to extend the geometrical definition of elliptic genus (2.22) to non-compact spaces and obtain a geometric derivation of the CFT result (2.17). Similarly, one should compute the geometrical twined (or equivariant) elliptic genera and compare them with the conjecture in Appendix B.2.
- The map (4.12) from the umbral moonshine function H_g^X (or equivalently ϕ_g^X) to the weak Jacobi form Z_g^X is a projection: the summing over the torsion points projects out terms that would have corresponded to states with fractional $U(1)$ charges. In particular, determining a G^X-module for the set of weak Jacobi forms Z_g^X is in general not sufficient to construct the G^X-module K^X underlying H_g^X. It is hence important to gain a better understanding about the physical origin of this projection. Its form is very reminiscent of the Landau–Ginzburg description of the non-linear sigma model and we are currently investigating the relation between umbral moonshine and Landau–Ginzburg type theories.
- The above fact suggests that the full content of umbral moonshine might go well beyond the realm of $K3$ sigma models, and to explain the origin of umbral moonshine we might need to go beyond CFT. It has been suggested that Mathieu moonshine has imprints in a variety of string theory setups (see for instance [18, 26, 34, 36, 37, 57]). Analogously, for all 23 cases of umbral moonshine, it would be interesting to explore the possible string

theoretic extension of the current result.

Acknowledgements

We would like to thank John Duncan, Sameer Murthy, Slava Nikulin, Anne Taormina, Jan Troost, Cumrun Vafa, Dan Whalen and in particular Shamit Kachru, for helpful discussions. MC would like to thank Stanford University and Cambridge University for hospitality. SH is supported by an ARCS Fellowship. We thank the Simons Center for Geometry and Physics for hosting the programme "Mock Modular Forms, Moonshine, and String Theory", where this project was initiated.

A Special Functions

First, we define the *Jacboi theta functions* $\theta_i(\tau, z)$ as follows.

$$\begin{aligned}
\theta_1(\tau,z) &= -iq^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}) \qquad \text{(A.1)}\\
\theta_2(\tau,z) &= q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^{n-1})\\
\theta_3(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1+y\,q^{n-1/2})(1+y^{-1}q^{n-1/2})\\
\theta_4(\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1-y\,q^{n-1/2})(1-y^{-1}q^{n-1/2})
\end{aligned}$$

In particular we will use the transformation of θ_1 under the Jacobi group

$$\begin{aligned}
\theta_1(\tau,z) &= -\theta_1(\tau,-z)\\
&= e(-\tfrac{1}{2}\tfrac{z^2}{\tau})(i\tau)^{-1/2}\theta_1(-\tfrac{1}{\tau},\tfrac{z}{\tau})\\
&= e(-1/8)\,\theta_1(\tau+1,z)\\
&= (-1)^{\lambda+\mu}e(\tfrac{1}{2}(\lambda^2\tau+2\lambda z))\theta_1(\tau,z+\lambda\tau+\mu). \qquad \text{(A.2)}
\end{aligned}$$

Second, we introduce the theta functions

$$\theta_{m,r}(\tau,z) = \sum_{k=r \pmod{2m}} q^{k^2/4m}y^k. \qquad \text{(A.3)}$$

for $m \in \mathbb{Z}_{>0}$ which satisfy

$$\theta_{m,r}(\tau,z) = \theta_{m,r+2m}(\tau,z) = \theta_{m,-r}(\tau,-z).$$

The theta function $\theta_m = (\theta_{m,r})$, $r \in \mathbb{Z}/2m\mathbb{Z}$, is a vector-valued Jacobi form of weight 1/2 and index m satisfying

$$\begin{aligned}\theta_m(\tau,z) &= \sqrt{\frac{1}{2m}}\sqrt{\frac{i}{\tau}}\ \mathrm{e}(-\tfrac{m}{\tau}z^2)\,\mathcal{S}_\theta.\theta_m(-\tfrac{1}{\tau},\tfrac{z}{\tau})\\ &= \mathcal{T}_\theta.\theta_m(\tau+1,z)\\ &= \theta_m(\tau,z+1) = \mathrm{e}(m(\tau+2z))\theta_m(\tau,z+\tau),\end{aligned} \tag{A.4}$$

where the $\mathcal{S}_\theta$ and $\mathcal{T}_\theta$ matrices are $2m \times 2m$ matrices with entries

$$(\mathcal{S}_\theta)_{r,r'} = \mathrm{e}(\tfrac{rr'}{2m})\,\mathrm{e}(\tfrac{-r+r'}{2}) \quad , \quad (\mathcal{T}_\theta)_{r,r'} = \mathrm{e}(-\tfrac{r^2}{4m})\,\delta_{r,r'}. \tag{A.5}$$

For later use we also introduce some weight two modular forms for the Hecke congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad - bcN = 1, \right\}. \tag{A.6}$$

including $\Lambda_N \in M_2(\Gamma_0(N))$ for all $N \in \mathbb{Z}_{>0}$

$$\begin{aligned}\Lambda_N(\tau) &= N\,q\partial_q \log\left(\frac{\eta(N\tau)}{\eta(\tau)}\right)\\ &= \frac{N(N-1)}{24}\left(1 + \frac{24}{N-1}\sum_{k>0}\sigma(k)(q^k - Nq^{Nk})\right),\end{aligned} \tag{A.7}$$

where $\sigma(k)$ is the divisor function $\sigma(k) = \sum_{d|k} d$. For $N = 44$ we will need the unique weight two newform

$$f^{44}_{new} = q + q^3 - 3q^5 + 2q^7 - 2q^9 - q^{11} - 4q^{13} - 3q^{15} + 6q^{17} + \dots$$

Finally we discuss Jacobi forms following [98]. For every pair of integers k and m, we say a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is an *(unrestricted) Jacobi form* of weight k and index m

for the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ if it satisfies

$$\phi(\tau, z) = e(m(\lambda^2\tau + 2\lambda z))\, \phi(\tau, z + \lambda\tau + \mu) \tag{A.8}$$

$$= e(-m\tfrac{cz^2}{c\tau+d})\, (c\tau + d)^{-k} \phi(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}). \tag{A.9}$$

The invariance of $\phi(\tau, z)$ under $\tau \to \tau + 1$ and $z \to z + 1$ implies a Fourier expansion

$$\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n, r) q^n y^r \tag{A.10}$$

for $q = e(\tau)$ and $y = e(z)$, and the elliptic transformation can be used to show that $c(n, r)$ depends only on the *discriminant* $D = r^2 - 4mn$ and on r mod $2m$. An unrestricted Jacobi form is called a *weak Jacobi form* when the Fourier coefficients satisfy $c(n, r) = 0$ whenever $n < 0$. See, for instance, [41] for an introduction of Jacobi forms following [98].

B Calculations and Proofs

B.1 Proof of (4.9)

The aim of this subsection is to provide more details on the elliptic genus computed in §2 and to prove the identity (4.9) for all 23 Niemeier lattices L^X.

As we mentioned in the main text, the Cappelli–Itzykson–Zuber matrices govern the spectrum of $\mathcal{N} = 2$ minimal models as well as the mock modularity of mock modular forms featuring in umbral moonshine. Explicitly, the matrices Ω^Φ labelled by the root system Φ is given in Table 3, where we have introduced for each divisor n of m the following matrices

$$\Omega_m(n)_{r,r'} = \begin{cases} 1 & \text{if } r + r' = 0 \text{ mod } 2n \text{ and } r - r' = 0 \text{ mod } 2m/n, \\ 0 & \text{otherwise,} \end{cases} \tag{B.1}$$

One significance of the Cappelli–Itzykson–Zuber matrices in our context is that it captures the action of the so-called Eichler–Zagier operator $\mathcal{W}_m(n)$, defined for every divisor n of m acting on a function $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ as [98]

$$(f|\mathcal{W}_m(n))\,(\tau, z) = \frac{1}{n} \sum_{a,b=0}^{n-1} e\left(m\left(\tfrac{a^2}{n^2}\tau + 2\tfrac{a}{n}z + \tfrac{ab}{n^2}\right)\right) f\left(\tau, z + \tfrac{a}{n}\tau + \tfrac{b}{n}\right). \tag{B.2}$$

Φ	Ω^Φ
A_{m-1}	$\Omega_m(1)$
$D_{m/2+1}$	$\Omega_m(1)+\Omega_m(m/2)$
E_6	$\Omega_{12}(1)+\Omega_{12}(4)+\Omega_{12}(6)$
E_7	$\Omega_{18}(1)+\Omega_{18}(6)+\Omega_{18}(9)$
E_8	$\Omega_{30}(1)+\Omega_{30}(6)+\Omega_{30}(10)+\Omega_{30}(15)$

Table 3: The ADE matrices Ω of Cappelli–Itzykson–Zuber [42].

To be more precise, acting on the theta function (A.3) it satisfies

$$\theta_m|\mathcal{W}_m(n)=\Omega_m(n)\cdot\theta_m\ . \tag{B.3}$$

In order to exploit this equality in the calculation, we define the operator $\mathcal{W}^\Phi$ by replacing $\Omega_m(n)$ with $\mathcal{W}_m(n)$ in the definition of Ω^Φ (cf. Table 3), with the understanding that $f|\sum_i\mathcal{W}_m(n_i)=\sum_i f|\mathcal{W}_m(n_i)$. Similarly, we define $\mathcal{W}^{\Phi'}=\sum_i\mathcal{W}^{\Phi_i}$ for a union of the simply-laced root systems $\Phi'=\cup_i\Phi_i$ where all Φ_i have the same Coxeter number. For later convenience, analogous to (3.7) we will also define

$$\phi^{\Phi',P}(\tau,z)=\frac{-i\theta_1(\tau,mz)\theta_1(\tau,(m-1)z)}{\eta^3(\tau)\theta_1(\tau,z)}(\mu_{m,0}|\mathcal{W}^{\Phi'}(\tau,z)) \tag{B.4}$$

where m denotes the Coxeter number of Φ as usual.

In [41] a meromorphic function

$$\psi^{X,P}=\mu_{m,0}|\mathcal{W}^X$$

was defined for every Niemeier root system X, where $\mu_{m,0}$ is given by the Appell–Lerch sum as in (2.16) and $\mathcal{W}^X$ is defined as above. Note that ψ^X as a function of z has in general poles at $z\in\frac{\mathbb{Z}}{m}+\frac{\mathbb{Z}}{m}\tau$. In [41], following [12] this meromorphic function has the interpretation as the polar part of the meromorphic Jacobi form

$$\psi^X=\mu_{m,0}|\mathcal{W}^X-\sum_{r\in\mathbb{Z}/2m\mathbb{Z}}H^X_r\theta_{m,r}$$

of weight 1 and index m.

First, we would like to prove

$$Z^{\Phi,S}(\tau,z)=\frac{1}{2m}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}} q^{a^2}y^{2a}\,\phi^{\Phi,P}\big(\tau,\frac{z+a\tau+b}{m}\big). \tag{B.5}$$

We will start by providing more details on the expression (2.8) of the minimal model elliptic genus, which is a building block of the elliptic genus of the ADE singularities (2.17).

Fix m and let $\bar{m}=m-2$. The $\hat{A}_1$ string functions (chiral parafermion partition function times $\eta(\tau)$, see [44]) are given by $c^r_s=0$ if $r=s \pmod 2$ and otherwise

$$c^r_s(\tau)=\frac{1}{\eta^3(\tau)}\sum_{\substack{-|\alpha|<\beta\leq|\alpha| \\ (\alpha,\beta)\,\mathrm{or}\,(\frac{1}{2}-\alpha,\frac{1}{2}+\beta)=(\frac{r}{2m},\frac{s}{2\bar{m}})\,\mathrm{mod}\,\mathbb{Z}^2}}\mathrm{sgn}(\alpha)\,q^{m\alpha^2-\bar{m}\beta^2}$$

Note that we have shifted r by one compared to the convention in, for instance, [44], [63]. Clearly, $r\in\mathbb{Z}/2m\mathbb{Z}$ and $s\in\mathbb{Z}/2\bar{m}\mathbb{Z}$, and $c^r_s(\tau)=-c^{-r}_s(\tau)=c^r_{-s}(\tau)$. They can also be defined through the branching relation

$$\sum_{s\in\mathbb{Z}/2\bar{m}\mathbb{Z}} c^r_s\theta_{\bar{m},s}=\frac{\theta_{m,r}-\theta_{m,-r}}{\theta_{2,1}-\theta_{2,-1}},$$

where we have used the theta function defined in (A.3). Define

$$\chi^r_{s,\epsilon}(\tau,z)=\sum_{k\in\mathbb{Z}/\bar{m}\mathbb{Z}} c^r_{s-\epsilon+4k}(\tau)\,\theta_{2m\bar{m},2s+(4k-\epsilon)m}\big(\tau,\frac{z}{2m}\big).$$

We have $\epsilon\in\mathbb{Z}/4\mathbb{Z}$, from which $\epsilon=0,2$ correspond to the NS and $\epsilon=1,3$ to the Ramond sector. Note that now both r and s in $\chi^r_{s,\epsilon}$ take value in $\mathbb{Z}/2m\mathbb{Z}$.

Now let

$$\tilde{\chi}^r_s(\tau,z)=\chi^r_{s,1}(\tau,z)-\chi^r_{s,-1}(\tau,z).$$

It is easy to check that it transforms under the elliptic transformation as

$$\tilde{\chi}^r_s(\tau,z+a\tau+b)=(-1)^{a+b}\,e(\tfrac{sb}{m})\,e(-\tfrac{\hat{c}}{2}(a^2\tau+2az))\tilde{\chi}^r_{s-2a}(\tau,z). \tag{B.6}$$

They are the Ramond sector superconformal blocks relevant for the $\mathcal{N}=2$ minimal models with $c=3\frac{m-2}{m}$.

Using these building blocks, the elliptic genus of the minimal model corresponding to the

simply-laced root system Φ is then given by

$$Z^{\Phi}_{\text{minimal}}(\tau,z)=\frac{1}{2}\sum_{0<r,r'<m}(\Omega^{\Phi}_{r,r'}-\Omega^{\Phi}_{r,-r'})\sum_{s\in\mathbb{Z}/2m\mathbb{Z}}\tilde{\chi}^{r}_{s}(\tau,z)\tilde{\chi}^{r'}_{s}(\bar{\tau},0).$$

Using $\Omega^{\Phi}_{r,r'}=\Omega^{\Phi}_{-r,-r'}$, $\tilde{\chi}^{r}_{r'}(\tau,z)=-\tilde{\chi}^{-r}_{r'}(\tau,z)$ and $\tilde{\chi}^{r}_{s}(\tau,0)=\delta_{r,s}-\delta_{r,-s}$, we arrive at

$$Z^{\Phi}_{\text{minimal}}(\tau,z)=\frac{1}{2}\sum_{r,r'\in\mathbb{Z}/2m\mathbb{Z}}\Omega^{\Phi}_{r,r'}\tilde{\chi}^{r}_{r'}(\tau,z)=\frac{1}{2}\mathrm{Tr}(\Omega^{\Phi}\cdot\tilde{\chi}).$$

Now we define for any $n\tilde{n}=m$, $n,\tilde{n}\in\mathbb{Z}$ and operator acting on a function $f:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ as

$$f\big|\tilde{\mathcal{W}}_m(n)(\tau,z)=\frac{1}{n}\sum_{\substack{a,b\in\mathbb{Z}/m\mathbb{Z}\\ a,b=0\ (\tilde{n})}}(-1)^{a+b+ab}\,\mathrm{e}\left(\tfrac{m-2}{2m}(a^2\tau+2az+ab)\right)f(\tau,z+a\tau+b)$$

Using (B.6) it is easy to check that

$$\tilde{\chi}^{r}_{s}\big|\tilde{\mathcal{W}}_m(n)=(\Omega_m(n)\cdot\tilde{\chi})^{r}_{s}=\sum_{s'\in\mathbb{Z}/2m\mathbb{Z}}\delta_{s-s',0\ (2\tilde{n})}\delta_{s+s',0\ (2n)}\tilde{\chi}^{r}_{s'}$$

Finally, one can verify that

$$\begin{aligned}&\sum_{\alpha,\beta=0}^{m-1}(-1)^{\alpha+\beta}q^{\alpha^2/2}y^{\alpha}\big(\tilde{\chi}^{r}_{s}\big|\tilde{\mathcal{W}}_m(n)\big)(\tau,z+\alpha\tau+\beta)\,\mu\big(\tau,\frac{z+\alpha\tau+\beta}{m}\big)\\ &=\sum_{\alpha,\beta=0}^{m-1}(-1)^{\alpha+\beta}q^{\alpha^2/2}y^{\alpha}\tilde{\chi}^{r}_{s}(\tau,z+\alpha\tau+\beta)\,\big(\mu\big|\mathcal{W}_m(n)\big)\big(\tau,\frac{z+\alpha\tau+\beta}{m}\big).\end{aligned}$$

Subsequently, the identity (B.5) follows from the above equality and

$$Z^{A_{m-1}}_{\text{minimal}}(\tau,z)=\tfrac{1}{2}\,\mathrm{Tr}\tilde{\chi}=\frac{\theta_1(t,z/m)}{\theta_1(t,z(m-1)/m)}.$$

Finally we are ready to prove (4.9), which can be re-expressed as

$$\mathbf{EG}(\tau,z;K3)=\frac{1}{2m}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}}q^{a^2}y^{2a}\ \phi^{X,T}\big(\tau,\frac{z+a\tau+b}{m}\big)\tag{B.7}$$

when combined with the identity (B.5) that we just verified and when we use the definition

$$\phi^{X,T}(\tau,z)=(\phi^{X,P}+\phi^{X})(\tau,z)=\frac{-i\theta_1(\tau,mz)\theta_1(\tau,(m-1)z)}{\eta^3(\tau)\theta_1(\tau,z)}\psi^{X}(\tau,z).$$

From the fact that ψ^X transforms as a weight 1, index m Jacobi form and using the transformation (A.2) of the Jacobi theta function, it is straightforward to show that the RHS of (B.7) transforms as a weight 0, index 1 Jacobi form. Moreover, the poles of ψ^X at m-torsion points are combined with the zeros of $\theta_1(\tau, mz)$ and as a result $\phi^{X,T}$ is a holomorphic function on $\mathbb{H}\times\mathbb{C}$ admitting a double-expansion in powers of q and y. In order to show that the RHS of (B.7) is a weight 0, index 1 weak Jacobi form, we need to prove that there is no term in its Fourier expansion with $q^n,\ n<0$. This can be shown by using the explicit formulas involving $\mu_{m,0}$ and θ_1, combining with the fact that $H_r^X = O(q^{-r^2/4m})$ and the fact that the sum over b projects out all terms with fractional powers of y. After showing that both sides of (B.7) are weight 0, index 1 weak Jacobi forms, using the fact that the space of such functions is one-dimensional, the equality is proven by comparing both sides at, say, $z=0$.

B.2 Computing Z_g^X

In this subsection we compute the twining function Z_g^X in (4.12). The results of the computation are recorded in Appendix C. In particular, we will give the details of the computation of $Z_g^{X,S}$. As a part of the computation, we also make conjectures for the elliptic genus $Z_h^{\Phi,S}$ of du Val singularities twined by certain automorphisms $\langle h\rangle$ of the corresponding Dynkin diagram Φ.

From the action of $g\in G^X$ on the Niemeier root lattice X, we can divide the conjugacy classes $[g]$ into the following two types. In the first type, there exists an element in the conjugacy class that only permutes the irreducible components of X. More precisely, there exists an element g in the class that descends from an element in $\bar{G}^X\subseteq G^X$, where $\bar{G}^X$ is a quotient of G^X and is defined by

$$\bar{G}^X = \mathrm{Aut}(L^X)/\hat{W}^X,$$

where $\hat{W}^X < \mathrm{Aut}(L^X)$ is the subgroup of lattice automorphisms that stabilize the irreducible components of X. See Table 2 for the list of $\bar{G}^X$. In the second type, the action of an element in $[g]$ necessarily involves certain non-trivial automorphisms of some of the irreducible components in X. See [41] for a more detailed discussion.

As mentioned in §4, the twined function $Z_g^{X,S}$ for a conjugacy class $[g]$ of the first type, pointwise fixing a (not necessarily non-empty) union $X_g=\cup_i\Phi_i\subset X$ of the irreducible components Φ_i, is simply given by

$$Z_g^{X,S} = \sum_i Z^{\Phi_i,S}.$$

In order to compute the twined function $Z_g^{X,S}$ for $[g]$ for the second type of conjugacy

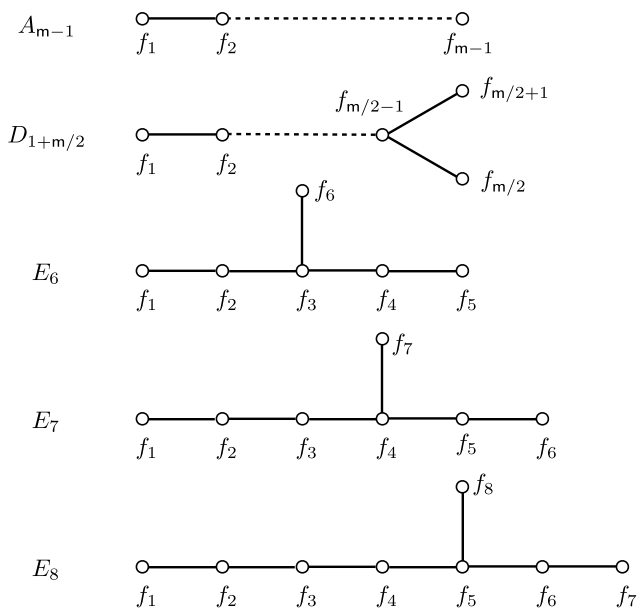


Figure 1: The ADE Dynkin diagrams

classes, we need to twine the elliptic genus of the (irreducible) ADE singularities by symmetries corresponding to the automorphisms of the Dynkin diagram Φ. In the rest of this appendix we will propose a conjectural answer.

For the A_{m-1} singularity with $m > 2$ we have the $\mathbb{Z}_2$ automorphism exchanging the simple root f_i with f_{m-i}, in the notation shown in Figure 1. We conjecture that the corresponding twined elliptic genus is

$$Z^{\Phi,S}_{\mathbb{Z}_2} = Z^{\Phi,S}|\mathcal{W}^{(-)}\ ,\ \Phi = A_{m-1},$$

where we have defined the operator acting on a function $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ as

$$f|\mathcal{W}^{(-)}(\tau, z) = -f(\tau, z + \tfrac{1}{2}).$$

In fact, the above expression for $Z^{\Phi,S}_{\mathbb{Z}_2}$ can be deduced from the action of $\mathbb{Z}_2$ on the eigenvectors of the appropriate Coxeter element, and similarly for the twined elliptic genus of the D- and E-type singularities discussed below.

For later use we also define the operators

$$f|\mathcal{W}^{(3)}(\tau,z)=\frac{1}{3}\sum_{a=0}^{2}f(\tau,z+\tfrac{a}{3})$$
$$f|\mathcal{W}^{(6)}(\tau,z)=\frac{1}{6}\sum_{a=0}^{5}f(\tau,z+\tfrac{a}{6}).$$

We remark that the above conjecture, if proven, provides a geometrical explanation of the following interesting property of the G^X-module K^X. It was observed and conjectured in [7,41] that in the cases where X has only A-type components (*i.e.* when $m-1|24$), the G^X-module K^X_r underlying the even components of the mock modular form $H^X_{g,r}$ (r even), are composed of irreducible faithful representations of G^X. On the other hand, the module K^X_r underlying the odd components of the mock modular form $H^X_{g,r}$ (r odd), are composed of G^X-representations that factor through $\bar{G}^X$. Similar considerations also apply to the cases when X contains also D- and E-type components.

For the D-type singularity different from D_4, we have the $\mathbb{Z}_2$ automorphism exchanging the simple root $f_{m/2}$ with $f_{1+m/2}$, in the notation shown in Figure 1. We conjecture that the corresponding twined elliptic genus is

$$Z^{\Phi,S}_{\mathbb{Z}_2}(\tau,z)=\frac{1}{m}\sum_{a,b\in\mathbb{Z}/m\mathbb{Z}}q^{a^2}y^{2a}\,\phi^{\Phi,P}_{\mathbb{Z}_2}\big(\tau,\frac{z+a\tau+b}{m}\big)$$

for $D_{1+m/2}$ for $m\neq 6$, where

$$\phi^{\Phi,P}_{\mathbb{Z}_2}(\tau,z)=\frac{-i\theta_1(\tau,mz)\theta_1(\tau,(m-1)z)}{\eta^3(\tau)\theta_1(\tau,z)}\big(-\mu_{m,0}|\mathcal{W}_m(m/2)+\mu_{m,0}|\mathcal{W}^{(-)}\big)(\tau,z). \tag{B.8}$$

The $\Phi=D_4$ Dynkin diagram permits a S_3 symmetry on the roots $\{f_1,f_3,f_4\}$. We conjecture that the corresponding twined elliptic genera are given by

$$Z^{\Phi,S}_{\mathbb{Z}_2}(\tau,z)=\frac{1}{6}\sum_{a,b\in\mathbb{Z}/6\mathbb{Z}}q^{a^2}y^{2a}\,\phi^{\Phi,P}_{\mathbb{Z}_2}\big(\tau,\frac{z+a\tau+b}{6}\big)$$
$$Z^{\Phi,S}_{\mathbb{Z}_3}(\tau,z)=\frac{1}{6}\sum_{a,b\in\mathbb{Z}/6\mathbb{Z}}q^{a^2}y^{2a}\,\phi^{\Phi,P}_{\mathbb{Z}_3}\big(\tau,\frac{z+a\tau+b}{6}\big)$$

where

$$\phi^{\Phi,P}_{\mathbb{Z}_2}(\tau,z) = \frac{-i\theta_1(\tau,6z)\theta_1(\tau,5z)}{\eta^3(\tau)\theta_1(\tau,z)}\mu_{6,0}|\mathcal{W}_{D_4,\mathbb{Z}_2}(\tau,z)$$
$$\phi^{\Phi,P}_{\mathbb{Z}_3}(\tau,z) = \frac{-i\theta_1(\tau,6z)\theta_1(\tau,5z)}{\eta^3(\tau)\theta_1(\tau,z)}\mu_{6,0}|\mathcal{W}_{D_4,\mathbb{Z}_3}(\tau,z)$$

and

$$\mathcal{W}_{D_4,\mathbb{Z}_2} = -\mathcal{W}_6(3) + \mathcal{W}^{(-)} + 2\mathcal{W}^{(6)}$$
$$\mathcal{W}_{D_4,\mathbb{Z}_3} = \mathcal{W}^{D_4} - 3\mathcal{W}^{(3)}.$$

The only E-type diagram with non-trivial automorphism is the $\mathbb{Z}_2$ generated by the action $f_i \mapsto f_{6-i}$ of E_6, for $1 \le i \le 5$. We conjecture that corresponding twined elliptic genus is

$$Z^{E_6,S}_{\mathbb{Z}_2}(\tau,z) = \frac{1}{12}\sum_{a,b\in\mathbb{Z}/12\mathbb{Z}} q^{a^2}y^{2a}\,\phi^{E_6,P}_{\mathbb{Z}_2}\big(\tau,\frac{z+a\tau+b}{12}\big)$$

where

$$\phi^{E_6,P}_{\mathbb{Z}_2}(\tau,z) = \frac{-i\theta_1(\tau,12z)\theta_1(\tau,11z)}{\eta^3(\tau)\theta_1(\tau,z)}\big(\mu_{12,0}|\mathcal{W}_{12}(6) + \mu_{12,0}|\mathcal{W}^{(-)} + \mu_{12,0}|\mathcal{W}_{12}(4)|\mathcal{W}^{(-)}\big)(\tau,z). \tag{B.9}$$

After giving the conjectural answer for the building blocks $Z^{\Phi,S}_h$ of the twining of $Z^{X,S}_g$, we need to know how such a $g \in G^X$ acts on the Niemeier root system X. This is encoded in the twisted Euler characters $\bar{\chi}^{X_A}$, χ^{X_A}, $\bar{\chi}^{X_D}$, χ^{X_D} ... attached to the A-, D-, and E-components of each X. See §2.4 of [41] for details and see Appendix B.2 of the same reference for the values of such twisted Euler characters for all 23 X. Combining these ingredients leads to the answer for $Z^{X,S}_g$ for all conjugacy classes $[g]$ for all the umbral groups G^X. This completes our computation of Z^X_g (4.12).

C The Twining Functions

In this appendix we provide the expression of $Z^X_g(\tau,z)$ (cf. (4.12)) in terms of the function h^X_g:

$$Z^X_g(\tau,z) = \frac{i\theta_1(\tau,z)^2}{\theta_1(\tau,2z)\eta^3(\tau)}\Big\{c^X_g\mu_{2,0}(t,z) + h^X_g(\tau)(\theta_{2,-1}(\tau,z) - \theta_{2,1}(\tau,z))\Big\}$$

where c^X_g is the number of fixed point in the 24-dimensional representation of G^X. In other words, for the cycle shape Π^X_g defined in (4.13), we have $c^X_g = m_1$ if $\ell_1 = 1$ and $c^X_g = 0$

otherwise. For instance, for $X = A_1^{24}$ and $[g]$ the identity class, the above formula gives the $\mathcal{N} = 4$ character decomposition of $\mathbf{EG}(K3)$ in (4.6).

For $X = A_1^{24}$, the functions $h_g^X(\tau)$ for all $[g] \subset G^X \cong M_{24}$ have been worked out in [18-21]. We refer to these papers, or the summary in [7,29]. For convenience we will denote $h_g^{A_1^{24}}$ simply by H_g for $[g] \subset M_{24}$. Recall that H_g is nothing but the function discussed in (1.2) when $[g] = 1A$ is the identity class of M_{24}. There are two cases, corresponding to $X = D_{24}$ and $X = D_{16}E_8$, with trivial G^X. As a result they are not included in the present appendix.

When h_g^X coincides with $H_{g'}$ for a certain g', we will simply use this identity to define h_g^X. When there does not exist such a $[g'] \subset M_{24}$, we write

$$h_g^X(\tau) = \frac{c_g^X}{24} H_{1A}(\tau) - \frac{\tilde{T}_g^X(\tau)}{\eta(\tau)^3} \tag{C.1}$$

and we will give the explicit expression for $\tilde{T}_g^X$ in the following tables using the functions given in Appendix A. We also use the short hand notation $(n)^k := \eta^k(n\tau)$.

Table 4: $X = A_2^{12}$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$4A$	4^6	H_{4C}
$2B$	$1^8 2^8$	H_{2A}
$2C$	2^{12}	H_{2B}
$3A$	$1^6 3^6$	H_{3A}
$6A$	$2^3 6^3$	$\tilde{T}^X_{6A} = 3\Lambda_2 + 2\Lambda_3 - \Lambda_4 - 3\Lambda_6 + \Lambda_{12}$
$3B$	3^8	$\tilde{T}^X_{3B} = 2(-4\Lambda_3 + \Lambda_9 - (1)^6/(3)^2)$
$6B$	6^4	$\tilde{T}^X_{6B} = 2\frac{(1)^5(3)}{(2)(6)}$
$4B$	$2^4 4^4$	H_{4A}
$4C$	$1^4 2^2 4^4$	H_{4B}
$5A$	$1^4 5^4$	H_{5A}
$10A$	$2^2 10^2$	H_{10A}
$12A$	12^2	$\tilde{T}^X_{12A} = 2\frac{(1)(2)^5(3)}{(4)^2(6)}$
$6C$	$1^2 2^2 3^2 6^2$	H_{6A}
$6D$	$2^3 6^3$	$h^X_{6D} = h^X_{6A}$
$8AB$	$4^2 8^2$	$\tilde{T}^X_{8AB} = 2(2)^4(4)^2/(8)^2$
$8CD$	$1^2 2^1 4^1 8^2$	H_{8A}
$20AB$	$4^1 20^1$	$\tilde{T}^X_{20AB} = 2\frac{(2)^7(5)}{(1)(4)^2(10)}$
$11AB$	$1^2 11^2$	$\tilde{T}^X_{11AB} = (2\Lambda_{11}(\tau) + 33(1)^2(11)^2)/5$
$22AB$	$2^1 22^1$	$\tilde{T}^X_{22AB} = (3\Lambda_2 - \Lambda_4 + 2\Lambda_{11} - 3\Lambda_{22} + \Lambda_{44})/15 - \frac{22}{3} f^{44}_{new} + \frac{11}{5}(1)^2(11)^2 + \frac{44}{5}(2)^2(22)^2 + \frac{88}{5}(4)^2(44)^2$

Table 5: $X = A_3^8$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}
$2B$	2^{12}	H_{2B}
$4A$	$2^4 4^4$	H_{4A}
$4B$	4^6	$\tilde{T}_{4B}^X = 2\Lambda_4 - 3\Lambda_8 + \Lambda_{16} - 2(1)^4(2)^2/(4)^2$
$2C$	$1^8 2^8$	H_{2A}
$3A$	$1^6 3^6$	H_{3A}
$6A$	$1^2 2^2 3^2 6^2$	H_{6A}
$6BC$	$2^3 6^3$	$h_{6BC}^X = h_{6A}^Y,\ Y = A_2^{12}$
$8A$	$4^2 8^2$	$\tilde{T}_{8A}^X = (2\Lambda_8 - 3\Lambda_{16} + \Lambda_{32})/8 + 8(4)^4(16)^4/(8)^4 - 8(4)^2(8)^2$
$4C$	$1^4 2^2 4^4$	H_{4B}
$7AB$	$1^3 7^3$	H_{7AB}
$14AB$	$1^1 2^1 7^1 14^1$	$\tilde{T}_{14AB}^X = (-\Lambda_2 - \Lambda_7 + \Lambda_{14} + 28(1)(2)(7)(14))/3$

Table 6: $X = A_4^6$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$2B$	2^{12}	H_{2B}
$2C$	$1^8 2^8$	H_{2A}
$3A$	3^8	H_{3B}
$6A$	6^4	H_{6B}
$5A$	$1^4 5^4$	H_{5A}
$10A$	$2^2 10^2$	$\tilde{T}_{10A}^X = (3\Lambda_2 - \Lambda_4 + 2\Lambda_5 - 3\Lambda_{10} + \Lambda_{20} + 40(2)^2(10)^2)/3$
$4AB$	4^6	$h_{4AB}^X = h_{4B}^Y,\ Y = A_3^8$
$4CD$	$1^4 2^2 4^4$	H_{4B}
$12AB$	12^2	$\tilde{T}_{12AB}^X = 2\frac{(2)^2(6)^4(1)^2}{(3)^2(12)^2}$

Table 7: $X = A_5^4 D_4$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}
$2B$	$1^8 2^8$	H_{2A}
$4A$	$1^4 2^2 4^4$	H_{4B}
$3A$	$1^6 3^6$	H_{3A}
$6A$	$1^2 2^2 3^2 6^2$	H_{6A}
$8AB$	$1^2 2^1 4^1 8^2$	H_{8A}

Table 8: $X = D_4^6$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$3A$	$1^6 3^6$	H_{3A}
$2A$	$1^8 2^8$	H_{2A}
$6A$	$1^2 2^2 3^2 6^2$	H_{6A}
$3B$	$1^6 3^6$	H_{3A}
$3C$	3^8	$h_{3C}^X = h_{3B}^Y,\ Y = A_2^{12}$
$4A$	$2^4 4^4$	H_{4A}
$12A$	$2^1 4^1 6^1 12^1$	$\tilde{T}_{12A}^X = (-2\Lambda_2 + 3\Lambda_4 + 2\Lambda_6 - \Lambda_8 - 3\Lambda_{12} + \Lambda_{24})/4 + 18(2)(4)(6)(12)$
$5A$	$1^4 5^4$	H_{5A}
$15AB$	$1^1 3^1 5^1 15^1$	$\tilde{T}_{15AB}^X = (-\Lambda_3 - \Lambda_5 + \Lambda_{15} + 45(1)(3)(5)(15))/4$
$2B$	$1^8 2^8$	H_{2A}
$2C$	2^{12}	H_{2B}
$4B$	$1^4 2^2 4^4$	H_{4B}
$6B$	$1^2 2^2 3^2 6^2$	H_{6A}
$6C$	6^4	$h_{6C}^X = h_{6B}^Y,\ Y = A_2^{12}$

Table 9: $X = A_6^4$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$4A$	4^6	H_{4C}
$3AB$	$1^6 3^6$	H_{3A}
$6AB$	$2^3 6^3$	$h_{6AB}^X = h_{6A}^Y = h_{6BC}^Z,\ Y = A_2^{12}, Z = A_3^8$

Table 10: $X = A_7^2 D_5^2$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}
$2B$	$1^8 2^8$	H_{2A}
$2C$	$1^8 2^8$	H_{2A}
$4A$	$2^4 4^4$	H_{4A}

Table 11: $X = A_8^3$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$2B$	$1^8 2^8$	H_{2A}
$2C$	2^{12}	H_{2B}
$3A$	3^8	$h_{3A}^X = h_{3B}^Y\ , Y = A_2^{12}$
$6A$	6^4	$\tilde{T}_{6A}^X = 2\frac{(1)^5(3)}{(2)(6)} + 24(6)^4$

Table 12: $X = A_9^2 D_6$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}
$4AB$	$1^4 2^2 4^4$	H_{4B}

Table 13: $X = D_6^4$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$3A$	$1^6 3^6$	H_{3A}
$2B$	$1^8 2^8$	H_{2A}
$4A$	4^6	H_{4C}

Table 14: $X = A_{11} D_7 E_6$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}

Table 15: $X = E_6^4$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}
$2B$	$1^8 2^8$	H_{2A}
$4A$	$2^4 4^4$	H_{4A}
$3A$	$1^6 3^6$	H_{3A}
$6A$	$1^2 2^2 3^2 6^2$	H_{6A}
$8AB$	$4^2 8^2$	$\tilde{T}^X_{8AB} = (2\Lambda_8 - 3\Lambda_{16} + \Lambda_{32})/8 + 24(4)^2(8)^2 + 8(4)^4(16)^4/(8)^4$

Table 16: $X = A_{12}^2$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}
$4AB$	4^6	$h^X_{4AB} = h^Y_{4B} = h^Z_{4AB}\ , Y = A_3^8,\ Z = A_4^6$

Table 17: $X = D_8^3$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}
$3A$	3^8	H_{3B}

Table 18: $X = A_{15}D_9$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}

Table 19: $X = A_{17}E_7$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}

Table 20: $X = D_{10}E_7^2$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}

Table 21: $X = D_{12}^2$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}

Table 22: $X = A_{24}$

$[g]$	Π_g^X	h_g^X
$1A$	1^{24}	H_{1A}
$2A$	2^{12}	H_{2B}

Table 23: $X = E_8^3$

$[g]$	$\tilde{\Pi}_g$	Z_g
$1A$	1^{24}	H_{1A}
$2A$	$1^8 2^8$	H_{2A}
$3A$	3^8	$h_{3B}^X = h_{3B}^Y,\ Y = A_2^{12}$

References

[1] J. H. Conway and S. P. Norton, "Monstrous Moonshine," *Bull. London Math. Soc.* **11** (1979) 308 339.

[2] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A natural representation of the Fischer-Griess Monster with the modular function J as character," *Proc. Nat. Acad. Sci. U.S.A.* **81** (1984) no. 10, Phys. Sci., 3256–3260.

[3] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A moonshine module for the Monster," in *Vertex operators in mathematics and physics (Berkeley, Calif., 1983)*, vol. 3 of *Math. Sci. Res. Inst. Publ.*, pp. 231–273. Springer, New York, 1985.

[4] R. E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras," *Invent. Math.* **109, No.2** (1992) 405–444.

[5] J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier, *The 1-2-3 of modular forms.* Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[6] T. Gannon, *Moonshine beyond the monster. The bridge connecting algebra, modular forms and physics.* Cambridge University Press, 2006.

[7] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine," arXiv:1204.2779 [math.RT].

[8] A. Folsom, "What is ... a mock modular form?," *Notices Amer. Math. Soc.* **57** (2010) no. 11, 1441–1443.

[9] D. Zagier, "Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)," *Astérisque* (2009) no. 326, Exp. No. 986, vii–viii, 143–164 (2010). Séminaire Bourbaki. Vol. 2007/2008.

[10] C. Vafa and E. Witten, "A strong coupling test of S-duality," *Nuclear Phys. B* **431** (1994) no. 1-2, 3–77. http://dx.doi.org/10.1016/0550-3213(94)90097-3.

[11] J. Troost, "The non-compact elliptic genus: mock or modular," *JHEP* **1006** (2010) 104, arXiv:1004.3649 [hep-th].

[12] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," arXiv:1208.4074 [hep-th].

[13] S. Alexandrov, J. Manschot, and B. Pioline, "D3-instantons, Mock Theta Series and Twistors," *JHEP* **1304** (2013) 002, arXiv:1207.1109 [hep-th].

[14] T. Eguchi and A. Taormina, "Unitary representations of the $N = 4$ superconformal algebra," *Phys. Lett. B* **196** (1987) no. 1, 75–81. http://dx.doi.org/10.1016/0370-2693(87)91679-0.

[15] T. Eguchi and A. Taormina, "Character formulas for the $N = 4$ superconformal algebra," *Phys. Lett. B* **200** (1988) no. 3, 315–322. http://dx.doi.org/10.1016/0370-2693(88)90778-2.

[16] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, "Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy," *Nucl. Phys.* **B315** (1989) 193.

[17] T. Eguchi, H. Ooguri, and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," *Exper.Math.* **20** (2011) 91–96, arXiv:1004.0956 [hep-th].

[18] M. C. Cheng, "K3 Surfaces, N=4 Dyons, and the Mathieu Group M_{24}," *Commun.Num.Theor.Phys.* **4** (2010) 623–658, arXiv:1005.5415 [hep-th].

[19] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu twining characters for K3," *JHEP* **1009** (2010) 058, arXiv:1006.0221 [hep-th].

[20] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," *JHEP* **1010** (2010) 062, arXiv:1008.3778 [hep-th].

[21] T. Eguchi and K. Hikami, "Note on Twisted Elliptic Genus of K3 Surface," *Phys.Lett.* **B694** (2011) 446–455, arXiv:1008.4924 [hep-th].

[22] M. C. Cheng and J. F. Duncan, "On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine," *Commun.Num.Theor.Phys.* **6** (2012) 697–758, arXiv:1110.3859 [math.RT].

[23] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch, and R. Volpato, "Generalised Mathieu Moonshine," *Commun.Num.Theor.Phys.* **7** (2013) 145–223, arXiv:1211.7074 [hep-th].

[24] M. C. N. Cheng and J. F. R. Duncan, "On the Discrete Groups of Mathieu Moonshine," *ArXiv e-prints* (Dec., 2012) , arXiv:1212.0906 [math.NT].

[25] M. R. Gaberdiel, D. Persson, and R. Volpato, "Generalised Moonshine and Holomorphic Orbifolds," arXiv:1302.5425 [hep-th].

[26] D. Persson and R. Volpato, "Second Quantized Mathieu Moonshine," arXiv:1312.0622 [hep-th].

[27] M. Raum, "M24-twisted product expansions are siegel modular forms," arXiv:1208.3453 [math.NT].

[28] T. Gannon, "Much ado about Mathieu," arXiv:1211.5531 [math.RT].

[29] M. C. N. Cheng and J. F. R. Duncan, "The Largest Mathieu Group and (Mock) Automorphic Forms," 1201.4140 [math.RT].

[30] A. Taormina and K. Wendland, "The symmetries of the tetrahedral kummer surface in the mathieu group M_{24}," 1008.0954.

[31] A. Taormina and K. Wendland, "The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}," *JHEP* **1308** (2013) 125, arXiv:1107.3834 [hep-th].

[32] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Symmetries of K3 sigma models," *Commun.Num.Theor.Phys.* **6** (2012) 1–50, arXiv:1106.4315 [hep-th].

[33] S. Govindarajan, "Unravelling Mathieu Moonshine," *Nucl.Phys.* **B864** (2012) 823–839, arXiv:1106.5715 [hep-th].

[34] M. C. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, *et al.*, "Mathieu Moonshine and N=2 String Compactifications," *JHEP* **1309** (2013) 030, arXiv:1306.4981 [hep-th].

[35] A. Taormina and K. Wendland, "Symmetry-surfing the moduli space of Kummer K3s," arXiv:1303.2931 [hep-th].

[36] S. Harrison, S. Kachru, and N. M. Paquette, "Twining Genera of (0,4) Supersymmetric Sigma Models on K3," *JHEP* **1404** (2014) 048, arXiv:1309.0510 [hep-th].

[37] J. A. Harvey and S. Murthy, "Moonshine in Fivebrane Spacetimes," *JHEP* **1401** (2014) 146, arXiv:1307.7717 [hep-th].

[38] T. Wrase, "Mathieu moonshine in four dimensional $\mathcal{N} = 1$ theories," *JHEP* **1404** (2014) 069, arXiv:1402.2973 [hep-th].

[39] T. Creutzig and G. Hoehn, "Mathieu Moonshine and the Geometry of K3 Surfaces," arXiv:1309.2671 [math.QA].

[40] J. Duncan and S. Mack-Crane, "Derived Equivalences of K3 Surfaces and Twined Ellipitic Genera," *to appear* .

[41] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine and the Niemeier Lattices," arXiv:1307.5793 [math.RT].

[42] A. Cappelli, C. Itzykson, and J. Zuber, "The ADE Classification of Minimal and A1(1) Conformal Invariant Theories," *Commun.Math.Phys.* **113** (1987) 1.

[43] A. Cappelli, C. Itzykson, and J. Zuber, "Modular Invariant Partition Functions in Two-Dimensions," *Nucl.Phys.* **B280** (1987) 445–465.

[44] D. Gepner, "Space-Time Supersymmetry in Compactified String Theory and Superconformal Models," *Nucl.Phys.* **B296** (1988) 757.

[45] C. Vafa and N. P. Warner, "Catastrophes and the Classification of Conformal Theories," *Phys.Lett.* **B218** (1989) 51.

[46] E. J. Martinec, "Algebraic Geometry and Effective Lagrangians," *Phys.Lett.* **B217** (1989) 431.

[47] H. Cartan, "Quotient d'un espace analytique par un groupe d'automorphismes," in *Algebraic geometry and topology*, pp. 90–102. Princeton University Press, Princeton, N. J., 1957. A symposium in honor of S. Lefschetz,.

[48] V. V. Nikulin, "Kählerian K3 Surfaces and Niemeier Lattices," `arXiv:1109.2879 [math.AG]`.

[49] V. V. Nikulin, "Degenerations of Kählerian K3 surfaces with finite symplectic automorphism groups ," `arXiv:1403.6061 [math.AG]`.

[50] A. H. Durfee, "Fifteen characterizations of rational double points and simple critical points," *Enseign. Math. (2)* **25** (1979) no. 1-2, 131–163.

[51] E. Witten, "String theory dynamics in various dimensions," *Nuclear Physics B* **443** (June, 1995) . `http://arxiv.org/abs/hep-th/9503124`.

[52] P. S. Aspinwall, "Enhanced gauge symmetries and K3 surfaces," *Phys.Lett.* **B357** (1995) 329–334, `arXiv:hep-th/9507012 [hep-th]`.

[53] S. H. Katz, A. Klemm, and C. Vafa, "Geometric engineering of quantum field theories," *Nucl.Phys.* **B497** (1997) 173–195, `arXiv:hep-th/9609239 [hep-th]`.

[54] H. Ooguri and C. Vafa, "Two-dimensional black hole and singularities of CY manifolds,". `http://arxiv.org/abs/hep-th/9511164`. Nucl.Phys.B463:55-72,1996.

[55] Y. Kazama and H. Suzuki, "New N=2 Superconformal Field Theories and Superstring Compactification," *Nucl.Phys.* **B321** (1989) 232.

[56] A. Giveon and D. Kutasov, "Little string theory in a double scaling limit," *JHEP* **9910** (1999) 034, `arXiv:hep-th/9909110 [hep-th]`.

[57] J. A. Harvey, S. Murthy, and C. Nazaroglu, "ADE Double Scaled Little String Theories, Mock Modular Forms and Umbral Moonshine," `arXiv:1410.6174 [hep-th]`.

[58] T. Gannon, "U$(1)^m$ modular invariants, $N = 2$ minimal models, and the quantum Hall effect," *Nuclear Phys. B* **491** (1997) no. 3, 659–688. `http://dx.doi.org/10.1016/S0550-3213(97)00032-1`.

[59] O. Gray, "On the complete classification of unitary $N = 2$ minimal superconformal field theories," *Comm. Math. Phys.* **312** (2012) no. 3, 611–654. http://dx.doi.org/10.1007/s00220-012-1478-z.

[60] E. Witten, "Phases of N=2 theories in two-dimensions," *Nucl.Phys.* **B403** (1993) 159–222, arXiv:hep-th/9301042 [hep-th].

[61] E. Witten, "On the Landau-Ginzburg Description of $N = 2$ Minimal Models," *Int.J.Mod.Phys.A* **9:4783-4800,1994** (Int.J.Mod.Phys.A9:4783-4800,1994) , hep-th/9304026.

[62] Z.-a. Qiu, "Modular Invariant Partition Functions for $N = 2$ Superconformal Field Theories," *Phys.Lett.* **B198** (1987) 497.

[63] T. Kawai, Y. Yamada, and S.-K. Yang, "Elliptic Genera and N=2 Superconformal Field Theory,". http://arxiv.org/abs/hep-th/9306096. Nucl.Phys.B414:191-212,1994.

[64] P. Di Francesco and S. Yankielowicz, "Ramond sector characters and N=2 Landau-Ginzburg models," *Nucl.Phys.* **B409** (1993) 186–210, arXiv:hep-th/9305037 [hep-th].

[65] E. Witten, "On string theory and black holes," *Phys.Rev.* **D44** (1991) 314–324.

[66] K. Hori and A. Kapustin, "Duality of the fermionic 2-D black hole and N=2 liouville theory as mirror symmetry," *JHEP* **0108** (2001) 045, arXiv:hep-th/0104202 [hep-th].

[67] D. Israel, C. Kounnas, A. Pakman, and J. Troost, "The Partition function of the supersymmetric two-dimensional black hole and little string theory," *JHEP* **0406** (2004) 033, arXiv:hep-th/0403237 [hep-th].

[68] T. Eguchi and Y. Sugawara, "SL(2,R)/U(1) supercoset and elliptic genera of noncompact Calabi-Yau manifolds," *JHEP* **0405** (2004) 014, arXiv:hep-th/0403193 [hep-th].

[69] L. J. Dixon, M. E. Peskin, and J. D. Lykken, "N=2 Superconformal Symmetry and SO(2,1) Current Algebra," *Nucl.Phys.* **B325** (1989) 329–355.

[70] A. Hanany, N. Prezas, and J. Troost, "The Partition function of the two-dimensional black hole conformal field theory," *JHEP* **0204** (2002) 014, arXiv:hep-th/0202129 [hep-th].

[71] R. Dijkgraaf, H. L. Verlinde, and E. P. Verlinde, "String propagation in a black hole geometry," *Nucl.Phys.* **B371** (1992) 269–314.

[72] T. Eguchi and Y. Sugawara, "Non-holomorphic Modular Forms and SL(2,R)/U(1) Superconformal Field Theory," *JHEP* **1103** (2011) 107, arXiv:1012.5721 [hep-th].

[73] S. K. Ashok and J. Troost, "A Twisted Non-compact Elliptic Genus," *JHEP* **1103** (2011) 067, arXiv:1101.1059 [hep-th].

[74] S. Zwegers, "Mock theta functions," arXiv:0807.4834 [math.RT].

[75] S. K. Ashok, N. Doroud, and J. Troost, "Localization and real Jacobi forms," arXiv:1311.1110 [hep-th].

[76] S. Murthy, "A holomorphic anomaly in the elliptic genus," arXiv:1311.0918 [hep-th].

[77] S. Ochanine *Topology. An International Journal of Mathematics* **26** (1987) 143.

[78] E. Witten, "ELLIPTIC GENERA AND QUANTUM FIELD THEORY," *Commun. Math. Phys.* **109** (1987) 525.

[79] P. S. Landweber, ed., *Elliptic curves and modular forms in algebraic topology.* Springer-Verlag, Berlin, 1988.

[80] T. Kawai, Y. Yamada, and S.-K. Yang, "Elliptic genera and n=2 superconformal field theory," *Nucl.Phys.B* **414:191-212,1994** (Nucl.Phys.B414:191-212,1994) , hep-th/9306096.

[81] A. Kapustin, "Chiral de Rham complex and the half-twisted sigma-model," arXiv:hep-th/0504074 [hep-th].

[82] H.-V. Niemeier, "Definite quadratische Formen der Dimension 24 und Diskriminante 1," *J. Number Theory* **5** (1973) 142–178.

[83] J. H. Conway, "A characterisation of Leech's lattice," *Invent. Math.* **7** (1969) 137–142.

[84] J. Leech, "Notes on sphere packings," *Canad. J. Math.* **19** (1967) 251–267.

[85] J. Leech, "Some sphere packings in higher space," *Canad. J. Math.* **16** (1964) 657–682.

[86] N.-P. Skoruppa, *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts.* Bonner Mathematische Schriften [Bonn Mathematical Publications], 159. Universität Bonn Mathematisches Institut, Bonn, 1985. Dissertation, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1984.

[87] T. Eguchi and A. Taormina, "On the unitary representations of $N = 2$ and $N = 4$ superconformal algebras," *Phys. Lett. B* **210** (1988) no. 1-2, 125–132. http://dx.doi.org/10.1016/0370-2693(88)90360-7.

[88] S. Kondō, "Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces," *Duke Math. J.* **92** (1998) no. 3, 593–603. http://dx.doi.org/10.1215/S0012-7094-98-09217-1. With an appendix by Shigeru Mukai.

[89] V. V. Nikulin, "Integral symmetric bilinear forms and some of their applications," *Mathematics of the USSR-Izvestiya* **14** (1980) no. 1, 103. http://stacks.iop.org/0025-5726/14/i=1/a=A06.

[90] V. V. Nikulin, "Finite groups of automorphisms of Kählerian surfaces of type $K3$," *Uspehi Mat. Nauk* **31** (1976) no. 2(188), 223–224.

[91] S. Mukai, "Finite groups of automorphisms of $K3$ surfaces and the Mathieu group," *Invent. Math.* **94** (1988) no. 1, 183–221. http://dx.doi.org/10.1007/BF01394352.

[92] V. V. Nikulin, "Finite groups of automorphisms of Kählerian $K3$ surfaces," *Trudy Moskov. Mat. Obshch.* **38** (1979) 75–137.

[93] K. Hashimoto, "Finite symplectic actions on the $K3$ lattice," *Nagoya Math. J.* **206** (2012) 99–153. http://projecteuclid.org/euclid.nmj/1337690053.

[94] V. V. Nikulin, "On Kummer Surfaces," *Izv. Akad. Nauk SSSR Ser. Mat.* .

[95] A. Fujiki, "Finite automorphism groups of complex tori of dimension 2," *Publ. RIMS. Kyoto Univ.* **24** (1988) 1–97.

[96] K. Wendland, "Consistency of orbifold conformal field theories on K3," *Adv.Theor.Math.Phys.* **5** (2002) 429–456, arXiv:hep-th/0010281 [hep-th].

[97] P. S. Aspinwall and D. R. Morrison, "String theory on K3 surfaces," arXiv:hep-th/9404151 [hep-th].

[98] M. Eichler and D. Zagier, *The theory of Jacobi forms.* Birkhäuser, 1985.

Symmetry, Integrability and Geometry: Methods and Applications SIGMA **15** (2019), 014, 27 pages

Generalised Umbral Moonshine

Miranda C.N. CHENG [†1†2], *Paul DE LANGE* [†3] *and Daniel P.Z. WHALEN* [†4]

[†1] *Korteweg-de Vries Institute for Mathematics, Amsterdam, The Netherlands*
E-mail: *mcheng@uva.nl*

[†2] *Institute of Physics, University of Amsterdam, Amsterdam, The Netherlands*

[†3] *Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA*
E-mail: *p.delange@uky.edu*

[†4] *Stanford Institute for Theoretical Physics, Department of Physics and Theory Group, SLAC, Stanford University, Stanford, CA 94305, USA*
E-mail: *dpzwhalen@gmail.com*

Received October 08, 2018, in final form January 30, 2019; Published online March 02, 2019
https://doi.org/10.3842/SIGMA.2019.014

arXiv:1608.07835v3 [math.RT] 2 Mar 2019

Abstract. Umbral moonshine describes an unexpected relation between 23 finite groups arising from lattice symmetries and special mock modular forms. It includes the Mathieu moonshine as a special case and can itself be viewed as an example of the more general moonshine phenomenon which connects finite groups and distinguished modular objects. In this paper we introduce the notion of generalised umbral moonshine, which includes the generalised Mathieu moonshine [Gaberdiel M.R., Persson D., Ronellenfitsch H., Volpato R., *Commun. Number Theory Phys.* **7** (2013), 145–223] as a special case, and provide supporting data for it. A central role is played by the deformed Drinfel'd (or quantum) double of each umbral finite group G, specified by a cohomology class in $H^3(G, U(1))$. We conjecture that in each of the 23 cases there exists a rule to assign an infinite-dimensional module for the deformed Drinfel'd double of the umbral finite group underlying the mock modular forms of umbral moonshine and generalised umbral moonshine. We also discuss the possible origin of the generalised umbral moonshine.

Key words: moonshine; mock modular form; finite group representations; group cohomology

2010 Mathematics Subject Classification: 11F22; 11F37; 20C34

1 Introduction

Moonshine is a relation between finite groups and modular objects. The study of this relation started with the so-called monstrous moonshine [22] and has recently been revived by the discovery of Mathieu moonshine [12, 35, 36, 41, 42] and umbral moonshine [17, 18], which recovers Mathieu moonshine as a special case. While in moonshine one considers functions indexed by elements of a finite group, in generalised moonshine the relevant functions are indexed by pairs of commuting elements. The latter is a generalisation in that it recovers the former when the first element of the pair is set to be the identity element of the group. In this work we introduce *generalised umbral moonshine*, recovering the generalised Mathieu moonshine [43] as a special case. With this we hope to contribute to the understanding of moonshine in the following two ways: 1) by providing more examples of the moonshine relation, in particular working towards a complete analysis of umbral moonshine, and 2) by shedding light on the underlying structure of umbral moonshine which is still obscure.

This paper is a contribution to the Special Issue on Moonshine and String Theory. The full collection is available at https://www.emis.de/journals/SIGMA/moonshine.html

The discovery of moonshine was initiated in the late 1970's with the realisation [22] that there must be a representation $V_n^\natural$ of the monster group responsible for each of the Fourier coefficients of a set of distinguished modular functions. These modular functions – the Hauptmoduls J_g – are indexed by elements g of the monster group:

$$J_g = q^{-1} + \sum_{n=1}^{\infty} a_g(n)\, q^n, \qquad a_g(n) = \mathrm{Tr}_{V_n^\natural} g.$$

Importantly, it was soon realised that this representation has an underlying algebraic structure: a vertex operator algebra (VOA), or a 2d chiral conformal field theory (CFT) in the physics language [4, 38, 39, 40]. In particular, the monster representation $V_n^\natural$ is nothing but the space of quantum states with energy (eigenvalue of $\hat{H} = L_0 - c/24$) n of a chiral CFT – the so-called monster CFT – that has monster symmetry.

Given a chiral conformal field theory with a discrete symmetry group G, one can also consider orbifolding the theory by subgroups of G. This leads to the concept of twisted sectors, which have symmetries given by elements of the group G that commute with the twisting subgroup. Considering the action of this remaining symmetry group on the twisted sector quantum states leads to the *twisted-twined partition functions* indexed by two commuting elements – the twisting and the twining elements – of the original group G. At least heuristically, the existence of these twisted-twined partition functions is believed to be a conceptual partial explanation for Norton's *generalised monstrous moonshine* [50, 51], which we will review in more detail in Section 2.1.

The landscape of moonshine has changed dramatically since the observation of Eguchi, Ooguri, and Tachikawa in 2010 [36]. These authors pointed out a surprising empirical relation between the elliptic genus of $K3$ surfaces and the sporadic Mathieu group M_{24}. The ensuing development has led to the discovery of 23 cases of *umbral moonshine*, which recovers the above-mentioned M_{24} relation as a special case. The important features of umbral moonshine include the following facts: 1) the relevant functions are so-called mock modular forms which have a modified transformation property under the modular group, and 2) the 23 cases are organised in terms of 23 special (Niemeier) lattices of rank 24. In particular, their lattice symmetries dictate the finite groups featuring in umbral moonshine.

In order to understand umbral moonshine, it is crucial to know what the underlying algebraic structure is. Despite various recent advances [21, 20, 33, 46, 47, 49], the nature of this structure is still obscure. An obvious first guess is that a 2d CFT will again be the relevant structure, especially given the CFT context in which the first case of umbral moonshine – the Mathieu moonshine – was uncovered. However, there is a salient incompatibility between the modular behaviour expected from a CFT partition function and that displayed in umbral moonshine that we will explain in Section 2.2. As a result, new ingredients other than a conventional CFT/VOA are believed to be necessary for a uniform understanding of umbral moonshine. The quest for this new structure constitutes one of our motivations to study generalised umbral moonshine, which establishes the central role played by the deformed Drinfel'd double. Note that the importance of the third group cohomology in moonshine has been suggested by Terry Gannon and nicely demonstrated in [43] in the context of generalised Mathieu moonshine.

The rest of the paper is organised as follows. To describe our results, we begin in Section 2 by recalling the (generalised) monstrous moonshine, umbral moonshine, and reviewing the basic properties of the deformed Drinfel'd double of a finite group and its representations. In Section 3 we conjecture that there is a way to assign an infinite-dimensional module for the deformed Drinfel'd double of the umbral group underlying each of the 23 cases of generalised umbral moonshine and unpack this conjecture in a more explicit form in terms of the six conditions of generalised umbral moonshine. To provide evidence and explicit data for this conjecture, in Section 4 we present the group theoretic underpinnings of the generalised umbral moonshine and

exposit our methodology and results. In Section 5 we turn our attention to the modular side of moonshine and present our proposal for the twisted-twined functions of generalised umbral moonshine. In Section 6 we close the paper with a summary and discussions. To illustrate the content of the generalised umbral moonshine conjecture, in the appendix we provide the expansion of three examples of the g-twisted generalised moonshine functions belonging to three different lambencies, as well as the decompositions of the first few homogeneous components of the corresponding (conjectural) generalised umbral moonshine module into irreducible projective representations.

We also include two files accompanying this article; the first describes the explicit construction of the umbral groups that we employ, and the second describes the construction of the 3-cocycles.

2 Background

In this section we provide the relevant background for generalised umbral moonshine by reviewing the theories of generalised monstrous moonshine and umbral moonshine. We also review the basic properties of the deformed Drinfel'd double of a finite group and its representation theory.

2.1 Generalised monstrous moonshine

The monster (or the Fischer–Griess monster) group $\mathbb{M}$ is the largest sporadic group. Monstrous moonshine connects the representation theory of the monster group and certain distinguished modular functions, namely the Hauptmoduls for genus zero subgroups of $\mathrm{PSL}_2(\mathbb{R})$.[1] In particular, the Fourier coefficients of the modular J-invariant are equated with the dimensions of the homogeneous components of a distinguished infinite-dimensional $\mathbb{Z}$-graded module for the monster group. This module was constructed (conjecturally) by Frenkel, Lepowsky, and Meurman [39, 40]. It possesses the structure of a VOA and is later proven by Borcherds to be the monstrous module [5]. In this way, the q-powers of the J-invariant acquire the interpretation as the L_0-eigenvalues of the states in the corresponding chiral conformal field theory shifted by -1. Here $-1 = -c/24$, where c is the central charge of the Virasoro algebra. More precisely, if we label by $V^\natural = \oplus_{n\geq -1} V_n^\natural$ the Hilbert space of quantum states of the monstrous chiral conformal field theory, we have

$$\mathrm{Tr}_{V^\natural}\, q^{L_0-c/24} := \sum_{n\geq -1} q^n \dim V_n^\natural = J(\tau) = q^{-1} + 196884q + 21493760q^2 + \cdots ,$$

where $q := \mathrm{e}^{2\pi \mathrm{i}\tau}$. More generally, for any $g \in \mathbb{M}$ the corresponding twined partition function, given by $\mathrm{Tr}_{V^\natural}\, g q^{L_0-c/24}$, coincides with the Hauptmodul $J_g(\tau) = q^{-1} + O(q)$ of a genus zero subgroup Γ_g of $\mathrm{PSL}_2(\mathbb{R})$.

An important way to construct new chiral CFTs is to "orbifold" a chiral CFT that has a discrete symmetry group G [25, 28, 29, 40] by combining g-twisted modules for $g \in G$. The g-twisted sector Hilbert space, denoted by $\mathcal{H}_g$, has remaining symmetries given by elements of G commuting with g. By considering these symmetries, one can define the twisted-twined partition function

$$Z_{(g,h)}(\tau) = \mathrm{Tr}_{\mathcal{H}_g}\, h q^{L_0-c/24}$$

for all commuting pairs (g, h) of G. Recall that in a (non-chiral) orbifold conformal field theory the above quantity is associated to a torus path integral interpretation with a g-twisted monodromy/defect for the spatial circle and h-boundary condition for the temporal circle. Hence,

[1] A discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is said to be genus zero if its fundamental domain on the upper-half plane, when suitabley compactified, is a genus zero Riemann surface. A function is said to be a Hauptmodul for Γ if it is an isomorphism from the compactified fundamental domain to the Riemann sphere.

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ of the torus complex structure is equivalent to the transformation $(g,h) \mapsto (g',h') = \left(g^a h^c, g^b h^d\right)$ of the twisting-twining elements. When working with a chiral theory, this consideration then leads to the expectation that $Z_{(g,h)}\left(\frac{a\tau+b}{c\tau+d}\right)$ and $Z_{(g',h')}(\tau)$ coincide up to a phase. Indeed, eight years after the monstrous moonshine conjecture, Norton proposed the following.

Generalised monstrous moonshine conjecture [50] (revised in [51]): There exists a rule that assigns to each element g of the monster group $\mathbb{M}$ a graded projective representation $V(g) = \bigoplus_{n\in\mathbb{Q}} V(g)_n$ of the centraliser group $C_{\mathbb{M}}(g)$, and to each pair (g,h) of commuting elements of $\mathbb{M}$ a holomorphic function $T_{(g,h)}$ on the upper-half plane $\mathbb{H}$, satisfying the following conditions:

I For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $T_{(g^a h^c, g^b h^d)}(\tau)$ is proportional to $T_{(g,h)}\left(\frac{a\tau+b}{c\tau+d}\right)$.

II The function $T_{(g,h)}$ is either constant or a Hauptmodul for some genus zero congruence group.

III The function $T_{(g,h)}$ is invariant up to constant multiplication under simultaneous conjugation of the pair (g,h) in $\mathbb{M}$.

IV There is some lift $\tilde{h}$ of h to a linear transformation on $V(g)$ such that

$$T_{(g,h)}(\tau) = \sum_{n\in\mathbb{Q}} \mathrm{Tr}_{V(g)_n} \tilde{h} q^{n-1}.$$

V The function $T_{(g,h)}$ coincides with the J-invariant if and only if $g = h = e \in \mathbb{M}$.

The proof of this conjecture has been recently announced [10], building on previous work [8, 9, 11, 30, 48]. See also [54] for a previous result and [10] for a more in-depth review of the literature. With the crucial exception of the genus zero property (**II**), all the above properties of generalised moonshine can be understood in the physical framework of holomorphic orbifolds along the lines discussed above [27]. See [52] for a recently proposed physical interpretation of the genus zero property.

2.2 Umbral moonshine

Historically, the study of umbral moonshine started with the discovery of Mathieu moonshine, initiated by a remarkable observation made in the context of $K3$ conformal field theories in [36]. The Mathieu moonshine can be phrased as the fact that the Fourier coefficients of a set of mock modular forms can be equated with the group characters of a distinguished infinite-dimensional $\mathbb{Z}$-graded module for M_{24}. Later it was realised in [17, 18] that the Mathieu moonshine is but an instance of a larger structure, which was named umbral moonshine. Umbral moonshine is labelled by Niemeier lattices, the unique (up to isomorphism) 23 even self-dual positive-definite lattices in 24 dimensions with a non-trivial root system. Recall that the root system is the sublattice generated by the root vectors, the lattice vectors of norm squared 2. These 23 Niemeier lattices are uniquely specified by their root systems, which are precisely the 23 unions of simply-laced (ADE) root systems with the same Coxeter number whose ranks are 24. The relevant finite groups, the so-called umbral groups, are given by the automorphism of the corresponding lattice quotiented by the Weyl reflections with respect to the lattice root vectors.

It turns out to be natural to associate a genus zero subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of the form $\Gamma_0(m) + e, f, \ldots$ to each of the 23 Niemeier lattices [15, 18], where $\Gamma_0(m) + e, f, \ldots$ denotes the subgroup of $\mathrm{SL}_2(\mathbb{R})$ obtained by attaching the corresponding Atkin–Lenher involutions $W_e, W_f, \ldots$ to the congruence subgroup $\Gamma_0(m)$. Following [18] we use the shorthand $\ell = m + e, f, \ldots$ for the genus zero group $\Gamma_0(m) + e, f, \ldots$ and call ℓ the lambency of the corresponding instance of umbral moonshine. The level m of the group $\Gamma_0(m) + e, f, \ldots$ coincides with the index of the

corresponding mock Jacobi forms, so we say that m is the index of that lambency. We also observe that m is the Coxeter number of the root system of the corresponding Niemeier lattice:

$$\boxed{\begin{array}{c}\text{meromorphic}\\ \text{Jacobi forms}\\ \psi^{(\ell)}_{(g,h)}\end{array}} \xrightarrow[\text{the polar part}]{\text{subtracting}} \boxed{\begin{array}{c}\text{mock}\\ \text{Jacobi forms}\\ \tilde{\psi}^{(\ell)}_{(g,h)}\end{array}} \xrightarrow[\text{decomposition}]{\theta\text{-function}} \boxed{\begin{array}{c}\text{vector-valued}\\ \text{mock modular forms}\\ H^{(\ell)}_{(g,h)} = \big(H^{(\ell)}_{(g,h),r}\big)\end{array}}$$

There are a few ways to view the modular objects of umbral moonshine, which are summarised in diagram (2.2) and which we will now explain. We start by defining a number of actions on functions defined on $\mathbb{H}\times\mathbb{C}$ that will be used in the modularity criteria. For a given $m\in\mathbb{Z}_+$ we defined the index m (generalised) elliptic action by[2]

$$(\phi\big|_m(\lambda,\mu))(\tau,\zeta) := \phi(\tau,\zeta+\lambda\tau+\mu)\,\mathrm{e}\big(m\big(\lambda^2\tau+2\lambda\zeta+\lambda\mu\big)\big) \tag{2.1}$$

for any $(\lambda,\mu)\in\mathbb{R}^2$. Here and in the rest of the paper we will use the shorthand notation $\mathrm{e}(x) := \mathrm{e}^{2\pi\mathrm{i}x}$. Moreover, for $2k\in\mathbb{Z}$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ we define the modular transformation

$$(\phi\big|_{k,m}\gamma)(\tau,\zeta) := \phi\left(\frac{a\tau+b}{c\tau+d},\frac{\zeta}{c\tau+d}\right)\frac{1}{(c\tau+d)^k}\,\mathrm{e}\left(-\frac{cm\zeta^2}{c\tau+d}\right). \tag{2.2}$$

It is easy to check that the elliptic and modular operators satisfy

$$\phi|_m(x,y)|_{k,m}\gamma = \phi|_{k,m}\gamma|_m(\gamma(x,y)). \tag{2.3}$$

for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ and $(x,y)\in\mathbb{R}^2$, where we have defined $\gamma(x,y) := (ax+cy,bx+dy)$.

Associated to each lambency $\ell=m+e,f,\ldots$ and each element g of the corresponding umbral group $G^{(\ell)}$ is a so-called meromorphic Jacobi form $\psi^{(\ell)}_g$. In particular, $\psi^{(\ell)}_g$ transforms as a weight one index m Jacobi form for a congruence subgroup $\Gamma_g\subseteq\mathrm{SL}_2(\mathbb{Z})$, possibly with a non-trivial multiplier (cf. Sections 3 and 5.2). Namely, we have

$$\psi^{(\ell)}_g\big|_m(\lambda,\mu) = \psi^{(\ell)}_g$$

and

$$\varsigma_g(\gamma)\psi^{(\ell)}_g\big|_{1,m}\gamma = \psi^{(\ell)}_g \tag{2.4}$$

for all $(\lambda,\mu)\in\mathbb{Z}^2$, $\gamma\in\Gamma_g$, and for some $\varsigma_g\colon\Gamma_g\to\mathbb{C}^\times$. Moreover, $\psi^{(\ell)}_g$ is meromorphic as a function $\zeta\mapsto\psi^{(\ell)}_g(\tau,\zeta)$ and can have poles at most at $2m$-torsion points, $\zeta\in\frac{1}{2m}\mathbb{Z}+\frac{\tau}{2m}\mathbb{Z}$.

As explained in [23, 57], meromorphic Jacobi forms such as $\psi^{(\ell)}_g$ lead to vector-valued mock modular forms. Recall that a holomorphic function $f\colon\mathbb{H}\to\mathbb{C}$ is said to be a weight k mock modular form for $\Gamma\subseteq\mathrm{SL}_2(\mathbb{Z})$ with shadow g if g is a holomorphic modular form of weight $2-k$ such that the non-holomorphic function

$$\hat{f}(\tau) := f(\tau)+(4\mathrm{i})^{k-1}\int_{-\bar{\tau}}^{\infty}(z+\tau)^{-k}\overline{g(-\bar{\tau})}\mathrm{d}z$$

transforms as a weight k modular form for Γ. To explain the relation between meromorphic Jacobi forms and vector-valued mock modular forms in the current context, we first define the holomorphic function $\tilde{\psi}^{(\ell)}_g$ from the meromorphic function $\psi^{(\ell)}_g$ by subtracting its "polar part"

$$\tilde{\psi}^{(\ell)}_g(\tau,\zeta) := \psi^{(\ell)}_g(\tau,\zeta)-\psi^{(\ell),P}_g(\tau,\zeta). \tag{2.5}$$

[2]Note that usually the elliptic action is only defined for $(\lambda,\mu)\in\mathbb{Z}^2$, in which case the factor $\mathrm{e}(m\lambda\mu)$ is unity and is not included in the standard definition.

The polar part $\psi_g^{(\ell),P}$ is

$$\psi_g^{(\ell),P} = \sum_{a,b\in\mathbb{Z}/2m\mathbb{Z}} \chi_g^{(a,b)} \mu_m|_m\big(\tfrac{a}{2m}, \tfrac{b}{2m}\big), \tag{2.6}$$

with $\chi_g^{(a,b)} \in \mathbb{C}$ given by the characters of certain representations of the umbral group $G^{(\ell)}$ [18]. In the above we have used the generalised Appell–Lerch sum

$$\mu_m(\tau,\zeta) = -\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{1+yq^k}{1-yq^k},$$

with $q = \mathrm{e}(\tau)$ and $y = \mathrm{e}(\zeta)$.

It follows from the mock modular property of the Appell–Lerch sum (see, for instance, [23, 57]) that $\tilde{\psi}_g^{(\ell)}$ is an example of the so-called weak mock Jacobi forms. Moreover, from the elliptic transformations of $\psi_g^{(\ell)}$ (cf. equation (3.1)) and μ_m we see that $\tilde{\psi}_g^{(\ell)}$ satisfies the same elliptic transformation as $\psi_g^{(\ell)}$, and therefore admits the following theta function decomposition [23]

$$\tilde{\psi}_g^{(\ell)}(\tau,\zeta) = \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} H_{g,r}(\tau)\theta_{m,r}(\tau,\zeta), \tag{2.7}$$

where

$$\theta_{m,r}(\tau,\zeta) = \sum_{k=r \bmod 2m} q^{k^2/4m} y^k. \tag{2.8}$$

Recall that, when regarded as a vector-valued function with $2m$ components, $\theta_m = (\theta_{m,r})$ transforms under $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ as

$$\rho_m(\gamma)\theta_m|_{\frac{1}{2},m}\gamma = \theta_m \tag{2.9}$$

with the multiplier system $\rho_m\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(2m,\mathbb{C})$ whose detailed description can be found in, for instance, [37, Section 5]. From the modular property of $\psi_g^{(\ell)}$ (2.4) and the mock modular property of the Appell–Lerch sum [23, 57]), we see that $H_g = (H_{g,r})$ is a weight 1/2 vector-valued mock modular form[3] for a certain $\Gamma_g \subseteq \mathrm{SL}_2(\mathbb{Z})$ that we will define in (3.5).

In the rest of the paper, we will generalise the above relation between meromorphic Jacobi forms, mock Jacobi forms, and vector-valued mock modular forms, to functions labelled by commuting pairs of elements of $G^{(\ell)}$. This relation between $\psi_{(g,h)}$, $\tilde{\psi}_{(g,h)}$ and $H_{(g,h)}$ is summarised in diagram (2.2).

Just like the Hauptmoduls in monstrous moonshine, the mock modular forms $H_g^{(\ell)}$ are also special as we shall explain now. Recall that Cappelli, Itzykson and Zuber found an ADE classification of modular invariant combinations of A_1 affine characters [7]. This classification, combined with the ADE data from the root systems of the Niemeier lattices, leads to a specification of the modularity property of the mock modular forms of umbral moonshine as described in [18]. The umbral moonshine conjecture states that these vector-valued mock modular forms $H_g^{(\ell)}$ for $g = e$, are in fact the unique such forms with the slowest possible growth in their Fourier coefficients given their modularity properties. In particular, $H_e^{(\ell)}$ coincides with the Rademacher sum specified by its modular property and its pole at the cusps [18]. See [13, 14, 19, 55] for a discussion on the cases where $g \neq e$.

[3] In general $H_{g,r}(\tau)$ grows exponentially near at most one cusp of the group Γ_g. Hence, it is *weakly holomorphic* vector-valued mock modular forms that we encounter in generalised umbral moonshine. To avoid clutter we will skip the adjective "weakly holomorphic" in the remaining of the paper.

After specifying the finite group $G^{(\ell)}$ and the vector-valued mock modular forms $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$ for all elements $g \in G^{(\ell)}$, the umbral moonshine conjecture states that there exists a $G^{(\ell)}$-module whose graded characters coincide with the Fourier expansion of $H_g^{(\ell)}$. Namely, there exists an infinite-dimensional $G^{(\ell)}$-module

$$K^{(\ell)} = \bigoplus_{\substack{r\in\{1,\ldots,m-1\}\\ \alpha\in\mathbb{Q}_{>0}}} K_{r,\alpha}^{(\ell)}$$

such that for all $g \in G$ and $r \in \{1, \ldots, m-1\}$ we have

$$H_{g,r}^{(\ell)}(\tau) = -2\delta_{r^2,1}^{[4m]} q^{-1/4m} + \sum_{\alpha\in\mathbb{Q}_{>0}} q^{\alpha} \mathrm{Tr}_{K_{r,\alpha}^{(\ell)}} g. \tag{2.10}$$

In the above equation we defined

$$\delta_{a,b}^{[C]} = \begin{cases} 1 & \text{if } a = b \bmod C, \\ 0 & \text{otherwise.} \end{cases}$$

Corresponding to the Niemeier lattice with root system A_1^{24} is the case of umbral moonshine with $\ell = 2$ and $G^{(\ell)} \cong M_{24}$. In this case, the vector-valued mock modular forms have just a single independent component since $H_{g,r}^{(2)} = \pm H_{g,1}^{(2)}$ for $r = \pm 1 \bmod 4$ and $H_{g,r}^{(2)} = 0$ for $r = 0 \bmod 2$, and the above conjecture recovers that of Mathieu moonshine.

The above conjecture was proven in [32, 44], in the sense that the existence of the module $K^{(\ell)}$ has been established using properties of (mock) modular forms. However, among the 23 cases of umbral moonshine, modules have only been constructed for the eight simpler cases [2, 16, 33, 34]. A uniform construction of the umbral moonshine modules is to the best of our knowledge not yet in sight and is expected to be the key to a true understanding of this new moonshine phenomenon. One of the most puzzling features in this regard is the non-vanishing weight of the umbral moonshine function: if one assumes that the umbral moonshine module, just like the monstrous moonshine module, possesses the structure of a conventional chiral CFT, than the usual physics lore mentioned in the last subsection as well as mathematical results on VOAs [56] lead to the expectation that one should be able to attach a weight zero modular object to the moonshine module. This process can be done in a straightforward manner for the eight cases of umbral moonshine which correspond to the eight Niemeier lattices with A-type root systems, but not for the remaining fifteen cases involving D- and/or E-type root systems, at least not if we require the modular object to be holomorphic as usual. Moreover, among all the A-type cases, it was argued in [17] that apart from the A_1^{24} case of Mathieu moonshine, the associated weight zero Jacobi forms cannot coincide with a partition function or elliptic genus of a "usual" (chiral) conformal field theory. This is due to the expectation that the NS-NS ground states lead to a non-vanishing $q^0 y^{m-1}$ term in the index m weight zero Jacobi form, which is absent in all cases other than the A_1^{24} case. In this regard, the A_1^{24} case of umbral moonshine is set apart from the other cases that the mock modular forms can be regarded as arising from the supersymmetric index of a physical conformal field theory – any $K3$ non-linear sigma model in fact. This more involved modular property suggests that a novel approach is likely to be needed in order to understand umbral moonshine in general.

2.3 Deformed Drinfel'd double and its representations

To set the stage for the generalised umbral moonshine conjecture, we will quickly review the definition of a deformed Drinfel'd (or quantum) double of a finite group and its representations. Drinfel'd introduced the quantum double construction, which associates to a Hopf algebra $\mathcal{A}$

a quasi-triangular Hopf algebra $D(\mathcal{A})$. This quasi-triangular Hopf algebra contains $\mathcal{A}$ as well as its dual algebra $\mathcal{A}^*$ as Hopf algebras and $D(\mathcal{A}) = \mathcal{A}^* \otimes A$ as a vector space [31]. Let G be a finite group and $\mathcal{A} = kG$ be the group algebra over field k. It was argued that the corresponding quantum double for the case $k = \mathbb{C}$, often denoted simply by $D(G)$, is relevant for 3d TQFTs [26] and relatedly 2d orbifold CFTs [24, 25]. The finite group G plays the role of the discrete gauge group in te former case and that of the orbifold group in the latter case. Here and in the rest of the paper we will work with $k = \mathbb{C}$ exclusively.

In [24], Dijkgraaf, Pasquier and Roche introduced an interesting generalisation of this construction. For G a finite group and M a G-module, one can define the group cohomology $H^n(G, M) := \operatorname{Ker} \partial^{(n)} / \operatorname{Im} \partial^{(n-1)}$ where the coboundary operator $\partial^{(n)}$ on the space of n-cochains is given by

$$\begin{aligned}\big(\partial^{(n)}\lambda\big)(h_1, \dots, x_n, x_{n+1}) = \lambda(x_2, \dots, x_{n+1}) + (-1)^{n+1}\lambda(h_1, \dots, x_n) \\ + \sum_{i=1}^{n} (-1)^i \lambda(h_1, \dots, x_i x_{i+1}, \dots, x_{n+1}).\end{aligned}$$

From now on we take $M = U(1)$ (on which G acts trivially), and to avoid an overload of notation we will often conflate $x \in \mathbb{R}/\mathbb{Z} \cong U(1)$ with $\mathrm{e}^{2\pi \mathrm{i} x} \in \mathbb{C}^\times$ when the meaning is clear from the context. Given G and a 3-cocycle $\omega\colon G \times G \times G \to \mathbb{C}^\times$, the deformed quantum double $D^\omega(G)$ is a quasi-triangular quasi-Hopf algebra generated by the elements[4] $P(x)$ and $Q(y)$, with $x, y \in G$ [24]. The multiplications are given by

$$P(x)P(y) = \begin{cases} P(x) & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases} \qquad Q(x)Q(y) = \theta(x, y)Q(xy),$$

while the co-multiplication are given by

$$\Delta P(x) = \sum_{z \in G} P(z) \otimes P\big(z^{-1}x\big), \qquad \Delta Q(x) = \eta(x)Q(x) \otimes Q(x).$$

The multiplication and co-multiplication rules are determined by the 3-cocycle ω as

$$\theta(x, y) = \sum_{z \in G} \theta_z(x, y)P(z), \qquad \eta(z) = \sum_{x, y \in G} \eta_z(x, y)P(x) \otimes P(y),$$

where

$$\theta_g(x, y) := \frac{\omega(x, g|^x, y)}{\omega(g, x, y)\omega(x, y, g|^{xy})}, \tag{2.11}$$

$$\eta_g(x, y) := \frac{\omega(x, g, y|^x)}{\omega(x, y, g)\omega(g, x|^g, y|^g)}. \tag{2.12}$$

Moreover, we have

$$Q(x)^{-1}P(y)Q(x) = P\big(y|^x\big),$$

where we use the notation $y|^x := x^{-1}yx$ to denote the conjugate of y by x. The identity element of $D^\omega(G)$ is $Q(1) = \sum_{z \in G} P(z)$.

[4] Here we use the notation in [3]. Some other common choices of notation for $P(x)Q(y)$ include $\delta_x y$, $x\underset{y}{\llcorner\!\!\!_\!\!\!_}$ and $\langle \overset{y}{\leftarrow} x \rangle$. We will work only with the so-called normalised cochains, namely functions $u\colon G^n \to \mathbb{C}^\times$ satisfying $u(x_1, \dots, x_n) = 1$ whenever at least one of the n group elements $x_1, \dots, x_n$ coincides with the identity element e. This corresponds to requiring $\rho(e) = \mathbf{1}_V \in \operatorname{End}(V)$ in the G-representation (ρ, V).

Note that $\theta_g(x,y) = \eta_g(x,y)$ when $x,y \in C_G(g)$, the centralizer of g in G. It is easy to check that the restriction $c_g := \theta_g|_{C_G(g)\times C_G(g)} = \eta_g|_{C_G(g)\times C_G(g)}$ is a 2-cocycle and moreover, that the corresponding map $Z^3(G,U(1)) \to Z^2(C_G(g),U(1))$ induces a map $H^3(G,U(1)) \to H^2(C_G(g),U(1))$.

The fact that the above defines a quasi-Hopf algebra is guaranteed by the 3-cocycle condition $\partial^{(3)}\omega = 0$. In particular, the co-multiplication, instead of satisfying $(\mathbf{1}\otimes\Delta)\cdot\Delta = (\Delta\otimes\mathbf{1})\cdot\Delta$ as in the case of a Hopf algebra, satisfies

$$(\mathbf{1}\otimes\Delta)\cdot\Delta(a) = \varphi(\Delta\otimes\mathbf{1})\cdot\Delta(a)\varphi^{-1} \qquad \forall a \in D^\omega(G)$$

in a quasi-Hopf algebra, where the intertwiner is given by

$$\varphi = \sum_{x,y,z\in G} \omega(x,y,z)P(x)\otimes P(y)\otimes P(z).$$

It is easy to see that changing ω without changing the cohomology class leads to an isomorphic quasi-Hopf algebra [24]. In the language of holomorphic orbifold the non-triviality of the 3-cocycle signals the failure of the fusion between operators belonging to different twisted sectors to be associative. See [24] and references therein for more details on $D^\omega(G)$ including the R-matrices and the antipode.

Next we will summarize the basic properties of the representations of $D^\omega(G)$. See, for instance, [1, 3, 24] for more information. Firstly, we will establish that given an ω-compatible projective representation of $C_G(g)$ for some $g \in G$ we can build a corresponding $D^\omega(G)$-representation via the so-called DPR induction.

Recall that a projective representation (ρ, V), where $\rho\colon H \to \mathrm{End}(V)$, satisfies

$$\rho(h_1)\rho(h_2) = c^\rho(h_1,h_2)\,\rho(h_1h_2)$$

for any $h_1,h_2 \in H$, where $c^\rho \in Z^2(H,U(1))$. Given a $g_A \in G$ and a projective representation (ρ_{g_A}, V_{g_A}) of $C_G(g_A)$ corresponding to the 2-cocycle $\theta_{g_A} \in Z^2(C_G(g_A),U(1))$, we can construct a *DPR-induced* representation of $D^\omega(G)$ in the following way [24]. Let $\mathcal{B}_{g_A}$ be the subalgebra of $D^\omega(G)$ spanned by elements of the form $P(g)Q(x)$ with $g \in G$ and $x \in C_G(g_A)$. We define the action of $\mathcal{B}_{g_A}$ on V_{g_A} by $\pi(P(g)Q(x)) = \delta_{g,g_A}\rho_{g_A}(x)$. The DPR-induced $D^\omega(G)$ representation, denoted $\big(\pi_{(\rho_{g_A},V_{g_A})}, \mathrm{Ind}^{\mathrm{DPR}}(V_{g_A})\big)$, is given by $\mathrm{Ind}^{\mathrm{DPR}}(V_{g_A}) := \mathbb{C}[G]\otimes_{\mathcal{B}_{g_A}} V_{g_A}$, where we identify $\mathbb{C}[G]$ as the subalgebra spanned by $\sum_{z\in G} P(z)Q(x)$ with $x \in G$. Explicitly, to describe the action of $D^\omega(G)$ we choose a set of representatives $\{x_1,\ldots,x_n\}$ of $G/C_G(g_A)$, and it can be checked that, for $v \in V_{g_A}$,

$$\pi_{(\rho_{g_A},V_{g_A})}(P(g)Q(x))(x_j\otimes v) = \frac{\theta_{g_A}(x,x_j)}{\theta_{g_A}(x_k,h)}\delta_{g|^x,g_j}(x_k\otimes\rho_{g_A}(h)v),$$

where $h \in C_G(g_A)$ is determined by $xx_j = x_kh$.

Secondly, any irreducible representation of $D^\omega(G)$ is labelled by a conjugacy class A of G and an irreducible projective representation of $C_G(g_A)$ with the 2-cocycle θ_{g_A}, where $g_A \in A$. Moreover, such a irreducible representation of $D^\omega(G)$ is equivalent to the corresponding DPR-induced representation described above. As a result, studying the representations of $D^\omega(G)$ is equivalent to studying the projective representations of all the centraliser subgroups with the 2-cocycles given by ω via (2.11), a fact that we will exploit in order to make explicit conjectures in the next subsection.

The representations of a deformed quantum double share many features with those of finite groups. For instance there is an orthogonality relation among irreducible representations. For

a given representation V of $D^\omega(G)$, we write its character as $\mathrm{Ch}^V_{(x,y)} := \mathrm{Tr}_V(P(x)Q(y))$. These characters satisfy the following properties,

$$\begin{aligned} &\mathrm{Ch}^V_{(x,y)} = 0 \qquad \text{unless} \quad xy = yx, \\ &\mathrm{Ch}^V_{(x|^z,y|^z)} = \frac{\theta_x(z,y|^z)}{\theta_x(y,z)}\,\mathrm{Ch}^V_{(x,y)}, \end{aligned} \tag{2.13}$$

which can also be checked explicitly for the DPR-induced representations. Moreover, $\mathrm{Rep}(\mathrm{D}^\omega(\mathrm{G}))$ is a modular category. In terms of the characters, the modular group acts as

$$T\,\mathrm{Ch}^V_{(x,y)} = \theta_x(x,y)\,\mathrm{Ch}^V_{(x,xy)}, \tag{2.14}$$

$$S\,\mathrm{Ch}^V_{(x,y)} = \frac{1}{\theta_y(x,x^{-1})}\,\mathrm{Ch}^V_{(y,x^{-1})}, \tag{2.15}$$

satisfying $S^2 = (ST)^3$, $S^4 = \mathrm{id}$. In particular, the fusion rules between different representations satisfy the Verlinde formula given by the above S-matrix.

3 Generalised umbral moonshine

The *generalised umbral moonshine conjecture* states that for each of the 23 lambencies of umbral moonshine with umbral group G there is a 3-cocycle $\omega\colon G\times G\times G\to\mathbb{C}^\times$ such that there exists an infinite-dimensional module for $D^\omega(G)$ underlying a set of distinguished meromorphic Jacobi forms that we will specify in Section 5. In the following we will re-formulate the conjecture in explicit terms, using the relation to the projective representations of the centraliser subgroups discussed in Section 2.3. This leads to the first five of the six conditions that we will discuss in this section. The sixth condition is the counterpart of the Hauptmodul condition and demonstrates the special property of the generalised umbral moonshine functions.

Explicitly, given the lambency ℓ with index m and the corresponding umbral group G, we propose in Section 5 the twisted-twined functions $\psi_{(g,h)}$ (and the corresponding twisted-twined mock modular forms $H_{(g,h)}$) for each commuting pair (g,h). We conjecture that they satisfy the six conditions listed below. The relation between $\psi_{(g,h)}$ and the mock modular forms $H_{(g,h)}$ is as described in (2.2).

I Modularity. The modularity condition first states that the function $\psi_{(g,h)}$, which is meromorphic as a function $\zeta\mapsto\psi_{(g,h)}(\tau,\zeta)$ and can have poles (at most) at $2m$-torsion points, $\zeta\in\frac{1}{2m}\mathbb{Z}+\frac{\tau}{2m}\mathbb{Z}$, transforms in the same way as an index m Jacobi form under the elliptic transformation (cf. equation (2.1)). Namely, for all $(\lambda,\mu)\in\mathbb{Z}^2$,

$$\boxed{\psi_{(g,h)}\big|_m(\lambda,\mu) = \psi_{(g,h)}.} \tag{3.1}$$

Second, under the weight one modular transformation they satisfy for any $g,h\in G$ and any $\gamma\in\mathrm{SL}_2(\mathbb{Z})$

$$\boxed{\varsigma_{(g,h)}(\gamma)\psi_{(g,h)}\big|_{1,m}\gamma = \psi_{\gamma(g,h)}} \tag{3.2}$$

for some $\varsigma_{(g,h)}\colon\mathrm{SL}_2(\mathbb{Z})\to\mathbb{C}^\times$. In the above we have defined $\gamma(g,h) := \big(g^ah^c, g^bh^d\big)$ for two commuting elements $g,h\in G$.

This condition reflects the modular properties (2.14), (2.15) of $D^\omega(G)$ representations, embodied by the meromorphic Jacobi forms $\psi_{(g,h)}$, and can be regarded as the counterpart of condition **I** of generalised monstrous moonshine.

II 3-cocycle. This condition states that there exists a 3-cocycle $\omega\colon G\times G\times G\to\mathbb{C}^\times$ compatible with the multiplier system $\varsigma_{(g,h)}$ in (3.2) in the following way. For any $g\in G$ let θ_g be defined as in (2.11). Then the compatibility conditions read

$$\boxed{\begin{aligned}&\varsigma_{(g,h)}\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)=\theta_g(g,h)^{-1},&(3.3a)\\&\varsigma_{(g,h)}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)=\theta_h\big(g,g^{-1}\big),&(3.3b)\end{aligned}}$$

which determine $\varsigma_{(g,h)}$ in terms of ω together with (3.2).

This condition captures the modular properties (2.14), (2.15) of representations of the quantum deformed double $D^\omega(G)$. It is believed that a 3-cocycle underlies the generalised monstrous moonshine in a similar way [43], although it was not mentioned in any of the five stated conditions of generalised monstrous moonshine.

III Projective class function. This condition states that $\psi_{(g,h)}$ is a projective class function, satisfying for all $k\in G$

$$\boxed{\psi_{(g|^k,h|^k)}=\xi_{(g,h)}(k)\psi_{(g,h)}}\tag{3.4}$$

with the phases $\xi_{(g,h)}(k)$ determined by the 3-cocycle ω as

$$\xi_{(g,h)}(k)=\frac{\theta_g(k,h|^k)}{\theta_g(h,k)}.$$

This property together with the modularity property (3.2) implies that the $\mathrm{SL}_2(\mathbb{Z})$ subgroup for which $H_{(g,h)}$ is a vector-valued mock modular form is given by

$$\Gamma_{g,h}:=\big\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\big|\gamma(g,h)=\big(g|^k,h|^k\big)\text{ for some }k\in G\big\}.\tag{3.5}$$

This condition captures the property (2.13) of representations of the quantum deformed double $D^\omega(G)$ and is the counterpart of condition **III** of generalised monstrous moonshine.

IV Finite group. The vector-valued mock modular forms $H_{(g,h)}=(H_{(g,h),r})$ have the following moonshine relation to the finite group $G=G^{(\ell)}$. Namely, there is a way to assign an infinite-dimensional projective $C_G(g)$-module

$$K^g=\bigoplus_{\substack{r\in\mathbb{Z}/2m\mathbb{Z}\\ \alpha\in\mathbb{Q}_{>0}}}K^g_{r,\alpha}$$

with the 2-cocycle given by $c^\rho=\theta_g$, such that for all $g\in G$, $h\in C_G(g)$ and $r\in\mathbb{Z}/2m\mathbb{Z}$,

$$\boxed{H_{(g,h),r}(\tau)=c_{(g,h),r}q^{-1/4m}+\sum_{\alpha\in\mathbb{Q}_{>0}}q^\alpha\mathrm{Tr}_{K^g_{r,\alpha}}h.}\tag{3.6}$$

The coefficient $c_{(g,h),r}$ equals ∓2 whenever $H_{(g,h),r}(\tau)=\pm H_{(e,h),1}(\tau)$ and vanishes otherwise.

The above condition describes the $D^\omega(G)$-module as DPR-induced modules as explained in Section 2.3 and can be regarded as the counterpart of condition **IV** of generalised monstrous moonshine.

V Consistency. This condition states that the generalised umbral moonshine is compatible with umbral moonshine in the sense that $\psi^{(\ell)}_{(e,h)}$ coincides with the weight one meromorphic Jacobi form $\psi^{(\ell)}_h$ given in [17, 18]. In particular, the resulting mock modular form $H_{(e,h)}$ coincides with the McKay–Thompson series $H^{(\ell)}_h = (H^{(\ell)}_{h,r})$ proposed in [17, 18].

This condition establishes the relation between generalised umbral moonshine and the original umbral moonshine, and can be regarded as the counterpart of condition **V** of generalised monstrous moonshine.

VI Pole structure. This condition states that the mock modular form $H_{(g,h)}$ is bounded at all but at most one orbit of the cusps $\mathbb{Q} \cup \{\mathrm{i}\infty\}$ under $\Gamma_{(g,h)}$. It serves as the counterpart of condition **II** (the Hauptmodul condition) of generalised monstrous moonshine.

An immediate consequence of the generalised umbral moonshine conjecture is certain non-trivial constraints on their twisted-twined functions. To illustrate this, let us consider a pair (g,h) of commuting elements in G that is conjugate to $\left(g^{-1},h^{-1}\right)$. The conditions (3.2) and (3.4) lead to the equality

$$\varsigma_{(g,h)}\left(\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right)\psi_{(g,h)} = \xi_{(g,h)}(k)\psi_{(g,h)}, \tag{3.7}$$

where k a group element satisfying $\left(g^{-1},h^{-1}\right) = \left(g|^k,h|^k\right)$. It follows that $\psi_{(g,h)}$ must vanish unless

$$\varsigma_{(g,h)}\left(\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right) = \xi_{g^{-1},h^{-1}}(k).$$

This is the analogue of the obstruction (2) in [43]. We will employ this obstruction[5] in our computation and list the obstructed twisted-twined functions in Table 2.

Note that in the case $\ell = 2$, the first five conditions are equivalent to the ones discussed in [43].

4 Groups, cohomologies, and representations

In this section we present the group-theoretic underpinnings of the theory. We introduce the umbral groups and discuss their cohomology, their rank-2 subgroups, and the projective representations of their centraliser subgroups.

4.1 Group cohomology

The computation of the twisted-twined functions for generalised umbral moonshine depends crucially on the computation of the third group cohomology $H^3\left(G^{(\ell)},U(1)\right)$.

The cohomology groups are computed for all of the umbral groups using GAP and the module HAP. The results are listed in Table 1, where we include the root system X, the lambency ℓ, the umbral group $G^{(\ell)}$, of the corresponding cases of umbral moonshine. Throughout this paper we write $\mathbb{Z}/N\mathbb{Z}$ as N. Also listed in the table is n, given by the group automorphism $G^{(\ell)} \cong n.\overline{G}^{(\ell)}$, where $\overline{G}^{(\ell)}$ is the group of permutations of the irreducible components of the root systems of the corresponding Niemeier lattice induced by the lattice automorphism group. See [18] for more details. Note that $n = 2$ or $n = 1$ for all lambencies except for $\ell = 6 + 3$ which has $n = 3$.

The subgroups $n \subset G^{(\ell)}$ play a special role in umbral moonshine: the corresponding twined functions are proportional to the untwined functions, $H_{(e,z),r} = c_r H_{(e,e),r}$ for a $c_r \in \mathbb{C}$ for all $z \in n$

[5] In principle there is another independent possible obstruction coming from the projective class function property of $\psi_{(g,h)}$, namely $\psi_{(g,h)}$ must vanish unless $\xi_{(g,h)}(k) = 1$ for all k commuting with both g and h. However, in all cases apart from the $\ell = 2$ case treated in [43], this does not lead to new constraints.

Table 1: Umbral groups and group cohomology.

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$	A_8^3
ℓ	2	3	4	5	6	7	8	9
G^X	M_{24}	$2.M_{12}$	$2.\mathrm{AGL}_3(2)$	$\mathrm{GL}_2(5)/2$	$\mathrm{GL}_2(3)$	$\mathrm{SL}_2(3)$	Dih_4	Dih_6
$\bar{G}^X$	M_{24}	M_{12}	$\mathrm{AGL}_3(2)$	$\mathrm{PGL}_2(5)$	$\mathrm{PGL}_2(3)$	$\mathrm{PSL}_2(3)$	2^2	Sym_3
$H^3(G^{(\ell)},\mathbb{C}^\times)$	12	$8\oplus 24$	$4\oplus 24$	$4\oplus 12$	$2\oplus 24$	24	$2\oplus 2\oplus 4$	$2\oplus 2\oplus 6$
n	1	2	2	2	2	2	2	2
|new|	34	12	8	3	0	0	0	0

X	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}	D_4^6	D_6^4
ℓ	10	12	13	16	18	25	6+3	10+5
G^X	4	2	4	2	2	2	$3.\mathrm{Sym}_6$	Sym_4
$\bar{G}^X$	2	1	2	1	1	1	Sym_6	Sym_4
$H^3(G^{(\ell)},\mathbb{C}^\times)$	4	1	4	1	1	1	$2\oplus 6\oplus 12$	$2\oplus 12$
n	2	2	2	2	2	2	3	2
|new|	0	0	0	0	0	0	8	0

X	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}	E_6^4	E_8^3
ℓ	14+7	18+9	22+11	30+15	46+23	12+4	$30{+}6, 10, 15$
G^X	Sym_3	2	2	1	1	$\mathrm{GL}_2(3)$	Sym_3
$\bar{G}^X$	Sym_3	2	2	1	1	$\mathrm{PGL}_2(3)$	Sym_3
$H^3(G^{(\ell)},\mathbb{C}^\times)$	6	1	1	1	1	$2\oplus 24$	6
n	1	1	1	1	1	2	1
|new|	0	0	0	0	0	0	0

and $r \in \mathbb{Z}/2m\mathbb{Z}$. More generally, we have $H_{(e,zg),r} = c_r\, H_{(e,g),r}$ for all $g \in G^{(\ell)}$. Together with the modularity condition (3.2), it suggests that the generalised umbral moonshine function $H_{(g,h)}$ is really "new" if (g,h) cannot be obtained from the pair (z,g') for some $z \in n$ via an $\mathrm{SL}_2(\mathbb{Z})$ transformation. This inspires the following terminology: we say that a subgroup $\langle g,h\rangle$ is *old* if it is isomorphic to $\langle z,g'\rangle$ for some $g' \in G^{(\ell)}, z \in n$, and is *new* otherwise. We will also call the corresponding twisted-twined function an old resp. new function. In Table 1 we also list |new|, the number of conjugacy classes of rank-2 Abelian subgroups of $G^{(\ell)}$ that are new.

The group theoretic computation for the case $\ell = 2$ has been performed in [43]. For $\ell = 3$, direct evaluation of cochains is computationally challenging. However, restriction of cochains to the p-Sylow subgroups $S_p(G)$ together with the inclusion map (cf. [6, Chapter III-10])

$$H^3\big(G^{(\ell)},U(1)\big) \to \bigoplus_{p|\exp(G)} H^3\big(S_p\big(G^{(\ell)}\big),U(1)\big),$$

allows us to simplify our calculations for $\ell = 3$. In the above, $\exp(G)$ denotes the exponent of the group, and $\exp(G^{(3)}) = 1320$.

4.2 The 3-cocycles

For most of the lambencies ℓ of umbral moonshine, the cohomology class of the 3-cocycle ω can be uniquely determined by the multiplier systems of the old functions $\psi_{(e,g)}$ in the following way. Fix a lambency ℓ. For any $g \in G$, $\varsigma_{(e,g)}$ (cf. equation (3.2)) can be restricted to $\Gamma_0(\operatorname{ord} g)$ to obtain a multiplier system $\varsigma_{(e,g)}\colon \Gamma_0(\operatorname{ord} g) \to \mathbb{C}^\times$, which we assume to arise from a 3-cocycle $\omega\colon G\times G\times G \to \mathbb{C}^\times$ as discussed in Section 3. At the same time, these multipliers are required to coincide with those of the known functions $\psi_{(e,g)}$ [17, 18] (cf. Section 3, condition V). In this way, we obtain a consistency condition on $[\omega]$. This consistency condition uniquely fixes $[\omega] \in H^3(G,U(1))$ for all umbral moonshine cases except for $\ell = 3,4,7$. In these cases, there are exactly two cohomology classes that are consistent with the multiplier phases of the known functions $\psi_{(e,g)}$ for all $g \in G$.

For the $\ell = 4$ case, the two cohomology classes provide different multiplier systems on a weight 1 Jacobi form associated to one of the new functions. One of the corresponding spaces of weight 1 weak Jacobi forms has dimension 0, but the vanishing of that specific new function would be incompatible with a decomposition of the characters into projective representations (cf. Section 3, condition IV). We therefore conclude that the new function must be non-zero and this allows us to uniquely select a 3-cohomology class.

For $\ell = 3$ and $\ell = 7$, both of the 3-cohomology classes that are consistent with the known functions $\psi_{(e,g)}$ are also consistent with all the remaining twisted-twined functions we propose. As a result, in this sense generalised umbral moonshine for these two cases is compatible with both choices.

4.3 The rank two subgroups

Given a 3-cocycle ω, the projective class function property (cf. equation (3.4)) guarantees that the generalised umbral moonshine function $\psi_{(g,h)}$ is determined by $\psi_{(g',h')}$ whenever the groups $\langle g, h\rangle$ and $\langle g', h'\rangle$ are conjugate to each other. It therefore suffices to specify the twisted-twined functions $\psi_{(g,h)}$ for a set of representatives in the coset $\{g, h \in G \,|\, gh = hg\}/\!\sim$, where the equivalence relation is generated by $\mathrm{SL}_2(\mathbb{Z})$ transformations and by conjugation. In other words, we let $(g, h) \sim \gamma\big(g|^k, h|^k\big)$ for all $k \in G$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

We tabulate these rank-2 subgroups in Table 2 and indicate the new rank-2 subgroups whose twisted-twined functions are obstructed by the condition discussed at the end of Section 3. In this table we also list the congruence subgroups $\Gamma_{(g,h)} \subseteq \mathrm{SL}_2(\mathbb{Z})$ stabilising the pair (g, h) up to simultaneous conjugation (cf. (3.5)). In this list we employed the following notation for the congruence subgroups

$$\mathrm{H}(p, q, r) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \,|\, \gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod \left(\begin{smallmatrix} p & q \\ r & p \end{smallmatrix}\right)\},$$

which is a group if and only if $p|qr$. We also use the familiar convention for specific congruence subgroups:

$$\Gamma(N) := \mathrm{H}(N, N, N), \qquad \Gamma_1(N) := \mathrm{H}(N, 1, N), \qquad \text{and} \qquad \Gamma_0(N) := \mathrm{H}(1, 1, N).$$

Finally, we have

$$\Gamma_{(4A,4c)} = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2) \,|\, a + b + c \equiv 1 \bmod 4\right\}.$$

Here and everywhere else we denote the conjugacy class names of group elements $g \in G$ with upper case letters, while the conjugacy class names of elements $h \in C_G(g)$ in centralizer subgroups are denoted with lower case letters. See the file accompanying this article for an explicit construction of the umbral groups.

4.4 Projective representations

In this subsection we briefly explain how we construct the relevant projective representations. Our discussion follows closely Appendices C and D of [43].

Recall the definition of a projective representation in Section 2.3. We say that two projective representations ρ and ρ' of a group H are equivalent if they differ by a 1-cochain $\xi \in Z^1(H, U(1))$, namely when $\rho(x) = \xi(x)\rho'(x)$ for all $x \in H$. As a result, different classes in the second group cohomology $H^2(H, U(1))$ lead to inequivalent projective representations. A convenient way to study the projective representations of a group H is to study the representations of a Schur cover X of H, defined as a central extension

$$1 \to M(H) \to X \xrightarrow{\phi} H \to 1,$$

where the normal subgroup is the Schur multiplier $M(H) \cong H_2(H,\mathbb{Z})$. As we will now explain, every representation of the Schur cover corresponds to a projective representation of H and vice versa.

Let $\phi^{-1}\colon H \to X$ be a lift of H. Let $\bar{\rho}\colon X \to \mathrm{End}(V)$ be a representation of X. Then $\rho := \bar{\rho}\circ\phi^{-1}$ is a projective representation of H with the associated cocycle c^ρ satisfying

$$c^\rho(h_1,h_2)\mathbf{1}_V = \bar{\rho}\left(\phi^{-1}(h_1)\phi^{-1}(h_2)\phi^{-1}\big((h_1h_2)^{-1}\big)\right).$$

As a result, the 2-cocycle c^ρ is fixed by $\bar{\rho}$ restricted to $M(H)$. Every projective representation of H is equivalent to a representation arising from its Schur cover in this way, and different (necessarily isoclinic) Schur covers of H lead to different but equivalent projective H-representations.

Given a two-cohomology class $[c] \in H^2(H,U(1))$, we would like to select the compatible representations from all the representations of X. In other words, we would like to pick out the X-representations that give rise to projective representations of H whose corresponding 2-cocycle is consistent with $[c]$. Now we describe how this can be achieved. For a representation $\bar{\rho}$ of X and $k \in M(H)$, if $h_1,h_2 \in H$ satisfy $k = \phi^{-1}(h_1)\phi^{-1}(h_2)\phi^{-1}\big(h_1^{-1}\big)\phi^{-1}\big(h_2^{-1}\big)$, then h_1 and h_2 necessarily commute, and so $c^\rho(h_1,h_2)/c^\rho(h_2,h_1)$ depends only on the cohomology $[c^\rho]$. Moreover, the quotient can be evaluated as

$$\begin{aligned} &c^\rho(h_1,h_2)/c^\rho(h_2,h_1)\mathbf{1}_V \\ &\quad= \bar{\rho}\left(\phi^{-1}(h_1)\phi^{-1}(h_2)\phi^{-1}(h_1h_2)^{-1}\right)\bar{\rho}\left(\phi^{-1}(h_2)\phi^{-1}(h_1)\phi^{-1}(h_2h_1)^{-1}\right)^{-1} = \bar{\rho}(k), \end{aligned} \tag{4.1}$$

which enables us to identify exactly the projective representations that are consistent with the given cohomology $[c^\rho]$. Finally, irreducible representations of X give rise to irreducible projective representations of H.

For our purpose, the finite groups whose projective representations we are interested in are the centralizer groups $C_G(g)$ where G is one of the umbral groups. The relevant 2-cocycle is determined by the 3-cocycle $\omega \in Z^3(G,U(1))$ as in (2.11). By considering the Schur cover of $C_G(g)$ we obtain all irreducible projective representations $C_G(g)$, and using (4.1) these projective representations can then be filtered to those consistent with $[\omega] \in H^3(G,U(1))$.

5 The generalised umbral mock modular forms

In this section, we employ the mock modularity property discussed in Section 3 to compute the mock modular forms of generalised umbral moonshine, which conjecturally arise from the generalised umbral moonshine module for the underlying deformed quantum double $D^\omega\big(G^{(\ell)}\big)$. In Section 5.1 we compute the g-twisted h-twined functions when $\langle g,h\rangle$ is an old group. In Section 5.2 we compute the remaining functions, the g-twisted h-twined functions when $\langle g,h\rangle$ is a new group.

5.1 Old groups

The untwisted twining functions of umbral moonshine were given in [12, 17, 18, 35, 41, 42], and explicit expressions for them can be found in [32]. For a given ℓ with index m, these functions, which are denoted $H_{(e,g)} = (H_{(e,g),r})$, are $2m$-dimensional vector-valued mock modular forms of weight-1/2 which are related to the meromorphic Jacobi forms $\psi_{(e,g)}$ as discussed in Sections 2.2 and 3. In this subsection we will discuss how to obtain the meromorphic Jacobi form $\psi_{(g,h)}$ and the vector-valued mock modular form $H_{(g,h)}$ from the untwisted function $\psi_{(e,g')}$ for these cases.

In the case there exists an $\mathrm{SL}_2(\mathbb{Z})$ element γ such that $\gamma(e,g') = (g,h)$, by the modularity condition (3.2), we have

$$\psi_{(g,h)} = \varsigma_{(e,g')}(\gamma)\psi_{(e,g')}|_{1,m}\gamma, \tag{5.1}$$

which can be evaluated explicitly from the relation between $\psi_{(e,g')}$ and $\psi_{(e,e)}$ and the explicit expression for $\psi_{(e,g')}$ given in [18, 32].

When $G^{(\ell)} \cong \bar{G}^{(\ell)}$ has $n = 1$, all old groups are isomorphic to a cyclic group generated by an element g', and the mock modular forms are found as described above. First consider umbral groups with $n = 2$. Let $z \in n$ be the nontrivial element and note that n is in the center of $G^{(\ell)}$. For the rank two old groups, namely those groups $\langle g, h\rangle$ that are isomorphic to $\langle z, g'\rangle$ for some $g' \in G^{(\ell)}$, we provide a conjecture and a justification for the corresponding twisted-twined functions $\psi_{(g,h)}$. From the modularity property (3.2), we see that it suffices to calculate $\psi_{(z,g)}$. Fix the lambency ℓ with index m. We conjecture that

$$\psi_{(z,g)} = \varsigma\psi_{(e,g)}|_{1,m}\left(\tfrac{1}{2}, 0\right) \tag{5.2}$$

for all $g \in G$ for some $\varsigma \in \mathbb{C}$.

To justify the conjecture, we start by verifying (5.2) whenever $\mathrm{ord}(g)$ is odd. Let γ be an element of $\mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(e, zg) = (z, g)$. For instance, we can take $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ \mathrm{ord}(g) & \mathrm{ord}(g)+1 \end{smallmatrix}\right)$. Since g is an element of the umbral group with odd order, the function $\psi_{(e,g)}$ transforms as a weight one Jacobi form for $\Gamma_0(\mathrm{ord}(g))$ with a multiplier system [18]. Hence we have

$$\psi_{(z,g)} = \varsigma_{(e,zg)}(\gamma)\psi_{(e,zg)}\big|_{1,m}\gamma = -\varsigma_{(e,zg)}(\gamma)\psi_{(e,g)}\big|_m\left(0, \tfrac{1}{2}\right)\big|_{1,m}\gamma = -\frac{\varsigma_{(e,zg)}(\gamma)}{\varsigma_{(e,g)}(\gamma)}\psi_{(e,g)}\big|_m\left(\tfrac{1}{2}, 0\right),$$

where we have used the operator relation (2.3) and the fact that

$$\psi_{(e,zg)} = -\psi_{(e,g)}\big|_m\left(0, \tfrac{1}{2}\right) \tag{5.3}$$

for all $g \in G^{(\ell)}$.

Using $\varsigma_{(z,g)}\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)\psi_{(z,g)}\big|_{1,m}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \psi_{(z,zg)}$, it is easy to also verify the validity of (5.2) for elements zg that are "paired" with the odd order elements g. Finally, we conjecture that (5.2) is also true when g and zg both have even order.

This conjecture is strongly motivated by the finite group property discussed in Section 3, which states that there exists an infinite dimensional module for $C_G(z) \cong G$ that underlies $H_{(z,g)}$. Since replacing θ_m by $\theta_m\big|_m\left(\frac{1}{2}, 0\right)$ simply reorders the components of the vector-valued mock modular forms $H_{(e,g)} = (H_{(e,g),r})$ by a shift $r \mapsto r + m$, the group theoretic consideration strongly suggests that this reordering has to happen for all $g \in G$, given that it is true for all odd order elements g as well as the elements zg that they are paired with.

For $\ell = 6 + 3$, the unique instance of umbral moonshine with $n = 3$, the conjecture is slightly different. First, instead of (5.3) we have now

$$\psi_{(e,zg)} = \psi_{(e,g)} - \frac{1}{2}\sum_{A\in\mathbb{Z}/3\mathbb{Z}} \psi_{(e,g)}|_{1,m}(0, A/3).$$

for $g \in G^{(\ell)}$ with order n_g not divisible by 3 that commutes with z. Choose any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying

$$c = 0 \bmod n_g = 1 \bmod 3, \qquad d = 1 \bmod n_g = 0 \bmod 3,$$

we have

$$\psi_{(z,g)} = \frac{\varsigma_{(e,zg)}(\gamma)}{\varsigma_{(e,g)}(\gamma)}\left(\psi_{(e,g)} - \frac{1}{2}\sum_{A\in\mathbb{Z}/3\mathbb{Z}} \psi_{(e,g)}|_{1,m}(A/3, 0)\right).$$

Repeating the same argument as before by evoking the projective representation property of moonshine, we conjecture that the above equality also holds for all g commuting with z, and hence for all rank two Abelian old groups $\langle z, g\rangle$.

5.2 New groups

Let $\langle g,h\rangle$ be a new group and consider the problem of evaluating $H_{(g,h)} = (H_{(g,h),r})$. First we discuss their analytic properties. We wish to show that for a new group $\langle g,h\rangle$, $H_{(g,h)}$ is bounded at all cusps.

Recall that the umbral moonshine conjecture [17, 18] states that, for all $g \in G$, the untwisted twined functions $H_{(e,g)}$ are bounded at all cusps inequivalent to $\{\mathrm{i}\infty\}$ under the action of $\Gamma_0(n_g)$. The number n_g is given by the order of the group element g as permutations of the irreducible components of the root system of the corresponding Niemeier lattice. See the appendices of [18] for the list of n_g. In particular, this means that they have no poles at the cusp $\tau \to 0$ for all $g \notin n$ and the modular property of generalised umbral moonshine (3.2) implies that the corresponding twisted untwined functions $H_{(g,e)}$ are bounded at the cusp $\mathrm{i}\infty$. This fact together with the finite group property of generalised umbral moonshine (3.6) then implies that all twisted twined functions $H_{(g,h)}$ are bounded at the cups $\mathrm{i}\infty$ for all $g \in G$, $g \notin n$ and for all $h \in C_G(g)$. Finally, evoking again the modular property (3.2) and considering $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $(g,h) = \gamma(g',h')$ to transform between cusps shows that the twisted twined functions $H_{(g,h)}$ are bounded at all cusps whenever $\langle g,h\rangle$ is a new group.

Next we discuss the modular properties of the new functions. From the above comment that $H_{(g,h)}$ is bounded at all cusps and the absence of the mock Jacobi form in the corresponding space with non-trivial shadow (see the proof of Proposition 4.1.1 of [15]), we expect that the shadow of $H_{(g,h)}$ vanishes. As a result, we conclude that $H_{(g,h)}$ is a weight 1/2 vector-valued modular form for $\Gamma_{(g,h)}$ with the multiplier system determined by the 3-cocyle ω (cf. equation (3.2)). Note that any congruence subgroup, for instance $\Gamma_{(g,h)}$, contains a subgroup of the form $\Gamma(N)$. Hence each individual component $H_{(g,h),r}$ is a weight 1/2 modular form (with trivial multiplier) for some $\Gamma(N)$. As a result, evoking Corollary 3 of [53], we conclude that each component $H_{(g,h),r}$ lies in the linear span of the theta functions

$$\theta^0_{m',s}(\tau) := \theta_{m',s}(\tau,0) = \sum_{k\in\mathbb{Z}} q^{\frac{(2m'k+s)^2}{4m'}}$$

for $m' \in \mathbb{Z}_+$, $s \in \mathbb{Z}/2m'\mathbb{Z}$.

Next we would like to further constrain the space of potential new functions of generalised umbral moonshine. Consider the congruence condition, which states that the set of q-powers of $H_{(g,h),r}$ with non-vanishing coefficients must be a subset of that of $H_{(g,e),r}$, a property that follows from the finite group property (3.6). This defines a subspace $\Theta_{(g,h),r}$ generated by all theta functions $\theta^0_{m',s}$ that $H_{(g,h),r}$ must lie in. Note that till now the space $\Theta_{(g,h),r}$ is independent of the twining element h, as long as it generates a new group with the given twisting element g. We can further restrict this space in two ways. First we require that also $H_{\gamma(g,h)}$ must have the correct q-powers for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Second, elements of $\Gamma_{(g,h)}$ act as a linear transformation on this space, denoted $\sigma\colon \Gamma_{(g,h)} \to \mathrm{End}(\Theta_{(g,h),r})$, and $H_{(g,h),r}$ must lie in the simultaneous eigenspace of $\sigma(\gamma)$ for all $\gamma \in \Gamma_{(g,h)}$, with the eigenvalues given by the multiplier system $\xi_{(g,h)}$ (cf. equation (3.4)). In all of the unobstructed new groups, these simultaneous eigenspaces have dimension either 0 or 1. Together with the requirement of the existence of a decomposition of the first homogeneous into projective representations, this fixes all the new functions.

In Table 2 we also record all the new functions up to a phase. To avoid cluttering we have adopted the shorthand notation

$$\hat{i} := \theta_{m,i}(\tau,\zeta) - \theta_{m,-i}(\tau,\zeta)$$

for $i \in \mathbb{Z}/2m\mathbb{Z}$.

Table 2: New umbral subgroup representatives $\langle g,h\rangle \subseteq G^{(\ell)}$.

ℓ	$\langle g,h\rangle$	$\Gamma_{(g,h)}$	Obs.	$\psi^{(\ell)}_{(g,h)}(\tau,\zeta)$
3	$\langle 2\mathrm{B},2\mathrm{a}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$	✓	0
	$\langle 2\mathrm{B},2\mathrm{c}\rangle$	$\Gamma_1(2)$	✓	0
	$\langle 2\mathrm{B},4\mathrm{a}\rangle$	$\mathrm{H}(2,2,4)$	✓	0
	$\langle 2\mathrm{B},4\mathrm{c}\rangle$	$\mathrm{H}(2,1,4)$	✓	0
	$\langle 2\mathrm{C},2\mathrm{b}\rangle$	$\Gamma(2)$	✓	0
	$\langle 2\mathrm{C},2\mathrm{e}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$	✓	0
	$\langle 3\mathrm{A},3\mathrm{a}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$		$-2\theta^0_{4,2}\cdot\hat{1}+\left(\theta^0_{4,0}+\theta^0_{4,4}\right)\cdot\hat{2}$
	$\langle 3\mathrm{A},3\mathrm{b}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$		$-2\theta^0_{4,2}\cdot\hat{1}+\left(\theta^0_{4,0}+\theta^0_{4,4}\right)\cdot\hat{2}$
	$\langle 3\mathrm{A},3\mathrm{c}\rangle$	$\Gamma_1(3)$		$\left(\theta^0_{9,3}-\theta^0_{9,9}\right)\cdot\hat{1}+\left(\theta^0_{9,0}-\theta^0_{9,6}\right)\cdot\hat{2}$
	$\langle 3\mathrm{A},6\mathrm{a}\rangle$	$\Gamma_1(6)$		$-\left(\theta^0_{9,3}-\theta^0_{9,9}\right)\cdot\hat{1}+\left(\theta^0_{9,0}-\theta^0_{9,6}\right)\cdot\hat{2}$
	$\langle 3\mathrm{A},6\mathrm{b}\rangle$	$\Gamma_1(2)$		$2\theta^0_{4,2}\cdot\hat{1}+\left(\theta^0_{4,0}+\theta^0_{4,4}\right)\cdot\hat{2}$
	$\langle 3\mathrm{A},6\mathrm{h}\rangle$	$\Gamma_1(2)$		$2\theta^0_{4,2}\cdot\hat{1}+\left(\theta^0_{4,0}+\theta^0_{4,4}\right)\cdot\hat{2}$
4	$\langle 2\mathrm{B},2\mathrm{a}\rangle$	$\Gamma_1(2)$	✓	0
	$\langle 2\mathrm{B},2\mathrm{b}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$	✓	0
	$\langle 2\mathrm{B},4\mathrm{b}\rangle$	$\Gamma_1(2)$		$2\left(\theta^0_{4,0}-\theta^0_{4,4}\right)\cdot\hat{2}$
	$\langle 2\mathrm{C},2\mathrm{a}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$	✓	0
	$\langle 2\mathrm{C},2\mathrm{c}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$		0
	$\langle 2\mathrm{C},2\mathrm{i}\rangle$	$\Gamma_1(2)$	✓	0
	$\langle 2\mathrm{C},4\mathrm{c}\rangle$	$\Gamma_1(2)$	✓	0
	$\langle 4\mathrm{A},4\mathrm{c}\rangle$	$\Gamma_{(4A,4c)}$		$(2-2\mathrm{i})\theta^0_{2,1}\cdot\hat{2}$
5	$\langle 2\mathrm{B},2\mathrm{d}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$	✓	0
	$\langle 2\mathrm{B},2\mathrm{e}\rangle$	$\Gamma_1(2)$	✓	0
	$\langle 2\mathrm{B},4\mathrm{a}\rangle$	$\Gamma_1(4)$		$2\mathrm{i}\left(\theta^0_{20,2}-\theta^0_{20,18}\right)\cdot\hat{2}+2\mathrm{i}\left(\theta^0_{20,6}-\theta^0_{20,14}\right)\cdot\hat{4}$
6+3	$\langle 2\mathrm{A},2\mathrm{a}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$	✓	0
	$\langle 2\mathrm{A},2\mathrm{b}\rangle$	$\Gamma_1(2)$	✓	0
	$\langle 2\mathrm{A},2\mathrm{c}\rangle$	$\Gamma_1(2)$	✓	0
	$\langle 2\mathrm{A},2\mathrm{b}\rangle$	$\Gamma(2)$	✓	0
	$\langle 2\mathrm{A},6\mathrm{a}\rangle$	$\Gamma_0(3)$	✓	0
	$\langle 2\mathrm{A},6\mathrm{a}\rangle$	$\Gamma_0(3)$	✓	0
	$\langle 2\mathrm{A},2\mathrm{a}\rangle$	$\mathrm{SL}_2(\mathbb{Z})$	✓	0
	$\langle 2\mathrm{B},4\mathrm{b}\rangle$	$\mathrm{H}(2,2,4)$	✓	0

6 Summary and discussion

In this paper we introduced the notion of generalised umbral moonshine. Generalised umbral moonshine can be regarded as an extension of umbral moonshine and is analogous to the generalised monstrous moonshine, albeit with significant conceptual and technical differences. When restricting to the A_1^{24} case it reduces to the generalised Mathieu moonshine analysed in [43].

We conjecture that the deformed Drinfel'd double of finite groups is the key structure underlying the (generalised) umbral moonshine phenomenon. More specifically, the generalised umbral moonshine conjecture states that for each of the 23 lambencies of umbral moonshine there is a way to assign an infinite-dimensional module for a deformed Drinfel'd double of the umbral group such that its graded character is given by the twisted-twined functions which we specified in Section 5. More explicitly, we proposed in Section 3 six conditions of generalised umbral moonshine conjecture and provided evidence in the form of proposing all the twisted-twined functions for all cases of generalised umbral moonshine. A proof of this conjecture should be attaianable following the methods of [32, 45].

It is interesting that one does not obtain new mock modular forms (with non-vanishing shadow) in this generalisation of umbral moonshine. As we reviewed in Section 2.2, the origin of the mockness of the umbral moonshine mock modular forms $H_g^{(\ell)}$ is the poles of the meromorphic Jacobi forms $\psi_g^{(\ell)}$ (2.5), and the residues of the poles of these functions are given by characters of the umbral group. From (3.2) and the conjecture in Section 5.1, we see that the shadow of the mock modular form $H_{(g,h)}^{(\ell)}$ when $\langle g,h\rangle$ is an old group can be obtained from that of certain umbral moonshine function $H_{g'}^{(\ell)}$. In the case when $\langle g,h\rangle$ is a new group, the shadow of $H_{(g,h)}^{(\ell)}$ vanishes as a result of the reasoning presented in Section 5.2. In fact, all the new functions, which do not play a role in umbral moonshine itself, are given by theta functions (see Table 2).

The existence of generalised umbral moonshine is particularly interesting in view of the absence (so far) of a clear relation between umbral moonshine and conventional CFTs[6]. As mentioned in Section 2.1, the generalised monstrous moonshine is believed to be a manifestation of the fact that one can define the twisted modules in view of the existence of the moonshine VOA. More generally, the physical context of the deformed Drinfel'd double of a finite group is usually assumed to be that of holomorphic orbifold CFTs or closely related structures such as 3d TQFTs. The important lesson generalised umbral moonshine has taught us is the following: whatever algebraic structure underlies umbral moonshine, it has to retain the intimate relation to the deformed Drinfel'd double as chiral conformal field theories do.

A Examples

In this appendix we focus on three examples of g-twisted generalised umbral moonshine functions for lambency $\ell = 3, 4, 6+3$ of index $m = 3, 4, 6$ respectively. In particular, we list the independent components of the vector-valued mock modular forms $H_{(g,h)} = (H_{(g,h),r})$, $r \in \mathbb{Z}/2m\mathbb{Z}$, for all $h \in C_{G^\ell}(g)$. Our notation for the conjugacy classes can be found in the attached file. In these tables, the rational numbers in the first column denotes the q-power of the corresponding terms in the Fourier expansion of $H_{(g,h),r}$. We also list the projective character table for the projective representations compatible with the 3-cocycle, as well as the decompositions of the first few homogeneous components $K_{r,\alpha}^g$ of the conjectural generalised umbral moonshine module for these cases (cf. equation (3.6)). To increase readability we use "." to denote zero.

[6]From this point of view, the fact that there is a generalised moonshine for each of the 23 cases of umbral moonshine is perhaps more surprising than the existence of generalised Mathieu moonshine alone. This is because, unlike the other 22 cases of umbral moonshine, Mathieu moonshine has an obvious relation to CFTs. Indeed, in [43] the existence of generalised Mathieu moonshine is argued to suggest that a holomorphic VOA may be underlying Mathieu moonshine.

A.1 $\ell = 3$, 4B twist

Table 3: Projective character table for $C_{G^{(3)}}(g)$.

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$8a$	$8b$	$8c$	$8d$	$8e$	$8f$	$8g$	$8h$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1
χ_3	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
χ_4	1	-1	1	-1	1	1	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1	-1
χ_5	1	-1	1	1	-1	-1	i	$-i$	$-i$	i	$-i$	i	ζ_8^7	ζ_8^3	ζ_8^5	ζ_8	ζ_8^3	ζ_8^7	ζ_8	ζ_8^5
χ_6	1	-1	1	1	-1	-1	i	$-i$	$-i$	i	$-i$	i	ζ_8^3	ζ_8^7	ζ_8	ζ_8^5	ζ_8^7	ζ_8^3	ζ_8^5	ζ_8
χ_7	1	-1	1	1	-1	-1	$-i$	i	i	$-i$	i	$-i$	ζ_8^5	ζ_8	ζ_8^7	ζ_8^3	ζ_8	ζ_8^5	ζ_8^3	ζ_8^7
χ_8	1	-1	1	1	-1	-1	$-i$	i	i	$-i$	i	$-i$	ζ_8	ζ_8^5	ζ_8^3	ζ_8^7	ζ_8^5	ζ_8	ζ_8^7	ζ_8^3
χ_9	1	1	1	-1	-1	-1	i	i	$-i$	$-i$	$-i$	i	ζ_8^7	ζ_8^7	ζ_8^5	ζ_8^5	ζ_8^3	ζ_8^3	ζ_8	ζ_8
χ_{10}	1	1	1	-1	-1	-1	i	i	$-i$	$-i$	$-i$	i	ζ_8^3	ζ_8^3	ζ_8	ζ_8	ζ_8^7	ζ_8^7	ζ_8^5	ζ_8^5
χ_{11}	1	1	1	-1	-1	-1	$-i$	$-i$	i	i	i	$-i$	ζ_8^5	ζ_8^5	ζ_8^7	ζ_8^7	ζ_8	ζ_8	ζ_8^3	ζ_8^3
χ_{12}	1	1	1	-1	-1	-1	$-i$	$-i$	i	i	i	$-i$	ζ_8	ζ_8	ζ_8^3	ζ_8^3	ζ_8^5	ζ_8^5	ζ_8^7	ζ_8^7
χ_{13}	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	$-i$	i	i	$-i$	$-i$	i	i	$-i$
χ_{14}	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	i	$-i$	$-i$	i	i	$-i$	$-i$	i
χ_{15}	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	$-i$	$-i$	i	i	$-i$	$-i$	i	i
χ_{16}	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	i	i	$-i$	$-i$	i	i	$-i$	$-i$
χ_{17}	2	.	-2	.	-2	2	2	.	2	.	-2	-2	.	.	.	.	.	.	.	.
χ_{18}	2	.	-2	.	-2	2	-2	.	-2	.	2	2	.	.	.	.	.	.	.	.
χ_{19}	2	.	-2	.	2	-2	$2i$	.	$-2i$	.	$2i$	$-2i$	.	.	.	.	.	.	.	.
χ_{20}	2	.	-2	.	2	-2	$-2i$	.	$2i$	.	$-2i$	$2i$	.	.	.	.	.	.	.	.

Table 4: Coefficients of $H_{(g,h),1}$.

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$8a$	$8b$	$8c$	$8d$	$8e$	$8f$	$8g$	$8h$
2/12	2	.	2	.	2	2	2	.	2	.	2	2	.	2	.	2	.	2	.	2
5/12	4	.	-4	.	4	-4	$4i$	.	$-4i$	.	$4i$	$-4i$	.	.	.	.	.	.	.	.
8/12	6	.	6	.	6	6	-6	.	-6	.	-6	-6	.	$2i$	.	$-2i$	.	$2i$	.	$-2i$
11/12	8	.	-8	.	8	-8	$-8i$	.	$8i$	.	$-8i$	$8i$	.	.	.	.	.	.	.	.
14/12	12	.	12	.	12	12	12	.	12	.	12	12	.	-4	.	-4	.	-4	.	-4
17/12	16	.	-16	.	16	-16	$16i$	.	$-16i$	.	$16i$	$-16i$	.	.	.	.	.	.	.	.
20/12	20	.	20	.	20	20	-20	.	-20	.	-20	-20	.	$-4i$	.	$4i$	.	$-4i$	.	$4i$
23/12	28	.	-28	.	28	-28	$-28i$	.	$28i$	.	$-28i$	$28i$	.	.	.	.	.	.	.	.
26/12	36	.	36	.	36	36	36	.	36	.	36	36	.	4	.	4	.	4	.	4
29/12	44	.	-44	.	44	-44	$44i$	.	$-44i$	.	$44i$	$-44i$	.	.	.	.	.	.	.	.

Table 5: Coefficients of $H_{(g,h),2}$.

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$8a$	$8b$	$8c$	$8d$	$8e$	$8f$	$8g$	$8h$
2/12	2	.	2	.	-2	-2	$2i$	.	$-2i$	.	$-2i$	$2i$	$2\zeta_8^7$	.	$2\zeta_8^5$	.	$2\zeta_8^3$	.	$2\zeta_8$	.
5/12	4	.	-4	.	-4	4	-4	.	-4	.	4	4	.	.	.	.	.	.	.	.
8/12	6	.	6	.	-6	-6	$-6i$	.	$6i$	.	$6i$	$-6i$	$2\zeta_8^5$	.	$2\zeta_8^7$	.	$2\zeta_8$	.	$2\zeta_8^3$	.
11/12	8	.	-8	.	-8	8	8	.	8	.	-8	-8	.	.	.	.	.	.	.	.
14/12	12	.	12	.	-12	-12	$12i$	.	$-12i$	.	$-12i$	$12i$	$4\zeta_8^3$	.	$4\zeta_8$	.	$4\zeta_8^7$	.	$4\zeta_8^5$	.
17/12	16	.	-16	.	-16	16	-16	.	-16	.	16	16	.	.	.	.	.	.	.	.
20/12	20	.	20	.	-20	-20	$-20i$	.	$20i$	.	$20i$	$-20i$	$4\zeta_8$	.	$4\zeta_8^3$	.	$4\zeta_8^5$	.	$4\zeta_8^7$	.
23/12	28	.	-28	.	-28	28	28	.	28	.	-28	-28	.	.	.	.	.	.	.	.
26/12	36	.	36	.	-36	-36	$36i$	.	$-36i$	.	$-36i$	$36i$	$4\zeta_8^7$	.	$4\zeta_8^5$	.	$4\zeta_8^3$	.	$4\zeta_8$	.
29/12	44	.	-44	.	-44	44	-44	.	-44	.	44	44	.	.	.	.	.	.	.	.

Table 6: Decompositions for the first few $K^g_{1,\alpha}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}
2/12	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.
8/12	.	.	.	.	.	.	.	.	.	.	.	.	2	1	1	2	.	.	.	.
11/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4
14/12	2	2	4	4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	.
20/12	.	.	.	.	.	.	.	.	.	.	.	.	4	6	6	4	.	.	.	.
23/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	14
26/12	10	10	8	8	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
29/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	22	.

Table 7: Decompositions for the first few $K^g_{2,\alpha}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}
2/12	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
5/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.
8/12	.	.	.	.	.	.	2	1	.	.	2	1	.	.	.	.	.	.	.	.
11/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	.	.	.
14/12	.	.	.	.	2	4	.	.	2	4	.	.	.	.	.	.	.	.	.	.
17/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	.	.
20/12	.	.	.	.	.	.	4	6	.	.	4	6	.	.	.	.	.	.	.	.
23/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	14	.	.	.
26/12	.	.	.	.	10	8	.	.	10	8	.	.	.	.	.	.	.	.	.	.
29/12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	22	.	.

A.2 $\ell = 4$, 2B twist

Table 8: Projective character table for $C_{G^{(4)}}(g)$.

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$4a$	$4b$	$4c$	$6a$	$6b$	$6c$	$6d$	$6e$	$6f$
χ_1	2	.	.	2	2	2	1	1	2	.	.	1	1	1	1	1	1
χ_2	2	.	.	-2	2	-2	1	1	.	.	2	1	-1	-1	-1	-1	1
χ_3	2	.	.	-2	-2	2	1	1	.	2	.	-1	-1	-1	1	1	-1
χ_4	2	.	.	2	2	2	ζ_3^2	ζ_3	2	.	.	ζ_3	ζ_3^2	ζ_3	ζ_3^2	ζ_3	ζ_3^2
χ_5	2	.	.	2	2	2	ζ_3	ζ_3^2	2	.	.	ζ_3^2	ζ_3	ζ_3^2	ζ_3	ζ_3^2	ζ_3
χ_6	2	.	.	-2	2	-2	ζ_3^2	ζ_3	.	.	2	ζ_3	ζ_6	ζ_6^5	ζ_6	ζ_6^5	ζ_3^2
χ_7	2	.	.	-2	2	-2	ζ_3	ζ_3^2	.	.	2	ζ_3^2	ζ_6^5	ζ_6	ζ_6^5	ζ_6	ζ_3
χ_8	2	.	.	-2	-2	2	ζ_3^2	ζ_3	.	2	.	ζ_6^5	ζ_6	ζ_6^5	ζ_3^2	ζ_3	ζ_6
χ_9	2	.	.	-2	-2	2	ζ_3	ζ_3^2	.	2	.	ζ_6	ζ_6^5	ζ_6	ζ_3	ζ_3^2	ζ_6^5
χ_{10}	4	.	.	4	-4	-4	-1	-1	.	.	.	1	-1	-1	1	1	1
χ_{11}	4	.	.	4	-4	-4	ζ_6	ζ_6^5	.	.	.	ζ_3	ζ_6	ζ_6^5	ζ_3^2	ζ_3	ζ_3^2
χ_{12}	4	.	.	4	-4	-4	ζ_6^5	ζ_6	.	.	.	ζ_3^2	ζ_6^5	ζ_6	ζ_3	ζ_3^2	ζ_3
χ_{13}	6	.	.	6	6	6	.	.	-2	.	.	.	.	.	.	.	.
χ_{14}	6	.	.	-6	6	-6	.	.	.	.	-2	.	.	.	.	.	.
χ_{15}	6	.	.	-6	-6	6	.	.	.	-2	.	.	.	.	.	.	.

Table 9: Coefficients of $H_{(g,h),1}$.

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$4a$	$4b$	$4c$	$6a$	$6b$	$6c$	$6d$	$6e$	$6f$
3/16	4	.	.	4	4	4	−1	−1	4	.	.	−1	−1	−1	−1	−1	−1
11/16	4	.	.	−4	4	−4	2	2	.	.	4	2	−2	−2	−2	−2	2
19/16	12	.	.	12	12	12	.	.	−4	.	.	.	.	.	.	.	.
27/16	16	.	.	−16	16	−16	−1	−1	.	.	.	−1	1	1	1	1	−1
35/16	24	.	.	24	24	24	.	.	8	.	.	.	.	.	.	.	.
43/16	36	.	.	−36	36	−36	.	.	.	.	4	.	.	.	.	.	.
51/16	56	.	.	56	56	56	−2	−2	−8	.	.	−2	−2	−2	−2	−2	−2
59/16	76	.	.	−76	76	−76	2	2	.	.	−4	2	−2	−2	−2	−2	2
67/16	108	.	.	108	108	108	.	.	12	.	.	.	.	.	.	.	.
75/16	148	.	.	−148	148	−148	−1	−1	.	.	4	−1	1	1	1	1	−1

Table 10: Coefficients of $H_{(g,h),2}$.

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$4a$	$4b$	$4c$	$6a$	$6b$	$6c$	$6d$	$6e$	$6f$
0/2	2	.	.	−2	−2	2	1	1	.	2	.	−1	−1	−1	1	1	−1
1/2	8	.	.	8	−8	−8	−2	−2	.	.	.	2	−2	−2	2	2	2
2/2	12	.	.	−12	−12	12	.	.	.	−4	.	.	.	.	.	.	.
3/2	16	.	.	16	−16	−16	2	2	.	.	.	−2	2	2	−2	−2	−2
4/2	32	.	.	−32	−32	32	−2	−2	.	.	.	2	2	2	−2	−2	2
5/2	48	.	.	48	−48	−48	.	.	.	.	.	.	.	.	.	.	.
6/2	64	.	.	−64	−64	64	2	2	.	.	.	−2	−2	−2	2	2	−2
7/2	96	.	.	96	−96	−96	.	.	.	.	.	.	.	.	.	.	.
8/2	132	.	.	−132	−132	132	.	.	.	4	.	.	.	.	.	.	.
9/2	184	.	.	184	−184	−184	2	2	.	.	.	−2	2	2	−2	−2	−2

Table 11: Coefficients of $H_{(g,h),3}$.

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$4a$	$4b$	$4c$	$6a$	$6b$	$6c$	$6d$	$6e$	$6f$
3/16	4	.	.	−4	4	−4	−1	−1	.	.	4	−1	1	1	1	1	−1
11/16	4	.	.	4	4	4	2	2	4	.	.	2	2	2	2	2	2
19/16	12	.	.	−12	12	−12	.	.	.	.	−4	.	.	.	.	.	.
27/16	16	.	.	16	16	16	−1	−1	.	.	.	−1	−1	−1	−1	−1	−1
35/16	24	.	.	−24	24	−24	.	.	.	.	8	.	.	.	.	.	.
43/16	36	.	.	36	36	36	.	.	4	.	.	.	.	.	.	.	.
51/16	56	.	.	−56	56	−56	−2	−2	.	.	−8	−2	2	2	2	2	−2
59/16	76	.	.	76	76	76	2	2	−4	.	.	2	2	2	2	2	2
67/16	108	.	.	−108	108	−108	.	.	.	.	12	.	.	.	.	.	.
75/16	148	.	.	148	148	148	−1	−1	4	.	.	−1	−1	−1	−1	−1	−1

Table 12: Decompositions for the first few $K^g_{1,\alpha}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
3/16	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.
11/16	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.
19/16	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.
27/16	.	.	.	.	.	1	1	.	.	.	.	.	.	2	.
35/16	2	.	.	2	2	.	.	.	.	.	.	.	2	.	.
43/16	.	2	.	.	.	2	2	.	.	.	.	.	.	4	.
51/16	.	.	.	2	2	.	.	.	.	.	.	.	8	.	.
59/16	.	4	.	.	.	2	2	.	.	.	.	.	.	10	.
67/16	6	.	.	6	6	.	.	.	.	.	.	.	12	.	.
75/16	.	6	.	.	.	7	7	.	.	.	.	.	.	18	.

Table 13: Decompositions for the first few $K^g_{2,\alpha}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
0/2	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
1/2	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.
2/2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
3/2	.	.	.	.	.	.	.	.	.	.	2	2	.	.	.
4/2	.	.	.	.	.	.	.	2	2	.	.	.	.	.	4
5/2	.	.	.	.	.	.	.	.	.	4	4	4	.	.	.
6/2	.	.	4	.	.	.	.	2	2	.	.	.	.	.	8
7/2	.	.	.	.	.	.	.	.	.	8	8	8	.	.	.
8/2	.	.	6	.	.	.	.	6	6	.	.	.	.	.	16
9/2	.	.	.	.	.	.	.	.	.	14	16	16	.	.	.

Table 14: Decompositions for the first few $K^g_{3,\alpha}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
3/16	.	.	.	.	.	1	1	.	.	.	.	.	.	.	.
11/16	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
19/16	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.
27/16	.	.	.	1	1	.	.	.	.	.	.	.	2	.	.
35/16	.	2	.	.	.	2	2	.	.	.	.	.	.	2	.
43/16	2	.	.	2	2	.	.	.	.	.	.	.	4	.	.
51/16	.	.	.	.	.	2	2	.	.	.	.	.	.	8	.
59/16	4	.	.	2	2	.	.	.	.	.	.	.	10	.	.
67/16	.	6	.	.	.	6	6	.	.	.	.	.	.	12	.
75/16	6	.	.	7	7	.	.	.	.	.	.	.	18	.	.

A.3 $\ell = 6 + 3$, 3B twist

Table 15: Projective character table for $C_{G^{(6+3)}}(g)$.

	$1a$	$2a$	$3a$	$3b$	$3c$	$3d$	$3e$	$6a$	$6b$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	1	-1	-1
χ_3	1	-1	ζ_3^2	ζ_3	ζ_3	1	ζ_3^2	ζ_6	ζ_6^5
χ_4	1	-1	ζ_3	ζ_3^2	ζ_3^2	1	ζ_3	ζ_6^5	ζ_6
χ_5	1	1	ζ_3^2	ζ_3	ζ_3	1	ζ_3^2	ζ_3^2	ζ_3
χ_6	1	1	ζ_3	ζ_3^2	ζ_3^2	1	ζ_3	ζ_3	ζ_3^2
χ_7	2	.	2	2	-1	-1	-1	.	.
χ_8	2	.	$2\zeta_3$	$2\zeta_3^2$	ζ_6	-1	ζ_6^5	.	.
χ_9	2	.	$2\zeta_3^2$	$2\zeta_3$	ζ_6^5	-1	ζ_6	.	.

Table 16: Coefficients of $H_{(g,h),1}$.

	$1a$	$2a$	$3a$	$3b$	$3c$	$3d$	$3e$	$6a$	$6b$
5/72	2	2	2	2	2	2	2	2	2
29/72	2	-2	$2\zeta_3$	$2\zeta_3^2$	$2\zeta_3^2$	2	$2\zeta_3$	$2\zeta_6^5$	$2\zeta_6$
53/72	4	.	$4\zeta_3^2$	$4\zeta_3$	$4\zeta_3$	4	$4\zeta_3^2$	.	.
77/72	2	-2	2	2	2	2	2	-2	-2
101/72	6	2	$6\zeta_3$	$6\zeta_3^2$	$6\zeta_3^2$	6	$6\zeta_3$	$2\zeta_3$	$2\zeta_3^2$
125/72	6	-2	$6\zeta_3^2$	$6\zeta_3$	$6\zeta_3$	6	$6\zeta_3^2$	$2\zeta_6$	$2\zeta_6^5$
149/72	8	.	8	8	8	8	8	.	.
173/72	8	.	$8\zeta_3$	$8\zeta_3^2$	$8\zeta_3^2$	8	$8\zeta_3$	.	.
197/72	12	4	$12\zeta_3^2$	$12\zeta_3$	$12\zeta_3$	12	$12\zeta_3^2$	$4\zeta_3^2$	$4\zeta_3$
221/72	14	-2	14	14	14	14	14	-2	-2

Table 17: Coefficients of $H_{(g,h),3}$.

	$1a$	$2a$	$3a$	$3b$	$3c$	$3d$	$3e$	$6a$	$6b$
5/72	4	.	$4\zeta_3$	$4\zeta_3^2$	$2\zeta_6$	-2	$2\zeta_6^5$	.	.
29/72	4	.	$4\zeta_3^2$	$4\zeta_3$	$2\zeta_6^5$	-2	$2\zeta_6$	.	.
53/72	8	.	8	8	-4	-4	-4	.	.
77/72	4	.	$4\zeta_3$	$4\zeta_3^2$	$2\zeta_6$	-2	$2\zeta_6^5$	.	.
101/72	12	.	$12\zeta_3^2$	$12\zeta_3$	$6\zeta_6^5$	-6	$6\zeta_6$	.	.
125/72	12	.	12	12	-6	-6	-6	.	.
149/72	16	.	$16\zeta_3$	$16\zeta_3^2$	$8\zeta_6$	-8	$8\zeta_6^5$	.	.
173/72	16	.	$16\zeta_3^2$	$16\zeta_3$	$8\zeta_6^5$	-8	$8\zeta_6$	.	.
197/72	24	.	24	24	-12	-12	-12	.	.
221/72	28	.	$28\zeta_3$	$28\zeta_3^2$	$14\zeta_6$	-14	$14\zeta_6^5$	.	.

Table 18: Decompositions for the first few $K^g_{1,\alpha}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
5/72	2	.	.	.	.	.	.	.	.
29/72	.	.	.	2	.	.	.	.	.
53/72	.	.	2	.	2	.	.	.	.
77/72	.	2	.	.	.	.	.	.	.
101/72	.	.	.	2	.	4	.	.	.
125/72	.	.	4	.	2	.	.	.	.
149/72	4	4	.	.	.	.	.	.	.
173/72	.	.	.	4	.	4	.	.	.
197/72	.	.	4	.	8	.	.	.	.
221/72	6	8	.	.	.	.	.	.	.

Table 19: Decompositions for the first few $K^g_{3,\alpha}$.

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
5/72	.	.	.	.	.	.	.	2	.
29/72	.	.	.	.	.	.	.	.	2
53/72	.	.	.	.	.	.	4	.	.
77/72	.	.	.	.	.	.	.	2	.
101/72	.	.	.	.	.	.	.	.	6
125/72	.	.	.	.	.	.	6	.	.
149/72	.	.	.	.	.	.	.	8	.
173/72	.	.	.	.	.	.	.	.	8
197/72	.	.	.	.	.	.	12	.	.
221/72	.	.	.	.	.	.	.	14	.

Acknowledgements

We are grateful to John Duncan, Simon Lentner, Terry Gannon, Erik Verlinde for helpful discussions. We especially John Duncan for many of the group descriptions and for helpful comments on an earlier version of the manuscript. The work of M.C. and D.W. was supported by ERC starting grant H2020 ERC StG 2014.

References

[1] Altschuler D., Coste A., Maillard J.M., Representation theory of twisted group double, *Ann. Fond. Louis de Broglie* **29** (2004), 681–694, arXiv:hep-th/0309257.

[2] Anagiannis V., Cheng M.C.N., Harrison S.M., $K3$ elliptic genus and an umbral moonshine module, arXiv:1709.01952.

[3] Bántay P., Orbifolds, Hopf algebras, and the moonshine, *Lett. Math. Phys.* **22** (1991), 187–194.

[4] Borcherds R.E., Vertex algebras, Kac–Moody algebras, and the monster, *Proc. Nat. Acad. Sci. USA* **83** (1986), 3068–3071.

[5] Borcherds R.E., Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992), 405–444.

[6] Brown K.S., Cohomology of groups, *Graduate Texts in Mathematics*, Vol. 87, Springer-Verlag, New York, 1994.

[7] Cappelli A., Itzykson C., Zuber J.B., The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories, *Comm. Math. Phys.* **113** (1987), 1–26.

[8] Carnahan S., Generalized moonshine I: Genus-zero functions, *Algebra Number Theory* **4** (2010), 649–679, arXiv:0812.3440.

[9] Carnahan S., Generalized moonshine, II: Borcherds products, *Duke Math. J.* **161** (2012), 893–950, arXiv:0908.4223.

[10] Carnahan S., Generalized moonshine, IV: Monstrous Lie algebras, arXiv:1208.6254.

[11] Carnahan S., Miyamoto M., Regularity of fixed-point vertex operator subalgebras, arXiv:1603.05645.

[12] Cheng M.C.N., $K3$ surfaces, $\mathcal{N}=4$ dyons and the Mathieu group M_{24}, *Commun. Number Theory Phys.* **4** (2010), 623–657, arXiv:1005.5415.

[13] Cheng M.C.N., Duncan J.F.R., On Rademacher sums, the largest Mathieu group and the holographic modularity of moonshine, *Commun. Number Theory Phys.* **6** (2012), 697–758, arXiv:1110.3859.

[14] Cheng M.C.N., Duncan J.F.R., On the discrete groups of Mathieu moonshine, in Perspectives in Representation Theory, *Contemp. Math.*, Vol. 610, Amer. Math. Soc., Providence, RI, 2014, 65–78, arXiv:1212.0906.

[15] Cheng M.C.N., Duncan J.F.R., Optimal mock Jacobi theta functions, arXiv:1605.04480.

[16] Cheng M.C.N., Duncan J.F.R., Meromorphic Jacobi forms of half-integral index and umbral moonshine modules, arXiv:1707.01336.

[17] Cheng M.C.N., Duncan J.F.R., Harvey J.A., Umbral moonshine, *Commun. Number Theory Phys.* **8** (2014), 101–242, arXiv:1204.2779.

[18] Cheng M.C.N., Duncan J.F.R., Harvey J.A., Umbral moonshine and the Niemeier lattices, *Res. Math. Sci.* **1** (2014), Art. 3, 81 pages, arXiv:1307.5793.

[19] Cheng M.C.N., Duncan J.F.R., Harvey J.A., Weight one Jacobi forms and umbral moonshine, *J. Phys. A: Math. Theor.* **51** (2018), 104002, 37 pages, arXiv:1703.03968.

[20] Cheng M.C.N., Ferrari F., Harrison S.M., Paquette N.M., Landau–Ginzburg orbifolds and symmetries of $K3$ CFTs, *J. High Energy Phys.* **2017** (2017), no. 1, 046, 49 pages, arXiv:1512.04942.

[21] Cheng M.C.N., Harrison S., Umbral moonshine and $K3$ surfaces, *Comm. Math. Phys.* **339** (2015), 221–261, arXiv:1406.0619.

[22] Conway J.H., Norton S.P., Monstrous moonshine, *Bull. London Math. Soc.* **11** (1979), 308–339.

[23] Dabholkar A., Murthy S., Zagier D., Quantum black holes, wall crossing, and mock modular forms, arXiv:1208.4074.

[24] Dijkgraaf R., Pasquier V., Roche P., Quasi Hopf algebras, group cohomology and orbifold models, *Nuclear Phys. B Proc. Suppl.* **18B** (1990), 60–72.

[25] Dijkgraaf R., Vafa C., Verlinde E., Verlinde H., The operator algebra of orbifold models, *Comm. Math. Phys.* **123** (1989), 485–526.

[26] Dijkgraaf R., Witten E., Topological gauge theories and group cohomology, *Comm. Math. Phys.* **129** (1990), 393–429.

[27] Dixon L., Ginsparg P., Harvey J., Beauty and the beast: superconformal symmetry in a Monster module, *Comm. Math. Phys.* **119** (1988), 221–241.

[28] Dixon L., Harvey J.A., Vafa C., Witten E., Strings on orbifolds, *Nuclear Phys. B* **261** (1985), 678–686.

[29] Dixon L., Harvey J.A., Vafa C., Witten E., Strings on orbifolds. II, *Nuclear Phys. B* **274** (1986), 285–314.

[30] Dong C., Li H., Mason G., Modular-invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1–56, arXiv:q-alg/9703016.

[31] Drinfel'd V.G., Quantum groups, in Proceedings of the International Congress of Mathematicians, Vols. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 798–820.

[32] Duncan J.F.R., Griffin M.J., Ono K., Proof of the umbral moonshine conjecture, *Res. Math. Sci.* **2** (2015), Art. 26, 47 pages, arXiv:1406.0571.

[33] Duncan J.F.R., Harvey J.A., The umbral moonshine module for the unique unimodular Niemeier root system, *Algebra Number Theory* **11** (2017), 505–535, arXiv:1412.8191.

[34] Duncan J.F.R., O'Desky A., Super vertex algebras, meromorphic Jacobi forms and umbral moonshine, *J. Algebra* **515** (2018), 389–407, arXiv:1705.09333.

[35] Eguchi T., Hikami K., Note on twisted elliptic genus of $K3$ surface, *Phys. Lett. B* **694** (2011), 446–455, arXiv:1008.4924.

[36] Eguchi T., Ooguri H., Tachikawa Y., Notes on the $K3$ surface and the Mathieu group M_{24}, *Exp. Math.* **20** (2011), 91–96, arXiv:1004.0956.

[37] Eichler M., Zagier D., The theory of Jacobi forms, *Progress in Mathematics*, Vol. 55, Birkhäuser Boston, Inc., Boston, MA, 1985.

[38] Frenkel I., Lepowsky J., Meurman A., Vertex operator algebras and the monster, *Pure and Applied Mathematics*, Vol. 134, Academic Press, Inc., Boston, MA, 1988.

[39] Frenkel I.B., Lepowsky J., Meurman A., A natural representation of the Fischer–Griess monster with the modular function J as character, *Proc. Nat. Acad. Sci. USA* **81** (1984), 3256–3260.

[40] Frenkel I.B., Lepowsky J., Meurman A., A moonshine module for the monster, in Vertex Operators in Mathematics and Physics (Berkeley, Calif., 1983), *Math. Sci. Res. Inst. Publ.*, Vol. 3, Springer, New York, 1985, 231–273.

[41] Gaberdiel M.R., Hohenegger S., Volpato R., Mathieu twining characters for $K3$, *J. High Energy Phys.* **2010** (2010), no. 9, 058, 20 pages, arXiv:1006.0221.

[42] Gaberdiel M.R., Hohenegger S., Volpato R., Mathieu moonshine in the elliptic genus of $K3$, *J. High Energy Phys.* **2010** (2010), no. 10, 062, 24 pages, arXiv:1008.3778.

[43] Gaberdiel M.R., Persson D., Ronellenfitsch H., Volpato R., Generalized Mathieu moonshine, *Commun. Number Theory Phys.* **7** (2013), 145–223, arXiv:1211.7074.

[44] Gannon T., Much ado about Mathieu, *Adv. Math.* **301** (2016), 322–358, arXiv:1211.5531.

[45] Griffin M.J., Mertens M.H., A proof of the Thompson moonshine conjecture, *Res. Math. Sci.* **3** (2016), Art. 36, 32 pages, arXiv:1607.03078.

[46] Harvey J.A., Murthy S., Moonshine in fivebrane spacetimes, *J. High Energy Phys.* **2014** (2014), no. 1, 146, 29 pages, arXiv:1307.7717.

[47] Harvey J.A., Murthy S., Nazaroglu C., ADE double scaled little string theories, mock modular forms and umbral moonshine, *J. High Energy Phys.* **2015** (2015), no. 5, 126, 57 pages, arXiv:1410.6174.

[48] Höhn G., Generalized moonshine for the baby monster, 2003.

[49] Kachru S., Paquette N.M., Volpato R., 3D string theory and umbral moonshine, *J. Phys. A: Math. Theor.* **50** (2017), 404003, 21 pages, arXiv:1603.07330.

[50] Mason G., Finite groups and modular functions (with an appendix by S.P. Norton), in The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), *Proc. Sympos. Pure Math.*, Vol. 47, Amer. Math. Soc., Providence, RI, 1987, 181–210.

[51] Norton S., From moonshine to the monster, in Proceedings on Moonshine and Related Topics (Montréal, QC, 1999), *CRM Proc. Lecture Notes*, Vol. 30, Amer. Math. Soc., Providence, RI, 2001, 163–171.

[52] Paquette N.M., Persson D., Volpato R., Monstrous BPS-algebras and the superstring origin of moonshine, *Commun. Number Theory Phys.* **10** (2016), 433–526, arXiv:1601.05412.

[53] Shimura G., On modular forms of half integral weight, *Ann. of Math.* **97** (1973), 440–481.

[54] Tuite M.P., Generalised moonshine and abelian orbifold constructions, in Moonshine, the Monster, and Related Topics (South Hadley, MA, 1994), *Contemp. Math.*, Vol. 193, Amer. Math. Soc., Providence, RI, 1996, 353–368, arXiv:hep-th/9412036.

[55] Whalen D., Vector-valued Rademacher sums and automorphic integrals, arXiv:1406.0571.

[56] Zhu Y., Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–302.

[57] Zwegers S., Mock theta functions, Ph.D. Thesis, Utrecht University, 2002, arXiv:0807.4834.

数理解析研究所講究録 第2086巻 2018年 91-98

MONSTROUS MOONSHINE OVER $\mathbb{Z}$?

SCOTT CARNAHAN

ABSTRACT. Monstrous Moonshine was extended in two complementary directions during the 1980s and 1990s, giving rise to Norton's Generalized Moonshine conjecture and Ryba's Modular Moonshine conjecture. Both conjectures have been unconditionally established in the last few years, so we describe some speculative conjectures that may extend and unify them.

1. MONSTROUS MOONSHINE AND ITS EXTENSIONS

Monstrous moonshine began in the 1970s with an examination of the modular J-invariant, which we normalize as:

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + \cdots$$

Here, τ ranges over the complex upper half-plane $\mathfrak{H}$, and the expansion in powers of $q = e^{2\pi i\tau}$ is the Fourier expansion. The J-function is a Hauptmodul for $SL_2(\mathbb{Z})$, meaning J is invariant under the standard action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$ by Möbius transformations, and induces a complex-analytic isomorphism from the quotient $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ to the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ minus finitely many points.

McKay first observed a possible relationship between J and the Monster sporadic simple group $\mathbb{M}$, pointing out that the irreducible representations of $\mathbb{M}$ have lowest dimension 1 and 196883, and that $196884 = 196883+1$. Thompson then showed that the next few coefficients of J also decompose into a small number of irreducible representations of $\mathbb{M}$ [Thompson 1979]. This suggested that the coefficients of J, which are all non-negative integers, describe a special class of representations of $\mathbb{M}$.

The observations by McKay and Thompson were explained by the construction of the Monster vertex operator algebra $V^\natural = \bigoplus_{n=0}^{\infty} V_n^\natural$ in [Frenkel-Lepowsky-Meurman 1988]. The appearance of J comes from the fact that $\sum_{n=0}^{\infty} \dim V_n^\natural q^{n-1} = J(\tau)$, and the appearance of the monster comes from the fact that $\mathrm{Aut}(V^\natural) \cong \mathbb{M}$.

Borcherds showed in [Borcherds 1992] that $V^\natural$ encoded a stronger relationship between modular functions and the monster, showing that for any $g \in \mathbb{M}$, the McKay-Thompson series

$$T_g(\tau) := \sum_{n=0}^{\infty} \mathrm{Tr}(g|V_n^\natural)q^{n-1}$$

is a Hauptmodul for some discrete group $\Gamma_g < SL_2(\mathbb{R})$.

This resolved the main "Monstrous Moonshine" conjecture posed in [Conway-Norton 1979], and it also partially resolved the "Jack Daniels" question posed in [Ogg 1974]. As Ogg noted, the 15 primes dividing the order of $\mathbb{M}$ are precisely the primes p such that the modular curve $X_0(p)^+$ is genus zero, and he offered a bottle of Jack Daniels for an explanation. The explanation for the monstrous primes being contained among the genus zero primes is: for each such prime p, there is a conjugacy class pA in the monster, and Borcherds's theorem established that for any $g \in p$A, the McKay-Thompson series $T_g(\tau)$ is a Hauptmodul for $X_0(p)^+$. The reverse containment is unlikely to be resolved without a good understanding for why $\mathbb{M}$ is as big as it is.

One Moonshine-like phenomenon that Borcherds's theorem did not explain was that for $g \in p$A, the q-expansion of $T_g(\tau)$ has non-negative integer coefficients, and furthermore, these coefficients

appear to decompose well into dimensions of representations of the centralizer $C_{\mathbb{M}}(g)$. For example, when $p = 2$, the Hauptmodul for $X_0(2)^+$ has the q-expansion

$$q^{-1} + 4372q + 96256q^2 + \cdots$$

while the centralizer $2.\mathbb{B}$, a central extension of the Baby monster sporadic group, has irreducible representations of dimension $1, 4371, 96255, 96256, \ldots$. This suggests that the Hauptmodul for g in the class pA is given by the graded dimension of some natural representation of $C_{\mathbb{M}}(g)$. In fact, there were two conjectures, both resolved recently, either of which could be called an explanation of this phenomenon:

- Generalized Moonshine (conjectured in [Norton 1987]): There is a rule that attaches to each $g \in \mathbb{M}$, a graded projective representation $V(g) = \bigoplus_{n \in \mathbb{Q}} V(g)_n$ of $C_{\mathbb{M}}(g)$, and to each commuting pair (g, h) in $\mathbb{M}$ a holomorphic function $Z(g, h; \tau)$ on $\mathfrak{H}$ such that:
  (1) The series $\sum_n \mathrm{Tr}(\tilde{h}|V(g)_n)q^{n-1}$ converges normally on $\mathfrak{H}$ to $Z(g, h; \tau)$, for $\tilde{h}$ some lift of h to a linear transformation of $V(g)$.
  (2) The assignment $(g, h) \mapsto Z(g, h; \tau)$ is invariant under simultaneous conjugation up to rescaling.
  (3) Each function $Z(g, h; \tau)$ is either constant or a Hauptmodul.
  (4) For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $Z(g, h; \frac{a\tau+b}{c\tau+d})$ is proportional to $Z(g^a h^c, g^b h^d; \tau)$.
  (5) $Z(g, h; \tau)$ is proportional to $J(\tau)$ if and only if $g = h = 1$.
- Modular Moonshine (conjectured in [Ryba 1996]): For $g \in p$A, there is a $\mathbb{Z}_{\geq 0}$-graded vertex algebra ${}^gV = \bigoplus_{n=0}^{\infty} {}^gV_n$ over $\mathbb{F}_p$ such that for any p-regular element $h \in C_{\mathbb{M}}(g)$, the graded Brauer character $\widetilde{\mathrm{Tr}}(h|{}^gV_n)q^{n-1}$ is equal to the McKay-Thompson series $T_{gh}(\tau)$.

These conjectures were solved along somewhat complementary paths. For Generalized Moonshine, the candidate representation $V(g)$ was more or less identified as the irreducible g-twisted $V^\natural$-module $V^\natural(g)$ in [Dixon-Ginsparg-Harvey 1988], and these twisted modules were proved to exist for all $g \in \mathbb{M}$ in [Dong-Li-Mason 1997], resolving claims 1, 2, and 5. However, the main conjectured properties were only resolved recently, with claim 4 proved in [Carnahan-Miyamoto 2016] and claim 3 in [Carnahan 2012]. For Modular Moonshine, the vertex algebras were identified as Tate cohomology groups $\hat{H}^0(g, V^\natural_{\mathbb{Z}})$, and the conjectured properties were proved in [Borcherds-Ryba 1996], [Borcherds 1998], and [Borcherds 1999] under the assumption of existence of a self-dual integral form $V^\natural_{\mathbb{Z}}$ of $V^\natural$. The existence of the essential object was only resolved recently, in [Carnahan 2017b].

In both cases, substantial progress was made in the late 1990s, but complete solutions were not possible due to the underdeveloped state of the theory of vertex operator algebras at the time. The most necessary piece of technology was a solid theory of cyclic orbifolds, worked out in [van Ekeren-Möller-Scheithauer 2015]. This in turn depended on many foundational developments in the intervening years, such as the Verlinde formula [Huang 2004], a theory of logarithmic tensor products (e.g., [Huang-Lepowsky-Zhang 2010]), C_2-cofiniteness for fixed-point subVOAs [Miyamoto 2013], and a theory of quantum dimension for VOAs [Dong-Jiao-Xu 2013].

2. Open refinements

After the Generalized Moonshine and Modular Moonshine conjectures were proposed, some refinements have been proposed based on experimental evidence and theoretical developments.

For Generalized Moonshine, it is natural to ask if the ambiguous constants in claims 2 and 4 can be refined by a good choice of functions $Z(g, h; \tau)$. Norton's original proposal in [Norton 1987] suggested that the constants can be refined to roots of unity, and a later proposal in [Norton 2001] asserted that these roots of unity may be taken to be 24th roots of unity. More recently, the constants were conjectured to be controlled by a nontrivial "Moonshine Anomaly" $\alpha^\natural \in H^3(\mathbb{M}, U(1))$ in [Gaberdiel-Persson-Volpato 2013], and Norton's assertion amounts to the claim that this anomaly

has order 24. The refined behavior in claims 2 and 4 are not particularly special in a moonshine sense, as it is expected for any holomorphic vertex operator algebra.

There is some progress on this refinement of Generalized Moonshine:

(1) In [Carnahan 2017a], Theorem 6.2 asserts that the constants relating Hauptmoduln may be taken to be roots of unity. However, there is still no control over the constant functions. Also, the techniques used to prove this theorem only work for $V^\natural$ (and other holomorphic vertex operator algebras with the same character, but these were conjectured to not exist in the introduction to [Frenkel-Lepowsky-Meurman 1988]). A similar result for other holomorphic vertex operator algebras would require additional development in the theory of regularity for fixed-points.

(2) In [Johnson-Freyd 2017], the main result is that the Moonshine Anomaly has order 24. However, at the moment I am writing this, the proof seems to rest on still-conjectural parts of a dictionary relating the theories of conformal nets and vertex operator algebras.

There have been some intriguing developments regarding connections between elliptic cohomology and monstrous moonshine since the 1980s, but there are still essential ingredients missing from the story. Essentially, there is a conjectural connection through conformal field theory (proposed in [Segal 1987], further developed in [Baker-Thomas 1999] and ongoing work of Stolz and Teichner), and a connection through higher characters (also mentioned in [Segal 1987], following the theory developed in [Hopkins-Kuhn-Ravenel 2000]). Based on the introduction of [Ganter 2007], it seems that all of the claims of the Generalized Moonshine conjecture except the Hauptmodul claim have had a known basis in elliptic cohomology theory since the 1990s. In particular, claims 2, 4, and 5 in the Generalized Moonshine conjecture (namely those that don't involve $V(g)$ or the Hauptmodul condition) amount to the claim that the collection of functions $Z(g,h;\tau)$ describe an element of twisted equivariant cohomology of a point, in the sense of [Ginzburg-Kapranov-Vasserot 1995]. That is, $Z \in \mathrm{Ell}^\alpha_{\mathbb{M}}(pt)$. If we ignore positivity, Claim 1, which connects Z to $V(g)$, amounts to the claim that Z is an element in twisted Devoto equivariant Tate K-theory of a point, i.e., $Z \in K^\alpha_{Dev,G}(pt)$.

There have been more refinements in recent years. First, Lurie has pointed out in [Lurie 2009] that elliptic cohomology can be made genuinely 2-equivariant, in the sense that we may consider a more functorial approach to equivariance, and also allow categorical groups. This would let us shift from an ad hoc definition of $Ell^\alpha_{\mathbb{M}}(pt)$ to a natural definition of $Ell_{\widetilde{\mathbb{M}}}(pt)$, where $\widetilde{\mathbb{M}}$ is a categorical group defined by a 3-cocycle representing $\alpha^\natural$, but unfortunately this new definition has yet to fully appear in public. Second, Ganter has employed equivariant Hecke operators, first seen as operations on elliptic cohomology in [Baker 1990], to explore the genus zero condition from a topological perspective in [Ganter 2007]. She defined an exponential operation

$$f \mapsto \Lambda^{(2)}_t(f) = \exp(-\sum_{k>0} T_k(f)t^k)$$

taking Ell^α_G to $\bigoplus_{n\geq 0} Ell^{-n\alpha}_G t^n$ by a formal sum of Hecke operators, and interpreted replicability as the equality $p(f(p) - f(q)) = \Lambda^{(2)}_{-p}(f(q))$. By expanding at a cusp, this formula comes very close to the twisted denominator formulas for Monstrous Lie algebras $\mathfrak{m}_g$ given in [Carnahan 2008] and [Carnahan 2009]. The main difference seems to be the fact that the Weyl denominator of $\mathfrak{m}_g$ is given by $Z(1,g;\sigma) - Z(g,1;\tau)$ rather than $Z(1,g;\sigma) - Z(1,g;\tau)$. Assuming this can be fixed in a natural way, this condition is essentially the statement that Z is "self-exponential" at chromatic level 2, i.e., the exponential operation $\Lambda^{(2)}$ basically takes Z to a slightly altered version of itself. At chromatic level 1, we have similar behavior with line bundles, since the exterior algebra of a line satisfies $\Lambda^*(L) = 1 + L$. That is, self-exponential objects are "exceptionally small" in a way that is highly nontrivial in the case of elliptic objects. We then reformulate Generalized Moonshine as:

Conjecture 2.1. *$V^\natural$ defines an element of* $\mathrm{Ell}^{\alpha^\natural}(B\mathbb{M})$ *that is self-exponential.*

The main reason this is still conjectural is because the words "defines" and "self-exponential" are still rather ill-defined.

For Modular Moonshine, the behavior that is now known to hold for Fricke-invariant classes of prime order is also expected to hold for all Fricke-invariant classes. That is, we have:

Conjecture 2.2. *(essentially* [Borcherds-Ryba 1996] *open problems 3 and 4) For each Fricke element $g \in \mathbb{M}$, we have $\hat{H}^1(g, V^\natural_{\mathbb{Z}}) = 0$. Furthermore, the vertex algebra $\hat{H}^0(g, V^\natural_{\mathbb{Z}})$ over $\mathbb{Z}/|g|\mathbb{Z}$ admits an action of $C_{\mathbb{M}}(g)$ by automorphisms, such that for any $h \in C_{\mathbb{M}}(g)$ satisfying $(|g|, |h|) = 1$, the graded Brauer character of h on $\hat{H}^0(g, V^\natural_{\mathbb{Z}})$ is equal to the q-expansion of the McKay-Thompson series $T_{gh}(\tau)$.*

The case where g has prime order amounts to the Modular Moonshine conjecture, which is settled. However, [Borcherds-Ryba 1996] Theorem 4.7 also yields the vanishing of $\hat{H}^1(g, V^\natural_{\mathbb{Z}})$ for g in the composite order classes 15A and 21A. This conjecture does not seem to be far beyond the reach of current technology.

3. Unifications

It is reasonable to ask whether the two conjectures that extend Monstrous Moonshine are really part of some larger, unified Moonshine. Borcherds proposed one possible unifying conjecture, which we rephrase slightly:

Conjecture 3.1. *(*[Borcherds 1998] *Conjecture 7.1) There is a rule that assigns to each element $g \in \mathbb{M}$ a $\frac{1}{|g|}\mathbb{Z}$-graded super-module over $\mathbb{Z}[e^{2\pi i/|g|}]$, equipped with an action of a central extension $(\mathbb{Z}/|g|\mathbb{Z}).C_{\mathbb{M}}(g)$, satisfying the following properties:*

1. *${}^1\hat{V}$ is a self-dual integral form of $V^\natural$.*
2. *If g and h commute, then $\hat{H}^*(\hat{g}, {}^h\hat{V}) \cong {}^{gh}\hat{V} \otimes \mathbb{Z}/|g|\mathbb{Z}$ for some lift $\hat{g}$ of g.*
3. *If g is Fricke (i.e., $T_g(\tau) = T_g(\frac{-1}{N\tau})$ for some $N \in \mathbb{Z}_{>0}$), then*
   (a) *${}^g\hat{V} \otimes \mathbb{Z}/|g|\mathbb{Z} \cong \hat{H}^0(g, {}^1\hat{V})$ as vertex algebras over $\mathbb{Z}/|g|\mathbb{Z}$ with $C_{\mathbb{M}}(g)$-action. In particular, $\hat{H}^1(g, {}^1\hat{V}) = 0$.*
   (b) *${}^g\hat{V} \otimes \mathbb{C} \cong V^\natural(g)$ as graded $(\mathbb{Z}/|g|\mathbb{Z}).C_{\mathbb{M}}(g)$-modules.*
4. *${}^g\hat{V}$ is "often" a vertex superalgebra with a conformal vector and a self-dual invariant symmetric bilinear form.*
5. *If g and h commute and have coprime order, then the graded Brauer character of some lift of g on ${}^h\hat{V}$ is equal to $T_{gh}(\tau)$.*

This conjecture is not a true unification, since it does not fully encompass Generalized Moonshine. In particular, the $SL_2(\mathbb{Z})$ and Hauptmodul claims are missing, and there are no claims about non-Fricke twisted modules. These can be added by hand, but without some additional structure, they do not seem to blend into the rest of the conjecture naturally.

It may be rewarding to consider how much of Borcherds's conjecture we can prove with current technology.

First, we may assign to ${}^1\hat{V}$ a self-dual integral form $V^\natural_{\mathbb{Z}}$ with $\mathbb{M}$-symmetry. We have existence from Theorem 2.4.2 of [Carnahan 2017b], but do not have a good uniqueness result yet.

Second, we can get pretty close to a candidate object for ${}^g\hat{V}$ when g is Fricke:

Theorem 3.2. *Let $g \in \mathbb{M}$ be a Fricke element, and let N be the level of $T_g(\tau)$. Then there is a $\mathbb{Z}[\frac{1}{N}, e^{2\pi i/N}]$-form of the irreducible twisted module $V^\natural(g)$, with an action of $V^\natural_{\mathbb{Z}} \otimes \mathbb{Z}[\frac{1}{N}, e^{2\pi i/N}]$, and compatible symmetries by $(\mathbb{Z}/|g|\mathbb{Z}).C_{\mathbb{M}}(g)$.*

Proof. By Theorem 1 of [Paquette-Persson-Volpato 2017] (in the anomaly-free case) and Proposition 3.2 of [Carnahan 2017a] (in the anomalous case), the cyclic orbifold dual $V^\natural/g$ is isomorphic to $V^\natural$. Then by Proposition 1.4.1 of [Carnahan 2017b], there is a self-dual $\mathbb{Z}[\frac{1}{N}, e^{2\pi i/N}]$-form of the abelian intertwining algebra ${}^g_N V^\natural$ with an action of $(\mathbb{Z}/N\mathbb{Z}).C_{\mathbb{M}}(g)$ by homogeneous automorphisms. The central $\mathbb{Z}/N\mathbb{Z}$ acts by scalar multiplication, and the description of cyclic orbifolds in [van Ekeren-Möller-Scheithauer 2015] allows us to reduce this to $\mathbb{Z}/|g|\mathbb{Z}$. This abelian intertwining algebra contains the $\mathbb{Z}[\frac{1}{N}, e^{2\pi i/N}]$-form of $V^\natural(g)$ as a sub-$(V^\natural)^g_{\mathbb{Z}} \otimes \mathbb{Z}[\frac{1}{N}, e^{2\pi i/N}]$-module. □

This theorem yields a candidate for ${}^g\hat{V} \otimes_{\mathbb{Z}[e^{2\pi i/|g|}]} \mathbb{Z}[\frac{1}{N}, e^{2\pi i/N}]$. To obtain ${}^g\hat{V}$ itself, we would need to choose a suitable stable lattice whose reduction to $\mathbb{Z}/|g|\mathbb{Z}$ is equivariantly isomorphic to $\hat{H}^0(g, {}^1\hat{V})$. Even assuming Conjecture 2.2, we would have to glue characteristic zero representations to characteristic p^k representations for various primes p. As Jeremy Rickard pointed out to me at [Rickard 2018], the fact that the characteristic zero characters are equal to the Brauer characters is far from sufficient for the existence of a compatible mixed characteristic representation.

The conjectured vertex superalgebra structure on ${}^g\hat{V}$ is a bit of a mystery. Borcherds claims in [Borcherds 1998] that we get such a structure for the classes 1A, 2B, 3B, 3C, 5B, 7B, and 13B, but apparently not 2A and 3A. Although I have not seen Borcherds's constructions, it is natural to extend the pattern of prime-order non-Fricke classes:

Conjecture 3.3. *If g is non-Fricke, then ${}^g\hat{V}$ is a vertex superalgebra.*

One feature that distinguishes the non-Fricke elements of $\mathbb{M}$ is the fact that the cyclic orbifold dual of $V^\natural$ along a non-Fricke element is always the Leech lattice vertex operator algebra, as conjectured in [Tuite 1993] and proved in Theorem 4.5 of [Carnahan 2017a]. It is therefore reasonable to expect the following:

Conjecture 3.4. *When g is non-Fricke, there is a natural construction of the vertex superalgebra structure on ${}^g\hat{V}$ using the Leech Lattice.*

Unfortunately, this is rather vague as stated. If we had a more precise formulation, we might be well on our way toward a construction.

4. Speculation about gluing

In the previous section, we noted that we don't have an obvious way to glue a $\mathbb{Z}[\frac{1}{N}, e^{2\pi i/N}]$-form of $V^\natural(g)$ to the Tate cohomology of $V^\natural_{\mathbb{Z}}$ to form ${}^g\hat{V}$. This problem could be resolved if we had a canonical way to lift Tate cohomology to mixed characteristic. Tate cohomology does not in general admit such canonical lifts, but we could hope that there is an additional structure on $V^\natural_{\mathbb{Z}}$, like a non-linear descent datum, that produces such a lift. I have been unable to find a structure of this precise form in the mathematical literature, but it bears some resemblance to various proposals for algebraic structures over the mythical "field of one element". While $\mathbb{F}_1$-structures do not have a universally recognized interpretations, one feature many proposals share is that they let one treat mixed-characteristic and equal-characteristic objects on a roughly equal footing.

Conjecture 4.1. *There is an $\mathbb{F}_1$-module structure on $V^\natural_{\mathbb{Z}}$ with $\mathbb{M}$-symmetry, where by "$\mathbb{F}_1$-module structure" we mean that for each $g \in \mathbb{M}$, there is a distinguished lift of the $(\mathbb{Z}/|g|\mathbb{Z}).C_{\mathbb{M}}(g)$-action on $\hat{H}^*(g, V^\natural_{\mathbb{Z}})$ (as a graded module) to the big Witt vectors $W_\infty \hat{H}^*(g, V^\natural_{\mathbb{Z}})$.*

In [Borger 2008], Λ-rings are proposed as an interpretation for the term "$\mathbb{F}_1$-algebra". It is then reasonably natural to presume modules ought to have a correspondingly non-linear interaction with Witt vectors. Conjecture 4.1 would yield candidates for the proposed modules ${}^g\hat{V}$, but it does not naturally lead to any exceptional modularity properties, such as the Hauptmodul property for p-singular elements.

5. Connections to newer moonshines

In 2014, an amateur mathematician, Tito Piezas III, posted a question proposing a Thompson moonshine phenomenon on `mathoverflow.net`, a question and answer site for mathematicians. The initial observation was that the weight 1/2 weakly holomorphic modular form

$$b(\tau) = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + \cdots$$

has coefficients that easily decompose into combinations of dimensions of irreducible representations of the Thompson sporadic group Th. It was quickly pointed out that there is already a Thompson moonshine, coming from Generalized Moonshine for the class 3C, where the twisted module has character $\sqrt[3]{j(\tau/3)} = q^{-1/9} + 248q^{2/9} + 4124q^{5/9} + \cdots$ and an action of $\mathbb{Z}/3\mathbb{Z} \times Th$. These two moonshines are connected by the Harvey-Moore-Borcherds correspondence, where up to some adjustment with eta functions and reparametrization, the Borcherds product attached to $b(\tau)$ is equal to the character of the irreducible 3C-twisted $V^\natural$-module. Generalized Moonshine then gives an explanation for the appearance of Thompson representations attached to powers of q in $b(\tau)$ that are squares. However, there is at this time no explanation for the representation-theoretic content of other coefficients.

Piezas's observation was refined to a concrete conjecture in [Harvey-Rayhoun 2015], and the conjecture was solved in [Griffin-Mertens 2016]. The function $b(\tau)$ was replaced with

$$2b(\tau) + 240\theta(\tau) = 2q^{-3} + 248 + 2 \cdot 27000q^4 - 2 \cdot 85995q^5 + \cdots$$

to remove sign ambiguities, and the theorem is that there is a graded Th-super-module $V = \bigoplus_{n \geq -3} V_n$ whose graded dimension is $2b(\tau) + 240\theta(\tau)$, such that for any $h \in Th$, the trace function $f_h := \sum_{n \geq -3} \mathrm{Tr}(h|V_n)q^n$ is a modular form of weight 1/2 for some congruence group controlled by the order of h. We find that the Borcherds product attached to $2b(\tau) + 240\theta(\tau)$ is equal to $\eta^{248} Z(1, g; \tau)^2$ for g an element of class 3C in $\mathbb{M}$. More generally, one may ask whether Borcherds products take f_h to suitably altered Generalized Moonshine functions, or perhaps Modular Moonshine functions, and there are some promising partial answers. However, the nature of this correspondence is still a complete mystery.

The presence of a square in $\eta^{248} Z(g, 1; \tau)^2$ is quite similar to the situation in Mathieu Moonshine, where the Borcherds lift of the elliptic genus of a K3 surface yields the Igusa cusp form Φ_{10}. In this case, $1/\Phi_{10}$ is the square of a function with representation-theoretic interest, namely the Weyl-Kac-Borcherds denominator of a hyperbolic-type Lie algebra [Gritsenko-Nikulin 1995]. I do not know a good explanation for the natural appearance of squares of characters of infinite dimensional algebras.

Another mystery comes from a conjecture of Borcherds that was resolved in [Griess-Lam 2013], asserting that the Modular Moonshine vertex algebra for 3C is isomorphic to the E_8-vertex algebra over $\mathbb{F}_3$. This means the vertex algebra has symmetries of $E_8(\mathbb{F}_3)$, which is far larger than Th. While Th embeds in $E_8(\mathbb{F}_3)$, it does not embed in $E_8(\mathbb{C})$. This seems to be evidence against the idea that the conjectural module ${}^g\hat{V}$ can be made by simply taking an integral form of the E_8 vertex algebra.

Finally, a generalization of Thompson moonshine to other weight 1/2 forms, called "skew-holomorphic moonshine", was announced by Duncan, Harvey, and Rayhoun in 2016. While we are still waiting for a paper, it appears that applying the Borcherds product construction to the certain vector-valued forms yields modular functions that can range over all Fricke-invariant McKay-Thompson series. This suggests the existence of a connection to modular moonshine for Fricke g. Once again, we don't seem to have any theoretical underpinning of the underlying objects.

References

[Baker 1990] A. Baker, *Hecke operators as operations in elliptic cohomology* J. Pure Appl. Algebra **63** (1990), 1–11.

[Baker-Thomas 1999] A. Baker, C. Thomas, *Classifying spaces, virasoro equivariant bundles, elliptic cohomology and moonshine* Glasgow University Mathematics Department preprint 99/39.

[Borcherds 1992] R. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras* Invent. Math. **109** (1992) 405–444.

[Borcherds 1998] R. Borcherds, *Modular moonshine III* Duke Math. J. **93** no. 1 (1998) 129–154.

[Borcherds 1999] R. Borcherds, *The fake monster formal group* Duke Math. J. **100** no. 1 (1999) 139–165.

[Borcherds-Ryba 1996] R. Borcherds, A. Ryba, *Modular moonshine II* Duke Math. J. **83** (1996) no. 2, 435–459.

[Borger 2008] J. Borger, *The basic geometry of Witt vectors, I The affine case* Algebra and Number Theory **5** no. 2 (2011) 231–285. Available as `https://arxiv.org/abs/0801.1691`

[Carnahan 2008] S. Carnahan, *Generalized Moonshine I: Genus zero functions* Algebra and Number Theory **4** no. 6 (2010) 649–679. Available as `https://arxiv.org/abs/0812.3440`

[Carnahan 2009] S. Carnahan, *Generalized Moonshine II: Borcherds products* Duke Math J. **161** no. 5 (2012) 893–950. Available as `https://arxiv.org/abs/0908.4223`

[Carnahan 2012] S. Carnahan, *Generalized Moonshine IV: Monstrous Lie algebras* ArXiv preprint: `https://arxiv.org/abs/1208.6254`

[Carnahan 2017a] S. Carnahan, *51 constructions of the moonshine module* ArXiv preprint `https://arxiv.org/abs/1707.02954`

[Carnahan 2017b] S. Carnahan, *Four self-dual integral forms of the moonshine module* ArXiv preprint: `https://arxiv.org/abs/1710.00737`

[Carnahan-Miyamoto 2016] S. Carnahan, M. Miyamoto, *Regularity of fixed-point vertex operator subalgebras* ArXiv preprint `https://arxiv.org/abs/1603.05645`

[Conway-Norton 1979] J. Conway, S. Norton, *Monstrous Moonshine* Bull. Lond. Math. Soc. **11** (1979) 308–339.

[Dixon-Ginsparg-Harvey 1988] L. Dixon, P. Ginsparg, J. Harvey, *Beauty and the Beast: Superconformal Symmetry in a Monster Module* Commun. Math. Phys. **119** (1988) 221–241.

[Dong-Jiao-Xu 2013] C. Dong, X. Jiao and F. Xu, *Quantum dimensions and quantum Galois theory* Trans. Amer. Math. Soc. **365** (2013), no. 12, 6441–6469.

[Dong-Li-Mason 1997] C. Dong, H. Li, G. Mason, *Modular invariance of trace functions in orbifold theory* Comm. Math. Phys. **214** (2000) no. 1, 1–56. Available at `https://arxiv.org/abs/q-alg/9703016`

[van Ekeren-Möller-Scheithauer 2015] J. van Ekeren, S. Möller, N. Scheithauer, *Construction and classification of vertex operator algebras* ArXiv preprint `https://arxiv.org/abs/1507.08142`

[Frenkel-Lepowsky-Meurman 1988] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster* Pure and Applied Mathematics **134** Academic Press, Inc., Boston, MA, (1988).

[Gaberdiel-Persson-Volpato 2013] M. R. Gaberdiel, D. Persson, R. Volpato, *Generalised Moonshine and Holomorphic Orbifolds* Proceedings, String-Math 2012, Bonn, Germany, July 16-21, 2012, (2013) 73–86. Available at `https://arxiv.org/abs/1302.5425`

[Ganter 2007] *Hecke operators in equivariant elliptic cohomology and generalized Moonshine* Harnad, John (ed.) et al., Groups and symmetries. From Neolithic Scots to John McKay. AMS CRM Proceedings and Lecture Notes **47** (2009) 173–209. Available at `https://arxiv.org/abs/0706.2898`

[Ginzburg-Kapranov-Vasserot 1995] V. Ginzburg, M. Kapranov, E. Vasserot, *Elliptic algebras and equivariant elliptic cohomology i.* ArXiv preprint `https://arxiv.org/abs/q-alg/9505012`

[Griess-Lam 2013] R. Griess, C. H. Lam, *Groups of Lie Type, Vertex Algebras, and Modular Moonshine* International Mathematics Research Notices **2015** Issue 21, 1 January 2015 10716–10755. Available at `https://arxiv.org/abs/1308.2270`

[Griffin-Mertens 2016] M. Griffin, M. Mertens *A proof of the Thompson moonshine conjecture* Research in the Mathematical Sciences **3** no. 36 Available at `https://arxiv.org/abs/1607.03078`

[Gritsenko-Nikulin 1995] V. A. Gritenko and V. V. Nikulin, *Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras* Amer. J. Math. **119** no. 1 (1997) 181–224. Available at `https://arxiv.org/abs/alg-geom/9504006`

[Harvey-Rayhoun 2015] J. Harvey, B. Rayhaun *Traces of Singular Moduli and Moonshine for the Thompson Group* Comm. in Number Theory and Physics **10** (2016) no. 1 pp. 23–62. Available at `https://arxiv.org/abs/1504.08179`

[Hopkins-Kuhn-Ravenel 2000] M. Hopkins, N. Kuhn, D. Ravenel, *Generalized group characters and complex oriented cohomology theories* J. Amer. Math. Soc., **13** no.3 (2000) 553–594.

[Huang 2004] Y.-Z. Huang, *Vertex operator algebras and the Verlinde conjecture*, Commun. Contemp. Math. **10** (2008), no. 1, 103–154. Available at `http://arxiv.org/abs/math/0406291`

[Huang-Lepowsky-Zhang 2010] Y.-Z. Huang, J. Lepowsky, L. Zhang. *Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms.* ArXiv preprint `http://arxiv.org/abs/1012.4202`

[Johnson-Freyd 2017] T. Johnson-Freyd, *The Moonshine Anomaly* ArXiv preprint `https://arxiv.org/abs/1707.08388`

[Lurie 2009] J. Lurie, *A Survey of Elliptic Cohomology* In: N. Baas, E. Friedlander, B. Jahren, P. Østvær (eds) Algebraic Topology. Abel Symposia, vol 4. Springer, Berlin, Heidelberg (2009) 219–277.

[Miyamoto 2013] M. Miyamoto, *C_2-cofiniteness of cyclic-orbifold models*, Comm. Math. Phys. **335** (2015), no. 3, 1279–1286. Available at `https://arxiv.org/abs/1306.5031`

[Norton 1987] S. Norton, *Generalized moonshine* Proc. Sympos. Pure Math. **47** Part 1, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 209–210, Amer. Math. Soc., Providence, RI (1987).

[Norton 2001] S. Norton, *From Moonshine to the Monster* Proceedings on Moonshine and related topics (Montréal, QC, 1999), 163–171, CRM Proc. Lecture Notes **30**, Amer. Math. Soc., Providence, RI (2001).

[Ogg 1974] A. Ogg, *Automorphisms de courbes modulaires* Sem. Delange-Pisot-Poitou, Théorie des nombres **16** no. 1 (1974-1975), exp. no. 7, 1–8.

[Paquette-Persson-Volpato 2017] N. Paquette, D. Persson, R. Volpato, *BPS algebras, genus zero, and the heterotic Monster* ArXiv preprint `https://arxiv.org/abs/1701.05169`

[Rickard 2018] Jeremy Rickard *Answer to MathOverflow question 295485* URL (version: 2018-03-18): `https://mathoverflow.net/q/295485`

[Ryba 1996] A. Ryba, *Modular Moonshine?* In"Moonshine, the Monster, and related topics", edited by Chongying Dong and Geoffrey Mason. Contemporary Mathematics **193** American Mathematical Society, Providence, RI (1996) 307–336.

[Segal 1987] G. Segal, *Elliptic cohomology* Séminaire N. Bourbaki, 1987–1988, exp. no 695, pp. 187–201.

[Thompson 1979] J. Thompson, *Some numerology between the Fischer-Griess Monster and the elliptic modular function* Bull. London Math. Soc. *11* no. 3 (1979) 352–353.

[Tuite 1993] M. Tuite, *On the relationship between monstrous Moonshine and the uniqueness of the Moonshine module* Comm. Math. Phys. **166** (1995) 495–532. Available as `https://arxiv.org/abs/hep-th/9305057`.

Distribution Agreement

In presenting this thesis or dissertation as a partial fulfillment of the requirements for an advanced degree from Emory University, I hereby grant to Emory University and its agents the non-exclusive license to archive, make accessible, and display my thesis or dissertation in whole or in part in all forms of media, now or hereafter known, including display on the world wide web. I understand that I may select some access restrictions as part of the online submission of this thesis or dissertation. I retain all ownership rights to the copyright of the thesis or dissertation. I also retain the right to use in future works (such as articles or books) all or part of this thesis or dissertation.

Signature:

_____ _____

Morad Ihab Hassan Date

Mathieu Moonshine: Mock Modular Lifts

By

Morad Ihab Hassan
Master of Science

Mathematics

_____

Dr. John Duncan
Advisor

_____

Dr. David Zureick-Brown
Committee Member

_____

Dr. Gregory J. Martin
Committee Member

Accepted:

_____

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

_____

Date

Mathieu Moonshine: Mock Modular Lifts

By

Morad Ihab Hassan

Advisor: John Duncan, PhD

An abstract of
A thesis submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Master of Science
in Mathematics
2018

Abstract

Mathieu Moonshine: Mock Modular Lifts
By Morad Ihab Hassan

Classical Moonshine describes the remarkable phenomenon that the coefficients of Hauptmoduln are graded traces for the action of the Monster group, the largest of the 26 sporadic groups, on a graded infinite-dimensional module. A similar phenomena has been shown to hold for other sporadic groups, particularly the Mathieu group M_{24} where instead of Hauptmoduln, the graded traces are shown to be coefficients of mock modular forms. In a recent paper, Ono, Rolen, and Trebat-Leder relate a monstrous moonshine function to one of the Mathieu moonshine functions by constructing products of rational functions of the monstrous Hauptmodul via *generalized Borcherds lifts* on mock modular forms. The lifting procedure is that introduced in a paper by Bruinier-Ono. We conjecture a generalization of this lift for all the mock modular forms of Mathieu moonshine. In particular, our generalization relates Mathieu moonshine mock modular forms to the modular forms of Conway moonshine by evaluating Heegner points corresponding to various congruence subgroups of the modular group. We present data that supports our conjecture.

Mathieu Moonshine: Mock Modular Lifts

By

Morad Ihab Hassan

Advisor: John Duncan, PhD

A thesis submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Master of Science
in Mathematics
2018

Acknowledgement

I would like to thank my advisor Professor John Duncan for his support in my producing this thesis. Without his help and guidance, I would have never been able to accomplish what I have. I am also grateful to Professors David Zureick-Brown and Shanshuang Yang who have been there to help me shape my thoughts, both mathematical and otherwise. Finally, thank you Emory University, and specifically, the Math and Computer Science Department for providing me the opportunity to grow in the ways I have and for allowing me to indulge in this incredible pursuit.

Contents

1 Introduction **1**
1.1 Monstrous Moonshine . . . 1
1.2 Mathieu Moonshine . . . 2
1.3 Conway Moonshine . . . 5

2 Our Conjecture **6**
2.1 Calculating Heegner Points . . . 11

3 Generalized Borcherds Products **15**
3.1 A Lattice Related to $\Gamma_0(2)$ . . . 15
3.2 Weil Representation . . . 17
3.3 Vector Valued Modular Forms . . . 18
3.4 Lifting H_e . . . 20
3.5 Generalized Borcherds Products . . . 22

4 Definitions Modular Forms, Jacobi Forms, Theta Functions, etc. **23**
4.1 Mathieu (Mock) Modular Functions . . . 24
4.2 Theta Functions . . . 25

List of Tables

1 Cycle Shapes and Eta-functions attached to M_{24} . . . 3
2 Modular Subgroups attached to elements of M_{24} . . . 12
3 Quadratic representatives given $\Delta = -7$ . . . 14
4 Quadratic representatives given $\Delta = -15$ . . . 14
5 Mathieu Moonshine Mock Modulars . . . 24

1 Introduction

1.1 Monstrous Moonshine

Monstrous Moonshine refers to the surprising connection between the representation theory of the largest sporadic simple group G, known as the *monster group*, and functions which are invariant under the action of certain subgroups of $SL_2(\mathbb{R})$ known as *modular functions*. For example, taking the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$ and $\gamma \in \Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N|c\}$, we define the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$$

Then, a meromorphic function $f : \mathbb{H} \mapsto \mathbb{C}$ is a modular function for $\Gamma_0(N)$ if $f(\tau) = f(\gamma\tau)$ where $\gamma \in \Gamma_0(N)$. We refer to N as the level of the modular subgroup.

This connection began between the irreducible representations of the monster G and the normalized modular Klein j-function:

$$J(\tau) := j(\tau) - 744 = \frac{(1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n)^3}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}} - 744 = \frac{1}{q} + 196884q + 21493760q^2 + \dots \quad (1)$$

where John McKay saw the first coefficient of the Klein j-function's Fourier expansion could be written as a linear combination of the dimensions of the irreducible representations of the G. Then Thompson noticed the same was true with other coefficients as well such as

$$1 + 196883 + 21296876 = 21493760.$$

In Thompson [1979a], Thompson conjectured the existence of a special infinite-dimensional graded representation $V^\natural$ of G for which the dimension of the grade n part was the n-th Fourier coefficient of $J(\tau)$.

This conjecture was further expanded by Thompson, Conway, and Norton in Conway [1979/80], Thompson [1979b] who considered this connection for arbitrary $g \in G$ and conjectured that there existed a representation $V^\natural$ for whose McKay-Thompson series, written,

$$\tilde{T}_g(\tau) := \sum_{n=-1}^{\infty} tr(g|V_n^\natural)q^n, \tag{2}$$

were principal moduli, or Hauptmoduln, for certain genus zero groups commensurable with $SL_2(\mathbb{Z})$. Conway and Norton's conjectures were proved in full by Richard Borcherds Borcherds [1998] in 1992 through developing the theory of vertex operator algebras Borcherds [1986].

1.2 Mathieu Moonshine

A similar phenomena occurs between certain mock modular forms and the Mathieu group M_{24}. The observation made in Eguchi et al. [2011] was given the q-series

$$H(\tau) = 2q^{\frac{-1}{8}}(-1 + 45q + 231q^2 + \ldots) = q^{\frac{-1}{8}}\left(-2 + \sum_{n=1}^{\infty} t_n q^n\right),$$

the first few coefficients $45, 231, 770, \ldots$ are in fact the dimensions of irreducible representations of M_{24}.

We say a holomorphic function $h(\tau)$ on $\mathbb{H}$ is a *weakly holomorphic mock modular form of weight* w for a discrete subgroup Γ of $SL_2(\mathbb{Z})$ if it has at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$, and if there exists a homolorphic modular form $f(\tau)$ of weight $2 - w$ on Γ such that the *completion* $\hat{h}(\tau)$, given by

$$\hat{h}(\tau) = h(\tau) + (4i)^{w-1} \int_{-\overline{\tau}}^{\infty} (z + \tau)^{-w} \overline{f(-\overline{z})} dz, \tag{3}$$

is a (non-holomorphic) modular form of weight w for Γ for some multiplier system ψ. In

Table 1: The cycle shapes and attached *eta-functions* of the 26 conjugacy classes of the sporadic group M_{24}. The naming of conjugacy classes follows the ATLAS convention Conway et al. [1985]

[g]	cycle shape	$\eta_g(\tau)$
1A	1^{24}	$\eta(\tau)^{24}$
2A	$1^8 2^8$	$\eta(\tau)^8\eta(2\tau)^8$
2B	2^{12}	$\eta(2\tau)^{12}$
3A	$1^6 3^6$	$\eta(\tau)^6\eta(3\tau)^6$
3B	3^8	$\eta(3\tau)^8$
4A	$2^4 4^4$	$\eta(2\tau)^4\eta(4\tau)^4$
4B	$1^4 2^2 4^4$	$\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4$
4C	4^6	$\eta(4\tau)^6$
5A	$1^4 5^4$	$\eta(\tau)^4\eta(5\tau)^4$
6A	$1^2 2^2 3^2 6^2$	$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$
6B	6^4	$\eta(6\tau)^4$
7AB	$1^3 7^3$	$\eta(\tau)^3\eta(7\tau)^3$
8A	$1^2 2^1 4^1 8^2$	$\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$
10A	$2^2 10^2$	$\eta(2\tau)^2\eta(10\tau)^2$
11A	$1^2 11^2$	$\eta(\tau)^2\eta(11\tau)^2$
12A	$2^1 4^1 6^1 12^1$	$\eta(2\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$
12B	12^2	$\eta(12\tau)^2$
14AB	$1^1 2^1 7^1 14^1$	$\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$
15AB	$1^1 3^1 5^1 15^1$	$\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$
21AB	$3^1 21^1$	$\eta(3\tau)\eta(21\tau)$
23AB	$1^1 23^1$	$\eta(\tau)\eta(23\tau)$

this case, the function f is called the *shadow* of the mock modular form h and ψ is called the multiplier system of h. For more detail, see Cheng and Duncan [2012].

This essentially states that a mock modular form can be adjusted by its *shadow* and multiplier system to obtain a modular form but loses its holomorphicity in the process.

It has been shown that the function $H(\tau)$ is a *weakly holomorphic mock modular form of weight* 1/2 on $SL_2(\mathbb{Z})$ with *shadow* $24\eta(\tau)^3$ Eguchi and Hikami [2009], Dabholkar et al. [2012]. This is very much like the case of Monstrous Moonshine. This occurrence of information being encoded in a function which expressed modular(-like) properties led to a similar exploration of moonshine with the M_{24} group and conjectures for the existence of certain infinite dimensional representations of M_{24} that will behave interestingly for all $g \in M_{24}$.

The McKay-Thompson series for each $g \in M_{24}$ were proposed and are listed in Section 4.1. We arrive to a conjecture similar to that of Conway and Norton: does there exist a $K = \oplus_{n=0}^{\infty} K_n$ module of M_{24} such that the McKay-Thomspon series, given by

$$H_g(\tau) = q^{\frac{-1}{8}} \left(-2 + \sum_{n=1}^{\infty} tr(g|K_{8n-1}) q^n \right), \tag{4}$$

also display special mock modular properties? Terry Gannon confirms the existence of such a module in Gannon [2016].

We conclude this subsection by recalling properties of M_{24} which are relevant for our results. The group M_{24} may be characterized as the automorphism group of the unique doubly even self-dual binary code of length 24 with no words of weight 4, also known as the *(extended) binary Golay Code* (see Cheng and Duncan [2012] for more information). This perspectives permits M_{24} a natural permutation representation of degree 24 which we denote R. Via R, we may assign cycle shapes to each of its elements. Any cycle shape arising from an element of M_{24} is of the form $i_1^{l_1} i_2^{l_2} \cdots i_r^{l_r}$ where

$$\sum_{s=1}^{r} l_s i_s = 24,$$

for some $l_s \in \mathbb{N}$ and $1 \leq i_1 < \ldots < i_r \leq 23$ with $r \geq 1$. To each of the elements $g \in M_{24}$, we can attach an *eta-product*, to be denoted η_g, which is the function on the upper half-plane given by

$$\eta_g(\tau) := \prod_s \eta(i_s \tau)^{l_s} \tag{5}$$

where $\prod_s i_s^{l_s}$ is the cycle shape attached to g, and $\eta(\tau)$ is the Dedekind eta function defined as $\eta(\tau) := q^{1/24} \prod_{n>0} (1 - q^n)$ for $q(\tau) := e(\tau)$ where $e(x) := e^{2\pi i x}$.

1.3 Conway Moonshine

We direct the reader's attention to one more significant case of moonshine, Conway Moonshine. Similar to Mosntrous Moonshine, in Conway [1979/80] Conway and Norton also describe an assignment of genus zero groups $\Gamma_g < SL_2(\mathbb{R})$, to elements g of the Conway group, C_{O_0}. In addition to this assignment, several important results display the qualities of Conway moonshine that are analogues to Monstrous moonshine. The first is recalled from Conway [1979/80], Queen [1981].

Theorem 1.1. *For any $g \in C_{O_0}$, regarded as a subgroup of $SO(\alpha)$, the function*

$$t_g := \frac{\eta_g(\tau)}{\eta_g(2\tau)} \tag{6}$$

is a principal modulus for a genus zero group $\Gamma_g < SL_2(\mathbb{R})$ containing some $\Gamma_0(N)$.

Here each η_g is an attachment similar to one described earlier in Mathieu moonshine. In fact, we will soon see that our choice to attach η_g products to M_{24} was partly inspired by Conway moonshine. We note C_{O_0} is not representable as a permutations on 24 elements, so the choice of $\eta_g(\tau)$ is determined by what is referred to in Duncan and Mack-Crane [2015] as the *Frame shape* of g.

Remark 1.2. It is a fact M_{24} is a subgroup of C_{O_0} (which can be remembered by reading the exposition in Duncan and Mack-Crane [2015]). Interestingly, our assignment η_g for some element $g \in M_{24}$ is equal to the assignment of η_g to this very same g as an element of C_{O_0}.

Let T_g^s be defined as

$$T_g^s := t_g(\tau/2) + C_g, \tag{7}$$

where C_g is constant added to normalize the function, or remove the non-vanishing constant term. Then, Duncan and Mack-Crane show in Duncan and Mack-Crane [2015] that a $\frac{1}{2}\mathbb{Z}$-graded infinite-dimensional C_{O_0} module, $V^{s\natural} = \bigoplus_{n\geq 0} V^{s\natural}_{n/2}$, can be constructed which obtains

the coefficients of these T_g^s's through the module's McKay-Thompson series. Furthermore, each of these T_g^s's are special, in that $T_g^s(2\tau)$ is the unique normalized principal modulus attached to the genus zero group Γ_g. As we will see, these T_g^s's will be directly involved in our research.

2 Our Conjecture

We propose generalizations of the generalized Borcherds lifts introduced in Bruinier and Ono [2010] and used in Ono et al. [2015] to construct meromorphic modular functions on $\Gamma_0(N_g)$ from certain mock modular forms arising from umbral moonshine. We conjecture an extension of this lift to the special case of Mathieu moonshine mock modulars, and in doing so, we establish a connection between the Mathieu group's mock-modular graded twists and rational products of modular functions arising from Conway moonshine.

To precisely state our generalization, we set up the following notation. Let Δ be a negative *fundamental discriminant* and $r^2 = \Delta \pmod 8$. Let the *order* of g be denoted by $ord(g)$. Note that a *fundamental discriminant* Δ is an integer such that $\Delta = 1 \pmod 4$ and is square-free or $D = 4m$, where $m = 2$ or $3 \pmod 4$ and m is square-free. Then $\Psi_{\Delta,r}$ is a function which takes H_g and produces a modular form, referred to as the *Lift* of H_g and defined in the following way,

$$\Psi_{\Delta,r}(\tau, H_g) := \prod_{n=0}^{\infty} P_{\Delta,r}(q^n, g), \tag{8}$$

for

$$P_{\Delta,r}(x, g) := \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} \exp\left(\ell_{\Delta,r}\left(e(b/\Delta)x, g\right)\left(\tfrac{\Delta}{b}\right)\right), \tag{9}$$

where

$$\ell_{\Delta,r}(y,g) := \sum_{k=1}^{\infty}(-1)^k \frac{y^k}{k} c_{g^k}(|\Delta|n^2, rn), \tag{10}$$

and $c_g(|\Delta|n^2, rn)$ is defined by

$$c_g(|\Delta|n^2, rn) := \begin{cases} tr(g|K_{|\Delta|n^2}) & \text{if } rn \equiv 1 \bmod 4, \\ -tr(g|K_{|\Delta|n^2}) & \text{if } rn \equiv -1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

This notation is lengthy, yet as we will see, is rather simple to follow in practice. Before continuing, We provide an example calculation for $\Psi_{\Delta,r}$ for an arbitrary (Δ, r).

Example 2.1. Let $[g] = 1A$, then we have $ord(g) = 1$. We have

$$\begin{aligned} \ell_{\Delta,r}(y,e) :=& \sum_{k=1}^{\infty}(-1)^k \frac{y^k}{k} c_{g^k}(|\Delta|n^2, rn) = \sum_{k=1}^{\infty}(-1)^k \frac{y^k}{k} c_e(|\Delta|n^2, rn) \\ =& c_e(|\Delta|n^2, rn)\log(1-y), \end{aligned}$$

and

$$\begin{aligned} P_{\Delta,r}(x,e) :=& \prod_{b\in\mathbb{Z}/|\Delta|\mathbb{Z}} \exp\left(\ell_{\Delta,r}\left(e(b/\Delta)x, e\right)\left(\tfrac{\Delta}{b}\right)\right) \\ =& \prod_{b\in\mathbb{Z}/|\Delta|\mathbb{Z}} (1-e(b/\Delta)x)^{c_e(|\Delta|n^2,rn)\left(\frac{\Delta}{b}\right)}. \end{aligned}$$

Thus,

$$\Psi_{\Delta,r}(\tau, H_e) := \prod_{n=0}^{\infty} P_{\Delta,r}(q^n, e)$$

$$= \prod_{n=0}^{\infty} \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} (1 - e(b/\Delta)q^n)^{c_e(|\Delta|n^2, rn)\left(\frac{\Delta}{b}\right)}$$

Now given a conjugacy class $[g]$, we define T_g in the following way,

$$T_g := \frac{\eta_g(\tau)}{\eta_g(2\tau)} + \chi(g), \tag{12}$$

where $\chi(g)$ is a constant added to normalize the rational form which can be found in Table 5. By defining T_g in this way, it is a meromorphic modular form on some discrete subgroup commensurable with $SL_2(\mathbb{Z})$ Duncan and Mack-Crane [2015]. We call a *Heegner point* a complex number the form $\alpha = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$ with $(a, b, c) = 1$, $N_g|a$ for $\Gamma_0(N_g)$, and $\Delta = b^2 - 4ac$ where Δ is a fundamental discriminant. Finally, we call a pair (Δ, r) *admissible* for g if Δ is a negative fundamental discriminant and $r^2 = \Delta \pmod{4N_g}$. Now, we can present our conjecture:

Conjecture 2.2. *Given an admissible pair (Δ, r) for conjugacy class $[g]$, then $\Psi_{\Delta,r}(\tau, H_g)$ satisfies the following:*

$$\Psi_{\Delta,r}(\tau, H_g) = \prod_{i=1}^{m} \frac{(T_g(\tau) - T_g(\alpha_i))^{\gamma_i}}{(T_g(\tau) - \overline{T_g(\alpha_i)})^{\gamma_i}}, \tag{13}$$

where each α_i is a satisfactory Heegner Point.

As a result of the conjecture, we have the following corollary.

Corollary 2.3. *The twisted graded trace satisfies the equality*

$$\frac{1}{\epsilon_\Delta} \sum_{i=1}^{m} \gamma_i (T_g(\alpha_i) - \overline{T_g(\alpha_i)}) = tr(g|K_\Delta), \tag{14}$$

with

$$\epsilon_\Delta = \sum_{b\in\mathbb{Z}/|\Delta|\mathbb{Z}} e(b/\Delta)\cdot\left(\frac{\Delta}{b}\right) \tag{15}$$

where α_i is the same Heegner point and γ_i is some integer.

Remark 2.4. This conjecture is inspired by an example calculation in Ono et al. [2015] for $[g] = 1A$ which utilizes the generalized Borcherds products formed in Bruinier and Ono [2010] to lift a weak harmonic Maass form of weight 1/2 to a meromorphic modular functions. We present an example calculation for $[g] = 2A$.

Example 2.5. Let $[g] = 2\text{A}$. Then the corresponding modular function is

$$T_g = \frac{\eta(\tau)^8}{\eta(4\tau)^8} + 8.$$

Picking admissible pair $(\Delta, r) = (-7, 1)$, then

$$\begin{aligned}\Psi_{-7,1}(\tau, H_g) =& \frac{(T_g(\tau) - T_g(\alpha))^2}{(T_g(\tau) - \overline{T_g(\alpha)})^2}\\ =& 1 + 6\sqrt{-7}q + (126 + 45\sqrt{-7})q^2 + \cdots,\end{aligned}$$

where $\alpha = \frac{3+\sqrt{-7}}{8}$.

We check Corollary 2.3,

$$\epsilon_\Delta = \sum_{b\in\mathbb{Z}/|-7|\mathbb{Z}} e(-b/7)\left(\frac{-7}{b}\right) = -\sqrt{-7}$$

and

$$\frac{1}{\epsilon_\Delta}\sum_i \gamma_i T_g(\alpha_i) = 6 = c_g(7,1) = tr(g|K_{7/8}).$$

We provide another example here with a different fundamental discriminant $\Delta = -15$.

The case when $[g] = 1\text{A}$ and $(\Delta, r) = (-15, 1)$ was already done in Ono et al. [2015], so again we proceed to the next case where $[g] = 2\text{A}$.

Example 2.6. Let $[g] = 2\text{A}$ and $(\Delta, r) = (-15, 1)$. Then

$$\Psi_{-15,1}(\tau, H_g) = \frac{(T_g(\tau) - T_g(\alpha_1))^2 (T_g(\tau) - T_g(\alpha_2))^2}{(T_g(\tau) - \overline{T_g(\alpha_1)})^2 (T_g(\tau) - \overline{T_g(\alpha_2)})^2}$$

$$= 1 + 14\sqrt{-15}q + (-1470 + 231\sqrt{-15})q^2 + \cdots,$$

where $\alpha_1 = \frac{-1+\sqrt{15}}{8}$ and $\alpha_2 = \frac{7+\sqrt{-15}}{16}$.

We get

$$\epsilon_\Delta = \sqrt{-15},$$

and

$$\frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 14 = c_g(15, 1) = tr(g|K_{1,15/8}).$$

Remark 2.7. Our ability to follow through with these computations relies on computing satisfactory Heegner points. Thus, given $[g]$, our conjecture only concerns pairs (Δ, r) for which Heegner points exist. For example, no Heegner points exist for $15A$ given the admissible pair $(-7, 1)$, yet they do exist for this conjugacy class when we set the admissible pair to $(-15, 1)$ (see Tables 3 and 4).

Remark 2.8. There does not seem to be any conjugacy class for which there does not exist Heegner points for all admissible pairs. Given the data presented in Tables 3 and 4, the most likely candidate for such a choice is $21A$. But even that class will admit a quadratic representative for the pair $(-47, 1)$ and, as a result, admit some collection of Heegner points.

2.1 Calculating Heegner Points

Using Theorem 2.9 and given an admissible pair, we can calculate the quadratic forms which contain Heegner points as its roots.

Consider integral binary quadratic forms $[a, b, c](x, y) = ax^2 + bxy + cy^2$. Then $\Gamma_0(N)$ acts on such forms as follows:

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x, y) = [a, b, c](\alpha x + \beta y, \gamma x + \delta y).$$

This action preserves the discriminant $\Delta = b^2 - 4ac$ and the greatest common divisor (a, b, c). We denote $\mathcal{Q}_\Delta$ and $\mathcal{Q}^0_\Delta$ to be the set of all quadratic forms of discriminant Δ and the subset of primitive forms, respectively. When defining a classification with respect to the subgroup $\Gamma_0(N)$, more invariants arise. Specifically, under the action by $\Gamma_0(N)$, the value of b modulo $2N$ which we shall denote as r remains unchanged and we always have $gcd(a, N) = N$. Then, for an integer r modulo $2N$ such that $\Delta = r^2 \pmod{4N}$, we set

$$\mathcal{Q}_{N,\Delta,r} = \{[a, b, c] \in \mathcal{Q}_\Delta : N|a, b = r \pmod{2N}\}$$

Then the subset of primitive forms, $\mathcal{Q}^0_{N,\Delta,r}$ is $\Gamma_0(N)$ invariant. We denote the quadratic forms from here on as $[aN, b, c]$.

We define $m := \left(N, r, \frac{r^2-\Delta}{4N}\right)$. For $[aN, b, c] \in \mathcal{Q}^0_{N,\Delta,r}$, define $m_1 := (N, b, a)$, $m_2 := (N, b, c)$, which are coprime and have product m. Then, the following is true Gross et al. [1987]:

Theorem 2.9. *There is a 1:1 correspondence between $\Gamma_0(N)$ equivalence classes of the forms $[aN, b, c] \in \mathcal{Q}^0_{N,\Delta,r}$ satisfying $(N, b, c) = m_1$, $(N, a, b) = m_2$ and the $SL_2(\mathbb{Z})$ equivalence classes of the forms in $\mathcal{Q}^0_{\Delta,r}$ given by $\mathcal{Q} = [aN, b, c] \mapsto \mathcal{Q}' = [aN_1, b, cN_2]$, where N_1N_2 is any decomposition of N into coprime positive factors satisfying $(m_1, N_2) = (m_2, N_1) = 1$. In*

$[g]_{M_{24}}$	Γ_g	$[g]_{C_{O_0}}$	Frame Shape
1A	$2-$	1A	1^{24}
2A	$4-$	2B	$1^8 2^8$
2B	$4\|2-$	2D	2^{12}
3A	$6+3$	3B	$1^6 3^6$
3B	$6\|3$	3D	3^8
4A	$8\|2-$	4G	$2^4 4^4$
4B	$8-$	4E	$1^4 2^2 4^4$
4C	$8\|4-$	4H	4^6
5A	$10+5$	5B	$1^4 5^4$
6A	$12+3$	6K	$1^2 2^2 3^2 6^2$
6B	$12\|6-$	6P	6^4
7AB	$14+7$	7B	$1^3 7^3$
8A	$16-$	8G	$1^2 2^1 4^1 8^2$
10A	$20\|2+5$	10J	$2^2 10^2$
11A	$22+11$	11A	$1^2 11^2$
12A	$24\|2+3$	12P	$2^1 4^1 6^1 12^1$
12B	$24\|12-$	12S	12^2
14AB	$28+7$	14C	$1^1 2^1 7^1 14^1$
15AB	$30+5, 5, 15$	15D	$1^1 3^1 5^1 15^1$
21AB	$42\|3+7$	21C	$3^1 21^1$
23AB	$46+23$	23AB	$1^1 23^1$

Table 2: Shows the modular subgroups corresponding to each $g \in M_{24}$. Note that since $M_{24} < C_{O_0}$, then we are able to look at Duncan and Mack-Crane [2015] to determine the appropriate modular subgroup. The modular subgroups are meant to be read $\Gamma_g(N_g)$ where the column below Γ_g lists the level N_g. Definitions for each of the levels can be found in Duncan and Mack-Crane [2015].

particular, $|\mathcal{Q}^0_{N,\Delta,r}/\Gamma_0(N)| = 2^v|\mathcal{Q}^0_{N,\Delta,r}/SL_2(\mathbb{Z})|$*, where* v *is the number of prime factors of* m*.*

Note that $2^v|\mathcal{Q}^0_{N,\Delta,r}/SL_2(\mathbb{Z})| = 2h(\Delta)$ for $\Delta < 0$, where $h(\Delta)$ is the class number of $\mathbb{Q}(\sqrt{\Delta})$ the factor 2 arises because $\mathcal{Q}^0_\Delta$ also contains negative semi-definite forms.

Thus, we obtain a simple method for constructing the quadratic forms that hold the desired Heegner points. Given Δ, we solve for r satisfying the condition $\Delta \equiv_{4N} r^2$. Then we calculate b such that $b \equiv_{2N} r$, and all $(Na \cdot c)$ such that $\frac{b^2-\Delta}{4} = aN \cdot c$. To reconstruct a polynomial given $[g], \Delta$, we can look at the tables below. We can select any pair, but we'd need select a, c such that the conditions defined above are satisfied.

Remark 2.10. By Duncan and Mack-Crane [2015], there is a Frame shape for each $g \in C_{O_0}$ which can be expressed in a fashion similar to the cycle shapes from the permutation

representation of elements in M_{24}. For each frame shape, the same T_g function is defined and attached to each $g \in C_{O_0}$. Since M_{24} is a subgroup of C_{O_0}, we can naturally associate each frame shape to each $g \in M_{24}$ which in turn enables us to attach a T_g to each $g \in M_{24}$. It is proven in Duncan and Mack-Crane [2015] that each of these T_g are principal modulus for a genus zero group $\Gamma_g < SL_2(\mathbb{Z})$. For this reason, we can tell the level of each of these subgroups by referring to Table 2.

Example 2.11. Let $[g] = 2A$ where $\Delta = -7$, we look to Table 3 below for viable representations on $\Gamma_0(4)$: $Q_{\pm 1} = 4x^2 \pm 3x + 1, Q_{\pm 2} = 8x^2 \pm 5x + 1, Q_{\pm 3} = 4x^2 \pm 5x + 2$. There are, however, only 2 quadratic forms for $[2A]$ up to equivalence since $m = 1$ and so $|\mathcal{Q}^0_{4,-7,1}/\Gamma_0(4)| = 2$. Extracting the roots with positive imaginary components from these polynomials and inputting them into $T_{2A}(\tau)$ determines the uniqueness of the polynomial. That is, only inequivalent representatives will provide roots that produce distinctive outputs. A quick calculation will show that $Q_{\pm 1}$ may serve as our polynomial representatives for [2A] and provides us with Heegner points $\alpha_{\pm} = \frac{\pm 3 + \sqrt{-7}}{8}$ which we used in our earlier example.

Remark 2.12. Not all conjugacy classes will necessarily have a quadratic representatives given Δ and N. If there is no such r, then $\mathcal{Q}_{N,\Delta,r}$ is empty. For example, when $[g] = 3A$ and $(\Delta, r) = (-7, 1)$, there are no quadratic representatives.

Remark 2.13. We only calculate the Heegner points for conjugacy classes whose cycle shapes fixed one point. That is, whenever the cycle shape contains a $(1)^l$ for some positive integer l. We do this because the modular subgroups are considerably simpler as can be seen above in Table 2.

Remark 2.14. To verify the choice of polynomial, it has always been fruitful to check Corollary 3.3 with the provided Heegner points.

Table 3: Quadratic representatives given $\Delta = -7$

$[g]_{M_{24}}$	$[r_{-7}]$	$\{(b, aN \cdot c)_{-7}\}$
1A	$[1, 3, 5, 7]$	$\{(1, 2), (3, 4)\}$
2A	$[3, 5, 11, 13]$	$\{(3, 4), (5, 8)\}$
2B		
3A	-	-
3B	-	-
4A	$[5, 11, 21, 27]$	$\{(5, 8), (11, 32)\}$
4B		
4C		
5A	-	-
6A	-	-
6B	-	-
7AB	$[7, 21, 35, 49]$	$\{(7, 14), (21, 112)\}$
8A	$[11, 21, 43, 53]$	$\{(11, 32), (21, 112)\}$
10A	-	-
11A	$[9, 13, 31, 35, 53, 57, 75, 79]$	$\{(9, 22), (13, 44), (31, 242), (35, 308)\}$
12A	-	-
12B	-	-
14AB	$[21, 35, 77, 91]$	$\{(21, 112), (35, 308)\}$
15AB	-	-
21AB	-	-
23AB	$[19, 27, 65, 73, 111, 119, 157, 165]$	$\{(19, 92), (27, 184), (65, 1058), (73, 1334)\}$

Table 4: Quadratic representatives given $\Delta = -15$

$[g]_{M_{24}}$	$[r_{-15}]$	$\{(b, aN \cdot c)\}$
1A	$[1, 3, 5, 7]$	$\{(1, 4), (3, 6)\}$
2A	$[1, 7, 9, 15]$	$\{(1, 4), (7, 16)\}$
2B		
3A	$[3, 9, 15, 21]$	$\{(3, 6), (9, 24)\}$
3B		
4A	$[7, 9, 23, 25]$	$\{(7, 16), (9, 24)\}$
4B		
4C		
5A	$[5, 15, 25, 35]$	$\{(5, 10), (15, 60)\}$
6A	[9,15,33,39]	$\{(9, 24), (15, 60)\}$
6B		
7AB	-	-
8A	$[7, 25, 39, 57]$	$\{(7, 16), (25, 160)\}$
10A	-	-
11A	-	-
12A	$[9, 39, 57, 87]$	$\{(9, 24), (39, 384)\}$
12B		
14AB	-	-
15AB	$[15, 45, 75, 105]$	$\{(15, 60), (45, 510)\}$
21AB	-	-
23AB	$[13, 33, 59, 79, 105, 125, 151, 171]$	$\{(13, 46), (33, 276), (59, 874), (79, 1564)\}$

3 Generalized Borcherds Products

Borcherds Products or *Borcherds Lifts* is a method to construct meromorphic modular forms on the special orthogonal group, $SO^+(2, n^-)$ from weakly holomorphic modular forms on $SL_2(\mathbb{Z})$. That is, given a weakly holomorphic modular form f, there is an associated Ψ_f which is a convergent infinite product of complex valued functions which display modular properties and are holormorphic except for poles. A succinct, yet revealing account of Borcherds Products can be found in Hill [2012].

The *Generalized Borcherds Products* are defined in terms of vector valued weak harmonic Maass forms in Bruinier and Ono [2010] and are used to to prove a special case of our conjecture for $[g] = 1A$ in Ono et al. [2015]. Its purpose is to take a *harmonic weak maass forms* of weight 1/2 and produce a meromorphic modular form. Following the examples of the papers previously mentioned, we will also provide the background on vector-valued modular forms and follow closely the results presented in Ono et al. [2015], Bruinier and Ono [2010], but in less generality as to only satisfy the lift for H_e. See Bruinier and Ono [2010] for more general levels and functional equations.

3.1 A Lattice Related to $\Gamma_0(2)$

We define a lattice L and a dual lattice L' related to $\Gamma_0(2)$ such that the components of our vector-valued modular forms will be labeled by the elements of L'/L Ono et al. [2015], Bruinier and Ono [2010]. This lattice L should be even. We consider the quadratic space

$$V := \{X \in Mat_2(\mathbb{Q}) : tr(X) = 0\}$$

with the quadratic form $P(X) := 2det(X)$. The, the corresponding bilinear form is $(X, Y) := -2tr(XY)$. Let L be the lattice

$$L := \left\{ \begin{pmatrix} b & -a/2 \\ c & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

The dual lattice is then given by

$$L := \left\{ \begin{pmatrix} b/4 & -a/2 \\ c & -b/4 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

It will be helpful to view elements of L' as both matrices and as quadratic forms, with the matrix

$$X = \begin{pmatrix} b/4 & -a/2 \\ c & -b/4 \end{pmatrix}$$

corresponding to the integral binary quadratic form

$$Q = [2c, b, a] = 2cx^2 + bxy + ay^2.$$

Note that $P(X) = -Disc(Q)/8 = -(b^2 - 8ac)/8$. We refer to the finite abelian group L'/L as a *discriminant group* and as the *discriminant form* of the lattice L when L'/L has a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form induced by P. We can identify L'/L with $\frac{1}{4}\mathbb{Z}/\mathbb{Z}$ by defining the isomorphism $\phi : L'/L \mapsto \frac{1}{4}\mathbb{Z}/\mathbb{Z}$ where

$$\phi\left(\begin{pmatrix} b/4 & -a/2 \\ c & -b/4 \end{pmatrix} + L \right) = b/4 + \mathbb{Z}.$$

The quadratic form P with the quadratic form $\frac{h}{4} \to \frac{-h^2}{8}$ on $\mathbb{Q}/\mathbb{Z}$. We will also occasionally identify $\frac{h}{4} \in \mathbb{Q}/\mathbb{Z}$ with $h \in \mathbb{Z}/4\mathbb{Z}$. For a fundamental discriminant D and $r/4 \in L'/L$ with

$r^2 = D \ (mod\ 8)$, let

$$Q_{D,r} := \{Q = [2c, b, a] : a, b, c \in \mathbb{Z}, Disc(Q) = D, b = r \pmod 4\}.$$

The action of $\Gamma_0(2)$ on this set is given by the usual action of congruence subgroups on binary quadratic forms: Write integral binary quadratic forms $[a, b, c](x, y) := ax^2+bxy+cy^2$. Then $\Gamma_0(N)$ acts on such forms as follows

$$[a, b, c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x, y) = [a, b, c](\alpha x + \beta y, \gamma x + \delta y).$$

3.2 Weil Representation

We write $Mp_2(\mathbb{R})$ for the metaplectic two-fold cover of $SL_2(\mathbb{R})$. The elements of this group are pairs $(M, \phi(\tau))$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\phi : \mathbb{H} \mapsto \mathbb{C}$ is a holomorphic function such that $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

We denote the integral metaplectic group, the inverse image of $\Gamma : SL_2(\mathbb{Z})$ under the covering map, by $\widetilde{\Gamma} := Mp_2(\mathbb{Z})$.

The generators of $\widetilde{\Gamma}$ are $T := (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$ and $S := (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau})$. Here, $\sqrt{\tau} = \sqrt{|\tau|}e^{\arg(\tau)i}$ for $-\pi < \arg(\tau) < \pi$ where $|\cdot|$ is the complex modulus.

The Weil Representation associated with a discriminant form L'/L is a representation of $\widetilde{\Gamma}$ on the group algebra $\mathbb{C}[L'/L]$. We denote the standard basis elements of $\mathbb{C}[L'/L]$ by ν_h, $h \in L'/L$, and write $\langle \cdot, \cdot \rangle$ for the standard scalar product such that $\langle \nu_h, \nu_{h'} \rangle = \delta_{h,h'}$. The Weil representation ρ_L associated with the discriminant form L'/L is the unitary representation of $\widetilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)(\nu_h) := e(h^2/2)\nu_h$$

and

$$\rho_L(S)(\nu_h) := \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h, h'))\nu_{h'},$$

where (b^+, b^-) is the signature of the vector space V defined earlier. That is, b^+ is the dimension of a maximal positive definite subspace of V and b^- is the dimension of a maximal negative definite subspace Here $e(x) := e^{2\pi i x}$.

3.3 Vector Valued Modular Forms

If $f : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is a function, then we can write $f = \sum_{h \in L'/L} f_h \nu_h$ for its decomposition into components with respect to the standard basis of $\mathbb{C}[L'/L]$. Let $k \in \frac{1}{2}\mathbb{Z}$, and let $M^!_{k,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$-valued weakly holomorphic modular forms. Modular forms are denoted by M_{k,ρ_L} and cusp forms are denoted by S_{k,ρ_L}. Cusp forms are modular forms which vanish at cusps.

Assume$k \leq 1$, and a twice continuously differentiable function $f : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form* (of weight k with respect to $\widetilde{\Gamma}$ and ρ_L) if it satisfies:

- $f(\gamma\tau) = \phi(\tau)^{2k}\rho_L(\gamma, \phi)f(\tau)$ for all $(\gamma, \phi) \in \widetilde{\Gamma}$,
- there is a $C > 0$ such that $f(\tau) = O(e^{Cw})$ as $w \mapsto \infty$,
- $\Delta_k f = 0$,

where

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial u}\right)$$

is the weight k hyperbolic Laplace operator.

We denote the vector space of these harmonic weak Maass forms by $\mathcal{H}_{k,\rho_L}$. Moreover, for $f \in \mathcal{H}_{k,\rho_L}$ we have the associated Fourier expansion

$$f(\tau) = P_f(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ -\infty \ll n \leq 0}} c^+(n,h) e(n\tau) \nu_h$$

and some $\epsilon > 0$. Here, P_f is uniquely determined by f and is called the *principal part* of f. The Fourier expansion of any $f \in H_{k,\rho_L}$ gives a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n,h) e(n\tau) \nu_h,$$

$$f^-(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n < 0}} c^-(n,h) W(2\pi n w)(e(n\tau) \nu_h,$$

and $W(x) = W_k(x) := \int_{-2x}^{\infty} e^{-t} t^{-k} dt = \Gamma(1-k, 2|x|)$ for $x < 0$.

Note that there is an antilinear differential operator defined by

$$\xi_k : H_{k,\rho_L} \mapsto S_{2-k,\overline{\rho_L}}, f(\tau) \mapsto \xi_k(f)(\tau) := 2iy^k \frac{\overline{\partial}}{\partial \overline{\tau}},$$

where $\overline{\rho_L}$ is the complex conjugate representation. This will be important in the following subsection. The Fourier expansion of $\xi_k(f)$ is given by

$$\xi_k(f) = -\sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n > 0}} (4\pi n)^{1-k} \overline{c^-(-n,h)} e(n\tau) \nu_h$$

The kernel of ξ_k is equal to $M^!_{k,\rho_L}$, and we have the following sequence:

$$0 \mapsto M^!_{k,\overline{\rho_L}} \mapsto H_{k,\overline{\rho_L}} \mapsto S_{2-k,\rho_L} \mapsto 0.$$

We call $\xi_k(f)$ the *shadow* of f. Note that $\xi_k(f)$ uniquely determines f^-, but f^+ is only determined up to addition of a weakly holomorphic modular form.

3.4 Lifting H_e

Here, we use all the earlier definitions and results to display the proof presented in Ono et al. [2015] which shows that H_e's generalized Borcherds Lift is a meromorphic modular form of the type stated earlier in the conjecture. We define a *mock modular form* of weight k to be the holomorphic part of some harmonic weak Maass form of weight k. The following will involve Jacobi forms and the reader may read 4.2 for review on Jacobi forms and definitions of $\varphi_1(\tau, z), \mu_{1,0}(\tau, z), \theta_r(\tau, z)$, and $R(u, \tau)$. We define the Jacobi form

$$\psi(\tau, z) := 2\varphi_1(\tau, z)\mu_{1,0}^g(\tau, z).$$

We can break up ψ into a finite part ψ_F and a polar part ψ_P. The polar part is given by

$$\psi_P(\tau, z) = 24\mu_{2,0}(\tau, z).$$

Then the mock modular form H_e is defined by $H_e := H_{e,1}$ where

$$\psi_F(\tau, z) = \psi(\tau, z) - \psi_P(\tau, z) = \sum_{h \in \mathbb{Z}/4\mathbb{Z}} H_{e,h}(\tau)\theta_{2,h}(\tau, z),$$

where

$$\theta_{2,h}(\tau, z) := \sum_{n=h \pmod 4} q^{n^2/8} y^k.$$

Note that ψ satisfies an optimal growth condition,

$$q^{1/8} H_{e,h}(\tau) = O(1)$$

as $\tau \to i\infty$ for all $h \in \mathbb{Z}/4\mathbb{Z}$.

We also define the shadow $S_e(\tau)$, the non-holomorphic part $F_{e,h}(\tau)$, and the harmonic weak Maass form $\hat{H}_e(\tau)$ corresponding to the mock modular form H_e via their components:

$$\begin{aligned}
S_{e,h}(\tau) :=& \sum_{n=h \pmod 4} nq^{n^2/8}, \\
F_{e,h}(\tau) :=& \int_{-\overline{\tau}}^{i\infty} \frac{S_h(z)}{\sqrt{-i(z+\tau)}} dz \\
=& -4q^{-(h-2)^2/8} R\left(\frac{h-2}{4}(4\tau) + \frac{1}{2}, 4\tau\right), \text{ and} \\
\hat{H}_{e,h}(\tau) :=& H_{e,h}(\tau) + F_{e,h}(\tau).
\end{aligned}$$

Note that by definition, $S_{e,h}(\tau) = S_{e,-h}(\tau)$. Therefore, $S_{e,0} = S_{e,2} = 0$. The same is true of $H_{e,h}$. We can write this in terms of Shimura's theta functions as $S_{e,h}(\tau) = \theta(\tau; h, 4, 4, x)$ by Shimura [1973]. Then using the transformation laws for his θ-functions, we get that $S_{e,h}$ transforms as follows:

$$\begin{aligned}
S_{e,h}(\tau)(\tau+1) =& e(h^2/8) S_{e,h}(\tau), \text{ and} \\
S_{e,h}(-1/\tau) =& \tau^{3/2} \frac{e(-1/8)}{\sqrt{4}} \sum_{k \pmod 4} e(kh/4) S_{e,k}(\tau).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
S_e(\tau+1) =& \rho_L(T) S_e(\tau), \text{ and} \\
S_e(-1/\tau) =& \tau^{3/2} \rho_L(S) S_e(\tau).
\end{aligned}$$

From these transformations, we see that $S_e(\tau) : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is a weight 3/2 vector-valued modular form transforming under the Weil representation ρ_L, i.e an element of the space $M_{3/2,\rho_L}$. From Cheng et al. [2014], we know that H_e is a mock modular form with shadow S_e. This gives the following theorem:

Theorem 3.1. *We have that $\hat{H}_e(\tau) : \mathbb{H} \mapsto \mathbb{C}[L'/L]$ is a weight 1/2 vector valued harmonic weak Maass form transforming under the restriction of the Weil representation $\overline{\rho_L}$ to the preimage of $\Gamma_0(2)$ in $\tilde{\Gamma}$. Moreover, it has shadow $S_e(\tau)$, non-holomorphic part F_e, and principal part $P_{\hat{H}}(\tau) = -2q^{-1/8}(\nu_1 - \nu_3)$.*

3.5 Generalized Borcherds Products

Now that we are able to rewrite the mock modular form H_e as a harmonic weak Maass form, we can introduce the application of the generalized Borcherds Lifts defined in Bruinier and Ono [2010] to obtain a meromorphic modular form. Recalling the result from Ono et al. [2015], we call a pair (Δ, r) admissible if Δ is a negative fundamental discriminant and $r^2 = \Delta \pmod 8$. We also let $e(a) := e^{2\pi i a}$.

Theorem 3.2. *Let $c(n,r)$ be the n-th Fourier coefficient of the r-th component of H_e or $tr(e|K_{n,r})$. Let (Δ, r) be an admissible pair. Then the twisted generalized Borcherds product is*

$$\Psi_{\Delta,r}(\tau, H_e) := \prod_{n=1}^{\infty} P_\Delta(q^n)^{c_e\left(\Delta n^2, rn\right)}$$

where

$$P_\Delta(x) := \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} \left[1 - e\left(b/\Delta\right) x\right]^{\left(\frac{\Delta}{b}\right)}$$

is a rational function in $T_e(\tau)$ with a discriminant Δ Heegner divisor.

Corollary 3.3. *By the preceding theorem,*

$$\Psi_{\Delta,r}(\tau, H_e) := \prod_i (T_e(\tau) - T_e(\alpha_i))^{\gamma_i}$$

for some discriminant Δ *and Heegner points* α_i*. Thus, we have*

$$c_e\left(\Delta n^2, rn\right) = \frac{1}{\epsilon_\Delta} \sum_i \gamma_i T_e(\alpha_i)$$

where

$$\epsilon_\Delta = \sum_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} e(b/\Delta) \left(\frac{\Delta}{b}\right)$$

Remark 3.4. *Generalized Borcherds Products* are the prime result used in this paper. In that, we propose a generalization of the Borcherds Lift applied here to the case of other H_g mock modular functions dependent on the conjugacy class of g.

4 Definitions Modular Forms, Jacobi Forms, Theta Functions, etc.

We conclude this paper with an appendix of the relevant functions involved in our computations. First beginning with the mock-modular McKay-Thompson series corresponding with Mathieu Moonshine. We define the following:

$$\Lambda_M(\tau) := Mq\frac{d}{dq}\left(log\frac{\eta(M\tau)}{\eta(\tau)}\right) = \frac{M(M-1)}{24} + M\sum_{k>0}\sum_{d|k} d\left(q^k - Mq^{Mk}\right) \tag{16}$$

which is a modular form of weight two for $\Gamma_0(N)$ if $M|N$.

[g]	$\chi(g)$	$\tilde{T}_g(\tau)$
$1A$	24	0
$2A$	8	$16\Lambda_2$
$2B$	0	$-24\Lambda_2 + 8\Lambda_4 = 2\eta(\tau)^8/\eta(2\tau)^4$
$3A$	6	$6\Lambda_3$
$3B$	0	$2\eta(\tau)^6/\eta(3\tau)^2$
$4A$	0	$4\Lambda_2 - 6\Lambda_4 + 2\Lambda_8 = 2\eta(2\tau)^8/\eta(4\tau)^4$
$4B$	4	$4(-\Lambda_2 + \Lambda_4)$
$4C$	0	$2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
$5A$	4	$2\Lambda_5$
$6A$	2	$2(-\Lambda_2 - \Lambda_3 + \Lambda_6)$
$6B$	0	$2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2/\eta(6\tau)^2$
$7AB$	3	Λ_7
$8A$	2	$-\Lambda_4 + \Lambda_8$
$10A$	0	$2\eta(\tau)^3\eta(2\tau)\eta(5\tau)/\eta(10\tau)$
$11A$	2	$2(\Lambda_{11} - 11\eta(\tau)^2\eta(11\tau)^2)/5$
$12A$	0	$2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(12\tau)$
$12B$	0	$2\eta(\tau)^4\eta(4\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(3\tau)\eta(12\tau)^2$
$14AB$	1	$(-\Lambda_2 - \Lambda_7 + \Lambda_{14} - 14\eta(\eta)\eta(2\tau)\eta(7\tau)\eta(14\tau))/3$
$15AB$	1	$(-\Lambda_3 - \Lambda_5 + \Lambda_{15} - 15\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau))/4$
$21AB$	0	$(7\eta(\tau)^3\eta(7\tau)^3/\eta(3\tau)\eta(21\tau) - \eta(\tau)^6/\eta(3\tau)^2)/3$
$23AB$	1	$(\Lambda_{23} - 23f_{23,1} + 23f_{23,3})/11$

Table 5: In this table, we list the relevant functions utilized in defining the various weight 1/2 mock modular forms of H_g described in Preposition 4.1. Here $f_{23,1}$ and $f_{23,3}$ are defined in Duncan et al. [2015]

4.1 Mathieu (Mock) Modular Functions

For M_{24}, we have the following candidates introduced mostly in Cheng [2010], Gaberdiel et al. [2010a,b], Eguchi and Hikami [2011] for the associated McKay-Thompson series H_g. From them, we have the following result:

Proposition 4.1. *Let $H : \mathbb{H} \mapsto \mathbb{C}$ given by*

$$H(\tau) = \frac{-2E_2(\tau) + 48F_2(\tau)}{\eta(\tau)^3} = 2q^{-\frac{1}{8}}(-1 + 45q + 231q^2 + \ldots) \tag{17}$$

where

$$F_2(\tau) = \sum_{\substack{r>s>0 \\ r-s=1 \ mod\ 2}} (-1)^r s q^{rs/2} = q + q^2 - q^3 + q^4 + \ldots. \tag{18}$$

and

$$E_2 = 1 - 24 \sum_{n>0} \sum_{k|n} q^n \tag{19}$$

Then for all $g \in M_{24}$, the function

$$H_g(\tau) = \frac{\chi(g)}{24} H(\tau) - \frac{\tilde{T}_g(\tau)}{\eta(\tau)^3} \tag{20}$$

is a mock modular form for $\Gamma_0(N_g)$ of weight $1/2$ with shadow $\chi(g)\eta(\tau)^3$. Moreover, we have

$$\hat{H}_g(\tau) = \psi(\gamma) jac(\gamma, \tau)^{1/4} \hat{H}_g(\gamma\tau),$$

for $\gamma \in \Gamma_0(n_g)$ where

$$\hat{H}_g(\tau) = H_g(\tau) + \chi(g)(4i)^{-1/2} \int_{-\overline{\tau}}^{\infty} (z+\tau)^{-1/2} \overline{\eta(-\overline{z})} dz,$$

and the multiplier system is given by $\psi(\gamma) = \epsilon(\gamma)^{-3} \rho_{n_g|h_g}(\gamma)$. Here

$$jac(\gamma, \tau) = (c\tau + d)^{-2}.$$

and

$$\rho_{n_g|h_g}(\gamma) = e(-cd/n_g h_g).$$

4.2 Theta Functions

We define the Jacobi theta functions $\theta_i(\tau, z)$ as follows for $q := e(\tau)$ and $y := e(z)$ for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ as in Ono et al. [2015]

$$\theta_2(\tau, z) := q^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1} q^{n-1})$$

$$\theta_3(\tau, z) := \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n-1/2}(1+y^{-1}q^{n-1/2}$$

$$\theta_4(\tau, z) := \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-1/2}(1-y^{-1}q^{n-1/2}$$

We use them to define weight zero index 1 Jacobi form φ_1.

$$\varphi := 4(f_2^2 + f_3^2 + f_4^2),$$

where $f_i(\tau, z) := \theta_i(\tau, z)/\theta_i(\tau, 0)$ for $i = 2, 3, 4$.

We define the Appell-Lerch sum as in Cheng et al. [2014] given by

$$\mu_{m,0}(\tau, z) := -\sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{1+yq^k}{1-yq^k}.$$

It is the holomorphic part of a weight 1 index 2 *real-analytic Jacobi form.*

Finally,

$$R(\tau, z) := \sum_{\nu\in 1/2+\mathbb{Z}} \left\{ sgn(\nu) - E(\nu+a)\sqrt{2t} \right\} (-1)^{\nu-1/2} q^{-\nu^2/2} y^{-\nu},$$

where $t := \Im(\tau)$, $a := \frac{\Im(u)}{\Im(\tau)}$, and $E(z) := 2\int_0^z e^{-\pi u^2} du$.

References

Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986. ISSN 0027-8424. doi: 10.1073/pnas.83.10.3068.

URL https://doi.org/10.1073/pnas.83.10.3068.

Richard E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998. ISSN 0020-9910. doi: 10.1007/s002220050232. URL https://doi.org/10.1007/s002220050232.

Jan Bruinier and Ken Ono. Heegner divisors, L-functions and harmonic weak Maass forms. *Ann. of Math. (2)*, 172(3):2135–2181, 2010. ISSN 0003-486X. doi: 10.4007/annals.2010.172.2135. URL https://doi.org/10.4007/annals.2010.172.2135.

Miranda C. N. Cheng. $K3$ surfaces, $\mathcal{N} = 4$ dyons and the Mathieu group M_{24}. *Commun. Number Theory Phys.*, 4(4):623–657, 2010. ISSN 1931-4523. doi: 10.4310/CNTP.2010.v4.n4.a2. URL https://doi.org/10.4310/CNTP.2010.v4.n4.a2.

Miranda C. N. Cheng and John F. R. Duncan. The largest Mathieu group and (mock) automorphic forms. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 53–82. Amer. Math. Soc., Providence, RI, 2012. doi: 10.1090/pspum/085/1374. URL https://doi.org/10.1090/pspum/085/1374.

Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral moonshine and the Niemeier lattices. *Res. Math. Sci.*, 1:Art. 3, 81, 2014. ISSN 2522-0144. doi: 10.1186/2197-9847-1-3. URL https://doi.org/10.1186/2197-9847-1-3.

J. H. Conway. Monsters and moonshine. *Math. Intelligencer*, 2(4):165–171, 1979/80. ISSN 0343-6993. doi: 10.1007/BF03028594. URL https://doi.org/10.1007/BF03028594.

J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups*. Oxford University Press, Eynsham, 1985. ISBN 0-19-853199-0. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

Atish Dabholkar, Sameer Murthy, and Don Zagier. Quantum Black Holes, Wall Crossing, and Mock Modular Forms. 2012.

John F. R. Duncan and Sander Mack-Crane. The moonshine module for Conway's group. *Forum Math. Sigma*, 3:e10, 52, 2015. ISSN 2050-5094. doi: 10.1017/fms.2015.7. URL https://doi.org/10.1017/fms.2015.7.

John F. R. Duncan, Michael J. Griffin, and Ken Ono. Proof of the umbral moonshine conjecture. *Res. Math. Sci.*, 2:Art. 26, 47, 2015. ISSN 2522-0144. doi: 10.1186/s40687-015-0044-7. URL https://doi.org/10.1186/s40687-015-0044-7.

Tohru Eguchi and Kazuhiro Hikami. Superconformal algebras and mock theta functions. II. Rademacher expansion for $K3$ surface. *Commun. Number Theory Phys.*, 3(3):531–554, 2009. ISSN 1931-4523. doi: 10.4310/CNTP.2009.v3.n3.a4. URL https://doi.org/10.4310/CNTP.2009.v3.n3.a4.

Tohru Eguchi and Kazuhiro Hikami. Note on twisted elliptic genus of $K3$ surface. *Phys. Lett. B*, 694(4-5):446–455, 2011. ISSN 0370-2693. doi: 10.1016/j.physletb.2010.10.017. URL https://doi.org/10.1016/j.physletb.2010.10.017.

Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the $K3$ surface and the Mathieu group M_{24}. *Exp. Math.*, 20(1):91–96, 2011. ISSN 1058-6458. doi: 10.1080/10586458.2011.544585. URL https://doi.org/10.1080/10586458.2011.544585.

Matthias R. Gaberdiel, Daniel Grumiller, and Dmitri Vassilevich. Graviton 1-loop partition function for 3-dimensional massive gravity. *J. High Energy Phys.*, (11):094, 18, 2010a. ISSN 1126-6708. doi: 10.1007/JHEP11(2010)094. URL https://doi.org/10.1007/JHEP11(2010)094.

Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu Moonshine in

the elliptic genus of $K3$. *J. High Energy Phys.*, (10):062, 24, 2010b. ISSN 1126-6708. doi: 10.1007/JHEP10(2010)062. URL https://doi.org/10.1007/JHEP10(2010)062.

Terry Gannon. Much ado about Mathieu. *Adv. Math.*, 301:322–358, 2016. ISSN 0001-8708. doi: 10.1016/j.aim.2016.06.014. URL https://doi.org/10.1016/j.aim.2016.06.014.

B. Gross, W. Kohnen, and D. Zagier. Heegner points and derivatives of L-series. II. *Math. Ann.*, 278(1-4):497–562, 1987. ISSN 0025-5831. doi: 10.1007/BF01458081. URL https://doi.org/10.1007/BF01458081.

Richard Hill. Notes on Borcherds Products. 2012. URL http://www.math.unt.edu/~richter/afbridges/sites/default/files/lecture_notes_hill.pdf.

Ken Ono, Larry Rolen, and Sarah Trebat-Leder. Classical and umbral moonshine: connections and p-adic properties. *J. Ramanujan Math. Soc.*, 30(2):135–159, 2015. ISSN 0970-1249.

Larissa Queen. Modular functions arising from some finite groups. *Math. Comp.*, 37(156): 547–580, 1981. ISSN 0025-5718. doi: 10.2307/2007446. URL https://doi.org/10.2307/2007446.

Goro Shimura. Modular forms of half integral weight. pages 57–74. Lecture Notes in Math., Vol. 320, 1973.

J. G. Thompson. Some numerology between the Fischer-Griess Monster and the elliptic modular function. *Bull. London Math. Soc.*, 11(3):352–353, 1979a. ISSN 0024-6093. doi: 10.1112/blms/11.3.352. URL https://doi.org/10.1112/blms/11.3.352.

J. G. Thompson. Finite groups and modular functions. *Bull. London Math. Soc.*, 11(3): 347–351, 1979b. ISSN 0024-6093. doi: 10.1112/blms/11.3.347. URL https://doi.org/10.1112/blms/11.3.347.

IITM/PH/TH/2011/3 arXiv:1106.5715 [hep-th]

v2.2; June 2018

arXiv:1106.5715v3 [hep-th] 29 Oct 2018

Unravelling Mathieu moonshine

Suresh Govindarajan*

Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India

Abstract

The $D1-D5-KK-p$ system naturally provides an infinite dimensional module graded by the dyonic charges whose dimensions are counted by the Igusa cusp form, $\Phi_{10}(\mathbf{Z})$. We show that the Mathieu group, M_{24}, acts on this module by recovering the Siegel modular forms that count twisted dyons as a trace over this module. This is done by recovering Borcherds product formulae for these modular forms using the M_{24} action. This establishes the correspondence ('moonshine') proposed in arXiv:0907.1410 that relates conjugacy classes of M_{24} to Siegel modular forms. This also, in a sense that we make precise, subsumes existing moonshines for M_{24} that relates its conjugacy classes to eta-products and Jacobi forms.

June 2018: Corrected some sign errors that appear in the published version as well.

*Email: suresh@physics.iitm.ac.in

1 Introduction

In ref. [1], a moonshine for the Mathieu group, M_{24}, relating its conjugacy classes to genus-two Siegel modular forms was proposed. Let $\rho = 1^{a_1}2^{a_2}\cdots N^{a_N}$ be the cycle shape for an M_{24} conjugacy class and $(k+2) = \frac{1}{2}\sum_i a_i$. Then the moonshine correspondence proposed in [1] (see also [2]) is as follows:

$$\rho \longrightarrow \Phi_k^\rho(\mathbf{Z}) \ , \tag{1.1}$$

where $\Phi_k^\rho(\mathbf{Z})$ is a genus two Siegel modular form at level N and $\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right) \in \mathbb{H}_2$. We shall restrict the considerations of this paper to the situation when the conjugacy class reduces to a conjugacy class of M_{23}: this implies that $a_1 \neq 0$. In these cases, it is known that the Siegel modular form counts $\frac{1}{4}$-BPS twisted dyons in the heterotic string compactified on a six-torus.[1]

It is natural to ask whether there exists an M_{24}-module $V^\natural$ graded by three integers (n, ℓ, m) corresponding to the dyonic charges $(\frac{1}{2}\mathbf{q}_e^2, \mathbf{q}_e \cdot \mathbf{q}_m, \frac{1}{2}\mathbf{q}_m^2)$ i.e.,

$$V^\natural = \bigoplus_{(n,\ell,m)} V_{n,\ell,m} \ , \tag{1.2}$$

such that the Igusa cusp form is given by the following trace[2]

$$\frac{1}{\Phi_{10}(\mathbf{Z})} = \sum_{(n,\ell,m)} \mathrm{sdim}\left[V_{(n,\ell,m)}\right] q^n r^\ell s^m \ , \tag{1.3}$$

with $q = \exp(2\pi i\tau)$, $r = \exp(2\pi i z)$, $s = \exp(2\pi i\sigma)$. Further,

$$\frac{1}{\Phi_k^\rho(\mathbf{Z})} = \sum_{(n,\ell,m)} \mathrm{STr}\left[g\big|_{V_{(n,\ell,m)}}\right] q^n r^\ell s^m \ , \tag{1.4}$$

counts g-twisted $\frac{1}{4}$-BPS states in the toroidally compactified heterotic string and ρ is the conjugacy class of $g \in M_{24}$ that reduces to a conjugacy class of M_{23}.

The $D1 - D5 - KK - p$ system [3] provides an obvious candidate for $V^\natural$. It has a natural decomposition into three distinct parts:

$$V^\natural = W \otimes \mathcal{B} \otimes \mathcal{S} \ , \tag{1.5}$$

[1] Such dyons are invariant under the action of a finite abelian group of order N – this group maps to a symplectic automorphism of $K3$ in the dual type II picture.

[2] The superdimension sdim of the graded vector space $V = V_+ \oplus V_-$ is defined to be $\mathrm{sdim}(V) = \mathrm{Tr}_V(-1)^F := \mathrm{STr}_V(1) = (d_+ - d_-)$ where d_+ is the dimension of the bosonic (even) subspace, V_+ and d_- is the dimension of the fermionic (odd) subspace, V_- of V.

with

$$\sum_{(n,\ell)} \text{sdim}\left[W_{(n,\ell,0)}\right] q^n r^\ell s^0 = \frac{\eta(\tau)^6}{\theta_1(\tau,z)^2} \ ,$$
$$\sum_{n} \text{sdim}\left[\mathcal{B}_{(n,0,0)}\right] q^n r^0 s^0 = \frac{1}{\eta(\tau)^{24}} \ ,$$
$$\sum_{(n,\ell,m)} \text{sdim}\left[\mathcal{S}_{(n,\ell,m)}\right] q^n r^\ell s^m = \frac{1}{\mathcal{E}(K3,\mathbf{Z})} \ ,$$

where $\mathcal{E}(K3,\mathbf{Z})$ is the second-quantized elliptic genus of $K3$ [4]. The problem thus reduces to showing the action of M_{24} on these three modules.

The paper is organized as follows. In section 2, we discuss the module $V^\natural$ in terms of its components. When possible, we show how some parts are M_{24}-modules connecting them to known moonshines for M_{24}. We reduce the problem to proving that the module $\mathcal{S}$, associated with the second-quantized elliptic genus of $K3$, is an M_{24} module. In section 3, we focus on the details of the moonshine proposed in [1]. We show that the module $\mathcal{S}$ is an M_{24} module by showing that the M_{24} action implies replication formulae for twisted elliptic genera of symmetric products of $K3$. It is proposed that a twisted Hecke-like operator generates all these replication formulae. In section 4, we prove that the proposal provides a Borcherds product formula that incorporates the M_{24}-action on the second-quantized elliptic genus of $K3$ and hence for the Siegel modular forms. This product formula agrees with existing formulae in the literature. We conclude with a few observations. Appendix A provides some information about the elliptic genus of Kähler and hyperKähler manifolds.

2 The module $V^\natural$ and its parts

The microscopic counting of $\frac{1}{4}$-BPS states in four-dimensional CHL compactifications of the heterotic string with $\mathcal{N}=4$ supersymmetry was carried out by David and Sen [5]. Using a chain of dualities as well the 4D-5D correspondence [6], the counting was carried out in a configuration of D1 and D5 branes moving in $K3\times S^1\times TN$ where TN is the Taub-NUT space. The geometry close to the center of TN is five-dimensional while the geometry far away from the center is four-dimensional with the appearance of an additional circle $\widetilde{S}^1$.

The result of David-Sen [5] for the type IIB dual of the toroidally compactified heterotic string expresses the Igusa cusp form, $\Phi_{10}(\mathbf{Z})$, as a product of three terms

$$\frac{64}{\Phi_{10}(\mathbf{Z})} = \underbrace{\left[\frac{4\,\eta(\tau)^6}{\theta_1(\tau,z)^2}\right]}_{\text{(i)}} \times \underbrace{\left[\frac{16}{\eta(\tau)^{24}}\right]}_{\text{(ii)}} \times \underbrace{\left[\frac{1}{\mathcal{E}(K3;\mathbf{Z})}\right]}_{\text{(iii)}} \ , \tag{2.1}$$

As indicated above the states that contribute to the counting arise from three distinct sectors (labeled (i)-(iii)) in the type IIB description:

(i) the overall motion of the $D1$-$D5$ branes in Taub-NUT space.

(ii) the excitations of the KK-monopole – following the chain of dualities, these excitations get mapped to the states of the heterotic string.

(iii) the motion of the $D1$-branes in the worldvolume of the $D5$-branes – this counting leads to the second-quantized elliptic genus of $K3$.

In terms of fermionic zero-modes, half of the supersymmetry (contributing sixteen zero-modes) is broken by the KK-monopole and its excitations. The remaining quarter of the supersymmetry is broken (contributing four zero-modes) by the center of mass motion. The motion of the $D1$-branes in the worldvolume of the $D5$-branes do not break any further supersymmetry. Let Q_1 and Q_5 denote the number of $D1$-branes and $D5$-branes respectively. Further, let n denote the momentum along S^1 (the KK monopole charge) and J the angular momentum in TN space. Then, the $D1-D5-KK-p$ system carries the following T-duality invariant charges [5]:

$$\tfrac{1}{2}\mathbf{q}_e^2 = n \ , \quad \mathbf{q}_e \cdot \mathbf{q}_m = J \ , \quad \tfrac{1}{2}\mathbf{q}_m^2 = (Q_1 - Q_5)Q_5 \ . \tag{2.2}$$

The module $V^\natural$ is graded by these three charges.

We shall now consider each one of these terms separately to further elucidate the structure of the module $V^\natural$. The rest of this section is based on ref. [5] and we refer the reader to it for a detailed description.

2.1 The overall motion in Taub-NUT space

The overall motion of the $D1$-$D5$ system in Taub-NUT space is described by a 1+1-dimensional supersymmetric conformal field theory(CFT) with Taub-NUT as its target space. The charge associated with $\mathbf{q}_e^2$ gets related to the L_0 eigenvalue while the charge associated with $\mathbf{q}_e \cdot \mathbf{q}_m$ gets related to the charge due to a $U(1)_L$ symmetry in the field theory. The field theory consists of four free left-moving fermions; four right-moving fermions charged under $U(1)_L$ and interacting with four scalars with Taub-NUT target space. The unbroken supersymmetry acts on the right-movers. Thus BPS states are obtained when the right-movers are in the ground state. The only permitted excitations are from the four left-moving fermions as well as the left-movers of the four scalars. The index of the supersymmetric states in the field theory which we interpret as a supertrace over W is given by

$$\sum_{(n,\ell)} \text{sdim}\left[W_{(n,\ell,0)}\right] q^n r^\ell s^0 = Z_{\text{free}}(\tau)\ Z_{\text{osc}}(\tau, z)\ Z_{\text{zero-mode}}(z) \ , \tag{2.3}$$

where $Z_{\text{free}}(\tau)$ is the contribution of the free left-moving fermions (which are z-independent due to their $U(1)_L$ invariance); $Z_{\text{osc}}(\tau, z)$ is the contribution from the left-moving oscillators of the four scalars and $Z_{\text{zero-mode}}(z)$ is the contribution from the bosonic zero-modes (which are τ independent). Thus, the space W is the direct product of three-spaces and the contributions are:

$$Z_{\text{free}}(\tau) = 4 \prod_{n=1}^{\infty} (1 - q^n)^4 \ , \tag{2.4}$$

$$Z_{\text{osc}}(\tau, z) = \prod_{n=1}^{\infty} (1 - rq^n)^{-2} (1 - r^{-1} q^n)^{-2} \ , \tag{2.5}$$

$$Z_{\text{zero-mode}}(z) = \frac{r}{(r-1)^2} \ . \tag{2.6}$$

The action of M_{24} on W

Let g denote a supersymmetry preserving symmetry of the conformal field theory associated with the sigma model with $K3$ target space of finite order N. Symplectic automorphisms of $K3$ furnish a large (but not all) class of such examples. A classification of such symmetries has appeared recently where it has been shown that such symmetries are subgroups of the Conway group, Co_1 [37]. Since M_{24} is a sub-group of Co_1, this includes all situations that implicitly or explicitly appear in this paper. It follows any such $g \in Co_1$ acts *trivially* on the space W. This is easy to understand since W arises from the dynamics of motion in the Taub-NUT space and does not 'see' the $K3$. We identify symplectic automorphisms of $K3$ with elements of M_{24}. Thus we conclude that M_{24} has no action on W. Thus,

$$\boxed{\sum_{(n,\ell,m)} \text{STr}\left[g\big|_{W_{(n,\ell,m)}} \right] q^n r^\ell s^m = \sum_{(n,\ell,m)} \text{sdim}\left[W_{(n,\ell,m)} \right] q^n r^\ell s^m \ .} \tag{2.7}$$

Remarks:

1. $Z_{\text{zero-mode}}(z)$ has different Fourier expansions depending on whether $|r| < 1$ and $|r| > 1$:

$$Z_{\text{zero-mode}}(z) = \begin{cases} r + 2r^2 + 3r^3 + \cdots & \text{for } |r| < 1 \ , \\ r^{-1} + 2r^{-2} + 3r^{-3} + \cdots & \text{for } |r| > 1 \ . \end{cases} \tag{2.8}$$

   Physically, this is related to the existence of a wall of marginal stability at $r = 1$, where there is a jump in dyon degeneracy due to additional contributions from two-centered black holes [7,8]. A mathematical interpretation can be given by using the connection of the square-root of the Igusa cusp form, $\Delta_5(\mathbf{Z})$, with a BKM Lie superalgebra [9]. The wall of marginal stability in the physical setting gets mapped to the wall of a Weyl chamber [10].

2. The module $V^\natural$ is thus dependent on the Weyl chamber. As in the case of finite Lie algebras, the Weyl chamber is characterized by the choice of Weyl vector.

2.2 The KK monopole

The low-energy dynamics of the D1-D5 system localized at the center of the Taub-NUT space can carry momentum along S^1 but not along $\widetilde{S}^1$.[3] The momentum along S^1 contributes to the electric charge $\frac{1}{2}\mathbf{q}_e^2$ with other charges vanishing. Following the chain of dualities, BPS states of this system get mapped to BPS states of the fundamental heterotic string. The BPS condition requires the supersymmetric right-movers of the heterotic string to be in the ground state and the various electric charges take values in the Narain lattice for heterotic string compactified on a six-torus. In the light-cone gauge, the space of BPS states is that of the oscillator modes of 24 left-moving chiral bosons. We will denote this space by $\mathcal{B}$.

The level matching condition relates the electric charge to the level, n, of the oscillator excitations [11]. One has

$$\tfrac{1}{2}\mathbf{q}_e^2 = n - 1 \ , \quad n = 0, 1, 2, \cdots \ . \tag{2.9}$$

One has

$$\mathrm{Tr}_{\mathcal{B}}\big(q^{L_0'-1}\big) = \frac{1}{\eta(\tau)^{24}} = q^{-1} + 24 + \cdots \ , \tag{2.10}$$

where L_0' is the contribution of the oscillator modes to L_0.

The action of M_{24} on $\mathcal{B}$: the first moonshine

Let g denote a symplectic automorphism of $K3$ of order N. Using the identification of the heterotic string as a NS5-brane wrapping $K3$, we can track the action of g to the fields in the 1+1 dimensional CFT of the heterotic string. In particular, one can show that g acts as a permutation of the 24 chiral bosons. Following the work of Mukai [12], it was shown in ref. [1] that the possible conjugacy classes are those of M_{24} that reduce to conjugacy classess of M_{23} and order$(g) \leq 8$ (and other conditions not stated here). M_{24} has precisely eight such conjugacy classes that we list in Table 1. As mentioned earlier, there exists symmetries of the $K3$ sigma model that are not symplectic automorphisms and do lead to other conjugacy classes that are not in the Table 1.

A straightforward computation (not explicitly shown in ref. [1] but given in ref. [2, see Appendix A]) leads to the following result:

$$\boxed{\mathrm{Tr}_{\mathcal{B}}\big(g\ q^{L_0'-1}\big) = \frac{1}{g_\rho(\tau)} = q^{-1} + a_1 + \cdots} \ , \tag{2.11}$$

[3] Recall that the geometry of the Taub-NUT space near the centre is that of $\mathbb{R}^4$ while it is $\mathbb{R}^3 \times \widetilde{S}^1$ far away from the center. Thus the geometry of spacetime is $K3 \times S^1 \times \mathbb{R}^4$.

N	1	2	3	4	5	6	7	8
ρ	1^{24}	$1^8 2^8$	$1^6 3^6$	$1^4 2^2 4^4$	$1^4 5^4$	$1^2 2^2 3^2 6^2$	$1^3 7^3$	$1^2 2^1 4^1 8^2$
Atlas	1A	2A	3A	4B	5A	6A	7A	8A

Table 1: Table of cycle shapes (along with their labels in the Atlas [13]) for symplectic automorphisms of $K3$.

where ρ is the conjugacy class of g and $g_\rho(\tau)$ is the η-product given by the map

$$\rho = 1^{a_1} 2^{a_2} \cdots N^{a_N} \longmapsto g_\rho(\tau) \equiv \prod_{j=1}^{N} \eta(j\tau)^{a_j} \ . \tag{2.12}$$

Remarks:

1. There exist symmetries of the K3 sigma model that are not symplectic automorphisms. In particular, all other conjugacy classes of M_{24} with at least four cycles, such as the cycle shape 2^{12} (conjugacy class 2B), arise this way [37]. The map Eq. (2.12) provides η-products for *all* conjugacy classes of M_{24}.

2. The η-products that appear are multiplicative i.e., they are Hecke eigenforms. They also furnish a moonshine for the group M_{24} [14,15]. This is the first moonshine for M_{24} that we will encounter – this has been dubbed the *additive* moonshine. As we will later see, this is the first of an infinite set of moonshines for M_{24} that imply that $V^\natural$ is an M_{24}-module.

3. $\mathcal{B}$ is obviously an S_{24}-module and hence an M_{24}-module. However, S_{24} is *not* a symmetry of the full $D1 - D5 - KK - p$ system. For instance, the requirement that g generate a symplectic automorphism needs us to choose special points in the moduli space of $K3$. This translates to a particular choice of the Narain lattice in the heterotic string – as the Narain lattice does not form part of $\mathcal{B}$, we see the larger group i.e., S_{24}.

4. If g has to generate a symplectic automorphism of $K3$, it forces us to restrict to conjugacy classes of M_{24} that reduce to those of M_{23} conjugacy. Other conjugacy classes also appear when one considers a pair of commuting symplectic automorphisms [2]. These lead to a generalized moonshine (in the sense of Norton [16, see appendix by Norton]) and will be discussed elsewhere.

2.3 The second-quantized elliptic genus

The third contribution arises from the relative motion of the $D1$-branes in the worldvolume of $D5$-branes (in the type IIB frame). Consider the situation when

one has Q_1 $D1$-branes wrapping S^1 and Q_5 D5-branes wrapping $K3 \times S^1$. Recall that the magnetic charge of the system is $\frac{1}{2}\mathbf{q}_m^2 = (Q_1 - Q_5)Q_5$. In particular, note that when $Q_5 = Q_1 = 1$, the magnetic charge is vanishing. The other charges arise as momentum of the D1-brane along S^1 and $\widetilde{S}^1$. The BPS condition combined with level-matching in the associated CFT relates the dyonic charges to the L_0 and J eigenvalues in the RR sector.

In the $U(Q_5)$ supersymmetric gauge theory on coincident $D5$-branes, the $D1$-branes appear as instantons and the instanton number of the configuration is $(Q_1 - Q_5)$ after taking into account the induced charge of a D5-brane wrapping $K3$ [17,18].[4] The moduli space of this system is $4[(Q_1 - Q_5)Q_5 + 1]$ dimensional and is identified with the symmetric product of $4[(Q_1 - Q_5)Q_5 + 1]$ copies of $K3$ [18]. We denote the symmetric product of m copies of $K3$ by $S^m(K3) \equiv K3^m/S_m$ and its smooth resolution by $K3^{[m]}$.

The dynamics of this relative motion is captured by the $\mathcal{N} = (4,4)$ superconformal sigma model with target space $S^{m+1}(K3)$ where $m = [(Q_1 - Q_5)Q_5]$. It was conjectured in ref. [19] and proved in ref. [4] that the elliptic genus of this sigma model can be written in terms of the partition function of a second-quantized string theory on $K3 \times S^1$. (See ref. [20] for a review on the relation between second-quantized string theory and symmetric products.) Let us denote the ellliptic genus of a Kähler manifold, M, by $\chi(M;\tau,z)$(see appendix A for definitions) . Then, the result of ref. [4] leads to a Borcherds product formula for the generating function of the elliptic genera of symmetric products of $K3$:

$$\frac{1}{\mathcal{E}(K3;\mathbf{Z})} := s^{-1} \times \Big(1 + \sum_{m=1}^{\infty} s^m \ \chi(S^m(K3);\tau,z)\Big) \tag{2.13}$$

$$= s^{-1} \times \prod_{\substack{n\geq 0\\ m>0,\ell}} \frac{1}{(1-q^n r^\ell s^m)^{c(nm,\ell)}} \ , \tag{2.14}$$

where $c(nm,\ell)$ are the Fourier-Jacobi coefficients of the elliptic genus of $K3$ which is a weight zero, index one Jacobi form that we denote by $\psi_{0,1}(\tau,z)$

$$\psi_{0,1}(\tau,z) := \chi(K3;\tau,z) = \sum_{n\geq 0,\ell} c(n,\ell)\ q^n r^\ell \ . \tag{2.15}$$

We refer to $\mathcal{E}(K3;\mathbf{Z})$ as the second-quantized elliptic genus of $K3$. Since we associate powers of s with the magnetic charge, we need to include an additional factor of s^{-1} in our definition of the second-quantized elliptic genus.

Remark: In the limit $z \to 0$, the elliptic genus, $\chi(M;\tau,z)$ reduces to the Euler

[4] When $Q_5 = 1$, one needs to turn on two-form flux to convert the $U(1)$ gauge theory to into a non-commutative theory which admits instantons.

characteristic of M which we denote by $\chi[M]$. One sees that[5]

$$\lim_{z\to 0}\frac{1}{\mathcal{E}(K3;\mathbf{Z})}=\frac{1}{s}+\sum_{m=1}^{\infty}\chi[S^m(K3)]\ s^{m-1}=\frac{1}{\eta(\sigma)^{24}}\ . \tag{2.16}$$

The appearance of the η-product is easily understood by carrying out electric-magnetic duality and the *magnetic* η-product gets mapped to the electric η-product. The condition that the ground state have charge $\mathbf{q}_m^2=-1$ then follows from level matching.

2.4 Decomposing the module $\mathcal{S}$

We decompose the module $\mathcal{S}$ using the magnetic charge as

$$\mathcal{S}:=\bigoplus_{m=-1}^{\infty}\mathcal{S}_m\ . \tag{2.17}$$

The $m=-1$ term is the *ground state* with $Q_1=Q_5=0$ and has magnetic charge $\mathbf{q}_m^2=-1$ with other charges vanishing. Thus, $\mathcal{S}_{-1}=\mathbb{C}$. The sub-module $\mathcal{S}_0=\mathcal{H}(K3)$ is special and we shall call it $\mathcal{K}$ and the elliptic genus of $K3$ is given by

$$\psi_{0,1}(\tau,z)=\mathrm{Tr}_{\mathcal{K}}\left((-1)^F q^{H_L} r^{J_L}\right)\ . \tag{2.18}$$

We will see that the modules $\mathcal{S}_m$ for $m\geq 1$ can be constructed from $\mathcal{K}$ [4]. Recall $\mathcal{K}$ is the Hilbert space of a $D1$-brane on $K3\times\widetilde{S}_1$ with winding number one. Denote by $\mathcal{K}_{(n)}$ the Hilbert space of a multiply wound $D1$-brane with winding number n – the oscillator modes of such a string have fractional moding $1/n$. Thus one has

$$\mathrm{Tr}_{\mathcal{K}_{(n)}}\left((-1)^F q^{H_L} r^{J_L}\right)=\psi_{0,1}\left(\tfrac{\tau}{n},z\right) \tag{2.19}$$

The work of DMVV [4] uses the structure of pemutation orbifolds and shows that the module $\mathcal{S}_{m-1}$ (for $m>1$) can be decomposed into a direct sum of twisted sectors of the orbifold CFT. Recall that twisted sectors of the S_m-orbifold are labelled by conjugacy classes of S_m. For instance, $\mathcal{S}_1$ has only two sectors – the untwisted sector (with conjugacy class 1^2) and a single twisted sector (with conjugacy class 2). Thus, one has

$$\mathcal{S}_1=\mathcal{H}_{1^2}^{\mathrm{inv}}\oplus\mathcal{H}_2^{\mathrm{inv}}\ , \tag{2.20}$$

where $\mathcal{H}_{1^2}=\mathcal{K}\otimes\mathcal{K}$ and $\mathcal{H}_2=\mathcal{K}_{(2)}$ and the superscript 'inv' indicates that we need to project onto S_2 invariant subspaces. Thus the elliptic genus of $S^2(K3)$

[5]For $m>1$, the manifold $S^m(K3)$ has orbifold singularities and by $\chi[S^m(K3)]$, we mean the orbifold Euler characteristic of $S^m(K3)$.

can be written in terms of the elliptic genus of $K3$ and leads to the following *doubling* formula.

$$\chi(S^2(K3);\tau,z) = \mathrm{Tr}_{\mathcal{K}\otimes\mathcal{K}}\Big(\mathcal{P}q^{L_0-1}r^{J_L}(-1)^F\Big) + \mathrm{Tr}_{\mathcal{K}_{(2)}}\Big(\mathcal{P}\ q^{L_0-1}r^{J_L}(-1)^F\Big)\ ,$$
$$= \frac{1}{2}\Big[\big(\psi_{0,1}(\tau,z)\big)^2 + \psi_{0,1}(2\tau,2z)\Big] + \frac{1}{2}\Big[\sum_{\ell=0}^{1}\psi_{0,1}(\tfrac{\tau+\ell}{2},z)\Big]\ , \quad (2.21)$$

where $\mathcal{P}$ denotes the projection on to the S^2-invariant sector.

More generally, one has

$$\mathcal{S}_{m-1} = \bigoplus_{[h]} \mathcal{H}_{[h]}^{C_h}\ , \quad (2.22)$$

where $[h]$ denotes the conjugacy class of an element $h \in S_m$, C_h is the centralizer group of h and the superscript indicates that we project onto a C_h-invariant subspace of $\mathcal{H}_{[h]}$. Thus, for a cycle shape $[h] = 1^{a_1}2^{a_2}3^{a_3}\cdots$ (with $\sum_j ja_j = m$), the centralizer group is

$$C_h = S_{a_1} \times (S_{a_2} \rtimes \mathbb{Z}_2^{a_2}) \times (S_{a_3} \rtimes \mathbb{Z}_3^{a_3}) \times \cdots\ ,$$

and $\mathcal{H}_{[h]} = \mathcal{K}^{\otimes a_1} \otimes \mathcal{K}_{(2)}^{\otimes a_2} \otimes \mathcal{K}_{(3)}^{\otimes a_3} \otimes \cdots$. Hence, one has

$$\mathcal{H}_{[h]}^{C_h} = S^{a_1}(\mathcal{K}) \otimes S^{a_2}(\mathcal{K}_{(2)}^{\mathbb{Z}_2}) \otimes S^{a_3}(\mathcal{K}_{(3)}^{\mathbb{Z}_3}) \otimes \cdots\ . \quad (2.23)$$

Thus, it is easy to see that the above structure leads to replication formulae i.e., all elliptic genera are expressible in terms of the elliptic genus of $K3$ i.e., $\psi_{0,1}(\tau,z)$.

DMVV show that the various replication formulae implied by the structure of the various $\mathcal{S}_m$ lead to the following remarkable formula:

$$\boxed{\Big(1 + \sum_{m=1}^{\infty} s^m\ \chi(S^m(K3);\tau,z)\Big) = \exp\Big(\sum_{m=1}^{\infty} s^m\ V_m \cdot \psi_{0,1}(\tau,z)\Big)}\ , \quad (2.24)$$

where V_m is the Hecke-like operator that maps weak Jacobi forms of weight zero and index 1 to weak Jacobi forms of weight zero and index m [21].[6]

$$V_m \cdot \psi_{0,1}(\tau,z) := \sum_{\substack{ad=m \\ b \bmod d}} \frac{1}{m}\ \psi_{0,1}\big(\tfrac{a\tau+b}{d}, az\big)\ . \quad (2.25)$$

It is easy to recover Eq. (2.21) by matching coefficients of s^2 on both sides of Eq. (2.24). On substituting the Fourier-Jacobi expansion of the elliptic genus of $K3$, we recover the Borcherds product formula given in Eq. (2.14).

[6]The Hecke-like operator is usually written as a sum over the coset $\Gamma_1\backslash S_m$ where S_m is an $GL(2,\mathbb{Z})$ with $\det(S_m) = m$. In the following, we have written the Hecke-like operator for a particular parametrization of the coset.

2.5 The action of M_{24} on $\mathcal{K}$: the second moonshine

In CFT's on a torus on considers the following traces

$$ {}_{g}\underset{h}{\boxed{}} = \mathrm{Tr}_{\mathcal{H}_h}\big(g \ \cdots\big) \ , \tag{2.26} $$

where g and h are symmetries of the CFT and $\mathcal{H}_h$ is a module twisted by h (i.e., a module in the h-twisted sector in the orbifold by a group containing h). We will also denote the same object by $[g, h]$, on occasion, in a more compact notation.

There is growing evidence that $\mathcal{K}$ is an M_{24}-module [22–24]. This is best seen by decomposing the elliptic genus of $K3$ in terms of characters of the $\mathcal{N} = 4$ superconformal algebra [25,26]. This is the *second moonshine* for the Mathieu group M_{24}. As before, let g be a symplectic automorphism of $K3$ of order N, then it has been shown that the g-twisted elliptic genus of $K3$ defined as follows

$$ \boxed{\psi_{0,1}^{[g,1]}(\tau, z) := \mathrm{Tr}_{\mathcal{K}}\Big(g \ q^{L_0-1} r^{J_L} (-1)^F\Big) \ ,} \tag{2.27} $$

can be expressed in terms of characters of M_{24} corresponding to the conjugacy class, ρ, of g. Several aspects have been numerically verified to fairly high powers by several authors [27,28]. In this paper, we shall assume that $\mathcal{K}$ is indeed an M_{24}-module though this has not been proven to the best of our knowledge. With this assumption, we shall proceed to show that $\mathcal{S}$ is then an M_{24} module in the next section.

3 The moonshine correspondence

The generating function of $\frac{1}{4}$-BPS states that are invariant under a symplectic automorphism g (of order N) whose M_{24} conjugacy class is $\rho = 1^{a_2} 2^{a_2} \cdots N^{a_N}$ is a Siegel modular form of weight k and level N: $\Phi_k^{\rho}(\mathbf{Z})$ or more precisely $\Phi_k^{[g,1]}(\mathbf{Z})$ [1]. As with $\Phi_{10}(\mathbf{Z})$, it can be written as product of three terms.[7]

$$ \frac{64}{\Phi_k^{[g,1]}(\mathbf{Z})} = \underbrace{\left[\frac{4\ \eta(\tau)^6}{\theta_1(\tau, z)^2}\right]}_{\text{(i)}} \times \underbrace{\left[\frac{16}{g_\rho(\tau)^{24}}\right]}_{\text{(ii)}} \times \underbrace{\left[\frac{1}{\mathcal{E}^g(K3; \mathbf{Z})}\right]}_{\text{(iii)}} \ , \tag{3.1} $$

Term (i) remains unchanged while term (ii) becomes the multiplicative η-product as we discussed earlier. Term (iii) is the g-twisted second-quantized elliptic genus:

$$ \frac{1}{\mathcal{E}^g(K3; \mathbf{Z})} := s^{-1} \times \Big(1 + \sum_{m=1}^{\infty} s^m \ \chi^g(S^m(K3); \tau, z)\Big) \ , \tag{3.2} $$

[7] This Siegel modular form was called $\Phi_k^{(N,1)}(\mathbf{Z})$ in ref. [2] and simply $\Phi_k(\mathbf{Z})$ in ref. [29].

where

$$\chi^g(S^{m+1}(K3);\tau,z) = \mathrm{Tr}_{\mathcal{S}_m}\Big(g\ (-1)^F q^{H_L} r^{J_L}\Big)\ , \tag{3.3}$$

is the g-twisted elliptic genus. There are symmetries of $K3$ that are not symplectic automorphisms that gives rise to other conjugacy classes of M_{24}. In these cases as well, we expect formula Eq. (3.1) to hold for the generating function of twisted $\frac{1}{4}$-BPS states. However, it is not known whether they are modular forms. The goal of this section is to provide evidence that $\mathcal{S}_m$ is an M_{24}-module graded by the L_0 and J_L eigenvalues. Hence the above formula can interpreted as a trace over this module with g taken to be an element of M_{24}.

We have already seen that the modules $\mathcal{S}_m$ can be expressed in terms of $\mathcal{K}$. The 'building blocks' of $\mathcal{S}_m$ are $S^m(\mathcal{K})$ and $\mathcal{K}_{(m)}$. If g acts on $\mathcal{K}$, then it acts naturally as $\otimes^m g$ on the tensor product space $\mathcal{K}^{\otimes m}$. This induces an action on $S^m(\mathcal{K})$.[8] Given that $\mathcal{K}_{(m)}$ differs from $\mathcal{K}$ only in the L_0-grading and g commutes with L_0, one may guess g acts exactly as it did on $\mathcal{K}$. With this in mind, we shall consider S_1 and compute the g-twisted elliptic genus for $S^2(K3)$.

$$\begin{aligned}\chi(S^2(K3);\tau,z) &= \mathrm{Tr}_{S^2(\mathcal{K})}\Big(g\ q^{L_0-1}r^{J_L}(-1)^F\Big) + \mathrm{Tr}_{\mathcal{K}^{\mathbb{Z}_2}_{(2)}}\Big(g\ q^{L_0-1}r^{J_L}(-1)^F\Big)\\ &= \frac{1}{2}\Big[\big(\psi^{[g,1]}_{0,1}(\tau,z)\big)^2 + \psi^{[g^2,1]}_{0,1}(2\tau,2z)\big) + \sum_{\ell=0}^{1}\psi^{[g,1]}_{0,1}(\tfrac{\tau+\ell}{2},z)\Big]\ .\end{aligned} \tag{3.4}$$

Notice the appearance of the g^2-twisted elliptic genus of $K3$ in the above formulae. Thus we see that the g-twisted elliptic genus for $S^2(K3)$ satisfies a doubling formula as for the untwisted elliptic genus once we have included the action of g on the various building blocks. We have experimentally verified that the above formulae holds when g has order 2 (conjugacy class 2A) and 3 (conjugacy class 3A) – this is possible as we already have explicit formulae for the second-quantized elliptic genus for all conjugacy classes that appear in Table 1 [1]. Eq. (3.4) holds for all conjugacy classes of M_{24} if we assume that $\mathcal{K}$ is a M_{24}-module. We can thus anticipate that replication formulae should be present for the g-twisted elliptic genus for all $S^m(K3)$. We propose that all such replication formulae can be written in terms of a *twisted* Hecke-like operator, which we denote by T_m, that naturally generalizes the DMVV formula.

$$\boxed{\Big(1+\sum_{m=1}^{\infty} s^m\ \chi^g(S^m(K3);\tau,z)\Big) = \exp\Big(-\sum_{m=1}^{\infty} s^m\ T_m\cdot\psi^{[g,1]}_{0,1}(\tau,z)\Big)}\ , \tag{3.5}$$

where the twisted Hecke-like operator, T_m defined below, generates a Jacobi form of weight 0 and index m when acting on a weak Jacobi form of weight 0 and index

[8] On occasion, we shall be somewhat cavalier and use the same symbol g to represent $\otimes^m g$ and its action can be inferred by the module on which it acts.

1.

$$\boxed{T_m \cdot \psi_{0,1}^{[g,h]}(\tau,z) \equiv \frac{1}{m} \sum_{ad=m} \sum_{b=0}^{d-1} \psi_{0,1}^{[g^a h^{-b}, h^d]}\left(\tfrac{a\tau+b}{d}, az\right)} \ . \tag{3.6}$$

The superscript $[g,h]$ indicates the boundary conditions. This extends the considerations of Tuite [30], who defined such an operator for modular forms, to Jacobi forms. The important ingredient is that the boundary conditions also change under the action of $GL(2,\mathbb{Z})$. One has

$$\underset{h}{g\,\square} \xrightarrow{\gamma} \underset{g^c h^d}{g^a h^{-b}\,\square}\ , \qquad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z}) \ . \tag{3.7}$$

We have experimentally verified that the replication formulae hold for twisted elliptic genera of $S^m(K3)$ for $m = 2,3$ and for some choices of g and h. In fact, we first obtained these replication formulae experimentally before realizing that these are captured by the obvious generalization of Tuite's considerations to Jacobi forms. Such twisted Hecke-like operators have also appeared in the considerations of Ganter [31,32].

3.1 An infinite number of moonshines

We have seen that each of the spaces $\mathcal{S}_m$ are M_{24}-modules. Traces over them lead to Jacobi forms of weight zero and index $(m+1)$, one for each conjugacy class of M_{24} appearing in Table 1. The first of these is the moonshine associated with the elliptic genus of $K3$. Of course, the Siegel modular form subsumes all these modular forms and hence the moonshine associated with $\Phi_{10}(\mathbb{Z})$ subsumes all these moonshines for M_{24}. It is important to note that on decomposing any of the Siegel modular forms into Jacobi forms and η-products, all of them are associated with the *same* conjugacy class.

4 Borcherds Product Formulae from moonshine

We shall now show that Eq. (3.5) leads to a Borcherds product product formula for second-quantized twisted elliptic genus and hence for the corresponding Siegel modular form. These product formulae will be shown to agree with known product formulae thus providing evidence for the conjectured form of the Hecke-like operator given in Eq. (3.6). For other situations, in particular to all $g \in M_{24}$ that either have order > 8 or have conjugacy classes that do not reduce conjugacy classes of M_{23}, we obtain Borcherds product formulae for such conjugally classes.

Let $g \in M_{24}$. The Fourier-Jacobi expansion of the weak Jacobi form $\psi_{0,1}^{[g^a,1]}(\tau,z)$ is

$$\psi_{0,1}^{[g^a,1]}(\tau,z) := \sum_{n=0}^{\infty} \sum_{\substack{\ell\in\mathbb{Z} \\ 4n-\ell^2\geq 0}} c^a(n,\ell)\ q^n r^\ell \ . \tag{4.1}$$

When g has order N, a is defined modulo N since $\psi_{0,1}^{[g^a,1]}(\tau,z)=\psi_{0,1}^{[g^b,1]}(\tau,z)$ for all $a\equiv b \bmod N$. One has

$$\sum_{b=0}^{d-1}\psi_{0,1}^{[g^a,1]}\Big(\tfrac{a\tau+b}{d},az\Big)=\sum_{\widetilde{n}=0}^{\infty}\sum_{\substack{\ell\in\mathbb{Z}\\4\widetilde{n}-\ell^2\geq 0}} c^a(\widetilde{n},\ell)\ q^{a\widetilde{n}/d}r^{a\ell}\Big(\sum_{b=0}^{d-1}\beta^{b\widetilde{n}}\Big) \tag{4.2}$$

$$=d\sum_{n=0}^{\infty}\sum_{\substack{\ell\in\mathbb{Z}\\4nd-\ell^2\geq 0}} c^a(dn,\ell)\ q^{an}r^{a\ell}\ , \tag{4.3}$$

where $\beta:=\exp(2\pi i/d)$ is a d-th root of unity. In the second line, we have used the following result to set $\widetilde{n}=dn$

$$\sum_{b=0}^{d-1}\beta^{b\widetilde{n}}=\begin{cases}0, & \widetilde{n}\neq 0 \mod d\\ d, & \widetilde{n}=0 \mod d\end{cases}\ .$$

Now consider the argument of the exponential in Eq. (3.5)

$$\begin{aligned}\sum_{\widetilde{m}=1}^{\infty}s^{\widetilde{m}}\ T_{\widetilde{m}}\cdot\psi_{0,1}^{[g,1]}(\tau,z)&=\sum_{\widetilde{m}=1}^{\infty}\frac{s^{\widetilde{m}}}{\widetilde{m}}\sum_{\substack{ad=\widetilde{m}\\a\in\mathbb{Z}_{>0}}}\sum_{b=0}^{d-1}\psi_{0,1}^{[g^a,1]}\Big(\tfrac{a\tau+b}{d},az\Big)\\&=\sum_{\widetilde{m}=1}^{\infty}\sum_{\substack{ad=\widetilde{m}\\a\in\mathbb{Z}_{>0}}}\sum_{n=0}^{\infty}\sum_{\substack{\ell\in\mathbb{Z}\\4nd-\ell^2\geq 0}} c^a(dn,\ell)q^{an}r^{a\ell}\frac{s^{ad}}{a}\\&=\sum_{a=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{\substack{\ell\in\mathbb{Z}\\4nm-\ell^2\geq 0}} c^a(nm,\ell)\ \frac{q^{an}r^{a\ell}s^{am}}{a}\ ,\end{aligned} \tag{4.4}$$

where we have reorganized the sum in the last row and relabeled d as m. Let the discrete Fourier transform of the Fourier-Jacobi coefficient be given by

$$c^a(n,\ell)=\sum_{\alpha=0}^{N-1}\omega^{\alpha a}c_\alpha(n,\ell)\ , \tag{4.5}$$

where $\omega=\exp(2\pi i/N)$ is an N-th root of unity. Using this transform, we can carry out the summation over a in the last line of Eq. (4.4) to obtain

$$\sum_{\widetilde{m}=1}^{\infty}s^{\widetilde{m}}T_{\widetilde{m}}\cdot\psi_{0,1}^{[g,1]}(\tau,z)=-\sum_{\alpha=0}^{N-1}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{\substack{\ell\in\mathbb{Z}\\4nm-\ell^2\geq 0}}\log\Big(1-\omega^\alpha q^n r^\ell s^m\Big)^{c_\alpha(nm,\ell)}\ . \tag{4.6}$$

Taking the exponential on both sides, we get

$$\exp\Big(-\sum_{\widetilde{m}=1}^{\infty} s^{\widetilde{m}} T_{\widetilde{m}} \cdot \psi_{0,1}^{[g,1]}(\tau,z)\Big) = \prod_{\alpha=0}^{N-1}\prod_{m=1}^{\infty}\prod_{n=0}^{\infty}\prod_{\substack{\ell\in\mathbb{Z}\\ 4nm-\ell^2\geq 0}} \Big(1-\omega^{\alpha}q^n r^{\ell} s^m\Big)^{c_\alpha(nm,\ell)} \ . \tag{4.7}$$

Note that the product over m runs from 1 to ∞. This implies a Borcherds product formula for the associated Siegel modular form after multiplying by $s\ \phi_{k,1}^{[g,1]}(\tau,z)$. We obtain

$$\boxed{\Phi_k^{[g,1]}(\mathbf{Z}) = s\ \phi_{k,1}^{[g,1]}(\tau,z) \times \prod_{\alpha=0}^{N-1}\prod_{m=1}^{\infty}\prod_{n=0}^{\infty}\prod_{\substack{\ell\in\mathbb{Z}\\ 4nm-\ell^2\geq 0}} \Big(1-\omega^{\alpha}q^n r^{\ell} s^m\Big)^{c_\alpha(nm,\ell)}} \ . \tag{4.8}$$

This formula is precisely the one obtained by David-Jatkar-Sen when g generates a symplectic automorphism [33, see Eq. (3.17)]. All one needs is to observe that

$$\psi_{0,1}^{[g^a,1]}(\tau,z) = NF^{0,a}(\tau,z) \ ,$$

in their notation. It can be shown that this formula is equivalent to alternate versions of the formulae due to Gritsenko-Nikulin [34] and Aoki-Ibukiyama [35] as well as an earlier formula also due to David-Jatkar-Sen [3]. In all other cases, we obtain a Borcherds product formula for the generating function of twisted $\frac{1}{4}$-BPS states.

This concludes the proof that the twisted Hecke operator and the formula (3.5) for the twisted second-quantized elliptic genus holds. Further, with the assumption that $\mathcal{K}$ is an M_{24}-module, we see that all modules $\mathcal{S}_m$ for $m>0$ are also M_{24} modules. Thus, we obtain a self-consistent picture that the module $\mathcal{S}$ is an M_{24}-module.

4.1 M_{23} vs M_{24}

The condition that any symplectic automorphism, g must be contained in a M_{23} subgroup of M_{24} might seem to suggest that $V^\natural$ must be a M_{23}-module rather than a M_{24} module. Recall that in the realization of M_{24} as a permutation group, M_{23} is a subgroup of M_{24} that preserves one element. The key point is that the action of a symplectic automorphism depends only on its conjugacy class – it is possible to find two different realizations of a symplectic automorphism, g, that preserve distinct elements of $H^*(K3,\mathbb{Z})$ – this is related to distinct symplectic structures.

There are two implications that arise from the claim that $V^\natural$ is a M_{24}-module. First, there are elements of M_{23} with order >8 – formally, one can insert such elements into the trace over $V^\natural$. Second, there exist elements of M_{24} that are not in M_{23}. While we do not have an answer in full generality, we will comment on specific instances of these two possibilities below.

1. For instance, let us consider the case when g has order 11. There is indeed a multiplicative eta product for the cycle shape $1^2 11^2$. The trace over the $V^\natural$ leads to the product formula given in Eq. (4.8) for a possible Siegel modular form of weight 0. However, the naive formula for the additive lift does not work. Eguchi and Hikami have recently shown that there exists a modification due to the appearance of a new form for $\Gamma_0(11)$ and matches the terms appearing in the product formula [36].

2. Since Jacobi forms have been constructed for all conjugacy classes of M_{24} [23,27,28], it appears that we must obtain a product formula for the corresponding elliptic genus and thence a product formula formula for a (potential) Siegel modular form for every conjugacy class of M_{24} including those that do not reduce to conjugacy classes of M_{24}. One such conjugacy class is 2^{12} that has been recently shown to be a symmetry of the conformal field theory albeit not a symplectic one [37]. The Jacobi form as well as the eta product for the conjugacy class 2^{12} are both modular forms of $\Gamma_0(4)$. Should one expect a Siegel modular form of weight 4 at level four? It has been argued in [2] that the same cycle shape is associated with a pair of symplectic automorphisms and that leads to a Siegel modular form of the paramodular group at level two. This appeared in the work of Clery and Gritsenko [38]. We do not have a definitive answer on this case. So we will conclude this discussion with a couple of questions that we hope to address in the future. *Should the twisted Hecke-like operator defined in Eq. (3.6) be modified for these conjugacy classes?* Indeed there is a modification that appears for the paramodular group in ref. [38] and one needs to study if that is relevant for these cases. *Is there no factor that converts the second-quantized elliptic genus to a Siegel modular form for conjugacy classes of M_{24} that do not reduce to M_{23} conjugacy classes?* It may be that the trace over the module $V^\natural$ is not a modular form – recall that in all situations, the second-quantized elliptic genus needs to be multiplied by a factor (called the Hodge anomaly by Gritsenko [39]) to become a modular form. Note that an affirmative answer doesn't contradict the existence of $V^\natural$. However, if the factor exists and we do obtain a modular form, it must arise as one of the modular forms constructed by Clery and Gritsenko [38].

5 Conclusion

In this paper, we have shown that the $D1-D5-KK-p$ system indeed provides M_{24}-module. As a by-product of our investigation, we have implicitly obtained product formulae when the element g has orders six and eight. These were already obtained using the additive (Saito-Kurokawa-Maaß) lift in ref. [1]. It would be nice to see if there is Lie algebraic structure underlying these two modular forms

of the kind when the order of g was less than six [40]. It is obvious that the replication formulae that we considered here also hold for the moonshine proposed by us in ref. [41] for the Mathieu group, M_{12}. This more or less follows from the observation in that paper that the twisted elliptic genus of $K3$ decomposes into to a sum of two terms along with a similar decomposition for the Siegel modular form. This implies that there exists an M_{12}-module, $\widehat{V}^\natural$ such that $V^\natural = \widehat{V}^\natural \otimes \widehat{V}^\natural$.

An obvious extension of our considerations is to consider the modular forms that arise in CHL orbifolds – these were denoted by $\widetilde{\Phi}_k(\mathbf{Z})$ by Jatkar-Sen [29]. In the notation of this paper, these modular forms should be denoted by $\Phi_k^{[1,h]}(\mathbf{Z})$ where h is an element of M_{24}. As argued by us elsewhere [2], these should be considered in the context of generalized moonshine in the sense of Norton [16, see appendix by Norton]. The analog of $V^\natural$ in this context should be a twisted version that we shall call $V_h^\natural$. Now consider insertions of another element g of M_{24} that commutes with h into the various trace. This should give rise the modular forms that count twisted dyons in the CHL orbifold [2]. The details of the generalized moonshine will be discussed elsewhere [41]. These also provided realizations of conjugacy classes of M_{24} that do not reduce to conjugacy classes of M_{23}. For instance, the class $\rho = 2^{12}$ is associated with two commuting elements of M_{24}, each of order two.

Acknowledgments: We thank Karthik Inbasekar, Dileep P. Jatkar and K. Gopala Krishna for comments on a draft of this paper. We also thank the organizers of the Conference on Modular Forms and Mock Modular Forms and their Applications in Arithmetic, Geometry and Physics held at ASICTP, Trieste during 14-18 March, 2011 as well as the organizers of the Workshop on Mathieu Moonshine (7-9 July 2011) at ETH, Zürich for an opportunity to present this work.

A The elliptic genus

Let M denote a Kähler manifold of complex dimension d and $\mathcal{H}(M)$ denote the Hilbert space in the RR sector of the two-dimensional $\mathcal{N} = (2,2)$ Super Conformal Field Theory (SCFT) that arises from the supersymmetric nonlinear sigma model with target space M. The elliptic genus is given by [4]

$$\chi(M;\tau,z) = \mathrm{Tr}_{\mathcal{H}(M)}\Big((-1)^F q^{H_L}\ r^{J_L}\Big)\ , \tag{A.1}$$

where $H_L = \big(L_0 - \frac{d}{8}\big)$, J_L is the $U(1)$ R-charge for left-movers and $F = F_L + F_R$ is the total fermion number. When M is a Calabi-Yau manifold as we shall assume henceforth, the elliptic genus is a Jacobi form of weight 0 and index $d/2$. Further, $\chi(M;\tau,0)$ is independent of τ and gives the Euler characteristic of M which we denote by χ_M. Let g denote the generator of a symmetry of the SCFT (of finite order) that commutes with the right-moving supersymmetry generators. One

defines the g-twisted elliptic genus as follows:

$$\chi^g(M;\tau,z) \equiv \mathrm{Tr}_{\mathcal{H}(M)}\Big(g\ (-1)^F q^{H_L}\ r^{J_L}\Big)\ . \tag{A.2}$$

As in the untwisted case, the $z=0$ limit is τ-independent and defines the twisted Euler characteristic, χ^g_M. This obtains contributions from g-invariant harmonic forms when g acts as a permutation on $H^*(M,\mathbb{Z})$. In such cases, the g-twisted Euler characteristic is equal to the number of one-cycles in the conjugacy class of g in the appropriate permutation group.

When M is hyper-Kähler (with $d=2k$), then the supersymmetry is enhanced to $\mathcal{N}=(4,4)$ and J_L is identified with twice the Cartan generator of $SU(2)$ R-symmetry. Eguchi and Hikami [25,26] have shown that the elliptic genus hyper-Kähler manifolds can be expanded in terms of massless and massive characters of the level k $\mathcal{N}=4$ SCA. In particular, one has (see [42] for precise definitions)

$$\chi(M;\tau,z) = \chi_M\ \mathcal{C}_k(\tau,z) + \sum_{a=1}^{k} \Sigma^{(a)}(\tau)\ \mathcal{B}_k^{(a)}(\tau,z)\ , \tag{A.3}$$

where $\mathcal{C}_k(\tau,z)$ is a massless Ramond character (with isospin zero) and $\mathcal{B}_k^{(a)}(\tau,z)$ are massive Ramond characters.

$$\begin{aligned}\mathcal{C}^{(k)}(\tau,z) &= \mathrm{ch}^{\tilde{R}}_{k,k/2,0}\\ &= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\frac{i}{\theta_1(\tau,2z)}\sum_{n\in\mathbb{Z}} q^{(k+1)n^2} r^{2(k+1)n}\frac{1+q^n r}{1-q^n r}\ .\end{aligned} \tag{A.4}$$

$$\mathcal{B}_a^{(k)}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\frac{\vartheta_{k+1,a}(\tau,z)-\vartheta_{k+1,-a}(\tau,z)}{\vartheta_{2,1}(\tau,z)-\vartheta_{2,-1}(\tau,z)}\ ,\quad a=1,2,\ldots,k\ . \tag{A.5}$$

In particular, one has $\mathcal{C}_k(\tau,0)=1$ and $\mathcal{B}_k^{(a)}(\tau,0)=0$.

A.1 A family of hyperKähler manifolds

Choose $M=K3$. The elliptic genus of $K3$ is a weight-zero index-one Jacobi form. One has

$$\chi(K3;\tau,z) = 8\sum_{i=2}^{4}\left(\frac{\theta_i(\tau,z)}{\theta_i(\tau,0)}\right)^2\ . \tag{A.6}$$

Setting $z=0$, we see that $\chi_{K3}=24$ as expected. Further the elliptic genus of $K3$ can be expanded in terms of characters of the level-one $\mathcal{N}=4$ SCA.

$$\chi(K3;\tau,z) = 24\ \mathcal{C}_1(\tau,z) + \text{massive characters}\ . \tag{A.7}$$

An infinite family of hyperKähler manifolds are obtained by considering symmetric products of $K3$ [4]. We will denote them by $S^n(K3)$ and are of complex

dimension $d = 2n$. The elliptic genus of $S^n(K3)$ is a Jacobi form of weight-zero and index n and can also be expanded in terms of characters of the level n $\mathcal{N} = 4$ SCA. In particular, one has

$$\chi(S^n(K3); \tau, z) = \chi_{S^n(K3)} \times \mathcal{C}_n(\tau, z) + \text{massive characters} \ . \tag{A.8}$$

The space $S^n(K3)$ naturally arises as the moduli space of n zero-brane probes or equivalently as the Hilbert scheme of n points on a $K3$ surface [43].

References

[1] S. Govindarajan and K. Gopala Krishna, "BKM Lie superalgebras from dyon spectra in $\mathbb{Z}_N$ CHL orbifolds for composite N," *JHEP* **05** (2010) 014, arXiv:0907.1410 [hep-th].

[2] S. Govindarajan, "BKM Lie superalgebras from counting twisted CHL dyons," *JHEP* **1105** (2011) 089, arXiv:1006.3472 [hep-th].

[3] J. R. David, D. P. Jatkar, and A. Sen, "Product representation of dyon partition function in CHL models," *JHEP* **06** (2006) 064, arXiv:hep-th/0602254.

[4] R. Dijkgraaf, G. W. Moore, E. P. Verlinde, and H. L. Verlinde, "Elliptic genera of symmetric products and second quantized strings," *Commun. Math. Phys.* **185** (1997) 197–209, arXiv:hep-th/9608096.

[5] J. R. David and A. Sen, "CHL dyons and statistical entropy function from D1-D5 system," *JHEP* **11** (2006) 072, arXiv:hep-th/0605210.

[6] D. Shih, A. Strominger, and X. Yin, "Recounting Dyons in N=4 string theory," *JHEP* **0610** (2006) 087, arXiv:hep-th/0505094 [hep-th].

[7] A. Sen, "Walls of Marginal Stability and Dyon Spectrum in $\mathcal{N} = 4$ Supersymmetric String Theories," *JHEP* **05** (2007) 039, arXiv:hep-th/0702141.

[8] A. Dabholkar, D. Gaiotto, and S. Nampuri, "Comments on the spectrum of CHL dyons," *JHEP* **01** (2008) 023, arXiv:hep-th/0702150.

[9] V. A. Gritsenko and V. V. Nikulin, "Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras," *Amer. J. Math.* **119** no. 1, (1997) 181–224, alg-geom/9504006.

[10] M. C. N. Cheng and E. P. Verlinde, "Wall Crossing, Discrete Attractor Flow, and Borcherds Algebra," *SIGMA* **4** no. 068, (2008) 33, arXiv:0806.2337 [hep-th].

[11] A. Sen, "Black holes and the spectrum of half-BPS states in $\mathcal{N} = 4$ supersymmetric string theory," *Adv. Theor. Math. Phys.* **9** (2005) 527–558, arXiv:hep-th/0504005.

[12] S. Mukai, "Finite groups of automorphisms of $K3$ surfaces and the Mathieu group," *Invent. Math.* **94** no. 1, (1988) 183–221.

[13] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, and R. Abbott, *ATLAS of Finite Group Representations v3*. http://brauer.maths.qmul.ac.uk/Atlas/v3/.

[14] D. Dummit, H. Kisilevsky, and J. McKay, "Multiplicative products of η-functions," in *Finite groups—coming of age (Montreal, Que., 1982)*, vol. 45 of *Contemp. Math.*, pp. 89–98. Amer. Math. Soc., Providence, RI, 1985.

[15] G. Mason, "M_{24} and certain automorphic forms," in *Finite groups—coming of age (Montreal, Que., 1982)*, vol. 45 of *Contemp. Math.*, pp. 223–244. Amer. Math. Soc., Providence, RI, 1985.

[16] G. Mason, "Finite groups and modular functions," in *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, vol. 47 of *Proc. Sympos. Pure Math.*, pp. 181–210. Amer. Math. Soc., Providence, RI, 1987. With an appendix by S. P. Norton.

[17] M. R. Douglas, "Branes within branes," arXiv:hep-th/9512077 [hep-th].

[18] C. Vafa, "Instantons on D-branes," *Nucl.Phys.* **B463** (1996) 435–442, arXiv:hep-th/9512078 [hep-th].

[19] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, "Counting Dyons in $\mathcal{N} = 4$ String Theory," *Nucl. Phys.* **B484** (1997) 543–561, arXiv:hep-th/9607026.

[20] R. Dijkgraaf, "Fields, strings, matrices and symmetric products," arXiv:hep-th/9912104 [hep-th].

[21] M. Eichler and D. Zagier, *The theory of Jacobi forms*, vol. 55 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[22] T. Eguchi, H. Ooguri, and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," arXiv:1004.0956 [hep-th].

[23] M. C. Cheng, "K3 Surfaces, N=4 Dyons, and the Mathieu Group M24," *Commun.Num.Theor.Phys.* **4** (2010) 659–708, arXiv:1005.5415 [hep-th].

[24] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu twining characters for K3," *JHEP* **09** (2010) 058, arXiv:1006.0221 [hep-th].

[25] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions," *J. Phys.* **A42** (2009) 304010, arXiv:0812.1151 [math-ph].

[26] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions 2. Rademacher Expansion for K3 Surface," *Commun. Number Theory Phys.* **3** no. 3, (2009) 531–554, arXiv:0904.0911 [math-ph].

[27] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," *JHEP* **1010** (2010) 062, arXiv:1008.3778 [hep-th].

[28] T. Eguchi and K. Hikami, "Note on Twisted Elliptic Genus of K3 Surface," *Phys.Lett.* **B694** (2011) 446–455, arXiv:1008.4924 [hep-th].

[29] D. P. Jatkar and A. Sen, "Dyon spectrum in CHL models," *JHEP* **04** (2006) 018, arXiv:hep-th/0510147.

[30] M. P. Tuite, "Monstrous and generalized Moonshine and permutation orbifolds," in *Moonshine: the first quarter century and beyond*, vol. 372 of *London Math. Soc. Lecture Note Ser.*, pp. 378–392. Cambridge Univ. Press, Cambridge, 2010. arXiv:0811.4525 [math.QA].

[31] N. Ganter, "Hecke operators in equivariant elliptic cohomology and generalized Moonshine," in *Groups and symmetries*, vol. 47 of *CRM Proc. Lecture Notes*, pp. 173–209. Amer. Math. Soc., Providence, RI, 2009. arXiv:0706.2898 [math].

[32] N. Ganter, *Orbifold genera, product formulas and power operations.* ProQuest LLC, Ann Arbor, MI, 2004. arXiv:math/0407021. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[33] J. R. David, D. P. Jatkar, and A. Sen, "Dyon spectrum in generic $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifolds," *JHEP* **01** (2007) 016, arXiv:hep-th/0609109.

[34] V. A. Gritsenko and V. V. Nikulin, "Automorphic forms and Lorentzian Kac-Moody algebras. II," *Internat. J. Math.* **9** no. 2, (1998) 201–275.

[35] H. Aoki and T. Ibukiyama, "Simple graded rings of Siegel modular forms, differential operators and Borcherds products," *Internat. J. Math.* **16** no. 3, (2005) 249–279.

[36] T. Eguchi and K. Hikami, "Twisted Elliptic Genus for K3 and Borcherds Product," arXiv:1112.5928 [hep-th]. 17 pages.

[37] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Symmetries of K3 sigma models," arXiv:1106.4315 [hep-th].

[38] F. Clery and V. Gritsenko, "The Siegel modular forms of genus 2 with the simplest divisor," arXiv:0812.3962 [math.NT].

[39] V. Gritsenko, "Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms," *Algebra i Analiz* **11** no. 5, (1999) 100–125, math/9906190v1.

[40] S. Govindarajan and K. Gopala Krishna, "Generalized Kac-Moody Algebras from CHL dyons," *JHEP* **04** (2009) 032, arXiv:0807.4451 [hep-th].

[41] S. Govindarajan, "Generalized Mathieu Moonshine from counting dyons,". (to appear).

[42] T. Eguchi and K. Hikami, "$\mathcal{N} = 4$ Superconformal Algebra and the Entropy of HyperKahler Manifolds," *JHEP* **02** (2010) 019, arXiv:0909.0410 [hep-th].

[43] L. Göttsche, "The Betti numbers of the Hilbert scheme of points on a smooth projective surface," *Math. Ann.* **286** no. 1-3, (1990) 193–207.

arXiv:2011.07922v3 [hep-th] 22 Jan 2021

Jan 2021 (v2.1)
arXiv:2011.07922 [hep-th]

Mathieu Moonshine and Siegel Modular Forms

Suresh Govindarajan[†] and Sutapa Samanta[*]
[†]Department of Physics, Indian Institute of Technology Madras , Chennai 600036 INDIA
[*]School of Physical Sciences, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700032, INDIA
Email: [†]suresh@physics.iitm.ac.in, [*]psss2238@iacs.res.in

Abstract

A second-quantized version of Mathieu moonshine leads to product formulae for functions that are potentially genus-two Siegel Modular Forms analogous to the Igusa Cusp Form. The modularity of these functions do not follow in an obvious manner. For some conjugacy classes, but not all, they match known modular forms. In this paper, we express the product formulae for all conjugacy classes of M_{24} in terms of products of standard modular forms. This provides a new proof of their modularity.

1 Introduction

Modular forms are important ingredients in string theory as well as in mathematical physics. The generating function of a $\frac{1}{2}$-BPS state of $\mathcal{N}=4$ superstring theory obtained by compactifying the heterotic string on a six-torus is a modular form. Using dualities, the $\mathcal{N}=4$ superstring theory in four dimensions can also be obtained by compactifying type II string theory on $K3\times T^2$. The generating function of the degeneracy of electrically charged $\frac{1}{2}$-BPS state is given by the twenty fourth power of the Dedekind η-function [1]. When the six-torus is twined by a symmetry element, then the generating function is given by a multiplicative η product for a cycle shape that is associated with a conjugacy class of M_{24} [2]. Given a cycle shape ρ, the η-product associated with the cycle shape are defined by the following map:

$$\rho = 1^{a_1}2^{a_2}\cdots n^{a_n} \quad \longrightarrow \quad \eta_\rho(\tau) = \eta(\tau)^{a_1}\eta(2\tau)^{a_2}\cdots\eta(n\tau)^{a_n} \ .$$

A cycle shape is called balanced if there exists a positive integer M such that

$$\rho = 1^{a_1}2^{a_2}\cdots n^{a_n} = \left(\tfrac{M}{1}\right)^{a_1}\left(\tfrac{M}{2}\right)^{a_2}\cdots\left(\tfrac{M}{n}\right)^{a_n} .$$

Conway and Norton [3] observed that all the conjugacy classes of the Mathieu group M_{24} are given by a balanced cycle shape. Thus there is a connection between the generating function of the $\frac{1}{2}$ BPS states and the conjugacy classes of the Mathieu group. This is known as Mathieu moonshine [2, 4, 5].

Another Mathieu moonshine was discovered by Eguchi, Ooguri, and Tachikawa [6] that relates the conjugacy class $1a$ of M_{24} with the elliptic genus of $K3$. This elliptic genus is a weight zero and index one Jacobi form. Later, the Jacobi forms for all other conjugacy classes are constructed [7–9]. Gannon proved the existence of a class function that evaluates to all these Jacobi forms [10]. We list the cycle shapes and the conjugacy classes of M_{24} in Appendix C.

Now, if we consider the $\frac{1}{4}$-BPS states of the same $\mathcal{N}=4$ superstring theory as mentioned earlier, the generating functions of the degeneracy are genus-two Siegel Modular Forms in some cases [2]. In [2], the authors proposed the following map relating cycle shapes to Siegel Modular Forms:

$$\rho = 1^{a_1}2^{a_2}\cdots n^{a_n} \quad \longrightarrow \quad \Phi_k^\rho(\mathbf{Z}) \ ,$$

where $\Phi_k^\rho(\mathbf{Z})$ is a Siegel Modular Form with weight $k=(-2+\sum_i a_i)$ of a suitable subgroup of $Sp(4,\mathbb{Z})$. For conjugacy class $1a$, this is the well-known weight ten Igusa cusp form [1]. The two Mathieu moonshines were combined in a second-quantized version leading to class function that implied a product formula for $\Phi_k^\rho(\mathbf{Z})$ for all conjugacy classes for M_{24} [11]. The proof that these product formulae are indeed Siegel Modular Forms for all conjugacy classes first appeared in [12]. It was also proven for a sub-class that appear in the context of $L_2(11)$ moonshine

in [13]. The square-root of some of these Siegel Modular Forms also appear in the denominator formula of Borcherds-Kac-Moody Lie super-algebra [2, 13–16].

In this paper, we provide a new and distinct proof that the $\Phi_k^\rho(\mathbf{Z})$ are Siegel Modular Forms. The various connections are described in Fig. 1.

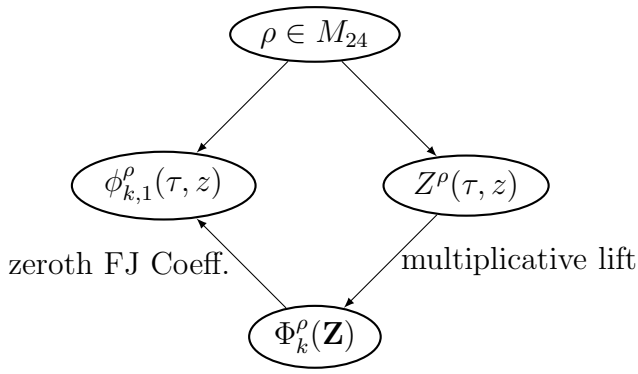


Figure 1: Moonshine for M_{24}. $\phi_{k,1}^\rho(\tau, z)$ is constructed from the multiplicative η-products and $Z^\rho(\tau, z)$ is the twined elliptic genus. $\Phi_k^\rho(\mathbf{Z})$ is the genus-two Siegel Modular Form.

The main result of this paper is the following proposition that we prove.

Proposition 2.2. *For $g \in M_{24}$, let $\rho_m = [g^m]$, $\rho = \rho_1$, $Z_{0,1}^{\rho_m}(\tau, z)$ be the Jacobi form of weight zero and index one corresponding to g^m and $\Phi_k^\rho(\mathbf{Z})$ be the modular form defined by the multiplicative lift given in Eq. (2.6). Then*

$$\left(\Phi_k^\rho(\mathbf{Z})\right)^{p_\rho} = \prod_{m|p_\rho} \left(B_{\frac{p_\rho}{m} Z^{\rho_m}}(m\,\mathbf{Z})\right) \ ,$$

where p_ρ is the length of the shortest cycle in the cycle shape for the conjugacy class ρ and $B_\psi(\mathbf{Z})$ is the Siegel Modular Form given by Theorem 2.1.

Remark: Our proof differs from the one by Persson-Volpato [12] in that we show that the product formula for the conjugacy class of an order N element of M_{24} correspond to Siegel Modular Forms at level N of $Sp(4, \mathbb{Z})$ with character corresponding to a p_ρ-th root of unity, where p_ρ is as defined above. In [12], it was shown that these products correspond to subgroup at higher level $p_\rho N$.[1] Thus our result complements theirs. Our proof also explicitly connects to Borcherds products that appear in number theory. In the process we obtain a simple looking formula that holds for all conjugacy classes.

After the introductory section, in Section 2, we discuss the ideas leading to Proposition 2.2 and a sketch of its proof using two illustrative examples. Section 3 summarises our results and concluding remarks. Appendix A provides the

[1] We thank Daniel Persson and Roberto Volpato for a useful email correspondence.

definitions and examples of modular forms and Jacobi forms. Appendix B lists the Jacobi forms for all conjugacy classes of M_{24}. A fairly long appendix D provides the details of the proof of Proposition 2.2.

2 Construction of Siegel Modular Forms for M_{24}

In this section, we construct genus two Siegel Modular Forms for all the conjugacy classes of M_{24}. We know that there's a weight zero and index one Jacobi form (see Appendix C) [6–9] for every conjugacy class of M_{24}. We apply the multiplicative lift to them to construct the Siegel Modular Forms. This multiplicative lift naturally leads to a product formula. We prove the modularity of the construction in two ways – (i) one is to construct the sum side using the additive lift and showing that this is equal to the product formula, (ii) the other is to compare the product formula with the Borcherds products. The first method works in most cases, while the second method works for all the cases.

2.1 The multiplicative lift

The Mathieu moonshine leads to a Siegel Modular Form which is denoted by $\Phi_k^\rho(\mathbf{Z})$. The formula for $\Phi_k^\rho(\mathbf{Z})$ is given by [11]

$$\Phi_k^\rho = s\,\phi_{k,1}^\rho \times \exp\left(-\sum_{m=1}^{\infty} s^m V_m \cdot Z_{0,1}^\rho(\tau,z)\right) \ , \tag{2.1}$$

where V_m is the generalized Hecke-like operator. This operator acts on a weight zero and index one Jacobi form as follows

$$V_m \cdot Z_{0,1}^\rho(\tau,z) \equiv \frac{1}{m}\sum_{ad=m}\sum_{b=0}^{d-1} Z_{0,1}^{\rho_a}\left(\frac{a\tau+b}{d}, az\right) \ . \tag{2.2}$$

where ρ denotes the conjugacy class of the element g of order N. Then ρ_a denotes the conjugacy class of the element g^a. We set $\rho_0 = 1^{24}$. The Fourier-Jacobi coefficients $f_a(n,\ell)$ of $Z_{0,1}^{\rho_a}(\tau,z)$ is defined as follows:

$$Z_{0,1}^{\rho_a}(\tau,z) = \sum_{n=0}^{\infty}\sum_{\ell\in\mathbb{Z}} f_a(n,\ell) q^n r^\ell \ . \tag{2.3}$$

Let us now define $c^\alpha(n,\ell)$ via following discrete Fourier transform.

$$f_a(n,\ell) = \sum_{\alpha=0}^{N-1} (\omega_N^a)^\alpha\, c^\alpha(n,\ell) \ , \tag{2.4}$$

where $\omega_N = \exp(2\pi i/N)$. The inverse Fourier transform of Eq. (2.4) is given by

$$c^\alpha = \frac{1}{N}\sum_{a=0}^{N-1} \exp(-2\pi i a\alpha/N) f_a \ . \tag{2.5}$$

Using (2.2), we obtain a product formula for (2.1) as follows:

$$\Phi_k^\rho(\mathbf{Z}) = s\,\phi_{k,1}^\rho \times \prod_{m=1}^{\infty}\prod_{\alpha=0}^{N-1}\prod_{n=0}^{\infty}\prod_{\substack{\ell\in\mathbb{Z}\\ 4nm-\ell^2\geq 0}} \left(1-\omega_n^\alpha q^n r^\ell s^m\right)^{c^\alpha(nm,\ell)} \ , \tag{2.6}$$

where the zeroth Fourier-Jacobi term, i.e., the coefficient of s^1 in the Fourier expansion is given by

$$\phi_{k,1}^\rho(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\ \eta_\rho(\tau) \ .$$

Remark: Let p_ρ (resp. N_ρ) be the length of the smallest (resp. largest) cycle in ρ. From the result of Cheng and Duncan [17], one has that $\eta_\rho(\tau)^{p_\rho}$ (and hence $[\phi_{k,1}^\rho(\tau,z)]^{p_\rho}$) is a modular (Jacobi) form for the subgroup $\Gamma_0(N_\rho)$ of $SL(2,\mathbb{Z})$. We thus anticipate that $[\Phi_k^\rho(\mathbf{Z})]^{p_\rho}$ will be a Siegel Modular Form for a level N_ρ subgroup of $Sp(4,\mathbb{Z})$.

For the cases when the order of $g(\in M_{24})$ is prime (i.e., $N = 2,3,5,7,11,23$), one has the conjugacy class g^a for $a \neq 0$ mod N is the same as that of g. Thus, one has $\rho_a = \rho$ for $a \neq 0$ mod N. Now using Eq. (2.5) we get

$$c^0 = \frac{1}{N}(f_0 + (N-1)f_1) \quad \text{and} \quad c^1 = \frac{1}{N}(f_0 - f_1) \ .$$

For these cases, on using the product representation for the theta and η-functions that appear in $\phi_{k,1}^\rho(\tau,z)$ and using $\prod_{n=0}^{N-1}(1-\omega^n x) = (1-x^N)$, the formula given in Eq. (2.6) simplifies to

$$\Phi_k^\rho = qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{c^0(nm,\ell)-c^1(nm,\ell)} \left(1-q^{nN} r^{\ell N} s^{mN}\right)^{c^1(nm,\ell)} \ , \tag{2.7}$$

$$= qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1-q^{nN} r^{\ell N} s^{mN}\right)^{\frac{f_0(nm,\ell)-f_1(nm,\ell)}{N}} \ , \tag{2.8}$$

where $(n,\ell,m) > 0$ implies $n > 0$, or $n = 0$ and $m > 0$, or $n = m = 0$ and $\ell < 0$. We will treat the cases of composite N separately.

2.2 The additive lift

The another construction of $\Phi^\rho(\mathbf{Z})$ is via additive lift. The additive seed is the weak Jacobi form of weight k and index t for the sub-group $\Gamma_0(Nq,q)$. This index

t can be integral or half integral. Let us consider a Jacobi form of weight k and index t, $\phi_{k,t}(\tau, z)$. Then for $k > 0$, the additive lift is given by

$$\Phi^{\rho}(\mathbf{Z}) = \sum_{m=1 \mod q} s^{mt} \phi^{\rho}|_{k,t} T_{-}(m)(\tau, z) \ , \tag{2.9}$$

where $T_{-}(m)$ is the Hecke-like operator introduced by Cléry and Gritsenko [18] and is given by

$$T_{-}(m) := \sum_{\substack{ad=m \\ (a,N)=1 \\ a>0}} \Gamma_1(Nq, q)\, \sigma_a \begin{pmatrix} a & bq \\ 0 & d \end{pmatrix} \ , \tag{2.10}$$

where $\sigma_a = \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \mod Nq$ and $(m, q) = 1$. For all M_{24} conjugacy classes (ρ), we have $t = 1$ and $q = 1$.

2.3 Modularity by comparing with the additive side

The additive seed for these cases are Jacobi forms of weight k and index one,

$$\phi^{\rho}_{k,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6}\ \eta_{\rho}(\tau) = \sum_{n,\ell} a(n, \ell) q^n r^{\ell} \ . \tag{2.11}$$

A necessary but not sufficient condition for the compatibility of the additive lift with the multiplicative lift is:

$$\left[\frac{\theta_1(\tau, z)^2}{\eta(\tau)^6}\eta_{\rho}(\tau)\right]\Bigg|_{k,1} T_{-}(2)(\tau, z) = -Z^{\rho}(\tau, z)\left[\frac{\theta_1(\tau, z)^2}{\eta(\tau)^6}\eta_{\rho}(\tau)\right] \ . \tag{2.12}$$

This is the coefficient of s in both the lifts, i.e., the ones given in Eq. (2.1) and Eq. (2.9). This condition holds for all the cycle shapes of M_{24} except for the following eight cycle shapes where we observe that [19]

$$\begin{aligned}
&-\frac{T_2\phi^{1^2 11^2}(\tau, z)}{\phi^{1^2 11^2}(\tau, z)} - Z^{1^2 11^2}(\tau, z) = \frac{11}{2}\phi^{1^2 11^2}(\tau, z) \ , \\
&-\frac{T_2\phi^{1^1 2^1 7^1 14^1}(\tau, z)}{\phi^{1^1 2^1 7^1 14^1}(\tau, z)} - Z^{1^1 2^1 7^1 14^1}(\tau, z) = 7\phi^{1^1 2^1 7^1 14^1}(\tau, z) \ , \\
&-\frac{T_2\phi^{1^1 3^1 5^1 15^1}(\tau, z)}{\phi^{1^1 3^1 5^1 15^1}(\tau, z)} - Z^{1^1 3^1 5^1 15^1}(\tau, z) = \frac{15}{2}\phi^{1^1 3^1 5^1 15^1}(\tau, z) \ , \\
&-\frac{T_2\phi^{2^2 10^2}(\tau, z)}{\phi^{2^2 10^2}(\tau, z)} - Z^{2^2 10^2}(\tau, z) = 10\phi^{2^2 10^2}(\tau, z) \ , \\
&-\frac{T_2\phi^{2^1 4^1 6^1 12^1}(\tau, z)}{\phi^{2^1 4^1 6^1 12^1}(\tau, z)} - Z^{2^1 4^1 6^1 12^1}(\tau, z) = 12\phi^{2^1 4^1 6^1 12^1}(\tau, z) \ ,
\end{aligned} \tag{2.13}$$

and

$$
\begin{aligned}
-\frac{T_2\phi^{4^6}(\tau,z)}{\phi^{4^6}(\tau,z)} - Z^{4^6}(\tau,z) &= 16\frac{\eta(2\tau)^4\eta(8\tau)^4}{\eta(4\tau)^4}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\ , \\
-\frac{T_2\phi^{3^8}(\tau,z)}{\phi^{3^8}(\tau,z)} - Z^{3^8}(\tau,z) &= 18\frac{\eta(\tau)^3\eta(9\tau)^3}{\eta(3\tau)^2}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\ , \\
-\frac{T_2\phi^{6^4}(\tau,z)}{\phi^{6^4}(\tau,z)} - Z^{6^4}(\tau,z) &= 2\left(\eta_{6^4}(\tau) + \frac{\eta(\tau)^3\eta(9\tau)^3}{\eta(3\tau)^2} + 6\frac{\eta(2\tau)^3\eta(18\tau)^3}{\eta(6\tau)^2}\right. \\
&\qquad \left. + \frac{\eta(4\tau)^3\eta(36\tau)^3}{\eta(12\tau)^2} + \cdots\right)\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\ ,
\end{aligned}
\tag{2.14}
$$

where $T_2\phi^\rho$ is short for $\phi^\rho|_{k,1}T_-(2)$. All the above Jacobi forms except the ones correspond to 3^8 and 4^6 are of weight $k=0$ and we have naïvely applied a formula which assumes $k>0$.

Even for the cases for which the compatibility condition Eq. (2.12) holds, we need to prove that all other terms also match. We do not pursue this method here. We prove the modularity using another method which we discuss next.

2.4 Modularity by comparing with a Borcherds formula

We begin with a theorem due to Cléry-Gritsenko (see also [20–22]) that leads to a Borcherds product formula for a meromorphic Siegel Modular Form starting from a nearly holomorphic Jacobi form of weight zero and index t.

Theorem 2.1 (Cléry-Gritsenko [18]). *Let ψ be a nearly holomorphic Jacobi form of weight 0 and index t of $\Gamma_0(N)$. Assume that for all cusps of $\Gamma_0(N)$ one has $\frac{h_e}{N_e}c_{f/e}(n,\ell)\in\mathbb{Z}$ if $4nt-\ell^2\leq 0$. Then the product*

$$
B_\psi(\mathbf{Z}) = q^A r^B s^C \prod_{f/e\in\mathcal{P}} \prod_{\substack{n,\ell,m\in\mathbb{Z}\\(n,\ell,m)>0}} \left(1-(q^n r^\ell s^{tm})^{N_e}\right)^{\frac{h_e}{N_e}c_{f/e}(nm,\ell)}\ ,
$$

with

$$
A=\frac{1}{24}\sum_{\substack{f/e\in\mathcal{P}\\ \ell\in\mathbb{Z}}} h_e\, c_{f/e}(0,\ell),\ \ B=\frac{1}{2}\sum_{\substack{f/e\in\mathcal{P}\\ \ell\in\mathbb{Z}_{>0}}} \ell h_e\, c_{f/e}(0,\ell),\ \ C=\frac{1}{4}\sum_{\substack{f/e\in\mathcal{P}\\ \ell\in\mathbb{Z}}} \ell^2 h_e\, c_{f/e}(0,\ell)\ ,
$$

defines a meromorphic Siegel Modular Form of weight

$$
k=\frac{1}{2}\sum_{\substack{f/e\in\mathcal{P}\\ \ell\in\mathbb{Z}}}\frac{h_e}{N_e}c_{f/e}(0,0)
$$

with respect to $\mathbf{\Gamma}_t(N)^+$ possibly with character. The character is determined by the zeroth Fourier-Jacobi coefficient of $B_\psi(\mathbf{Z})$, i.e., the coefficient of s^C in the

Fourier expansion, which is a Jacobi form of weight k and index C of the Jacobi subgroup of $\mathbf{\Gamma}_t(N)^+$.

Here $\mathcal{P}$ are the set of cusps for $\Gamma_0(N)$, h_e is the width of the cusp at $f/e \in \mathcal{P}$ and $N_e = N/e$. For any congruence subgroup Γ, the cusps are defined as the Γ-equivalent classes of $\mathbb{Q} \cup \{\infty\}$ where $\mathbb{Q}$ denotes the set of rational numbers. The width of the cusp $\alpha \in \mathcal{P}$ is the minimal h_e such that $\left(\begin{smallmatrix} 1 & h_e \\ 0 & 1 \end{smallmatrix}\right) \in \gamma^{-1}\Gamma\gamma$, where $\gamma \in SL(2,\mathbb{Z})$ such that $\gamma(\infty) = \alpha$.

Remark: As discussed in [13], the condition on $c_{f/e}(n,\ell)$ for $4nt - \ell^2 \leq 0$ that $\frac{h_e}{N_e} c_{f/e}(n,\ell) \in \mathbb{Z}$ will be relaxed as follows. For all cusps that have identical values of (N_e, h_e), we require that the sum of $\frac{h_e}{N_e} c_{f/e}(n,\ell)$ (with $4nt - \ell^2 \leq 0$) for all such cusps be integral. This ensures that B_ψ has only zeros or poles at all divisors – this is the reason for the condition in the theorem.

Define the projection [23], π_{FE}, as follows

$$\pi_{FE}\left(\phi|M_{f/e}\right)(\tau,z) := \frac{1}{h_e}\sum_{b=0}^{h_e-1} \phi|M_{f/e}(\tau+b,z) \ , \tag{2.15}$$

where $M_{f/e} = \left(\begin{smallmatrix} f & * \\ e & * \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ maps the cusp at $i\infty$ to f/e. It is the Fourier coefficients of the projected Jacobi form defined above that appears in the product formula.

Raum proved the modularity of the products such as those given in Eq. (2.6) by considering products of rescaled Borcherds products [23]. His results were however restricted to only conjugacy classes associated with elements of M_{24} that were primes or powers of primes. Further, there were some computational errors in his work that we fix as well. We not only extend his results but also provide a systematic method of obtaining the precise rescaled Borcherds products that are needed for all conjugacy classes. For all type-I conjugacy classes of M_{24}, we find that the product formula is equivalent to a single Borcherds formula. More generally, we find that the following holds.

Proposition 2.2. *For $g \in M_{24}$, let $\rho_m = [g^m]$, $\rho = \rho_1$, $Z^{\rho_m}_{0,1}(\tau,z)$ be the Jacobi form of weight zero and index one corresponding to g^m and $\Phi^\rho_k(\mathbf{Z})$ be the modular form defined by the multiplicative lift given in Eq. (2.6). Then*

$$\left(\Phi^\rho_k(\mathbf{Z})\right)^{p_\rho} = \prod_{m|p_\rho} \left(B_{\frac{p_\rho}{m} Z^{\rho m}}(m\,\mathbf{Z})\right) \ ,$$

where p_ρ is the length of the shortest cycle in the cycle shape for the conjugacy class ρ and $B_\psi(\mathbf{Z})$ is the Siegel Modular Form given by Theorem 2.1.

Proof. As the proof is mostly done by exhaustion, we illustrate this theorem for two examples, one with $p_\rho = 1$ when N is prime and one with $p_\rho \neq 1$ here. All other conjugacy classes are worked out in the Appendix E.

$\boxed{\mathbf{p}_\rho = \mathbf{1}}$

For all type-I conjugacy classes of M_{24} i.e with $p_\rho = 1$, we see that the multiplicative lift can be expressed in terms single Borcherds product. It is very easy to see for all the conjugacy classes with $N = 1, 2, 3, 5, 7, 11, 23$ where N is the order of the element $g \in M_{24}$. For prime N, there are only two cusps, one is at $i\infty$ (which is $\Gamma_0(N)$ equivalent to $1/N$) and another is at $0/1$. To prove the equality of multiplicative lift and the Borcherds product, we need to show that

$$\pi_{FE}(Z^\rho) = Z^\rho \ , \tag{2.16}$$

$$\pi_{FE}(Z^\rho|S) = \frac{1}{N}\left(Z^{1^{24}} - Z^\rho\right) \ . \tag{2.17}$$

The first equation holds trivially since Z^ρ has only integral powers of q in its Fourier-Jacobi expansion. The second part follows from a calculation.

$$\pi_{FE}(Z^\rho|S)(\tau, z) = \frac{2}{N+1}\phi_{0,1}(\tau, z) + \pi_{FE}(\alpha^{(N)}|S)(\tau)\phi_{-2,1}(\tau, z) \ ,$$

where $\alpha^{(N)}(\tau) = \frac{2N}{N+1}E_2^{(N)}(\tau)$ for $N = 2, 3, 5, 7$. For $N = 11$ and 23, one has

$$\alpha^{(11)}(\tau) = \frac{11}{6}E_2^{(11)}(\tau) - \frac{22}{5}\eta_{1^2 11^2}(\tau) \ ,$$
$$\alpha^{(23)}(\tau) = \tfrac{23}{12}E_2^{(23)}(\tau) - \tfrac{23}{22}f_{23,1}(\tau) - \tfrac{161}{22}f_{23,2}(\tau) \ ,$$

where $f_{23,a}(\tau)$ are defined in Appendix B.1. For $N = 2, 3, 5, 7$, computing $\pi_{FE}\left(\alpha^{(N)}|S\right)(\tau)$, we obtain

$$\pi_{FE}(\alpha^{(N)}|S)(\tau) = -\tfrac{2}{N(N+1)}\sum_{b=0}^{N-1} E_2^{(N)}(\tfrac{\tau+b}{N}) = -\tfrac{2}{(N+1)}E_2^{(N)}\big|_2 U_N = -\tfrac{2}{(N+1)}E_2^{(N)}(\tau) \ ,$$

where U_N is the Hecke operator for $\Gamma_0(N)$ (defined by Atkin and Lehner [24]) and $E_2^{(N)}$ is its eigenform with eigenvalue $+1$. Thus, we get

$$\begin{aligned}\pi_{FE}(Z^\rho|S)(\tau, z) &= \frac{2}{N+1}\phi_{0,1}(\tau, z) - \frac{1}{N}\alpha^{(N)}(\tau)\phi_{-2,1}(\tau, z) \\ &= \frac{2}{N}\phi_{0,1}(\tau, z) - \frac{1}{N}Z^\rho(\tau, z) \\ &= \frac{1}{N}\left(Z^{1^{24}} - Z^\rho\right)(\tau, z) \ ,\end{aligned}$$

which establishes Eq. (2.17) for prime $N = 2, 3, 5, 7$. A similar computation holds for $N = 11$ and 23. For $N = 11$, one can show that

$$\pi_{FE}(\alpha^{(11)}|S)(\tau) = -(1/11)\alpha^{(11)}(\tau) \ ,$$

and for $N = 23$, one can show that

$$\pi_{FE}(\alpha^{(23)}|S)(\tau) = -(1/23)\alpha^{(23)}(\tau) \ .$$

Thus Eq. (2.17) holds for $N = 11$ and 23 as well.

$\boxed{\mathbf{p}_\rho \neq \mathbf{1}}$

Let us consider the cycle shape 12^2. The multiplicative lift leads to the product formula given by (data about the cusps are given in E.2.8)

$$\Phi_{-1}^{12^2}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \Big(1 - q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^2\Big)^{x_1} \Big(1 - (q^n r^\ell s^m)^3\Big)^{x_2} \Big(1 - (q^n r^\ell s^m)^4\Big)^{x_3} \Big(1 - (q^n r^\ell s^m)^6\Big)^{x_4} \Big(1 - (q^n r^\ell s^m)^{12}\Big)^{x_5} \ , \quad (2.18)$$

where

$$\begin{aligned} x_1 &= \frac{1}{2}(f_2(nm,\ell) - f_1(nm,\ell)), \quad x_2 = \frac{1}{3}(f_3(nm,\ell) - f_1(nm,\ell)), \\ x_3 &= \frac{1}{4}(-f_2(nm,\ell) + f_4(nm,\ell)), \\ x_4 &= \frac{1}{3}(f_1(nm,\ell) - f_2(nm,\ell) - f_3(nm,\ell) + f_6(nm,\ell)), \\ x_5 &= \frac{1}{12}(f_0(nm,\ell) + f_2(nm,\ell) - f_4(nm,\ell) - f_6(nm,\ell)) \ . \end{aligned}$$

Now to see if this can be written as the Borcherds product as stated in Theorem 2.1 we need to compute the coefficients at different cusps. We need to combine cusps with same (h_e, N_e) value and one can show that the *only* non-zero contributions are the following:

$$\begin{aligned} \pi_{FE}\Big(Z^{12^2}|M_{\frac{1}{72}}\Big) &= -Z^{12^2}, \\ \pi_{FE}\Big(Z^{12^2}|M_{\frac{5}{48}} + Z^{12^2}|M_{\frac{1}{48}}\Big) &= -Z^{12^2}, \\ \pi_{FE}\Big(Z^{12^2}|M_{\frac{5}{24}} + Z^{12^2}|M_{\frac{1}{24}}\Big) &= Z^{12^2} \ . \end{aligned}$$

Using these and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{12^2}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} \Big(1 - q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^2\Big)^{-\frac{1}{2}f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^3\Big)^{-\frac{1}{3}f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^6\Big)^{\frac{1}{6}f_1(nm,\ell)} \ . \quad (2.19)$$

This does not give all the terms that appear in the product form given in Eq. (2.18). The missing terms can be accounted for by additional terms leading to

$$\left(\Phi_{-1}^{12^2}(\mathbf{Z})\right)^{12} = B_{12Z^{12^2}}(\mathbf{Z})B_{6Z^{6^4}}(2\mathbf{Z})B_{4Z^{4^6}}(3\mathbf{Z})B_{3Z^{3^8}}(4\mathbf{Z})B_{2Z^{2^{12}}}(6\mathbf{Z})B_{Z^{1^{24}}}(12\mathbf{Z}) \ . \tag{2.20}$$

$\left(\Phi_{-1}^{12^2}(\mathbf{Z})\right)^{12}$ is a meromorphic Siegel Modular Form of weight (-12) at level 24. The zeroth Fourier-Jacobi term is given by the additive seed $\phi_{-1,1}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(12\tau)^2$. □

3 Conclusions

We proved that there is a genus-two Siegel Modular Form (of a level N subgroup of $Sp(2,\mathbb{Z})$) for every conjugacy class of M_{24}. Each of these arise on evaluating the class function for M_{24} in Eq. (2.1) on the conjugacy class. This was achieved by expressing a product formula implied by Mathieu moonshine as rescaled product of the Borcherds products for all conjugacy classes. This makes their modularity manifest. This completes the proof that was initiated by Raum [23] as well as resolving puzzles raised in [11]. Our result also corrects Raum's incorrect results for the cycle shapes $2^{12}, 3^8, 2^4 4^4, 4^6$, and $1^2 2^1 4^1 8^2$.

Generalized Mathieu moonshine deals with pairs of commuting elements of $(g,h) \in M_{24}$. To each pair, we have an eta product as well as a Jacobi form [25]. Again, one has a product formula implied by second-quantized moonshine [11,12]. Their modularity has been proved in [12]. Our methods can be used to provide explicit formulae similar to the ones obtained in this paper.

Umbral moonshine is a generalization of Mathieu moonshine [26,27]. These connect finite groups, vector valued mock modular forms to the Niemeier lattices in 24 dimensions. In a recent paper [16], we have shown that there exist Siegel Modular Forms for the A-series of umbral moonshine. It is of interest if such Siegel Modular Forms appear for all other cases.

It has been shown that there is no point in the space of CFTs associated with the K3 sigma model where M_{24} appears as a symmetry [28]. The best known example obtains $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry [29]. There has been an interesting attempt to combine symmetries at different points in the CFT moduli space and obtain all of M_{24} [30]. It remains to be seen if there is a natural construction of a M_{24} modules that leads to the class functions associated with the Jacobi and Siegel Modular Forms.

Acknowledgments: We thank Mohammed Shabbir and S. Viwanath for useful discussions. SS was supported by the Ramanujan Fellowship of Ayan Mukhopadhyay (IIT Madras) when some of this work was carried.

A The modular group and congruence subgroups

Modular group: It is a group of all fractional linear transformations on the complex upper half plane, i.e, the transformations $z \mapsto \frac{az+b}{cz+d}$, such that a, b, c, and d are all integers and obey $ad - bc = 1$.

These transformations are isomorphic to the group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm\mathbf{1}\}$. The projective special linear group which we denote by Γ, is defined as follows:

$$\Gamma = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} .$$

The complex upper half plane is defined as

$$\mathbb{H}_1 = \{z \in \mathbb{C}; \mathrm{Im}\ (z) > 0\} .$$

The modular group is generated by following two transformation:

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \text{ such that } T \cdot z = z + 1$$
$$S = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \text{ such that } S \cdot z = -\frac{1}{z}$$

Now we define various subgroups of $SL(2,\mathbb{Z})$ which will be useful for later discussions. These subgroups are obtained by imposing certain constrains on the matrix elements and are called congruence subgroups. For an integer $N > 1$, the *principal congruence subgroup* of level N is defined as follows:

$$\Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) | \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N \right\} .$$

Another subgroup known as *Hecke congruence subgroup* of level N denoted by $\Gamma_0(N)$, is defined as

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) | \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod N \right\} .$$

Symplectic group: The symplectic group, $Sp(2,\mathbb{Q})$ is a set of 4×4 matrices written in terms of four 2×2 matrices A, B, C, D (with A, B, C, D taking values in $\mathbb{Q}$) as $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ satisfying $AB^T = BA^T$, $CD^T = DC^T$ and $AD^T - BC^T = I$. This group naturally acts on the Siegel upper half space $\mathbb{H}_2$, as

$$\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \longmapsto M \cdot \mathbf{Z} \equiv (A\mathbf{Z} + B)(C\mathbf{Z} + D)^{-1} . \tag{A.1}$$

The Siegel upper half space is defined as follows:

$$\mathbb{H}_2 = \left\{ \mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in M_2(\mathbb{C}),\ \mathrm{Im}(\mathbf{Z}) > 0 \right\} . \tag{A.2}$$

The paramodular group at level N, $\mathbf{\Gamma}_t(N)$, is a subgroup of $Sp(2,\mathbb{Q})$ is defined as follows:

$$\mathbf{\Gamma}_t(N) = \left\{ \left(\begin{smallmatrix} * & *t & * & * \\ * & * & * & *t^{-1} \\ N & *Nt & * & * \\ *Nt & *Nt & *t & * \end{smallmatrix}\right) \in Sp(2,\mathbb{Q}),\ \text{with } N, t \in \mathbb{Z}_{>0} \text{ and } * \in \mathbb{Z} \right\} . \tag{A.3}$$

The $t = 1, N = 1$ case of the general paramodular group $\mathbf{\Gamma}_t(N)$ is the usual symplectic group over integers, i.e., $\mathbf{\Gamma}_1 = Sp(2,\mathbb{Z})$. The embedding of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ in $\Gamma_t(N)$ is given by

$$\widetilde{\gamma} = \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \equiv \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \ c = 0 \mod N \ . \tag{A.4}$$

The normal double extension of $\mathbf{\Gamma}_t(N)$ in $Sp(2,\mathbb{R})$ is defined as $\mathbf{\Gamma}_t(N)^+ = \mathbf{\Gamma}_t(N) \cup \mathbf{\Gamma}_t(N) V_t$ where

$$V_t = \frac{1}{\sqrt{t}} \left(\begin{smallmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \end{smallmatrix}\right) \in Sp(2,\mathbb{R}) \ . \tag{A.5}$$

As shown by Cléry and Gritsenko [31], this $\mathbf{\Gamma}_t(N)^+$ is generated by V_t and by its parabolic subgroup

$$\mathbf{\Gamma}_t^\infty(N) = \left\{ \pm \left(\begin{smallmatrix} * & 0 & * & * \\ * & 1 & * & *t^{-1} \\ N & 0 & * & * \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \in \Gamma_t(N), \ \forall * \in \mathbb{Z} \right\} \ . \tag{A.6}$$

B Modular forms and Jacobi forms

Modular form: A modular form of weight k, with k being integer or half integer, and character v with respect to any of the subgroups of the modular group Γ is a holomorphic function $f : \mathbb{H}_1 \to \mathbb{C}$ satisfying,

$$(f|_k M)(z) = v(M) f(z) \ ,$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}_1$ and the slash operation is given by

$$(f|_k M)(z) = (c\tau + d)^{-k} f(M \cdot z) \ .$$

Jacobi form: A holomorphic function $\phi_{k,m}(\tau, z) : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$ is called a Jacobi form of weight k and index m if the function

$$\widetilde{\phi}_k(\mathbf{Z}) = \exp(2\pi i m \tau') \phi_{k,m}(\tau, z) \ ,$$

on $\mathbb{H}_2$ such that $\widetilde{\phi}_k(\mathbf{Z})$ is a modular form of weight k with respect to the Jacobi group $\mathbf{\Gamma}^J(N) \subseteq Sp(2,\mathbb{Z})$ with character v, i.e., it satisfies

$$\widetilde{\phi}|_k \ M(\mathbf{Z}) = v(M) \ \widetilde{\phi}_k(\mathbf{Z}) \qquad \forall M \in \Gamma^J(N) \ , \tag{B.1}$$

and it is holomorphic at all cusps, i.e., let $\gamma \in SL(2,\mathbb{Z})$ and $\tilde{\gamma}$, its embedding in $Sp(2,\mathbb{Z})$ as given in Eq. (A.4), then it has a Fourier expansion

$$\widetilde{\phi}|_k\tilde{\gamma}(\mathbf{Z}) = s^m \sum_{\substack{n,\ell \\ 4nm-\ell^2\geq 0}} c_\gamma(n,\ell)\ q^n r^\ell \tag{B.2}$$

where $q = e^{2\pi i\tau}$, $r = e^{2\pi i z}$ and $s = e^{2\pi i\tau'}$, $n, \ell \in \mathbb{Q}$.

Siegel Modular Form: A Siegel Modular Form of weight k for the subgroup $\mathbf{\Gamma}_t(N)$ with a character v is a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ which satisfies

$$(F|_k M)(\mathbf{Z}) = v(M)F(\mathbf{Z}) \ , \tag{B.3}$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{\Gamma}_t(N)$.
Here $|_k$ is the standard slash operator on the space of functions on $\mathbb{H}_2$:

$$(F|_k M)(\mathbf{Z}) := \det(C\mathbf{Z} + D)^{-k} F(M \cdot \mathbf{Z}) \ . \tag{B.4}$$

B.1 Examples of Modular and Jacobi forms

We discuss some examples of modular and Jacobi forms that enter our discussions. The Dedekind eta function is defined by

$$\eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m) \ .$$

Its modular properties are given in Eq. (D.1). $E_2^*(\tau)$ is the weight two non-holomorphic modular form of $SL(2,\mathbb{Z})$. It is given by

$$E_2^*(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\ q^n \ - \frac{3}{\pi\ \mathrm{Im}\tau} \ , \tag{B.5}$$

where $\sigma_\ell(n) = \sum_{1\leq d|n} d^\ell$. Then for $N > 1$,

$$E_2^{(N)}(\tau) := \frac{1}{N-1}\Big(NE_2^*(N\tau) - E_2^*(\tau)\Big) \tag{B.6}$$

is a weight two holomorphic modular form of $\Gamma_0(N)$ with constant coefficient equal to 1. For $\Gamma_0(23)$, we need the following weight-two modular forms that we denote by $f_{23,1}(\tau)$ and $f_{23,2}(\tau)$. The first few terms in their q-series are [8]

$$\begin{aligned} f_{23,1}(\tau) &= 2q - q^2 - q^4 - 2q^5 - 5q^6 + 2q^7 + 4q^9 + 6q^{10} - 6q^{11} + \cdots \ , \\ f_{23,2}(\tau) &= q^2 - 2q^3 - q^4 + 2q^5 + q^6 + 2q^7 - 2q^8 - 2q^{10} - 2q^{11} + \cdots \ . \end{aligned}$$

For $a.b \in (0,1) \mod 2$, the genus-one theta functions are defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau, z) = \sum_{l \in \mathbb{Z}} q^{\frac{1}{2}(l+\frac{a}{2})^2} \; r^{(l+\frac{a}{2})} \; e^{i\pi l b} \; , \tag{B.7}$$

with $r = \exp(2\pi i z)$. Let $\vartheta_1(\tau, z) \equiv \theta \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] (\tau, z)$, $\vartheta_2(\tau, z) \equiv \theta \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] (\tau, z)$, $\vartheta_3(\tau, z) \equiv \theta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] (\tau, z)$ and $\vartheta_4(\tau, z) \equiv \theta \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] (\tau, z)$. Using these, we can construct two index one Jacobi forms:

$$\phi_{0,1}(\tau, z) = 4 \sum_{a=3}^{4} \left[\frac{\vartheta_a(\tau, z)}{\vartheta_a(\tau, 0)} \right]^2 \; , \tag{B.8}$$

$$\phi_{-2,1}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6} \; . \tag{B.9}$$

Any Jacobi form, $Z_{0,1}(\tau, z)$, of weight zero and index one of $\Gamma_0(N)$ $(N > 1)$ can be written as follows [22]

$$Z_{0,1}(\tau, z) = A \; \phi_{0,1}(\tau, z) + B(\tau) \; \phi_{-2,1}(\tau, z) \; ,$$

where A is a constant and $B(\tau)$ is a weight two modular form of $\Gamma_0(N)$.

C List of M_{24} Jacobi forms

There are 26 conjugacy classes of M_{24}. The ATLAS nomenclature of the conjugacy classes are as follows:

$$\text{Type I} : 1a, 2a, 3a, 5a, 4b, 7a, 7b, 8a, 6a, 11a, 15a, 15b, 14a, 14b, 23a, 23b,$$
$$\text{Type II} : 12b, 6b, 4c, 3b, 2b, 10a, 21a, 21b, 4a, 12a.$$

The cycle shapes associated with type I conjugacy classes have at least one one-cycle.

Conj. class	ρ	$Z^\rho(\tau,z)$
1a	1^{24}	$2\ \phi_{0,1}(\tau,z)$
2a	$1^8 2^8$	$\frac{2}{3}\phi_{0,1}(\tau,z)+\frac{4}{3}E_2^{(2)}(\tau)\phi_{-2,1}(\tau,z)$
3a	$1^6 3^6$	$\frac{1}{2}\phi_{0,1}(\tau,z)+\frac{3}{2}E_2^{(3)}(\tau)\ \phi_{-2,1}(\tau,z)$
5a	$1^4 5^4$	$\frac{1}{3}\phi_{0,1}(\tau,z)+\frac{5}{3}E_2^{(5)}(\tau)\ \phi_{-2,1}(\tau,z)$
4b	$1^4 2^2 4^4$	$\frac{1}{3}\phi_{0,1}(\tau,z)+(-\frac{1}{3}E_2^{(2)}(\tau)+2E_2^{(4)}(\tau))\ \phi_{-2,1}(\tau,z)$
7a/b	$1^3 7^3$	$\frac{1}{4}\phi_{0,1}(\tau,z)+\frac{7}{4}E_2^{(7)}(\tau)\phi_{-2,1}(\tau,z)$
8a	$1^2 2^1 4^1 8^2$	$\frac{1}{6}\phi_{0,1}(\tau,z)+(-\frac{1}{2}E_2^{(4)}(\tau)+\frac{7}{3}E_2^{(8)}(\tau))\ \phi_{-2,1}(\tau,z)$
6a	$1^2 2^2 3^2 6^2$	$\frac{1}{6}\phi_{0,1}(\tau,z)+(-\frac{1}{6}E_2^{(2)}(\tau)-\frac{1}{2}E_2^{(3)}(\tau)+\frac{5}{2}E_2^{(6)}(\tau))\ \phi_{-2,1}(\tau,z)$
11a	$1^2 11^2$	$\frac{1}{6}\phi_{0,1}(\tau,z)+(\frac{11}{6}E_2^{(11)}(\tau)-\frac{22}{5}\eta_{1^2 11^2}(\tau))\ \phi_{-2,1}(\tau,z)$
15a/b	$1^1 3^1 5^1 15^1$	$\frac{1}{12}\phi_{0,1}(\tau,z)+(-\frac{1}{16}E_2^{(3)}(\tau)-\frac{5}{24}E_2^{(5)}(\tau)$ $+\frac{35}{16}E_2^{(15)}(\tau)-\frac{15}{4}\eta_{1^1 3^1 5^1 15^1}(\tau))\ \phi_{-2,1}(\tau,z)$
14a/b	$1^1 2^1 7^1 14^1$	$\frac{1}{12}\phi_{0,1}(\tau,z)+(-\frac{1}{36}E_2^{(2)}(\tau)-\frac{7}{12}E_2^{(7)}(\tau)$ $+\frac{91}{36}E_2^{(14)}(\tau)-\frac{14}{3}\eta_{1^1 2^1 7^1 14^1}(\tau))\ \phi_{-2,1}(\tau,z)$
23a/b	$1^1 23^1$	$\frac{1}{12}\phi_{0,1}(\tau,z)+(\frac{23}{12}E_2^{(23)}(\tau)-\frac{23}{22}f_{23,1}(\tau)$ $-\frac{161}{22}f_{23,2}(\tau))\ \phi_{-2,1}(\tau,z)$
12b	12^2	$2\,\eta_{1^4 2^{-1} 4^1 6^1 12^{-1}}(\tau)\ \phi_{-2,1}(\tau,z)$
6b	6^4	$2\,\eta_{1^2 2^2 3^2 6^{-2}}(\tau)\ \phi_{-2,1}(\tau,z)$
4c	4^6	$2\,\eta_{1^4 2^2 4^{-2}}(\tau)\ \phi_{-2,1}(\tau,z)$
3b	3^8	$2\,\eta_{1^6 3^{-2}}(\tau)\ \phi_{-2,1}(\tau,z)$
2b	2^{12}	$2\,\eta_{1^8 2^{-4}}(\tau)\ \phi_{-2,1}(\tau,z)$
10a	$2^2 10^2$	$2\,\eta_{1^3 2^1 5^1 10^{-1}}(\tau)\ \phi_{-2,1}(\tau,z)$
21a/b	$3^1 21^1$	$\left(\frac{7}{3}\eta_{1^3 3^{-1} 7^3 21^{-1}}(\tau)-\frac{1}{3}\eta_{1^6 3^{-2}}(\tau)\right)\ \phi_{-2,1}(\tau,z)$
4a	$2^4 4^4$	$2\,\eta_{2^8 4^{-4}}(\tau)\ \phi_{-2,1}(\tau,z)$
12a	$2^1 4^1 6^1 12^1$	$2\,\eta_{1^3 2^{-1} 3^{-1} 4^2 6^3 12^{-2}}(\tau)\ \phi_{-2,1}(\tau,z)$

Table 1: The Jacobi forms for all conjugacy classes of M_{24}.

D Rules for computing cusps

The S and T-transformation of η-function with simple arguments are as follows (see Chapter 2 of [32])

$$\begin{aligned} \eta(N\tau)|_S &= \tfrac{1}{\sqrt{N}} e^{-i\pi/4} \eta\left(\tfrac{\tau}{N}\right), \\ \eta\left(\frac{\tau}{N}\right)\Big|_S &= \sqrt{N} e^{-i\pi/4} \eta(N\tau), \\ \eta(\tau + a) &= e^{i\pi a/12} \eta(\tau) \ . \end{aligned} \tag{D.1}$$

For more complicated arguments of η-function, let $\psi_j := \eta\left(\frac{\tau+j}{N} + j\right)$. Consider the following cases

1. If $(j, N) = 1$, then choose a j' such that $jj' = N + 1$ we have,

$$\psi_j|_S(\tau) = \text{sign}(-j') e^{-3\pi i/4} \psi_{-j'}(\tau) \ . \tag{D.2}$$

2. $(j, N) \neq 1$, but a simple change in variable of τ leads to the condition $(j, N) = 1$. In that case, we make the necessary change in the variable. Then we choose a j' such that $jj' = N + 1$. An example of such case is the following.

$$\eta\left(\frac{\tau - 3}{6}\right)\Big|_S = \tau^{-1/2} \eta\left(\frac{-\frac{1}{\tau} - 3}{6}\right) = \tau^{-1/2} \eta\left(\frac{-\frac{1}{3\tau} - 1}{2}\right) = \tau^{-1/2} \eta\left(\frac{-1 - \tilde{\tau}}{2\tilde{\tau}}\right) \ ,$$

where $\tilde{\tau} = 3\tau$. Now let us consider $\psi^{(2)}_{-1}(\tilde{\tau}) = \eta\left(\frac{\tilde{\tau}-1}{2} - 1\right)$ and do the S-transformation.

$$\begin{aligned} \eta\left(\frac{\tilde{\tau} - 1}{2} - 1\right)\Big|_S = e^{-i\pi/12}\, \eta\left(\frac{\tilde{\tau} - 1}{2}\right)\Big|_S &= e^{-i\pi/12} \tilde{\tau}^{-1/2} \eta\left(\frac{-\frac{1}{\tilde{\tau}} - 1}{2}\right) \\ &= e^{-i\pi/12} \tilde{\tau}^{-1/2} \eta\left(\frac{-1 - \tilde{\tau}}{2\tilde{\tau}}\right) \ . \end{aligned}$$

Again using Eq. (D.2) we get

$$\eta\left(\frac{\tau - 1}{2} - 1\right)\Big|_S = e^{-3\pi i/4} \eta\left(\frac{\tau + 3}{2} + 3\right) \ .$$

Comparing these two and putting $\tilde{\tau} = 3\tau$ back we get

$$\eta\left(\frac{\tau - 3}{6}\right)\Big|_S = \sqrt{3} e^{-i\pi/3} \eta\left(\frac{3\tau + 1}{2}\right) \ .$$

3. $(j,N) \neq 1$, and there exists no change of variable in τ that can lead us to the condition $(j,N)=1$. In that case, we need to use a more general formula. Given $\psi_j(\tau)$, choose j' such that $jj' = 1 \bmod N$ and

$$G = \begin{pmatrix} j & \frac{jj'-1}{N} \\ N & j' \end{pmatrix} \in \Gamma_0(N) \ .$$

Let $G = ST^aST^bST^cS$, then

$$\psi_j\Big|_S(\tau) = e^{-\pi i} e^{(j+j'+a+b+c)\pi i/12}\, i^J\, \psi_{-j'}(\tau) \ ,$$

where J is the number of changes of sign in the sequence $1, c, bc-1, abc-a-c=N$.

E Proving Proposition 2.2

E.1 Type-I conjugacy classes

E.1.1 Cycle shape $1^4 \cdot 2^2 \cdot 4^4$

The twining genera $Z^{\rho_a}(\tau,z)$, $a=0,\dots 3$, which contribute to the multiplicative lifts are the one corresponding to conjugacy class $4b$ and its various cusps.

$$\begin{aligned} &Z^{\rho_0}(\tau,z) = Z^{1^{24}}(\tau,z), \\ &Z^{\rho}(\tau,z) = Z^{\rho_3}(\tau,z) = Z^{1^4 2^2 4^4}(\tau,z), \\ &Z^{\rho_2}(\tau,z) = Z^{1^8 2^8}(\tau,z) \ . \end{aligned}$$

All these Jacobi forms are listed in table 1. The multiplicative seed is $Z^{1^4 2^2 4^4}(\tau,z)$, and this is a Jacobi form of $\Gamma_0(4)$. The Borcherds lift (multiplicative lift), given by Eq. (2.6) leads to the following product formula:

$$\Phi_3^{1^4 2^2 4^4}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{c^0(nm,\ell)-c^2(nm,\ell)} \left(1-(q^n r^\ell s^m)^2\right)^{c^2(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^4\right)^{c^1(nm,\ell)} \ . \quad \text{(E.1)}$$

Then using Eq. (2.5) we get,

$$c^0 = \frac{1}{4}(f_0+2f_1+f_2), \ \ c^1 = \frac{1}{4}(f_0-f_2), \ \ c^2 = \frac{1}{4}(f_0-2f_1+f_2) \ . \quad \text{(E.2)}$$

Using these we can rewrite the product formula (E.1) as

$$\Phi_3^{1^4 2^2 4^4}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1-(q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell)-f_1(nm,\ell))} \left(1-(q^n r^\ell s^m)^4\right)^{\frac{1}{4}(f_0(nm,\ell)-f_2(nm,\ell))} \ . \quad \text{(E.3)}$$

To see if it can be written as the Borcherds product as stated in theorem 2.1, we need to compute the Fourier coefficients at different cusps. The cusps and other data for $\Gamma_0(4)$ are as follows:[2]

f/e	$i\infty$	$1/2$	$0/1$
h_e	1	1	4
N_e	1	2	4

Using $M_{i\infty} = 1, M_0 = S$, and $M_{1/p} = -ST^{-p}S$ and using the transformation rules under S and T as listed above in Section D, we can show that

$$\pi_{FE}\left(Z^{1^4 2^2 4^4}|M_{\frac{1}{2}}\right) = Z^{1^8 2^8} - Z^{1^4 2^2 4^4},$$
$$\pi_{FE}\left(Z^{1^4 2^2 4^4}|M_{\frac{0}{1}}\right) = \frac{1}{4}\left(Z^{1^{24}} - Z^{1^8 2^8}\right) .$$

The Fourier expansion of the cusp about zero does not have integral power in q and hence does not contribute to the product formula. Using these identities, we see that the Borcherds product as given by Clery-Gritsenko in 2.1 takes the form same as Eq. (E.3), i.e.,

$$B_{Z^{1^4 2^2 4^4}}(\mathbf{Z}) = \Phi_3^{1^4 2^2 4^4}(\mathbf{Z}) . \tag{E.4}$$

The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{3,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4$. Hence $\Phi_3^{1^4 2^2 4^4}(\mathbf{Z})$ is a modular form of weight 3 at level 4.

E.1.2 Cycle shape $1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$

This is an element of order 6. The twining genera that contribute to the multiplicative lifts are the following:

$$Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z),$$
$$Z^{\rho}(\tau, z) = Z^{\rho_5}(\tau, z) = Z^{1^2 2^2 3^2 6^2}(\tau, z),$$
$$Z^{\rho_2}(\tau, z) = Z^{\rho_4}(\tau, z) = Z^{1^6 3^6}(\tau, z),$$
$$Z^{\rho_3}(\tau, z) = Z^{1^8 2^8}(\tau, z) .$$

The multiplicative seed $Z^{1^2 2^2 3^2 6^2}(\tau, z)$ is a Jacobi form of $\Gamma_0(6)$. The cusps and other data for $\Gamma_0(6)$ are the following:

[2] Link to compute cusps of congruence subgroups $\Gamma_0(N)$: http://magma.maths.usyd.edu.au/calc/ and the code is
> G := CongruenceSubgroup(0,N);
> Cusps(G);
> Widths(G);

f/e	$i\infty$	$1/2$	$1/3$	$0/1$
h_e	1	3	2	6
N_e	1	3	2	6

The multiplicative lift given by Eq. (2.6) leads to the product formula,

$$\Phi_2^{1^2 2^2 3^2 6^2}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{c^0(nm,\ell)+c^1(nm,\ell)-c^2(nm,\ell)-c^3(nm,\ell)}$$
$$\left(1 - (q^n r^\ell s^m)^2\right)^{c^3(nm,\ell)-c^1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^3\right)^{c^2(nm,\ell)-c^1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^6\right)^{c^1(nm,\ell)} . \tag{E.5}$$

Then using Eq. (2.5) we express the c^α in terms of the Fourier-Jacobi coefficients of Z^{ρ_s} as follows:

$$c^0 = \frac{1}{6}(f_0 + 2f_1 + 2f_2 + f_3), \;\; c^1 = \frac{1}{6}(f_0 + f_1 - f_2 - f_3),$$
$$c^2 = \frac{1}{6}(f_0 - f_1 - f_2 + f_3), \;\; c^3 = \frac{1}{6}(f_0 - 2f_1 + 2f_2 - f_3) . \tag{E.6}$$

Then the above product formula becomes:

$$\Phi_2^{1^2 2^2 3^2 6^2}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell)-f_1(nm,\ell))}$$
$$\left(1 - (q^n r^\ell s^m)^3\right)^{\frac{1}{3}(f_3(nm,\ell)-f_1(nm,\ell))} \left(1 - (q^n r^\ell s^m)^6\right)^{\frac{1}{6}(f_0(nm,\ell)+f_1(nm,\ell)-f_2(nm,\ell)-f_3(nm,\ell))} . \tag{E.7}$$

Now we need to check if it can be written as the Borcherds product as stated in theorem 2.1. For that we need to compute the Fourier-Jacobi coefficients at cusps $i\infty, 1/2, 1/3$ and $0/1$. Straightforward computation shows that

$$\pi_{FE}\left(Z^{1^2 2^2 3^2 6^2}|M_{\frac{1}{2}}\right) = \frac{1}{3}\left(Z^{1^8 2^8} - Z^{1^2 2^2 3^2 6^2}\right),$$
$$\pi_{FE}\left(Z^{1^2 2^2 3^2 6^2}|M_{\frac{1}{3}}\right) = \frac{1}{2}\left(Z^{1^6 3^6} - Z^{1^2 2^2 3^2 6^2}\right),$$
$$\pi_{FE}\left(Z^{1^2 2^2 3^2 6^2}|M_{\frac{0}{1}}\right) = \frac{1}{6}\left(Z^{1^{24}} + Z^{1^2 2^2 3^2 6^2} - Z^{1^6 3^6} - Z^{1^8 2^8}\right) .$$

Using these and using the product formula for Borcherds product given by Cléry-Gritsenko, we get expression for $B_Z^{1^2 2^2 3^2 6^2}(\mathbf{Z})$ which is same as Eq. (E.7), i.e.,

$$B_{Z^{1^2 2^2 3^2 6^2}}(\mathbf{Z}) = \Phi_2^{1^2 2^2 3^2 6^2}(\mathbf{Z}) . \tag{E.8}$$

Hence $\Phi_2^{1^2 2^2 3^2 6^2}(\mathbf{Z})$ is a modular form of weight 2 and level 6. The zeroth coefficient of this Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{2,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$.

E.1.3 Cycle shape $1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$

This is a M_{24} element of order 8 and the conjugacy class is $8a$. The twining genera that contribute to the multiplicative lift are the following:

$$\begin{aligned}
&Z^{\rho_0}(\tau,z) = Z^{1^{24}}(\tau,z),\\
&Z^{\rho_1}(\tau,z) = Z^{\rho_3}(\tau,z) = Z^{\rho_5}(\tau,z) = Z^{\rho_7}(\tau,z) = Z^{1^2 2^1 4^1 8^2}(\tau,z),\\
&Z^{\rho_2}(\tau,z) = Z^{\rho_6}(\tau,z) = Z^{1^4 2^2 4^4}(\tau,z),\\
&Z^{\rho_4}(\tau,z) = Z^{1^8 2^8}(\tau,z)\ .
\end{aligned}$$

The multiplicative seed is given by $Z^{1^2 2^1 4^1 8^2}$ and this is a Jacobi form of $\Gamma_0(8)$. The cusps at level 8 are the following:

f/e	$i\infty$	$1/2$	$1/4$	$0/1$
h_e	1	2	1	8
N_e	1	4	2	8

The Borcherds lift given by Eq. (2.6) leads to the following product formula:

$$\begin{aligned}
\Phi_1^{1^2 2^1 4^1 8^2}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} &\left(1-q^n r^\ell s^m\right)^{c^0(nm,\ell)-c^4(nm,\ell)} \left(1-(q^n r^\ell s^m)^2\right)^{c^4(nm,\ell)-c^2(nm,\ell)}\\
&\left(1-(q^n r^\ell s^m)^4\right)^{c^2(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^8\right)^{c^1(nm,\ell)}\ . \qquad \text{(E.9)}
\end{aligned}$$

Then using Eq. (2.5), i.e.,

$$\begin{aligned}
&c^0 = \frac{1}{8}(f_0+4f_1+2f_2+f_4),\ \ c^1 = \frac{1}{8}(f_0-f_4),\\
&c^2 = \frac{1}{8}(f_0-2f_2+f_4),\ \ c^4 = \frac{1}{8}(f_0-4f_1+2f_2+f_4)\ , \qquad \text{(E.10)}
\end{aligned}$$

we rewrite the product formula in terms of the Fourier-Jacobi coefficients $f_a(n,\ell)$ as

$$\begin{aligned}
\Phi_1^{1^2 2^1 4^1 8^2}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} &\left(1-q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1-(q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell)-f_1(nm,\ell))}\\
&\left(1-(q^n r^\ell s^m)^4\right)^{\frac{1}{4}(f_4(nm,\ell)-f_2(nm,\ell))} \left(1-(q^n r^\ell s^m)^8\right)^{\frac{1}{8}(f_0(nm,\ell)-f_4(nm,\ell))}\ . \qquad \text{(E.11)}
\end{aligned}$$

To check if it can be written as the Borcherds product of Cléry-Gritsenko 2.1, we need to compute the coefficients at cusps about $i\infty, 1/2, 1/4$ and $0/1$. Computing $Z^{1^2 2^1 4^1 8^2}$ at these cusps one can show that

$$\begin{aligned}
&\pi_{FE}\left(Z^{1^2 2^1 4^1 8^2}|M_{\frac{1}{2}}\right) = \frac{1}{2}\left(Z^{1^8 2^8} - Z^{1^4 2^2 4^4}\right),\\
&\pi_{FE}\left(Z^{1^2 2^1 4^1 8^2}|M_{\frac{1}{4}}\right) = \left(Z^{1^4 2^2 4^4} - Z^{1^2 2^1 4^1 8^2}\right),\\
&\pi_{FE}\left(Z^{1^2 2^1 4^1 8^2}|M_{\frac{0}{1}}\right) = \frac{1}{8}\left(Z^{1^{24}} - Z^{1^8 2^8}\right)\ .
\end{aligned}$$

Using these and using the product formula for Borcherds product given by Cléry-Gritsenko, we get expression for $B_{Z^{1^2 2^1 4^1 8^2}}(\mathbf{Z})$ same as Eq. (E.11), i.e.,

$$B_{Z^{1^2 2^1 4^1 8^2}}(\mathbf{Z}) = \Phi_1^{1^2 2^1 4^1 8^2}(\mathbf{Z}) \ . \tag{E.12}$$

Hence $\Phi_1^{1^2 2^1 4^1 8^2}(\mathbf{Z})$ is a meromorphic Siegel Modular Form of weight 1 at level 8. The zeroth coefficient of this Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{1,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$.

E.1.4 Cycle shape $1 \cdot 2 \cdot 7 \cdot 14$

This is an element of order 14. The following twining genera that contribute to the multiplicative lift :

$Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z)$,

$Z^{\rho}(\tau, z) = Z^{\rho_3}(\tau, z) = Z^{\rho_5}(\tau, z) = Z^{\rho_9}(\tau, z) = Z^{\rho_{11}}(\tau, z) = Z^{\rho_{13}}(\tau, z) = Z^{1^1 2^1 7^1 14^1}(\tau, z)$,

$Z^{\rho_2}(\tau, z) = Z^{\rho_4}(\tau, z) = Z^{\rho_6}(\tau, z) = Z^{\rho_8}(\tau, z) = Z^{\rho_{10}}(\tau, z) = Z^{\rho_{12}}(\tau, z) = Z^{1^3 7^3}(\tau, z)$,

$Z^{\rho_7}(\tau, z) = Z^{1^8 2^8}(\tau, z)$.

The multiplicative seed for this case, i.e., $Z^{1^1 2^1 7^1 14^1}(\tau, z)$ is a Jacobi form of $\Gamma_0(14)$. The cusps and other data at level 14 are the following:

f/e	$i\infty$	$1/2$	$1/7$	$0/1$
h_e	1	7	2	14
N_e	1	7	2	14

The Borcherds lift given by Eq. (2.6) leads to the following product formula:

$$\Phi_0^{1^1 2^1 7^1 14^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{c^0(nm,\ell)+c^1(nm,\ell)-c^2(nm,\ell)-c^7(nm,\ell)}$$
$$\left(1 - (q^n r^\ell s^m)^2\right)^{c^7(nm,\ell)-c^1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^7\right)^{c^2(nm,\ell)-c^1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^{14}\right)^{c^1(nm,\ell)} . \tag{E.13}$$

Again we rewrite $c^\alpha(n, \ell)$ in terms of $f_a(n, \ell)$ using Eq. (2.5),

$$c^0 = \frac{1}{14}(f_0 + 6f_1 + 6f_2 + f_7), \ \ c^1 = \frac{1}{14}(f_0 + f_1 - f_2 - f_7),$$
$$c^2 = \frac{1}{14}(f_0 - f_1 - f_2 + f_7), \ \ c^7 = \frac{1}{14}(f_0 - 6f_1 + 6f_2 - f_7) \ . \tag{E.14}$$

Then the above product formula becomes

$$\Phi_0^{1^1 2^1 7^1 14^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell)-f_1(nm,\ell))}$$
$$\left(1 - (q^n r^\ell s^m)^7\right)^{\frac{1}{7}(f_7(nm,\ell)-f_1(nm,\ell))} \left(1 - (q^n r^\ell s^m)^{14}\right)^{\frac{1}{14}(f_0(nm,\ell)+f_1(nm,\ell)-f_2(nm,\ell)-f_7(nm,\ell))} . \tag{E.15}$$

To see if it can be written as the Borcherds product as stated in theorem 2.1 we need to compute the Fourier-Jacobi coefficients of the multiplicative seed at cusps about $i\infty, 1/2, 1/7$ and $0/1$. Computing the cusps, one can show that

$$\begin{aligned}
\pi_{FE}\left(Z^{1^1 2^1 7^1 14^1}|M_{\frac{1}{2}}\right) &= \frac{1}{2}\left(Z^{1^3 7^3} - Z^{1^1 2^1 7^1 14^1}\right), \\
\pi_{FE}\left(Z^{1^1 2^1 7^1 14^1}|M_{\frac{1}{7}}\right) &= \frac{1}{7}\left(Z^{1^8 2^8} - Z^{1^1 2^1 7^1 14^1}\right), \\
\pi_{FE}\left(Z^{1^1 2^1 7^1 14^1}|M_{\frac{0}{1}}\right) &= \frac{1}{14}\left(Z^{1^{24}} + Z^{1^1 2^1 7^1 14^1} - Z^{1^8 2^8} - Z^{1^3 7^3}\right) .
\end{aligned}$$

Then the Borcherds product formula given by Cléry-Gritsenko leads to product formula Eq. (E.15), i.e.,

$$B_{Z^{1^1 2^1 7^1 14^1}}(\mathbf{Z}) = \Phi_0^{1^1 2^1 7^1 14^1}(\mathbf{Z}) . \tag{E.16}$$

Hence $\Phi_0^{1^1 2^1 7^1 14^1}(\mathbf{Z})$ is a modular form of weight zero at level 14. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{0,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$.

E.1.5 Cycle shape $1 \cdot 3 \cdot 5 \cdot 15$

This is an M_{24} element of order 15. The twining genera that contribute to the multiplicative lift are the following:

$$\begin{aligned}
&Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z), \\
&Z^{\rho}(\tau, z) = Z^{\rho_2}(\tau, z) = Z^{\rho_4}(\tau, z) = Z^{\rho_7}(\tau, z), \\
&\quad = Z^{\rho_8}(\tau, z) = Z^{\rho_{11}}(\tau, z) = Z^{\rho_{13}}(\tau, z) = Z^{\rho_{14}}(\tau, z) = Z^{1^1 3^1 5^1 15^1}(\tau, z), \\
&Z^{\rho_3}(\tau, z) = Z^{\rho_6}(\tau, z) = Z^{\rho_9}(\tau, z) = Z^{\rho_{12}}(\tau, z) = Z^{1^4 5^4}(\tau, z), \\
&Z^{\rho_5}(\tau, z) = Z^{\rho_{10}}(\tau, z) = Z^{1^6 3^6}(\tau, z) .
\end{aligned}$$

The multiplicative seed is a Jacobi form of $\Gamma_0(15)$. The cusps and other data for $\Gamma_0(15)$ are the following:

f/e	$i\infty$	$1/3$	$1/5$	$0/1$
h_e	1	5	3	15
N_e	1	5	3	15

The Borcherds lift given by Eq. (2.6) leads to the following product formula:

$$\begin{aligned}
\Phi_0^{1^1 3^1 5^1 15^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} &\left(1 - q^n r^\ell s^m\right)^{c^0(nm,\ell)+c^1(nm,\ell)-c^3(nm,\ell)-c^5(nm,\ell)} \\
&\left(1 - (q^n r^\ell s^m)^3\right)^{c^5(nm,\ell)-c^1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^5\right)^{c^3(nm,\ell)-c^1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^{15}\right)^{c^1(nm,\ell)} .
\end{aligned} \tag{E.17}$$

Then using Eq. (2.5), i.e.,

$$c^0 = \frac{1}{15}(f_0 + 8f_1 + 4f_3 + 2f_5), \;\; c^1 = \frac{1}{15}(f_0 + f_1 - f_3 - f_5),$$
$$c^3 = \frac{1}{15}(f_0 - 2f_1 - f_3 + 2f_5), \;\; c^5 = \frac{1}{15}(f_0 - 4f_1 + 4f_3 - f_5) \;, \tag{E.18}$$

we can rewrite the above product formula as

$$\Phi_0^{1^1 3^1 5^1 15^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^3\right)^{\frac{1}{3}(f_3(nm,\ell) - f_1(nm,\ell))}$$
$$\left(1 - (q^n r^\ell s^m)^5\right)^{\frac{1}{5}(f_5(nm,\ell) - f_1(nm,\ell))} \left(1 - (q^n r^\ell s^m)^{15}\right)^{\frac{1}{15}(f_0(nm,\ell) + f_1(nm,\ell) - f_3(nm,\ell) - f_5(nm,\ell))} . \tag{E.19}$$

To see if it can be written as the Borcherds product as stated in theorem 2.1 we need to compute the Fourier coefficients at cusps about $i\infty, 1/3, 1/5$ and $1/15$. The cusp about zero does not have any integer power in q and hence does not contribute. Computing the seed, i.e., $Z^{1^1 3^1 5^1 15^1}(\tau, z)$ at these cusps one can show that

$$\pi_{FE}\left(Z^{1^1 3^1 5^1 15^1} | M_{\frac{1}{3}}\right) = \frac{1}{3}\left(Z^{1^4 5^4} - Z^{1^1 3^1 5^1 15^1}\right),$$
$$\pi_{FE}\left(Z^{1^1 3^1 5^1 15^1} | M_{\frac{1}{5}}\right) = \frac{1}{5}\left(Z^{1^6 3^6} - Z^{1^1 3^1 5^1 15^1}\right),$$
$$\pi_{FE}\left(Z^{1^1 3^1 5^1 15^1} | M_{\frac{0}{1}}\right) = \frac{1}{15}\left(Z^{1^{24}} + Z^{1^1 3^1 5^1 15^1} - Z^{1^6 3^6} - Z^{1^4 5^4}\right) \;.$$

Using these and using the product formula for Borcherds product given by Cléry-Gritsenko, we get expression for $B_{Z^{1^1 3^1 5^1 15^1}}(\mathbf{Z})$ which is identical to Eq. (E.19), i.e.,

$$B_{Z^{1^1 3^1 5^1 15^1}}(\mathbf{Z}) = \Phi_0^{1^1 3^1 5^1 15^1}(\mathbf{Z}) \;. \tag{E.20}$$

Hence $\Phi_0^{1^1 3^1 5^1 15^1}(\mathbf{Z})$ is a meromorphic Siegel Modular Form of weight zero at level 15. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{0,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$.

So we see that for all the type-I conjugacy classes of M_{24}, the length of the lowest cycle in the balanced cycle-shape is one and all the product formula obtained via the multiplicative lift matches with the Borcherds product formula given by Cléry-Gritsenko.

E.2 Type-II conjugacy classes

Now we consider the cases for which there is no cycle shape of length one. The corresponding conjugacy classes are called type-II conjugacy classes.

E.2.1 Cycle shape 2^{12}

This is an element of order 2. There are only two twining genera which contribute to the multiplicative lift. They are

$$Z^{\rho_0}(\tau,z) = Z^{1^{24}}(\tau,z)\ , \quad Z^{\rho}(\tau,z) = Z^{2^{12}}(\tau,z)\ .$$

The multiplicative seed corresponding to this conjugacy class, i.e., $Z^{2^{12}}(\tau,z)$ is a Jacobi form of $\Gamma_0(4)$. The cusps and other data of $\Gamma_0(4)$ are the following:

f/e	$i\infty$	$1/2$	$0/1$
h_e	1	1	4
N_e	1	2	4

The Borcherds lift given by Eq. (2.6) leads to the following product formula:

$$\Phi_4^{2^{12}}(\mathbf{Z}) \;=\; qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{c^0(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^2\right)^{c^1(nm,\ell)} \ . \tag{E.21}$$

Then using Eq. (2.5), i.e.,

$$c^0 = \frac{1}{2}(f_0+f_1),\ \ c^1 = \frac{1}{2}(f_0-f_1)\ , \tag{E.22}$$

we can rewrite the above product formula as

$$\Phi_4^{2^{12}} = qrs \prod_{(n,\ell,m)>0} \Big(1-q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1-(q^n r^\ell s^m)^2\Big)^{\frac{1}{2}(f_0(nm.\ell)-f_1(nm,\ell))} \ . \tag{E.23}$$

To see if it can be written as the Borcherds product as stated in theorem 2.1, we need to compute the coefficients at the cusps $i\infty$ and $1/2$. The other cusps $0/1$ does not contribute as there are no integer power in q in the Fourier expansion. Then we get

$$\pi_{FE}\Big(Z^{2^{12}}|M_{\frac{1}{2}}\Big) = -Z^{2^{12}}\ .$$

Using this and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{2^{12}}}(\mathbf{Z}) \;=\; \prod_{(n,\ell,m)>0} \Big(1\,-\,q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1\,-\,(q^n r^\ell s^m)^2\Big)^{-\frac{1}{2}f_1(nm,\ell)} \ . \tag{E.24}$$

This does not give all the terms that appear in the product form obtained from the Borcherds lift given in Eq. (E.23). The missing terms can be accounted for by additional terms leading to

$$\left(\Phi_4^{2^{12}}(\mathbf{Z})\right)^2 = B_{2Z^{2^{12}}}(\mathbf{Z})B_{Z^{1^{24}}}(\mathbf{2Z}) \ . \tag{E.25}$$

Therefore $\left(\Phi_4^{2^{12}}(\mathbf{Z})\right)^2$ is a meromorphic Siegel Modular Form of weight 8 at level 4. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{4,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(2\tau)^{12}$.

E.2.2 Cycle shape $2^4 \cdot 4^4$

This M_{24} element is of order 4. The twining genera that contribute to the multiplicative lifts are,

$$\begin{aligned}
&Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z) \ , \\
&Z^{\rho}(\tau, z) = Z^{\rho_3}(\tau, z) = Z^{2^4 4^4}(\tau, z) \ , \\
&Z^{\rho_2}(\tau, z) = Z^{1^8 2^8}(\tau, z) \ .
\end{aligned}$$

The multiplicative seed, i.e., $Z^{2^4 4^4}(\tau, z)$ is a Jacobi form of $\Gamma_0(8)$. The cusps and other data at level 8 are the following:

f/e	$i\infty$	$1/2$	$1/4$	$0/1$
h_e	1	2	1	8
N_e	1	4	2	8

Following the computation similar to the case of cycle shape $1^4 2^2 4^4$, the Borcherds lift leads to the following product formula:

$$\Phi_2^{2^4 4^4}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell) - f_1(nm,\ell))} \left(1 - (q^n r^\ell s^m)^4\right)^{\frac{1}{4}(f_0(nm,\ell) - f_2(nm,\ell))} \ . \tag{E.26}$$

To see if it can be written as the Borcherds product as stated in theorem 2.1 we need to compute the Fourier-Jacobi coefficients of the multiplicative seed at the cusps about $i\infty$ and $1/4$ as the Jacobi forms have integer coefficients for these cusps.

$$\pi_{FE}\left(Z^{2^4 4^4}|M_{\frac{1}{4}}\right) = -Z^{2^4 4^4} \ .$$

Using this and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{2^4 4^4}}(\mathbf{Z}) \;=\; \prod_{(n,\ell,m)>0} \Big(1 - q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^2\Big)^{-\frac{1}{2} f_1(nm,\ell)} \;. \quad \text{(E.27)}$$

This product formula does not give all the terms that appear in the product form obtained from the Borcherds lift given in Eq. (E.26). The missing terms can be accounted for by additional terms leading to

$$\Big(\Phi_2^{2^4 4^4}(\mathbf{Z})\Big)^2 = B_{2Z^{2^4 4^4}}(\mathbf{Z}) B_{Z^{1^8 2^8}}(2\mathbf{Z}) \;. \quad \text{(E.28)}$$

Therefore $\Big(\Phi_2^{2^4 4^4}(\mathbf{Z})\Big)^2$ is a meromorphic Siegel Modular Form of weight 4 at level 8. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{2,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \eta(2\tau)^4 \eta(4\tau)^4$.

E.2.3 Cycle shape 3^8

This is an element of order 3. The twining genera which contribute to the multiplicative lifts are

$$Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z),$$
$$Z^{\rho}(\tau, z) = Z^{\rho_2}(\tau, z) = Z^{3^8}(\tau, z) \;.$$

The corresponding multiplicative seed $Z^{3^8}(\tau, z)$ is a Jacobi form of $\Gamma_0(9)$. The cusps and other data for $\Gamma_0(9)$ are the following:

f/e	$i\infty$	$1/2$	$1/3$	$0/1$
h_e	1	1	1	9
N_e	1	3	3	9

Cusps about $2/3$ can be computed by $M_{\frac{2}{3}} = ST^{-1}ST^2S$. The cusp about 0 does not have integer power in q and hence does not contribute. The Borcherds lift given leads to the product formula given by

$$\Phi_2^{3^8} = qrs \prod_{(n,\ell,m)>0} \Big(1 - q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^3\Big)^{\frac{1}{3}(f_0(nm.\ell) - f_1(nm,\ell)} \;. \quad \text{(E.29)}$$

To see if it can be written as the Borcherds product as stated in theorem 2.1 we need to compute the coefficients at the cusps about $i\infty$, $1/3$ and $2/3$. The contributions from the cusps at $1/3$ and $2/3$ have cube roots of unity. However,

the contributions of the two cusps add in the product formula to give integral coefficients. One gets

$$\pi_{FE}\Big(Z^{3^8}|M_{\frac{1}{3}} + Z^{3^8}|M_{\frac{2}{3}}\Big) = -Z^{3^8} \ .$$

Using this and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{3^8}}(\mathbf{Z}) \ = \ \prod_{(n,\ell,m)>0} \Big(1 \ - \ q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1 \ - \ (q^n r^\ell s^m)^3\Big)^{-\frac{1}{3}f_1(nm,\ell)} \ . \quad \text{(E.30)}$$

This does not give all the terms that appear in the product form obtained from the Borcherds lift given in Eq. (E.29). The missing terms can be accounted for by additional terms leading to

$$\Big(\Phi_2^{3^8}(\mathbf{Z})\Big)^3 = B_{3Z^{4^6}}(\mathbf{Z}) B_{Z^{1^{24}}}(\mathbf{3Z}) \ . \quad \text{(E.31)}$$

Therefore $\Big(\Phi_2^{3^8}(\mathbf{Z})\Big)^3$ is a meromorphic Siegel Modular Form of weight 6 at level 9. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{2,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(3\tau)^8$.

E.2.4 Cycle shape 4^6

This is an element of order 4. The twining genera which contribute to the Siegel Modular Forms are

$$\begin{aligned} &Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z), \\ &Z^{\rho}(\tau, z) = Z^{\rho_3}(\tau, z) = Z^{4^6}(\tau, z), \\ &Z^{\rho_2}(\tau, z) = Z^{2^{12}}(\tau, z) \ . \end{aligned}$$

The multiplicative seed $Z^{4^6}(\tau, z)$ is a Jacobi form of $\Gamma_0(16)$. The cusps and other data at level 16 are the following:

f/e	$i\infty$	1/2	1/4	3/4	1/8	0/1
h_e	1	4	1	1	1	16
N_e	1	8	4	4	2	16

Cusp about 3/4 can be computed via following transformation:

$$M_{\frac{3}{4}} = ST^{-1}ST^2ST^{-1}S \ .$$

Following the computation similar to the case of cycle shape $1^4 2^2 4^4$, the Borcherds lift leads to the following product formula:

$$\Phi_1^{4^6}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell) - f_1(nm,\ell))} \left(1 - (q^n r^\ell s^m)^4\right)^{\frac{1}{4}(f_0(nm,\ell) - f_2(nm,\ell))} . \quad \text{(E.32)}$$

To see if it can be written as the Borcherds product as stated in theorem 2.1, we need to compute the coefficients at cusps about $i\infty$, $(1/4, 3/4)$ and $1/8$. We need to pair up cusps with identical (h_e, N_e). Computing these cusps, we get

$$\left(Z^{4^6}|M_{\frac{1}{4}}\right) = -iZ^{4^6}, \text{ and } \left(Z^{4^6}|M_{\frac{3}{4}}\right) = iZ^{4^6} .$$

Then one obtains

$$\pi_{FE}\left(Z^{4^6}|M_{\frac{1}{4}} + Z^{4^6}|M_{\frac{3}{4}}\right) = 0,$$
$$\pi_{FE}\left(Z^{4^6}|M_{\frac{1}{8}}\right) = -Z^{4^6} .$$

Using these and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{4^6}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{-\frac{1}{2} f_1(nm,\ell)} . \quad \text{(E.33)}$$

This does not give all the terms that appear in the product form obtained from the Borcherds lift given in Eq. (E.32). The missing terms can be accounted for by additional terms leading to

$$\left(\Phi_1^{4^6}(\mathbf{Z})\right)^4 = B_{4Z^{4^6}}(\mathbf{Z}) B_{2Z^{2^{12}}}(2\mathbf{Z}) B_{Z^{1^{24}}}(\mathbf{4Z}) . \quad \text{(E.34)}$$

Therefore $\left(\Phi_1^{4^6}(\mathbf{Z})\right)^4$ is a modular form of weight 4 at level 16. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{1,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(4\tau)^6$.

E.2.5 Cycle shape 6^4

This is an element of order 6 corresponding to the conjugacy class $6b$. The twining genera which contribute to the multiplicative lift are

$$Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z),$$
$$Z^{\rho}(\tau, z) = Z^{\rho_5}(\tau, z) = Z^{6^4}(\tau, z),$$
$$Z^{\rho_2}(\tau, z) = Z^{\rho_4}(\tau, z) = Z^{3^8}(\tau, z),$$
$$Z^{\rho_3}(\tau, z) = Z^{2^{12}}(\tau, z) .$$

The corresponding multiplicative seed, $Z^{6^4}(\tau, z)$ is a Jacobi form of $\Gamma_0(36)$. The cusps and other data at level 36 are the following:

c	$i\infty$	0	1/2	1/3	2/3	1/4	1/6	5/6	1/9	1/12	5/12	1/18
h_e	1	36	9	4	4	9	1	1	4	1	1	1
N_e	1	36	18	12	12	9	6	6	4	3	3	2

The cusps about 5/6 and 5/12 are respectively given by $M_{\frac{5}{6}} = -ST^{-1}ST^5S$ and $M_{\frac{5}{12}} = ST^{-2}ST^2ST^{-2}S$. Following the computation similar to the cycle shape $1^2 2^2 3^2 6^2$, the multiplicative lift leads to the product formula given by

$$\Phi_0^{6^4}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell) - f_1(nm,\ell))}$$
$$\left(1 - (q^n r^\ell s^m)^3\right)^{\frac{1}{3}(f_3(nm,\ell) - f_1(nm,\ell))} \left(1 - (q^n r^\ell s^m)^6\right)^{\frac{1}{6}(f_0(nm,\ell) + f_1(nm,\ell) - f_2(nm,\ell) - f_3(nm,\ell))} . \tag{E.35}$$

Now to see if can be written as the Borcherds product as stated in theorem 2.1 we need to compute the coefficients at cusps about $i\infty$, $(1/6, 5/6)$, $(1/12, 5/12)$ and $1/18$. We need to pair up cusps with identical values of h_e and N_e to get integral coefficients. Then from straightforward calculations one obtains

$$\pi_{FE}\left(Z^{6^4}|M_{\frac{1}{6}} + Z^{6^4}|M_{\frac{5}{6}}\right) = Z^{6^4},$$
$$\pi_{FE}\left(Z^{6^4}|M_{\frac{1}{12}} + Z^{6^4}|M_{\frac{5}{12}}\right) = -Z^{6^4},$$
$$\pi_{FE}\left(Z^{6^4}|M_{\frac{1}{18}}\right) = -Z^{6^4} .$$

Using these and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{6^4}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{-\frac{1}{2}f_1(nm,\ell)}$$
$$\left(1 - (q^n r^\ell s^m)^3\right)^{-\frac{1}{3}f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^6\right)^{\frac{1}{6}f_1(nm,\ell)} . \tag{E.36}$$

This does not give all the terms that appear in the product form given in Eq. (E.35). The missing terms can be accounted for by additional terms leading to

$$\left(\Phi_0^{6^4}(\mathbf{Z})\right)^6 = B_{6Z^{6^4}}(\mathbf{Z}) B_{3Z^{3^8}}(2\mathbf{Z}) B_{2Z^{2^{12}}}(3\mathbf{Z}) B_{Z^{1^{24}}}(6\mathbf{Z}) . \tag{E.37}$$

$\left(\Phi_0^{6^4}(\mathbf{Z})\right)^6$ is a meromorphic Siegel Modular Form of weight zero at level 36. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{0,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(6\tau)^4$.

E.2.6 Cycle shape $2^2 \cdot 10^2$

This is an element of order 10. The twining genera which contribute to the multiplicative lift are

$$\begin{aligned}
&Z^{\rho_0}(\tau,z) = Z^{1^{24}}(\tau,z),\\
&Z^{\rho}(\tau,z) = Z^{\rho_3}(\tau,z) = Z^{\rho_7}(\tau,z) = Z^{\rho_9}(\tau,z) = Z^{2^2 10^2}(\tau,z),\\
&Z^{\rho_2}(\tau,z) = Z^{\rho_4}(\tau,z) = Z^{\rho_6}(\tau,z) = Z^{\rho_8}(\tau,z) = Z^{1^4 5^4}(\tau,z),\\
&Z^{\rho_5}(\tau,z) = Z^{2^{12}}(\tau,z)\ .
\end{aligned}$$

The multiplicative seed, $Z^{2^2 10^2}(\tau,z)$ is a Jacobi form of $\Gamma_0(20)$. The cusps at level 20 are the following:

c	$i\infty$	1/2	1/4	1/5	1/10	0/1
h_e	1	5	5	4	1	20
N_e	1	10	5	4	2	20

The Borcherds lift given by Eq. (2.6) leads to the following product formula:

$$\Phi_0^{2^2 10^2}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{c^0(nm,\ell)+c^1(nm,\ell)-c^2(nm,\ell)-c^5(nm,\ell)}$$
$$\left(1-(q^n r^\ell s^m)^2\right)^{c^5(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^5\right)^{c^2(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^{10}\right)^{c^1(nm,\ell)}\ . \tag{E.38}$$

Then using Eq. (2.5), i.e.,

$$c^0 = \frac{1}{10}(f_0+4f_1+4f_2+f_5),\ \ c^1 = \frac{1}{10}(f_0+f_1-f_2-f_5)\ , \tag{E.39}$$
$$c^2 = \frac{1}{10}(f_0-f_1-f_2+f_5),\ \ c^5 = \frac{1}{10}(f_0-4f_1+4f_2-f_5)\ , \tag{E.40}$$

we can rewrite the above product formula as

$$\Phi_0^{2^2 10^2}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1-(q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell)-f_1(nm,\ell))}$$
$$\left(1-(q^n r^\ell s^m)^5\right)^{\frac{1}{5}(f_5(nm,\ell)-f_1(nm,\ell))} \left(1-(q^n r^\ell s^m)^{10}\right)^{\frac{1}{10}(f_0(nm,\ell)+f_1(nm,\ell)-f_2(nm,\ell)-f_5(nm,\ell))}\ . \tag{E.41}$$

Now to see if can be written as the Borcherds product as stated in theorem 2.1 we need to compute the coefficients at the cusps about $i\infty$, 1/2, 1/4 and

1/10 as other cusps do not have integral power in q. Then from straightforward calculations one obtains

$$\pi_{FE}\Big(Z^{2^2 10^2}|M_{\frac{1}{2}}\Big) = \frac{1}{5}\left(Z^{2^2 10^2} - Z^{2^{12}}\right),$$
$$\pi_{FE}\Big(Z^{2^2 10^2}|M_{\frac{1}{4}}\Big) = -\frac{1}{5}\left(Z^{2^2 10^2} - Z^{2^{12}}\right),$$
$$\pi_{FE}\Big(Z^{2^2 10^2}|M_{\frac{1}{10}}\Big) = -Z^{2^2 10^2} \ .$$

Using these and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{2^2 10^2}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} \Big(1 - q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^2\Big)^{-\frac{1}{2}f_1(nm,\ell)}$$
$$\Big(1 - (q^n r^\ell s^m)^5\Big)^{\frac{1}{5}(f_5(nm,\ell)-f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^{10}\Big)^{\frac{1}{10}(f_1(nm,\ell)-f_5(nm,\ell))} \ . \quad \text{(E.42)}$$

This does not give all the terms that appear in the product form given in Eq. (E.41). The missing terms can be accounted for by additional terms leading to

$$\Big(\Phi_0^{2^2 10^2}(\mathbf{Z})\Big)^2 = B_{2Z^{2^2 10^2}}(\mathbf{Z}) B_{Z^{1^4 5^4}}(2\mathbf{Z}) \ . \quad \text{(E.43)}$$

$\Big(\Phi_0^{2^2 10^2}(\mathbf{Z})\Big)^2$ is a meromorphic Siegel Modular Form of weight zero at level 20. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 1$ term is given by the additive seed $\phi_{0,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(2\tau)^2\eta(10\tau)^2$.

E.2.7 Cycle shape $2 \cdot 4 \cdot 6 \cdot 12$

This is an element of order 12. The twining genera which contribute to the multiplicative lift are

$$Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z),$$
$$Z^{\rho}(\tau, z) = Z^{\rho_5}(\tau, z) = Z^{\rho_7}(\tau, z) = Z^{\rho_{11}}(\tau, z) = Z^{2^1 4^1 6^1 12^1}(\tau, z),$$
$$Z^{\rho_2}(\tau, z) = Z^{\rho_{10}}(\tau, z) = Z^{1^2 2^2 3^2 6^2}(\tau, z),$$
$$Z^{\rho_3}(\tau, z) = Z^{\rho_9}(\tau, z) = Z^{2^4 4^4}(\tau, z),$$
$$Z^{\rho_4}(\tau, z) = Z^{\rho_8}(\tau, z) = Z^{1^6 3^6},$$
$$Z^{\rho_6}(\tau, z) = Z^{1^8 2^8} \ .$$

The corresponding multiplicative seed, $Z^{2^1 4^1 6^1 12^1}(\tau, z)$ is a Jacobi form of $\Gamma_0(24)$. The cusps at level 24 are the following:

c	$i\infty$	0	1/2	1/3	1/4	1/6	1/8	1/12
h_e	1	24	6	8	3	2	3	1
N_e	1	24	12	8	6	4	3	2

The Borcherds lift given by Eq. (2.6) leads to the following product formula:

$$\Phi_0^{2^1 4^1 6^1 12^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{c^0(nm,\ell)+c^2(nm,\ell)-c^4(nm,\ell)-c^6(nm,\ell)}$$
$$\left(1-(q^n r^\ell s^m)^2\right)^{c^6(nm,\ell)-c^2(nm,\ell)-c^3(nm,\ell)+c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^3\right)^{c^4(nm,\ell)-c^2(nm,\ell)}$$
$$\left(1-(q^n r^\ell s^m)^4\right)^{c^3(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^6\right)^{c^2(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^{12}\right)^{c^1(nm,\ell)} . \tag{E.44}$$

Then using Eq. (2.5), i.e.,

$$c^0 = \frac{1}{12}(f_0+4f_1+2f_2+2f_3+2f_4+f_6), \;\; c^1 = \frac{1}{12}(f_0+f_2-f_4-f_6) \; ,$$
$$c^2 = \frac{1}{12}(f_0+2f_1-f_2-2f_3-f_4+f_6), \;\; c^3 = \frac{1}{12}(f_0-2f_2+2f_4-f_6) \; ,$$
$$c^6 = \frac{1}{12}(f_0-4f_1+2f_2-2f_3+2f_4+f_6), \;\; c^4 = \frac{1}{12}(f_0-2f_1-f_2+2f_3-f_4+f_6) \; ,$$

we can rewrite the above product formula as

$$\Phi_0^{2^1 4^1 6^1 12^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} \left(1-q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1-(q^n r^\ell s^m)^2\right)^{\frac{1}{2}(f_2(nm,\ell)-f_1(nm,\ell))}$$
$$\left(1-(q^n r^\ell s^m)^3\right)^{\frac{1}{3}(f_3(nm,\ell)-f_1(nm,\ell))} \left(1-(q^n r^\ell s^m)^4\right)^{\frac{1}{4}(-f_2(nm,\ell)+f_4(nm,\ell))}$$
$$\left(1-(q^n r^\ell s^m)^6\right)^{\frac{1}{3}(f_1(nm,\ell)-f_2(nm,\ell)-f_3(nm,\ell)+f_6(nm,\ell))}$$
$$\left(1-(q^n r^\ell s^m)^{12}\right)^{\frac{1}{12}(f_0(nm,\ell)+f_2(nm,\ell)-f_4(nm,\ell)-f_6(nm,\ell))} . \tag{E.45}$$

Now to see if can be written as the Borcherds product as stated in theorem 2.1 we need to compute the coefficients at different cusps. Only the cusps about $1/4, 1/8$ and $1/12$ has integer power of q in the Fourier expansion. The other cusps do not contribute. Then from straightforward calculations one obtains

$$\pi_{FE}\left(Z^{2^1 4^1 6^1 12^1}|M_{\frac{1}{4}}\right) = \frac{1}{3}\left(Z^{2^1 4^1 6^1 12^1} - Z^{2^4 4^4}\right),$$
$$\pi_{FE}\left(Z^{2^1 4^1 6^1 12^1}|M_{\frac{1}{8}}\right) = -\frac{1}{3}\left(Z^{2^1 4^1 6^1 12^1} - Z^{2^4 4^4}\right),$$
$$\pi_{FE}\left(Z^{2^1 4^1 6^1 12^1}|M_{\frac{1}{12}}\right) = -Z^{2^1 4^1 6^1 12^1} \; .$$

Using these and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$B_{Z^{2^1 4^1 6^1 12^1}}(\mathbf{Z}) = \prod_{(n,\ell,m)>0} \left(1 - q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1 - (q^n r^\ell s^m)^2\right)^{-\frac{1}{2} f_1(nm,\ell)}$$
$$\left(1 - (q^n r^\ell s^m)^3\right)^{-\frac{1}{3}(f_1(nm,\ell) - f_3(nm,\ell))} \left(1 - (q^n r^\ell s^m)^6\right)^{\frac{1}{6}(f_1(nm,\ell) - f_3(nm,\ell))} . \quad \text{(E.46)}$$

This does not give all the terms that appear in the product form given in Eq. (E.45). The missing terms can be accounted for by additional terms leading to

$$\left(\Phi_0^{2^1 4^1 6^1 12^1}(\mathbf{Z})\right)^2 = B_{2Z^{2^1 4^1 6^1 12^1}}(\mathbf{Z}) B_{Z^{1^2 2^2 3^2 6^2}}(2\mathbf{Z}) . \quad \text{(E.47)}$$

$\left(\Phi_0^{2^1 4^1 6^1 12^1}(\mathbf{Z})\right)^2$ is a meromorphic Siegel Modular Form of weight zero at level 24. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{0,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$.

E.2.8 Cycle shape 12^2

This is an element of order 12. The twining genera which contribute to the multiplicative lift are

$$\begin{aligned}
&Z^{\rho_0}(\tau, z) = Z^{1^{24}}(\tau, z), \\
&Z^{\rho}(\tau, z) = Z^{\rho_5}(\tau, z) = Z^{\rho_7}(\tau, z) = Z^{\rho_{11}}(\tau, z) = Z^{12^2}(\tau, z), \\
&Z^{\rho_2}(\tau, z) = Z^{\rho_{10}}(\tau, z) = Z^{6^4}(\tau, z), \\
&Z^{\rho_3}(\tau, z) = Z^{\rho_9}(\tau, z) = Z^{4^6}(\tau, z), \\
&Z^{\rho_4}(\tau, z) = Z^{\rho_8}(\tau, z) = Z^{3^8}, \\
&Z^{\rho_6}(\tau, z) = Z^{2^{12}} .
\end{aligned}$$

The corresponding multiplicative seed, $Z^{12^2}(\tau, z)$ is a Jacobi form of $\Gamma_0(144)$. The cusps and other data at level 144 are the following:

c	$i\infty$	0	1/2	1/3	2/3	1/4	3/4	1/6	5/6	1/8	1/9	1/12	5/12
h_e	1	144	36	16	16	9	9	4	4	9	16	1	1
N_e	1	144	72	48	48	36	36	24	24	18	16	12	12

c	7/12	11/12	1/16	1/18	1/24	5/24	1/36	7/36	1/48	5/48	1/72
h_e	1	1	9	4	1	1	1	1	1	1	1
N_e	12	12	9	8	6	6	4	4	3	3	2

E.2.9 Cycle shape $3 \cdot 21$

This is an element of order 21. The twining genera that contribute to the multiplicative lift are

$$
\begin{aligned}
&Z^{\rho_0}(\tau,z) = Z^{1^{24}}(\tau,z),\\
&Z^{\rho}(\tau,z) = Z^{\rho_2}(\tau,z) = Z^{\rho_4}(\tau,z) = Z^{\rho_5}(\tau,z) = Z^{\rho_8}(\tau,z) = Z^{\rho_{10}}(\tau,z) = Z^{\rho_{11}}(\tau,z)\\
&\qquad = Z^{\rho_{13}}(\tau,z) = Z^{\rho_{16}}(\tau,z) = Z^{\rho_{17}}(\tau,z) = Z^{\rho_{19}}(\tau,z) = Z^{\rho_{20}}(\tau,z) = Z^{3^1 21^1}(\tau,z),\\
&Z^{\rho_3}(\tau,z) = Z^{\rho_6}(\tau,z) = Z^{\rho_9}(\tau,z) = Z^{\rho_{12}}(\tau,z) = Z^{\rho_{15}}(\tau,z) = Z^{\rho_{18}}(\tau,z) = Z^{1^3 7^3}(\tau,z),\\
&Z^{\rho_7}(\tau,z) = Z^{\rho_{14}}(\tau,z)) = Z^{3^8}(\tau,z)\ .
\end{aligned}
$$

In this case the multiplicative seed, $Z^{3^1 21^1}(\tau,z)$ is a Jacobi form of $\Gamma_0(63)$. The cusps and other data for $\Gamma_0(63)$ are the following:

c	$i\infty$	1/3	2/3	1/7	1/9	1/21	2/21	0/1
h_e	1	7	7	9	7	1	1	63
N_e	1	21	21	9	7	3	3	63

The Borcherds lift given by Eq. (2.6) leads to the following product formula:

$$
\begin{aligned}
\Phi_{-1}^{3^1 21^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} &\left(1-q^n r^\ell s^m\right)^{c^0(nm,\ell)+c^1(nm,\ell)-c^3(nm,\ell)-c^7(nm,\ell)}\\
&\left(1-(q^n r^\ell s^m)^3\right)^{c^7(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^7\right)^{c^3(nm,\ell)-c^1(nm,\ell)} \left(1-(q^n r^\ell s^m)^{21}\right)^{c^1(nm,\ell)}\ .
\end{aligned}
\tag{E.48}
$$

Then using Eq. (2.5), i.e.,

$$
\begin{aligned}
c^0 &= \frac{1}{21}(f_0+12f_1+6f_3+2f_7),\ \ c^1 = \frac{1}{21}(f_0+f_1-f_3-f_7)\ ,\\
c^3 &= \frac{1}{21}(f_0-2f_1-f_3+2f_7),\ \ c^7 = \frac{1}{21}(f_0-6f_1+6f_3-f_7)\ ,
\end{aligned}
$$

we can rewrite the above product formula as

$$
\begin{aligned}
\Phi_{-1}^{3^1 21^1}(\mathbf{Z}) = qrs \prod_{(n,\ell,m)>0} &\left(1-q^n r^\ell s^m\right)^{f_1(nm,\ell)} \left(1-(q^n r^\ell s^m)^3\right)^{\frac{1}{3}(f_3(nm,\ell)-f_1(nm,\ell))}\\
&\left(1-(q^n r^\ell s^m)^7\right)^{\frac{1}{7}(f_7(nm,\ell)-f_1(nm,\ell))} \left(1-(q^n r^\ell s^m)^{21}\right)^{\frac{1}{21}(f_0(nm,\ell)+f_1(nm,\ell)-f_3(nm,\ell)-f_7(nm,\ell))}\ .
\end{aligned}
\tag{E.49}
$$

Now to see if can be written as the Borcherds product as stated in theorem 2.1 we need to compute the coefficients at different cusps. We need to consider the contributions from the cusps at $i\infty$, $(1/3, 2/3)$, $1/9$, and $(1/21, 2/21)$ as other

cusps do not have integral power in q. Here again, we need to pair up cusps with identical h_e and N_e to obtain integral coefficients. Then from straightforward calculations one obtains

$$\begin{aligned}
\pi_{FE}\Big(Z^{3^1 21^1}|M_{\frac{1}{3}} + Z^{3^1 21^1}|M_{\frac{2}{3}}\Big) &= \frac{1}{7}(Z^{3^1 21^1} - Z^{3^8}), \\
\pi_{FE}\Big(Z^{3^1 21^1}|M_{\frac{1}{9}}\Big) &= -\frac{1}{7}(Z^{3^1 21^1} - Z^{3^8}), \\
\pi_{FE}\Big(Z^{3^1 21^1}|M_{\frac{1}{21}} + Z^{3^1 21^1}|M_{\frac{2}{21}}\Big) &= -Z^{3^1 21^1} \ .
\end{aligned}$$

Using these and the product formula for Borcherds product given by Cléry- Gritsenko, we get

$$\begin{aligned}
B_{Z^{3^1 21^1}}(\mathbf{Z}) = &\prod_{(n,\ell,m)>0} \Big(1 - q^n r^\ell s^m\Big)^{f_1(nm,\ell)} \Big(1 - (q^n r^\ell s^m)^2\Big)^{-\frac{1}{3}f_1(nm,\ell)} \\
&\Big(1 - (q^n r^\ell s^m)^7\Big)^{\frac{1}{7}(f_7(nm,\ell)-f_1(nm,\ell))} \Big(1 - (q^n r^\ell s^m)^{21}\Big)^{\frac{1}{21}(f_1(nm,\ell)-f_7(nm,\ell))} \ . \qquad \text{(E.50)}
\end{aligned}$$

This does not give all the terms that appear in the product form given in Eq. (E.49). The missing terms can be accounted for by additional terms leading to

$$\Big(\Phi_{-1}^{3^1 21^1}(\mathbf{Z})\Big)^3 = B_{3Z^{3^1 21^1}}(\mathbf{Z}) B_{Z^{1^3 7^3}}(3\mathbf{Z}) \ . \qquad \text{(E.51)}$$

$\Big(\Phi_{-1}^{3^1 21^1}(\mathbf{Z})\Big)^3$ is a meromorphic Siegel Modular Form of weight (-3) at level 63. The zeroth coefficient of the Siegel Modular Form, i.e., the $m = 0$ term is given by the additive seed $\phi_{-1,1}(\tau, z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\eta(3\tau)\eta(21\tau)$.

Unlike the type-I case, none of the conjugacy class has a cycle shape with cycle length one. In all these cases, the multiplicative lift can be expressed as a product of multiple rescaled Borcherds products. This proves that the multiplicative lift can be used to construct a genus-two Siegel Modular Form for all the conjugacy classes of M_{24}, including those for which the additive construction is not known.

References

[1] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, "Counting dyons in $\mathcal{N} = 4$ string theory," *Nucl. Phys.* **B484** (1997) 543–561, arXiv:hep-th/9607026 [hep-th].

[2] S. Govindarajan and K. Gopala Krishna, "Generalized Kac-Moody Algebras from CHL dyons," *JHEP* **04** (2009) 032, arXiv:0807.4451 [hep-th].

[3] J. H. Conway and S. P. Norton, "Monstrous moonshine," *Bulletin of the London Mathematical Society* **11** no. 3, 308–339.

[4] D. Dummit, H. Kisilevsky, and J. McKay, "Multiplicative products of η-functions," *Contemp. Math.* **045** (1985) 89–98.

[5] G. Mason, "M_{24} and certain automorphic forms," *Contemp. Math.* **45** (1985) 89–98.

[6] T. Eguchi, H. Ooguri, and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," *Exper. Math.* **20** (2011) 91–96, arXiv:1004.0956 [hep-th].

[7] M. C. N. Cheng, "K3 Surfaces, $\mathcal{N} = 4$ Dyons, and the Mathieu Group M24," *Commun. Num. Theor. Phys.* **4** (2010) 623–658, arXiv:1005.5415 [hep-th].

[8] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu twining characters for K3," *JHEP* **09** (2010) 058, arXiv:1006.0221 [hep-th].

[9] T. Eguchi and K. Hikami, "Note on twisted elliptic genus of $K3$ surface," *Phys. Lett.* **B694** (2011) 446–455, arXiv:1008.4924 [hep-th].

[10] T. Gannon, "Much ado about Mathieu," *Adv. Math.* **301** (2016) 322–358, arXiv:1211.5531 [math.RT].

[11] S. Govindarajan, "Unravelling Mathieu Moonshine," *Nucl. Phys.* **B864** (2012) 823–839, arXiv:1106.5715 [hep-th].

[12] D. Persson and R. Volpato, "Second Quantized Mathieu Moonshine," *Commun. Num. Theor. Phys.* **08** (2014) 403–509, arXiv:1312.0622 [hep-th].

[13] S. Govindarajan and S. Samanta, "Two moonshines for $L_2(11)$ but none for M_{12}," *Nucl. Phys.* **B939** (2019) 566–598, arXiv:1804.06677 [hep-th].

[14] S. Govindarajan and K. Gopala Krishna, "BKM Lie superalgebras from dyon spectra in Z(N) CHL orbifolds for composite N," *JHEP* **05** (2010) 014, arXiv:0907.1410 [hep-th].

[15] S. Govindarajan, "BKM Lie superalgebras from counting twisted CHL dyons," *JHEP* **05** (2011) 089, arXiv:1006.3472 [hep-th].

[16] S. Govindarajan and S. Samanta, "BKM Lie superalgebras from counting twisted CHL dyons – II," *Nucl. Phys. B* **948** (2019) 114770, arXiv:1905.06083 [hep-th].

[17] M. C. N. Cheng and J. F. R. Duncan, "On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine," *Commun. Num. Theor. Phys.* **6** (2012) 697–758, arXiv:1110.3859 [math.RT].

[18] V. Gritsenko and F. Clery, "The Siegel modular forms of genus 2 with the simplest divisor," arXiv:0812.3962 [math.NT].

[19] T. Eguchi and K. Hikami, "Twisted Elliptic Genus for K3 and Borcherds Product," *Lett. Math. Phys.* **102** (2012) 203–222, arXiv:1112.5928 [hep-th].

[20] V. A. Gritsenko and V. V. Nikulin, "Automorphic Forms and Lorentzian Kac–Moody Algebras. Part I," arXiv:alg-geom/9610022 [math.AG].

[21] V. A. Gritsenko and V. V. Nikulin, "Automorphic forms and Lorentzian Kac-Moody algebras. Part 2," arXiv:alg-geom/9611028 [alg-geom].

[22] H. Aoki and T. Ibukiyama, "Simple graded rings of siegel modular forms, differential operators and borcherds products," *International Journal of Mathematics* **16** (03, 2005) 249–279.

[23] M. Raum, "M_{24}-twisted product expansions are Siegel modular forms," *Commun. Number Theory Phys.* **7** no. 3, (2013) 469–495. https://doi.org/10.4310/CNTP.2013.v7.n3.a3.

[24] A. O. L. Atkin and J. Lehner, "Hecke operators on $\gamma_0(m)$," *Math. Ann.* **185** (1970) 134–160.

[25] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch, and R. Volpato, "Generalized Mathieu Moonshine," *Commun. Num. Theor Phys.* **07** (2013) 145–223, arXiv:1211.7074 [hep-th].

[26] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine," *Commun. Num. Theor. Phys.* **08** (2014) 101–242, arXiv:1204.2779 [math.RT].

[27] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine and the Niemeier Lattices," arXiv:1307.5793 [math.RT].

[28] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Symmetries of K3 sigma models," *Commun. Num. Theor. Phys.* **6** (2012) 1–50, arXiv:1106.4315 [hep-th].

[29] M. R. Gaberdiel, A. Taormina, R. Volpato, and K. Wendland, "A K3 sigma model with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry," *JHEP* **02** (2014) 022, arXiv:1309.4127 [hep-th].

[30] A. Taormina and K. Wendland, "The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}," *JHEP* **08** (2013) 125, arXiv:1107.3834 [hep-th].

[31] F. Cléry and V. Gritsenko, "Siegel modular forms of genus 2 with the simplest divisor," *Proceedings of the London Mathematical Society* **102** no. 6, (2011) 1024–1052.

[32] P. Niemann, "Some Generalized Kac-Moody Algebras With Known Root Multiplicities," arXiv:math/0001029 [math.QA].

arXiv:2208.00574v1 [math.NT] 1 Aug 2022

MATHIEU MOONSHINE AND BORCHERDS PRODUCTS

HAOWU WANG AND BRANDON WILLIAMS

Abstract. The twisted elliptic genera of a $K3$ surface associated with the conjugacy classes of the Mathieu group M_{24} are known to be weak Jacobi forms of weight 0. In 2010, Cheng constructed formal infinite products from the twisted elliptic genera and conjectured that they define Siegel modular forms of degree two. In this paper we prove that for each conjugacy class of level N_g the associated product is a meromorphic Borcherds product on the lattice $U(N_g) \oplus U \oplus A_1$ in a strict sense. We also compute the divisors of these products and determine for which conjugacy classes the product can be realized as an additive (generalized Saito–Kurokawa) lift.

1. Introduction

Monstrous moonshine, proposed by Conway–Norton [12] and proved by Borcherds [3], is the phenomenon that the Fourier coefficients of the modular J-function and the Hauptmoduls of other genus zero subgroups of $\mathrm{SL}_2(\mathbb{R})$ have simple expressions involving the dimensions of representations of the monster group.

Eguchi, Ooguri and Tachikawa observed [19] that the Fourier coefficients of the elliptic genus of a $K3$ surface have similar expressions, now involving the dimensions of representations of the largest Mathieu group M_{24}. This phenomenon has been generalized to twists by all elements of M_{24} and is called *Mathieu moonshine*. Cheng, Duncan and Harvey [9, 10] further generalized this phenomenon to the umbral moonshine, which relates Niemeier lattices and Ramanujan's mock theta functions.

The elliptic genus of a $K3$ surface is a weak Jacobi form of weight 0 and index 1 on $\mathrm{SL}_2(\mathbb{Z})$. By the work of Cheng [6], Gaberdiel–Hohenegger–Volpato [21, 22] and Eguchi–Hikami [17], one can associate to each conjugacy class of M_{24} a twist of the elliptic genus which plays the role of the McKay–Thompson series in Monstrous moonshine. If $[g]$ is a conjugacy class of M_{24} of order n_g and level N_g, then the twisted elliptic genus corresponding to $[g]$ has the expression

$$\begin{aligned}\phi_g(\tau,z) &= \frac{\chi(g)}{12}\phi_{0,1}(\tau,z) + \tilde{T}_g(\tau)\phi_{-2,1}(\tau,z)\\ &= \sum_{n\geq 0}\sum_{r\in\mathbb{Z}} c_g(4n-r^2)q^n\zeta^r,\end{aligned} \tag{1.1}$$

where $\chi(g) := \phi_g(\tau,0)$ is the Witten index, the character of a certain permutation representation of M_{24} on 24 points; $\tilde{T}_g$ is a certain holomorphic modular form of weight 2 and trivial character on $\Gamma_0(N_g)$; $\phi_{0,1}$ and $\phi_{-2,1}$ are the basic weak Jacobi forms introduced by Eichler–Zagier [20]; and $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z}$ for $(\tau,z) \in \mathbb{H}\times\mathbb{C}$. Altogether ϕ_g is a weak Jacobi form of weight 0 and index 1 on $\Gamma_0(N_g)$ with trivial character. The forms $\tilde{T}_g$ involve mock modular forms of weight $\frac{1}{2}$ appearing in Mathieu moonshine [8, 7]. Gannon [23] has proved that the Fourier coefficients of these mock modular forms are true characters of M_{24}.

The Igusa cusp form Φ_{10} [31] is related to the $K3$ elliptic genus $2\phi_{0,1}$ through the Borcherds multiplicative lift [4, 28, 5]. The inverse of Φ_{10} gives the second-quantized elliptic genus of a $K3$

Date: August 2, 2022.
2020 *Mathematics Subject Classification.* 11F50, 11F46, 81T30.
Key words and phrases. Borcherds products, Jacobi forms, Mathieu group M_{24},Twisted genera of $K3$ surfaces.

surface [15] and the partition function for $\frac{1}{4}$-BPS states of type II superstring theory compactified on $K3\times T^2$ [16], and the square root of Φ_{10} is the denominator of a generalized Kac–Moody algebra [28]. Cheng [6] observed that an action of M_{24} on a certain algebraic object related to this algebra would lead to expressions for the twists by $g\in M_{24}$ as infinite products involving the $K3$ elliptic genera twisted by g^k for $k|n_g$. Based on this observation, Cheng and Duncan [7, §4] associated to each conjugacy class g of M_{24} a formal infinite product

$$\Phi_g(\tau,z,\omega)=q\zeta s\prod_{(n,r,m)>0}\exp\left(-\sum_{a=1}^{\infty}\frac{c_{g^a}(4nm-r^2)}{a}\left(q^n\zeta^r s^m\right)^a\right), \tag{1.2}$$

where $s=e^{2\pi i\omega}$ and $(n,r,m)>0$ means that either $m>0$, or $m=0$ and $n>0$, or $m=n=0$ and $r<0$. They further made the following precise conjecture.

Conjecture 1.1 (Conjecture 4.1 of [7])**.** *For each conjugacy class g of M_{24}, the product Φ_g defines a Siegel modular form with some multiplier on the congruence subgroup*

$$\Gamma_0^{(2)}(N_g)=\left\{\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in \mathrm{Sp}_4(\mathbb{Z}):C=0\bmod N_g\right\}.$$

This conjecture was also motivated by the connection via wall-crossing between the $\frac{1}{4}$-BPS spectrum and the $\frac{1}{2}$-BPS spectrum in string theory. The regularized value of $1/\Phi_g$ at $z=0$ is a product of two copies of the twisted $\frac{1}{2}$-BPS partition functions $1/\eta_g$, where η_g is the eta product determined by the cycle shape of g. As notations in [7, Table 1], for g of type $2A,3A,5A,7AB$ the products Φ_g have interpretations as partition functions for $\frac{1}{4}$-BPS states twisted by a $\mathbb{Z}/p$ symmetry for $p=2,3,5,7$ [32, 14, 13, 38, 39, 26, 24]. As pointed out by Cheng [6], a positive solution to this conjecture would provide strong evidence for the existence of the underlying M_{24} symmetry and a general theory of twisted $\frac{1}{4}$-BPS spectrum. A modular construction of Φ_g has also appeared in [25, 18] for some particular conjugacy classes.

When g is of type $1A$, Φ_g is exactly the Igusa cusp form Φ_{10}. Aoki and Ibukiyama [1] proposed a construction of Siegel modular forms on $\Gamma_0^{(2)}(N)$ as Borcherds products of Jacobi forms of weight 0 and index 1 on $\Gamma_0(N)$ under certain assumptions. This construction was given in general by Cléry and Gritsenko [11] who also extended this construction to congruence subgroups of paramodular groups of degree two. In this way Cléry and Gritsenko constructed the square roots of Φ_g for g of type $2A,3A,4B$ and proved that they and Φ_{10} are the only Siegel modular forms on groups of type $\Gamma_0^{(2)}(N)$ which vanish precisely on the diagonal. Raum [34] proved Conjecture 1.1 for all conjugacy classes whose level is a prime power; more precisely, Raum showed that these Φ_g are products of rescalings of Borcherds products in the sense of Cléry and Gritsenko. Persson and Volpat [33] gave a full proof of Conjecture 1.1 by generalizing Φ_g to an analogue $\Phi_{g,h}$ for each commuting pair $g,h\in M_{24}$, and Govindarajan and Samanta [27] extended Raum's method of proof to all conjugacy classes.

In this paper we give a different proof of the conjecture. One important point is that we prove Φ_g extends to a single-valued meromorphic function on all $\mathbb{H}_2$, which does not seem to be clear a priori as the zeros and poles of Φ_g do not necessarily intersect the region where the infinite product converges. We represent Φ_g as the Borcherds lift of an explicit vector-valued modular form for the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$. This allows us to explicitly compute the divisor of Φ_g and to determine when Φ_g can be constructed as an additive lift. This was unexpected before (e.g. [33]).

Our approach is based on an isomorphism recently established in [40]:

Theorem 1.2 (Theorem 5.1 of [40])**.** *Let U be the unique even unimodular lattice of signature $(1,1)$ and L be an even positive-definite lattice of rank $\mathrm{rk}(L)$. Let N be a positive integer. There is an isomorphism between the space of weakly holomorphic modular forms of weight $k-\frac{1}{2}\mathrm{rk}(L)$ for*

the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ *attached to* $U(N) \oplus L$ *and the space of tuples of Jacobi forms*

$$\bigoplus_{d|N} J^!_{k,L}(\Gamma_1(N/d)),$$

where $J^!_{k,L}(\Gamma_1(t))$ *is the space of weakly holomorphic Jacobi forms of weight* k *and index* L *on* $\Gamma_1(t)$.

This isomorphism is used in [40] to classify Borcherds products of singular weight and to prove the moonshine conjecture for the Conway group proposed by Borcherds [2, 3] and Scheithauer [35, 36]: all twists of the denominator identity of the fake monster Lie algebra by elements of Conway's group Co_0 are the Fourier expansions of Borcherds products of singular weight on certain symmetric domains of type IV.

It is well known that $\mathrm{Sp}_4(\mathbb{Z})$ can be realized as a subgroup of the orthogonal group $\mathrm{O}^+(2U \oplus A_1)$ and Siegel modular forms of degree two can be viewed as modular forms on orthogonal groups of signature $(3, 2)$ (see e.g. [29]). By applying Theorem 1.2 to the A_1 root lattice we identify the inputs into Borcherds' lift to $\Gamma_0^{(2)}(N_g)$ with certain sequences of weakly holomorphic Jacobi forms (ϕ_d), $d|N_g$, where each ϕ_d has level $\Gamma_0(N_g/d)$.

We prove the following form of Conjecture 1.1:

Theorem 1.3. *Let* g *be a conjugacy class of* M_{24} *of cycle shape* $\prod t^{b_t}$ *and level* N_g. *Then* Φ_g *is a meromorphic Borcherds product of weight* $k_g = \frac{1}{2}\sum_{t=1}^{\infty} b_t - 2$ *on the lattice* $U(N_g) \oplus U \oplus A_1$. *The preimage of* Φ_g *under Borcherds' lift is the family*

$$\phi_{g^d}, \quad d|N_g.$$

The divisor of Φ_g *is computed explicitly in Appendix A. In particular we find the following:*

(1) Φ_g *is holomorphic if and only if* $N_g \leq 4$, *i.e. if* g *is of type* $1A$, $2A$, $2B$, $3A$, $4B$;
(2) Φ_g *has a meromorphic square root on* $\mathbb{H}_2$ *if and only if* g *is of type* $1A$, $2A$, $2B$, $3A$, $3B$, $4A$, $4B$, $4C$, $5A$, $6A$, $6B$, $10A$, $11A$.

Since each ϕ_{g^d} has rational Fourier expansion at infinity, [40] shows that the associated vector-valued modular form also has rational Fourier coefficients and therefore there exists a denominator $d_g \in \mathbb{N}$ such that $\Phi_g^{d_g}$ is a meromorphic Borcherds product. The nontrivial point of Theorem 1.3 is that we can take $d_g = 1$. The proof involves calculating the Fourier expansions of all ϕ_g at all cusps. Note that the divisors of Φ_g can be rather complicated, (e.g. for g of type $12B$ and $21AB$), but these zeros and poles have quite low multiplicities.

It was asked by Cheng–Duncan [7] and Eguchi–Hikami [18] whether Φ_g can be constructed as an additive lift of a Jacobi form when g is one of the 12 cycle shapes for which Φ_g has positive weight. We find a complete answer to this question here. Since most Φ_g have poles, we have to interpret this as the singular additive lift due to Borcherds [5, Theorem 14.3]. The natural candidate for the input into the additive lift is the leading Fourier–Jacobi coefficient of Φ_g,

$$\varphi_g(\tau, z) := \eta_g(\tau)\phi_{-2,1}(\tau, z), \tag{1.3}$$

which is a weak Jacobi form of weight k_g and index 1 on $\Gamma_0(N_g)$ with a Dirichlet character [7]. By comparing divisors we show that the additive lift of φ_g is a Borcherds product, which is related to (but does not always equal) Φ_g.

Theorem 1.4. *The product* Φ_g *can be constructed as a Borcherds additive lift if and only if* g *is of type* $1A$, $2A$, $2B$, $3A$, $4A$, $4B$, $5A$, $6A$, $7AB$, $8A$. *The input of the associated additive lift is the image of* $(\varphi_d)_{d|N_g}$ *under the isomorphism of Theorem 1.2, where* $\varphi_1 = \varphi_g$ *and* $\varphi_d = 0$ *for* $d > 1$. *When* g *is of type* $4C$ *or* $3B$, *the following identity holds:*

$$\Phi_g \cdot \mathbf{B}(f_g) = \mathbf{G}(\varphi_g),$$

where f_g is a Jacobi form appearing in an identity established by Eguchi and Hikami [18, §4]

$$\phi_g(\tau,z) = -\frac{T_2\varphi_g(\tau,z)}{\varphi_g(\tau,z)} - f_g(\tau,z), \tag{1.4}$$

the form $\mathbf{B}(f_g)$ is the Borcherds multiplicative lift of the family ($\phi_1 = f_g;\ \phi_d = 0,\ d > 1$), and $\mathbf{G}(\varphi_g)$ is the Borcherds additive lift of the family ($\varphi_1 = \varphi_g;\ \varphi_d = 0,\ d > 1$).

The paper is organized as follows. In Section 2 we collect necessary materials to prove the theorems, including the isomorphism between Jacobi forms and vector-valued modular forms, and the Jacobi forms representations of Borcherds additive lifts and Borcherds products for Siegel modular forms of degree two. Theorems 1.3 and 1.4 are proved in Section 3. We formulate the principal parts of the inputs of Φ_g and $\mathbf{B}(f_g)$ in Appendix A and Appendix B, respectively.

2. Preliminaries

2.1. **Modular forms for the Weil representation.** Let M be an even integral lattice of signature (l^+, l^-) with dual lattice M'. The lattice M induces a discriminant form $D_M := (M'/M, Q)$ with the quadratic form

$$Q : M'/M \to \mathbb{Q}/\mathbb{Z},\ Q(x+M) = (x,x)/2 + \mathbb{Z}.$$

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the metaplectic group consisting of pairs $A = (A, \psi_A)$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and ψ_A is a holomorphic square root of $\tau \mapsto c\tau + d$ on $\mathbb{H}$. The product in $\mathrm{Mp}_2(\mathbb{Z})$ is

$$(A,\psi_A)(B,\psi_B) = (AB, \psi_A(B\tau)\psi_B(\tau)).$$

The group $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $T = (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$ and $S = (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau})$. The representation of $\mathrm{Mp}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[D_M] = \mathrm{span}(\mathfrak{e}_x :\ x \in D_M)$ defined by

$$\rho_M(T)\mathfrak{e}_x = \mathbf{e}(-Q(x))\mathfrak{e}_x \quad \text{and} \quad \rho_M(S)\mathfrak{e}_x = \frac{\mathbf{e}(\mathrm{sign}(M)/8)}{\sqrt{|D_M|}} \sum_{y\in D_M} \mathbf{e}((x,y))\mathfrak{e}_y$$

is called the *Weil representation* ρ_M, where $\mathbf{e}(t) = e^{2\pi i t}$ for $t \in \mathbb{C}$, and $\mathrm{sign}(M) = l^+ - l^- \mod 8$.

A *weakly holomorphic modular form* of weight $k \in \frac{1}{2}\mathbb{Z}$ for the Weil representation ρ_M is a holomorphic function $f : \mathbb{H} \to \mathbb{C}[D_M]$ that satisfies

$$f(A\cdot\tau) = \psi_A(\tau)^{2k}\rho_M(A)f(\tau), \quad \text{for all } A \in \mathrm{Mp}_2(\mathbb{Z}),$$

and which is meromorphic at infinity; that is, f has a Fourier expansion of the form

$$f(\tau) = \sum_{x\in D_M} \sum_{\substack{n\in\mathbb{Z}-Q(x) \\ n\gg-\infty}} \alpha(n,x)q^n\mathfrak{e}_x.$$

The finite sum

$$\sum_{x\in D_M}\sum_{n<0} \alpha(n,x)q^n\mathfrak{e}_x$$

is called the *principal part* of f. The form f is called a *holomorphic modular form* if it is holomorphic at infinity, i.e. its principal part is zero. The $\mathbb{C}$-vector spaces of weakly holomorphic and holomorphic modular forms of weight k for ρ_M are respectively denoted by $M_k^!(\rho_M)$ and $M_k(\rho_M)$.

2.2. **Jacobi forms.** Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. A Jacobi form of weight $k \in \mathbb{Z}$, index $m \in \mathbb{N}$, level Γ and character $\chi : \Gamma \to \mathbb{C}^\times$ is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the transformations

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \exp\left(2\pi i \frac{mcz^2}{c\tau+d}\right)\phi(\tau, z)$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = \exp\Big(-2\pi i m\lambda^2\tau - 4\pi i m\lambda z\Big)\phi(\tau, z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\lambda, \mu \in \mathbb{Z}$, as well as certain growth conditions at "cusps". For any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the Fourier expansion of ϕ at the cusp $A \cdot \infty$ is determined up to a root of unity through

$$\begin{aligned} \phi\Big|_{k,m} A(\tau) &= (c\tau+d)^{-k} e^{-2\pi i \frac{mcz^2}{c\tau+d}} \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) \\ &= \sum_{n,r} c_A(n,r) q^n \zeta^r, \end{aligned}$$

where $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi i z}$ and $c_A(n,r) \in \mathbb{C}$. We call ϕ a *weakly holomorphic* Jacobi form if $c_A(n,r) = 0$ for all A and all sufficiently small n; a *weak* Jacobi form if $c_A(n,r) = 0$ whenever $n < 0$; and a *holomorphic* Jacobi form if $c_A(n,r) = 0$ whenever $r^2 > 4mn$.

The spaces of weakly holomorphic, weak, and holomorphic Jacobi forms of weight k, index m and level Γ are written

$$J^!_{k,m}(\Gamma, \chi), \quad J^{\mathrm{w}}_{k,m}(\Gamma, \chi), \quad J_{k,m}(\Gamma, \chi).$$

2.3. **The isomorphism between Jacobi forms and vector-valued modular forms.** Modular forms for the Weil representation have a useful equivalent description in terms of sequences of Jacobi forms. This was worked out for Jacobi forms of lattice index in [40] and we describe the special case of scalar-index Jacobi forms below.

Let t and N be positive integers. We fix the lattice $M := U(N) \oplus A_1(t)$, such that $M'/M \cong \mathbb{Z}/2t\mathbb{Z} \times \mathbb{Z}^2/N\mathbb{Z}$. We write elements of M'/M as tuples $(a/N, b/2t, c/N)$ with $a, c \in \mathbb{Z}/N\mathbb{Z}$ and $b \in \mathbb{Z}/2t\mathbb{Z}$, such that the quadratic form on M'/M is

$$Q(a/N, b/2t, c/N) = b^2/4t + ac/N.$$

Theorem 2.1. *Let $k \in \mathbb{Z}$. There is an isomorphism*

$$\hat{\mathbb{J}} : \bigoplus_{d|N} J^!_{k,t}(\Gamma_1(N/d)) \xrightarrow{\sim} M^!_{k-\frac{1}{2}}(\rho_M)$$

which sends a sequence of weakly holomorphic Jacobi forms

$$(\phi_d)_{d|N}, \quad \phi_d \in J^!_{k,t}(\Gamma_1(N/d)) \quad \text{for } d|N$$

to the weakly holomorphic modular form of weight $k - \frac{1}{2}$ for ρ_M defined as

$$\hat{\mathbb{J}}((\phi_d)_{d|N}) = \frac{1}{N} \sum_{a,b,c \in \mathbb{Z}/N\mathbb{Z}} e^{-2\pi i ac/N} \sum_{n \in \mathbb{Q}} \sum_{r \in \mathbb{Z}/2t\mathbb{Z}} c_{a,b}(n,r) q^{n - r^2/4t} \mathfrak{e}_{(c/N, r/2t, b/N)},$$

where for any $(a,b) \in \mathbb{Z}^2$ the numbers $c_{a,b}(n,r)$ are defined by the Fourier expansion

$$\phi_d\Big|_{k,t} A_{a,b}(\tau, z) = \sum_{n \in \mathbb{Q}} \sum_{r \in \mathbb{Q}} c_{a,b}(n,r) q^n \zeta^r,$$

where $d = \gcd(a,b,N)$ and $A_{a,b} \in \mathrm{SL}_2(\mathbb{Z})$ with $A_{a,b}(a,b) = (d,0)$ mod N.

Proof. This is a special case of [40]. Note that $\hat{\mathbb{J}}$ is the inverse of the isomorphism labelled $\mathbb{J}$ there. $\square$

The above map $\hat{\mathbb{J}}$ induces an isomorphism between spaces of holomorphic forms. The image of the subspace $\bigoplus_{d|N} J^!_{k,t}(\Gamma_0(N/d))$ under $\hat{\mathbb{J}}$ consists of those vector-valued modular forms

$$F(\tau) = \sum_{x \in M'/M} f_x(\tau)\mathfrak{e}_x$$

with the property $f_{(a/N,b/2t,c/N)} = f_{(au/N,b/2t,cu^{-1}/N)}$ for every $u \in (\mathbb{Z}/N\mathbb{Z})^\times$.

2.4. **Siegel modular forms of degree two.** Let $\Gamma^{(2)}$ be an arithmetic subgroup of the symplectic group $\mathrm{Sp}_4(\mathbb{R})$. An important class of examples for $\Gamma^{(2)}$ will be the following congruence subgroups of the paramodular group of level t:

$$K_0(t,N) = \left\{ \begin{pmatrix} * & *t & * & * \\ * & * & * & */t \\ *N & *Nt & * & * \\ *Nt & *Nt & *t & * \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Q}) : \text{all } * \text{ are integers} \right\}.$$

Note that $K_0(1,1) = \mathrm{Sp}_4(\mathbb{Z})$, $K_0(1,N) = \Gamma_0^{(2)}(N)$ is the congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ appearing in Conjecture 1.1, and $K_0(t,1) = K(t)$ is the usual paramodular group of level t. When $t > 1$, the normal double extension $K_0(t,N)^+ = K_0(t,N) \cup K_0(t,N)V_t$ also gives an example of $\Gamma^{(2)}$, where V_t is the Fricke involution

$$V_t = \frac{1}{\sqrt{t}} \begin{pmatrix} 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \end{pmatrix}, \quad t \geq 1.$$

$\Gamma^{(2)}$ acts on the Siegel upper half-space

$$\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}) : \mathrm{Im}(Z) > 0 \right\}$$

by Möbius transformations. A *meromorphic Siegel modular form* of weight $k \in \frac{1}{2}\mathbb{Z}$ for $\Gamma^{(2)}$ with a character (or multiplier system) $\chi : \Gamma^{(2)} \to \mathbb{C}^\times$ is a meromorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ satisfying

$$F(A \cdot Z) = \chi(A)\det(cZ+d)^k F(Z), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^{(2)}.$$

If $V_t \in \Gamma^{(2)}$ then we have

$$F\left(\begin{pmatrix} t\omega & z \\ z & \tau/t \end{pmatrix}\right) = (-1)^k \chi(V_t) F\left(\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\right).$$

2.5. **Singular Gritsenko lifts and Borcherds products.** In [5] Borcherds defined the singular theta lift and used it to find two constructions of meromorphic modular forms on orthogonal groups of lattices of signature $(l,2)$. The first construction yields modular forms, called *Borcherds products*, with special divisors and an infinite product expansion at every 0-dimensional cusp. The second construction is a generalization of the Gritsenko lift which produces meromorphic modular forms of general weight $k \in \mathbb{N}$, each with poles of order exactly k along special divisors. Both of these lifts take weakly holomorphic modular forms for the Weil representation as inputs.

In this subsection we want to describe the two singular theta lifts in special case of Siegel modular forms of degree two which can be identified with modular forms on orthogonal groups of signature $(3,2)$. Through the isomorphism $\hat{\mathbb{J}}$ in Theorem 2.1, the input forms into both lifts are understood as sequences of weakly holomorphic Jacobi forms.

The poles of singular Gritsenko lifts, and the zeros and poles of Borcherds products, are known explicitly and (for general lattices) lie on hyperplanes. Through the identification with the Siegel

upper half-space these divisors become the classical Humbert surfaces. Let a, n, r, m and b be integers satisfying $\gcd(a,n,r,m,b)=1$ and

$$\delta = \frac{ab}{N} - nm + \frac{r^2}{4t} > 0.$$

The Humbert surface (for $K_0(t,N)$) associated to this data is

$$H(a,n,r,m,b) = \Big\{ \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2 : \frac{a}{N} + n\tau + rz + tm\omega + bt \cdot \det(Z) = 0 \Big\}.$$

The number δ is called the discriminant of $H(a,n,r,m,b)$. Note that $K_0(t,N)$ maps Humbert surfaces to Humbert surfaces and preserves the discriminant.

We now describe the two theta lifts. We will only consider the special case that all Jacobi forms ϕ_d are $\Gamma_0(N/d)$-modular, because only this case is needed later. Under this assumption, the lifts are modular forms for the paramodular group $K_0(t,N)$ (possibly with characters). Additionally, the Fourier expansions of the lifts (but not the divisors) can be read off of the Fourier expansions of ϕ_d directly. Both theorems below are a special case of [40] which describes theta lifts more generally in terms of Jacobi forms of lattice index.

The Fourier–Jacobi expansions of the theta lifts involve certain Hecke operators. Let χ be a Dirichlet character modulo D and $\phi \in J^!_{k,t}(\Gamma_0(D),\chi)$ be a weakly holomorphic Jacobi form with Fourier expansion at infinity

$$\phi(\tau,z) = \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} c(n,r) q^n \zeta^r.$$

For any positive integer m the *index-raising Hecke operators* $T_-^{(D)}(m)$ are defined by

$$\phi\Big|_{k,t} T_-^{(D)}(m)(\tau,z) = \frac{1}{m} \sum_{\substack{a,d\in\mathbb{N} \\ ad=m \\ (a,D)=1}} \sum_{b\in\mathbb{Z}/d\mathbb{Z}} a^k \chi(a) \phi\Big(\frac{a\tau+b}{d}, az\Big).$$

In particular

$$\phi\Big|_{k,t} T_-^{(D)}(m)(\tau,z) = \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} \sum_{\substack{a\in\mathbb{N} \\ (a,D)=1 \\ a|\gcd(n,r,m)}} a^{k-1} \chi(a) c\left(\frac{nm}{a^2}, \frac{r}{a}\right) q^n \zeta^r.$$

By [11, Lemma 2.1], we have

$$\phi\Big|_{k,t} T_-^{(D)}(m) \in J^!_{k,mt}(\Gamma_0(D),\chi), \quad m \geq 1.$$

Following [4, Theorems 7.1, 9.3] and [5, Theorem 14.3], when $k \geq 0$ we define the index-zero Hecke operator by

$$\phi\Big|_{k,t} T_-^{(D)}(0)(\tau,z) = c_\phi + \sum_{\substack{n,r\in\mathbb{Z} \\ (n,r)>0}} \sum_{\substack{a\in\mathbb{N} \\ (a,D)=1 \\ a|\gcd(n,r)}} a^{k-1} \chi(a) c(0,r/a) q^n \zeta^r,$$

where c_ϕ is a certain constant (depending on ϕ, involving Bernoulli numbers), such that the above sum defines a meromorphic Jacobi form of weight k and index 0 for $\Gamma_0(D)$ with character χ. If all coefficients $c(0,r)$ of ϕ are zero, then $\phi\big|_{k,t} T_-^{(D)}(0)$ is identically zero.

Let $\phi \in J^!_{k,t}(\Gamma_0(D),\chi)$ with $k \geq 0$. We define the *formal additive lift* of ϕ as

$$\mathbf{G}(\phi)\left(\begin{pmatrix} \tau & z \\ z & w \end{pmatrix}\right) = \sum_{m=0}^{\infty} \phi\Big|_{k,t} T_-^{(D)}(m)(\tau,z) e^{2\pi i m t \omega}.$$

Theorem 2.2. *Let $M = U(N) \oplus A_1(t)$ with quadratic form Q. Let $\phi \in J^!_{k,t}(\Gamma_0(N), \chi)$, where k is a positive integer and χ is a Dirichlet character modulo N. Then the additive lift $\mathbf{G}(\phi)$ defines a meromorphic Siegel modular form of weight k for $K_0(t,N)^+$ with a character induced by χ, and this form is always symmetric, i.e. $\mathbf{G}(\phi)(V_t \cdot Z) = \mathbf{G}(\phi)(Z)$.*

We define $\phi_1 = \phi$, $\phi_d = 0$ for $d > 1$ and write

$$\hat{\mathbb{J}}(\phi) := \hat{\mathbb{J}}((\phi_d)_{d|N}) = \sum_{\gamma \in M'/M} \sum_{n \in \mathbb{Z} - Q(\gamma)} \alpha(n,\gamma) q^n \mathfrak{e}_\gamma.$$

Then the only singularities of $\mathbf{G}(\phi)$ are poles of order k along the Humbert surfaces $H(a,n,r,m,b)$ of discriminant $\delta = ab/N - nm + r^2/4t$ for which

$$\sum_{\lambda=1}^{\infty} \lambda^{-k} \alpha\left(-\lambda^2 \delta, \left(\frac{\lambda a}{N}, \frac{\lambda r}{2t}, \frac{\lambda b}{N}\right)\right) \neq 0.$$

Proof. By [40, Corollary 5.2], the vector-valued modular form $\hat{\mathbb{J}}(\phi)$ is invariant under $(\mathbb{Z}/N\mathbb{Z})^\times$ up to some character induced by χ. It is known that $K_0(t,N)/\{\pm I\}$ is isomorphic to the group generated by $(\mathbb{Z}/N\mathbb{Z})^\times$ and the discriminant kernel of the signature $(3,2)$ lattice $U(N) \oplus U \oplus A_1(t)$ (see e.g. [30, §2]). Therefore, we obtain the modularity of $\mathbf{G}(\phi)$ by applying [40, Remark 5.15] to the rank-one lattice $L = A_1(t)$. The property of singularities follows from [5, Theorem 14.3]. □

Theorem 2.3. *Let $M = U(N) \oplus A_1(t)$ with quadratic form Q. Let $\phi_d \in J^!_{0,t}(\Gamma_0(N/d))$, $d|N$ be weakly holomorphic Jacobi forms of weight zero with Fourier expansions*

$$\phi_d(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} c_d(n,r) q^n \zeta^r,$$

and suppose the principal part of

$$\hat{\mathbb{J}}((\phi_d)_{d|N}) = \sum_{\gamma \in M'/M} \sum_{n \in \mathbb{Z} - Q(\gamma)} \alpha(n,\gamma) q^n \mathfrak{e}_\gamma$$

is integral, i.e. $\alpha(n,r) \in \mathbb{Z}$ for $n < 0$. Then there is a meromorphic modular form $\Psi = \mathbf{B}((\phi_d)_{d|N})$ for the group $K_0(t,N)^+$, the Borcherds product*, with the following properties:*

(1) *When τ and w have sufficiently large imaginary part, Ψ is given by the infinite product*

$$\begin{aligned}\Psi(Z) &= q^A \zeta^B s^C \prod_{(n,r,m)>0} (1 - q^n \zeta^r s^{tm})^{\mathrm{mult}(n,r,m)} \\ &= q^A \zeta^B s^C \prod_{d|N} \prod_{(n,r,m)>0} (1 - q^{dn} \zeta^{dr} s^{dtm})^{\mathrm{mult}_d(nm,r)}\end{aligned}$$

where $s = e^{2\pi i \omega}$ and $(n,r,m) > 0$ means either $n > 0$, or $n = 0$ and $m > 0$, or $n = m = 0$ and $r < 0$; let μ denote the Möbius function;

$$\mathrm{mult}(n,r,m) = \sum_{\substack{bd>0 \\ bd|\gcd(n,r,m,N)}} \frac{\mu(b)}{bd} c_d\left(\frac{nm}{b^2 d^2}, \frac{r}{bd}\right);$$

$$\mathrm{mult}_d(n,r) = \sum_{t|d} \frac{\mu(d/t)}{d} c_t(n,r);$$

and where

$$A = \frac{1}{24} \sum_{r \in \mathbb{Z}} c_N(0,r), \quad B = \frac{1}{2} \sum_{r>0} c_N(0,r) r, \quad C = \frac{1}{4} \sum_{r \in \mathbb{Z}} c_N(0,r) r^2$$

is the Weyl vector.

(2) Ψ *has the Fourier–Jacobi expansion*

$$\Psi(Z) = \Theta(\tau, z) s^C \cdot \exp\Big(- \sum_{d|N} d^{-1} \mathbf{G}(\phi_d)(dZ) \Big),$$

in which the leading Fourier–Jacobi coefficient Θ *is the generalized theta block*

$$\Theta(\tau, z) = \prod_{d|N} \Big[\eta(d\tau)^{\mathrm{mult}_d(0,0)} \prod_{r=1}^{\infty} \Big(\frac{\vartheta(d\tau, drz)}{\eta(d\tau)} \Big)^{\mathrm{mult}_d(0,r)} \Big],$$

where as usual

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=1}^{\infty} \left(\frac{12}{n} \right) q^{n^2/24}$$

and

$$\begin{aligned} \vartheta(\tau, z) &= q^{1/8} (\zeta^{1/2} - \zeta^{-1/2}) \prod_{n=1}^{\infty} (1 - q^n \zeta)(1 - q^n)(1 - q^n \zeta^{-1}) \\ &= \sum_{n=1}^{\infty} \left(\frac{4}{n} \right) q^{n^2/8} (\zeta^{n/2} - \zeta^{-n/2}) \end{aligned}$$

(3) Ψ *has weight*

$$\frac{\alpha(0,0)}{2} = \frac{1}{2} \sum_{d|N} \mathrm{mult}_d(0,0).$$

(4) *All poles and zeros of* Ψ *lie on Humbert surfaces. The multiplicity of* Ψ *along the Humbert surface* $H(a, n, r, m, b)$ *of discriminant* δ *is the sum*

$$\sum_{\lambda=1}^{\infty} \alpha \left(-\lambda^2 \delta, \left(\frac{\lambda a}{N}, \frac{\lambda r}{2t}, \frac{\lambda b}{N} \right) \right).$$

(5) *We have the duality*

$$\Psi\left(\left(\begin{smallmatrix} t\omega & z \\ z & \tau/t \end{smallmatrix} \right) \right) = (-1)^{D_0} \Psi\left(\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix} \right) \right),$$

where D_0 *is the multiplicity of* Ψ *along the divisor* $H(0, -1, 0, 1, 0)$*, i.e.*

$$D_0 = \frac{1}{N} \sum_{d|N} \sum_{\lambda=1}^{\infty} \varphi(N/d) c_d(-\lambda^2, 0),$$

here $\varphi(-)$ *is the Euler's totient function.*

Proof. This is a special case of [40, Theorem 5.9]. □

If every ϕ_d has rational Fourier expansion at infinity then $\hat{\mathbb{J}}((\phi_d)_{d|N})$ also has rational Fourier expansion by [40, Lemma 5.6]. Therefore, there exists an integer D such that $\hat{\mathbb{J}}((D \cdot \phi_d)_{d|N})$ has integral principal part.

We also have the following expression for Ψ which is formally similar to the infinite product proposed by Cheng–Duncan (Equation (1.2) of the introduction).

Proposition 2.4 ((5.7) in §5.2 of [40])**.** *Suppose the conditions of Theorem 2.3 hold. For any positive integer* a *we define*

$$\phi_a = \phi_d, \quad d = \gcd(a, N).$$

Then the Borcherds product Ψ is represented on an open subset of $\mathbb{H}_2$ by the infinite product

$$\Psi(Z) = q^A \zeta^B s^C \prod_{(n,r,m)>0} \exp\left(-\sum_{a=1}^{\infty} c_a(nm,r) \frac{(q^n \zeta^r s^{tm})^a}{a} \right). \tag{2.1}$$

Remark 2.5. Cléry–Gritsenko ([11], especially Theorem 3.1) have considered Borcherds products on the groups $K_0(t,N)$ associated to Jacobi forms. In our notation, their result concerns the special case that all ϕ_d, $d>1$ are traces of ϕ_1, i.e.

$$\phi_d = \sum_{M \in \Gamma_0(N)\backslash\Gamma_0(N/d)} \phi_1\Big|_{k,t} M.$$

2.6. **The Mathieu group and eta products.** The largest Mathieu group M_{24} is a sporadic simple group of order

$$244823040 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$$

with a natural linear action on the Leech lattice, the unique unimodular lattice of rank 24 without roots. Through this action, every $g \in M_{24}$ has a characteristic polynomial of the form

$$\det(X-g) = \prod_{k=1}^{\infty} (1-X^k)^{b_k}, \quad b_k \in \mathbb{N}_0.$$

The symbol $\prod_{b_k \neq 0} k^{b_k}$ is called the *cycle shape* of g and it depends only on the conjugacy class of g in M_{24}.

There are 26 conjugacy classes in M_{24} with 21 distinct cycle shapes (see e.g. [7, Table 1]). For a conjugacy class $[g]$ with cycle shape $\prod k^{b_k}$, the largest $k \in \mathbb{N}$ for which $b_k \neq 0$ is the order of g, denoted n_g. The product of n_g and the smallest k for which $b_k \neq 0$ is the *level* of g, denoted N_g; it is the level of the associated eta product

$$\eta_g(\tau) = \prod_k \eta(k\tau)^{b_k} = q^{\frac{1}{24}\sum_k k b_k} \prod_{n=1}^{\infty} \prod_{k=1}^{\infty} (1-q^{nk})^{b_k}.$$

More precisely, η_g is a cusp form of weight $\frac{1}{2}\sum_k b_k$ for the group $\Gamma_0(N_g)$ and a Dirichlet character, which is trivial if the weight is even.

We write $\chi(g) := \mathrm{tr}(g)$ for the trace of g acting on the Leech lattice; if g has cycle shape $\prod_k k^{b_k}$ then $\chi(g) = b_1$. For any $d \in \mathbb{N}$, the cycle shape of g^d is

$$\prod_{k=1}^{\infty} \Big(k/\gcd(k,d) \Big)^{\gcd(k,d)\cdot b_k},$$

and in particular $\chi(g^d) = \sum_{k|d} k b_k$. By Möbius inversion, the exponents in the cycle shape are

$$b_d = \frac{1}{d} \sum_{k|d} \mu(d/k) \chi(g^k). \tag{2.2}$$

Each conjugacy class $[g]$ in M_{24} determines a twisted elliptic genus of $K3$ surfaces. This is a weak Jacobi form of weight 0 and index 1 and level $\Gamma_0(N_g)$ with trivial character, given by the expression

$$\begin{aligned} \phi_g(\tau,z) &= \frac{\chi(g)}{12} \phi_{0,1}(\tau,z) + \tilde{T}_g(\tau) \phi_{-2,1}(\tau,z) \\ &= \sum_{n=0}^{\infty} \sum_{r \in \mathbb{Z}} c_g(4n-r^2) q^n \zeta^r \\ &= 2\zeta^{-1} + (\chi(g)-4) + 2\zeta + O(q). \end{aligned}$$

Here, $\phi_{-2,1}$ and $\phi_{0,1}$ are the basic weak Jacobi forms

$$\phi_{-2,1}(\tau,z) = \frac{\vartheta(\tau,z)^2}{\eta(\tau)^6} = \zeta^{-1} - 2 + \zeta + O(q) \in J^{\mathrm{w}}_{-2,1}(\mathrm{SL}_2(\mathbb{Z})),$$
$$\phi_{0,1}(\tau,z) = -\frac{3}{\pi^2}\wp(\tau,z)\cdot\phi_{-2,1}(\tau,z) = \zeta^{-1} + 10 + \zeta + O(q) \in J^{\mathrm{w}}_{0,1}(\mathrm{SL}_2(\mathbb{Z})),$$

where $\wp$ is the Weierstrass $\wp$-function. The function $\tilde{T}_g$ is a particular modular form of weight 2 which can be found in [18, Table 2] or [27, Table 1].

Lemma 2.6. *Fix $g \in M_{24}$. Let a be a positive integer and $d = \gcd(a, n_g)$. Then $\phi_{g^a} = \phi_{g^d}$. Moreover, for any $d|N_g$, ϕ_{g^d} is a weak Jacobi form on $\Gamma_0(N_g/d)$ with trivial character.*

Proof. The form ϕ_h depends only the cycle shape of h. Since g^a and g^d have the same cycle shape, the equality $\phi_{g^a} = \phi_{g^d}$ is proved. To prove the last claim it suffices to show that the level of g^d divides N_g/d. This is verified by direct calculation. □

3. Proofs of the main theorems

We first employ Theorem 2.3 to prove Conjecture 1.1, that is, Φ_g is a meromorphic Siegel modular form on $\Gamma_0^{(2)}(N_g)$ which can be constructed as a Borcherds product on $U(N_g) \oplus U \oplus A_1$.

Proof of Theorem 1.3. It is clear from the definition that each ϕ_g has rational Fourier coefficients at infinity. By [40, Lemma 5.6], the image of $(\phi_{g^d})_{d|N_g}$ under the isomorphism $\hat{\mathbb{J}}$ (see Theorem 2.1), which we denote F_g, also has rational Fourier expansion. Therefore, there exists a denominator $d_g \in \mathbb{N}$ such that $d_g \cdot F_g$ has integral principal part. It follows that $\mathbf{B}(d_g F_g)$ is a well-defined meromorphic Borcherds product. Since

$$\phi_{g^{N_g}}(\tau,z) = 2\phi_{0,1}(\tau,z) = 2(\zeta + \zeta^{-1} + 10) + O(q),$$

the Weyl vector of $\mathbf{B}(d_g F_g)$ is $(A,B,C) = (d_g, d_g, d_g)$. Using Proposition 2.4 and Lemma 2.6 we see that

$$\mathbf{B}(d_g F_g) = \Phi_g^{d_g}.$$

By Theorem 2.3, the weight of Φ_g is

$$\begin{aligned} k_g &= \frac{1}{2}\sum_{d|N_g} \mathrm{mult}_d(0,0) \\ &= \frac{1}{2}\sum_{d|N_g}\frac{1}{d}\sum_{k|d}\mu(d/k)c_{g^k}(0) \\ &= \frac{1}{2}\sum_{d|N_g}\frac{1}{d}\sum_{k|d}\mu(d/k)\mathrm{tr}(g^k) - 2\sum_{d|N_g}\frac{1}{d}\sum_{k|d}\mu(d/k) \\ &= -2 + \frac{1}{2}\sum_k b_k, \end{aligned}$$

where we have used Equation (2.2) in the last equality.

It remains to show that $\mathbf{B}(F_g)$ itself is single-valued, i.e. the principal part of F_g is integral. By Theorem 2.1, the principal part of F_g is completely determined by the Fourier coefficients of index (n,r) with $n < \frac{1}{4}$ of ϕ_{g^d} at all cusps for all $d|n_g$. The forms $\tilde{T}_g$ are expressions in eta quotients and modular forms of the type

$$E_2^{(N)}(\tau) := \frac{1}{N-1}\Big(NE_2(N\tau) - E_2(\tau)\Big) = 1 + O(q),$$

where $E_2(\tau) = 1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n$. The Fourier expansion of $E_2^{(N)}$ at any cusp is easy to determine using the fact that E_2 is quasimodular for the full group $\mathrm{SL}_2(\mathbb{Z})$, and the Fourier expansion of an eta quotient at any cusp was given explicitly by Scheithauer [37, Propositions 6.1 and 6.2]. Using these formulas, we were able to find the principal parts by a computer calculation; they are listed in Appendix A and are indeed integral. We have thus proved that Φ_g is a meromorphic Siegel modular form of weight k_g for $\Gamma_0^{(2)}(N_g)$. □

Theorem 1.3 yields short proofs of three previously-known results which have applications to mathematical physics (see e.g. [6, 7, 33]):

Corollary 3.1. *The form Φ_g has the Fourier–Jacobi expansion*

$$\Phi_g(Z) = \varphi_g(\tau, z)\cdot s\cdot \exp\Big(-\sum_{d|N_g}\mathbf{G}\big(d^{-1}\phi_{g^d}\big)(dZ)\Big), \tag{3.1}$$

where the leading Fourier–Jacobi coefficient φ_g is

$$\varphi_g(\tau, z) = \eta_g(\tau)\phi_{-2,1}(\tau, z). \tag{3.2}$$

Proof. Except for the explicit description of φ_g, this follows immediately from Theorem 2.3. The Jacobi form φ_g is a theta block involving the coefficients $\mathrm{mult}_d(0,0)$ and $\mathrm{mult}_d(0,1)$ for $d|N_g$. We find

$$\mathrm{mult}_d(0,1) = \frac{2}{d}\sum_{t|d}\mu(d/t) = \begin{cases} 2: & d=1; \\ 0: & d>1; \end{cases}$$

and similarly

$$\mathrm{mult}_d(0,0) = \frac{1}{d}\sum_{t|d}\mu(d/t)\mathrm{tr}(g^t) - \frac{4}{d}\sum_{t|d}\mu(d/t) = \begin{cases} b_1-4: & d=1; \\ b_d: & d>1. \end{cases}$$

Therefore,

$$\varphi_g(\tau, z) = \frac{\prod_{d|N_g}\eta(d\tau)^{b_d}}{\eta(\tau)^4}\cdot\frac{\vartheta(\tau,z)^2}{\eta(\tau)^2} = \eta_g(\tau)\phi_{-2,1}(\tau, z).$$ □

Corollary 3.2. *The following duality holds:*

$$\Phi_g(\tau, z, \omega) = \Phi_g(\omega, z, \tau). \tag{3.3}$$

Proof. This follows from Theorem 2.3 (5) and the number $D_0 = 0$. □

Corollary 3.3. *The following identity holds:*

$$\lim_{z\to 0}\frac{\Phi_g(\tau, z, w)}{(2\pi i z)^2} = \eta_g(\tau)\eta_g(\omega). \tag{3.4}$$

Proof. Note that Φ_g has a double zero at $z = 0$. The limit can be viewed as the quasi-pullback of Φ_g along the embedding

$$U(N_g)\oplus U \hookrightarrow U(N_g)\oplus U\oplus A_1.$$

This is itself the Borcherds lift of the sequence of Jacobi forms of index $\{0\}$ (i.e. modular forms),

$$\phi_{g^d}(\tau, 0), \quad d|N_g.$$

Here $\phi_g(\tau, 0) = \chi(g) = \mathrm{tr}(g)$. The identity now follows from [40, Remark 5.12]. □

Now we investigate when Φ_g is an additive lift. Eguchi and Hikami [18] have represented the twisted elliptic genus ϕ_g in terms of the associated eta product η_g. More precisely, they find that

$$\phi_g(\tau,z) = -\frac{\varphi_g\Big|_{k_g,1} T_-^{(N_g)}(2)(\tau,z)}{\varphi_g(\tau,z)} - f_g(\tau,z), \tag{3.5}$$

where $\varphi_g = \eta_g\phi_{-2,1}$ is the leading Fourier–Jacobi coefficient and where f_g is identically zero if and only if g is of type $1A$, $2A$, $2B$, $3A$, $4A$, $4B$, $5A$, $6A$, $7AB$, $8A$. In addition, they found

$$f_{4C}(\tau,z) = 16\frac{\eta(2\tau)^4\eta(8\tau)^4}{\eta(4\tau)^4}\phi_{-2,1}(\tau,z), \tag{3.6}$$

$$f_{3B}(\tau,z) = 18\frac{\eta(\tau)^3\eta(9\tau)^3}{\eta(3\tau)^2}\phi_{-2,1}(\tau,z). \tag{3.7}$$

These 12 cycle shapes correspond to the only conjugacy classes of M_{24} for which φ_g has positive weight. For each of these, by Theorem 2.2 we have

$$\mathbf{G}(\varphi_g)(Z) = \sum_{m=1}^{\infty} \varphi_g\Big|_{k_g,1} T_-^{(N_g)}(m)(\tau,z)\cdot s^m, \quad s = e^{2\pi i\omega}. \tag{3.8}$$

Let $\hat{F}_g$ be the $\hat{\mathbb{J}}$-image of $(\phi_d)_{d|N_g}$ where $\phi_1 = f_g$ and $\phi_d = 0$ for $d > 1$. By Theorem 2.3,

$$\mathbf{B}(\hat{F}_g)(Z) = \exp\left(-\sum_{m=1}^{\infty} f_g\Big|_{0,1} T_-^{(N_g)}(m)(\tau,z)\cdot s^m\right) \tag{3.9}$$

has weight 0. Note that $\mathbf{B}(\hat{F}_g) = 1$ if $f_g = 0$.

Theorem 1.4 is equivalent to the following result.

Theorem 3.4. *Let $[g]$ be a conjugacy class of M_{24} with positive-weight φ_g. Then we have*

$$\Phi_g(Z)\cdot\mathbf{B}(\hat{F}_g)(Z) = \mathbf{G}(\varphi_g)(Z). \tag{3.10}$$

Proof. By Koecher's principle it is enough to show that $\frac{\mathbf{G}(\varphi_g)}{\Phi_g\cdot\mathbf{B}(\hat{F}_g)}$ is holomorphic, since $\mathbf{G}(\varphi_g)$ and $\Phi_g\cdot\mathbf{B}(\hat{F}_g)$ have the same weight and the same leading Fourier–Jacobi coefficient φ_g. The divisors of Φ_g and $\mathbf{B}(\hat{F}_g)$ can be read off of the data in the appendices.

When g is of type $1A$, $2A$, $3A$, $4B$, the product Φ_g has only a double zero on the diagonal $z = 0$. On the other hand, the double zero of $\varphi_g(\tau,z)$ at $z = 0$ leads to a double zero of $\mathbf{G}(\varphi_g)$ at $z = 0$, and we are done.

The other cases require more work. Firstly, the Gritsenko lift of $\varphi_g = \eta_g\phi_{-2,1}$ may have poles, but these are easily described. If the theta-decomposition of $\phi_{-2,1}$ is written $f_0\mathfrak{e}_0 + f_{1/2}\mathfrak{e}_{1/2}$, where

$$f_0(\tau) = 10 + 108q + 808q^2 + 4016q^3 + \ldots$$
$$f_{1/2}(\tau) = q^{-1/4} - 64q^{3/4} - 513q^{7/4} - 2752q^{11/4} - \ldots$$

then we have

$$\hat{\mathbb{J}}(\varphi_g) = \sum_{\substack{c,d\in\mathbb{Z}/N_g\mathbb{Z}\\ \gcd(c,d,N_g)=1}} \left(\eta_g\Big|\begin{pmatrix} * & * \\ c & d\end{pmatrix}\right)\left(f_0\cdot\mathfrak{e}_{(c/N_g,0,d/N_g)} + f_{1/2}\cdot\mathfrak{e}_{(c/N_g,1/2,d/N_g)}\right),$$

so the principal part of $\hat{\mathbb{J}}(\varphi_g)$ can be determined from the cusps of $\Gamma_0(N_g)$ at which η_g vanishes to order strictly less than $1/4$.

Secondly, $\Phi_g\cdot\mathbf{B}(\hat{F}_g)$ generally has zeros on Humbert surfaces $r^\perp$ other than the diagonal $z = 0$, and $\mathbf{G}(\varphi_g)$ must be shown to vanish on these zeros to at least the correct order. Following section 5

of [42], to show that $\mathbf{G}(\varphi_g)$ vanishes on $r^\perp$ to order N it is sufficient to show that the "development coefficients" $D_k^{r^\perp}\hat{\mathbb{J}}(\varphi_g)$ of $\hat{\mathbb{J}}(\varphi_g)$ along $r^\perp$ vanish for $0 \le k \le N$. The latter are weakly holomorphic vector-valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$ and they vanish identically as soon as all Fourier coefficients are zero up to an appropriate Sturm bound.

As soon as we have shown that every pole of $\mathbf{G}(\varphi_g)$ appears as a pole of $\Phi_g \cdot \mathbf{B}(\hat{F}_g)$, and that every zero of $\Phi_g \cdot \mathbf{B}(\hat{F}_g)$ appears as zero of $\mathbf{G}(\varphi_g)$, we can apply Koecher's principle as in the first paragraph. All of the steps above can be carried out using the Sage package [41].

As an example, we will describe the computations for the conjugacy class $3B$ of level 9. The form $\eta_g(\tau) = \eta(3\tau)^8$ vanishes at the cusps $1/3, 2/3, \infty$ of $\Gamma_0(9)$ to order 1, but it vanishes at the cusp 0 to order only 1/9; indeed,

$$\tau^{-4}\eta_g(-1/\tau) = \frac{1}{81}\eta(\tau/3)^8 = \frac{1}{81}q^{1/9} - \frac{8}{81}q^{4/9} \pm \dots$$

Therefore the associated vector-valued modular form has Fourier expansion beginning

$$\begin{aligned}\hat{\mathbb{J}}(\varphi_g) =& \frac{1}{81}q^{-5/36}\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times}\mathfrak{e}_{(a/9,1/2,-a^{-1}/9)} - \frac{2}{81}q^{1/9}\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times}\mathfrak{e}_{(a/9,0,-a^{-1}/9)}\\ &- \frac{8}{81}q^{7/36}\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times}\mathfrak{e}_{(a/9,1/2,5a^{-1}/9)} + \frac{16}{81}q^{4/9}\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times}\mathfrak{e}_{(a/9,0,5a^{-1}/9)}\\ &+ O(q^{19/36}),\end{aligned}$$

where a^{-1} is the inverse of a modulo 9. This shows that $\mathbf{G}(\varphi_g)$ has double poles only on Humbert surfaces of discriminant 5/36. Consulting Appendix A and Appendix B shows that Φ_g is holomorphic there and $\mathbf{B}(\hat{F}_g)$ contributes double poles on those divisors as well. We also see that $\Phi_g \cdot \mathbf{B}(\hat{F}_g)$ has only double zeros on the Humbert surfaces of discriminants 1/4 and 1/36. We verify that the first two development coefficients of $\hat{\mathbb{J}}(\varphi_g)$ along these divisors vanish to at least the respective Sturm bounds $\frac{1}{12}\cdot 2 = 1/6$ and $\frac{1}{12}\cdot 3 = 1/4$ using the Fourier expansion above, and we are done. □

For the other conjugacy classes, the form φ_g has non-positive weight and there seems to be no reasonable way to define its additive lift.

Acknowledgements H. Wang is supported by the Institute for Basic Science (IBS-R003-D1).

Appendix A: Data for Theorem 1.3

For each cycle shape of level N_g, the preimage of Φ_g under the Borcherds lift is expressed as a sequence of weak Jacobi forms $(\phi_d)_{d|N_g}$. We list the input forms ϕ_d as well as the principal part of the vector-valued modular form $\hat{\mathbb{J}}((\phi_d)_{d|N_g})$.

Class 1A. cycle shape 1^{24}, order 1, level 1. The input Jacobi form is

$$\begin{aligned}\phi_1 &:= \phi_{1A}(\tau, z)\\ &= 2\zeta^{\pm 1} + 20 + (20\zeta^{\pm 2} - 128\zeta^{\pm 1} + 216)q + (2\zeta^{\pm 3} + 216\zeta^{\pm 2} - 1026\zeta^{\pm 1} + 1616)q^2 + O(q^3).\end{aligned}$$

Principal part:

$$20\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)}.$$

Class 2A. cycle shape $1^8 2^8$, order 2, level 2. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{2A}(\tau, z)\\ &= 2\zeta^{\pm 1} + 4 + (4\zeta^{\pm 2} - 8)q + (2\zeta^{\pm 3} - 8\zeta^{\pm 2} - 2\zeta^{\pm 1} + 16)q^2 + O(q^3)\end{aligned}$$

and $\phi_2 = \phi_{1A}$.

Principal part:

$$12\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)}.$$

Class 2B. cycle shape 2^{12}, order 2, level 4. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{2B}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 8)q + (2\zeta^{\pm 3} + 8\zeta^{\pm 2} - 2\zeta^{\pm 1} - 16)q^2 + O(q^3)\end{aligned}$$

and $\phi_2 = \phi_4 = \phi_{1A}$.

Principal part:

$$8\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} + 2q^{-1/4}\mathfrak{e}_{(1/2,1/2,1/2)}.$$

Class 3A. cycle shape $1^6 3^6$, order 3, level 3. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{3A}(\tau, z)\\ &= 2\zeta^{\pm 1} + 2 + (2\zeta^{\pm 2} - 2\zeta^{\pm 1})q + (2\zeta^{\pm 3} - 4)q^2 + O(q^3)\end{aligned}$$

and $\phi_3 = \phi_{1A}$.

Principal part:

$$8\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)}.$$

Class 3B. cycle shape 3^8, order 3, level 9. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{3B}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 4\zeta^{\pm 1})q + (2\zeta^{\pm 3} - 4)q^2 + O(q^3)\end{aligned}$$

and $\phi_3 = \phi_9 = \phi_{1A}$.

Principal part:

$$\begin{aligned}&4\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} + 2q^{-1/4}\mathfrak{e}_{(1/3,1/2,2/3)} + 2q^{-1/4}\mathfrak{e}_{(2/3,1/2,1/3)}\\ &\quad - 6\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(a/9,1/2,-2a^{-1}/9)}.\end{aligned}$$

Class 4A. cycle shape $2^4 4^4$, order 4, level 8. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{4A}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 16\zeta^{\pm 1} - 24)q + (2\zeta^{\pm 3} - 24\zeta^{\pm 2} + 62\zeta^{\pm 1} - 80)q^2 + O(q^3);\end{aligned}$$

$\phi_2 = \phi_{2A}$ and $\phi_4 = \phi_8 = \phi_{1A}$.

Principal part:

$$4\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} + 2q^{-1/4}\mathfrak{e}_{(1/2,1/2,1/2)} - 2\sum_{a\in(\mathbb{Z}/8\mathbb{Z})^\times} q^{-1/8}\mathfrak{e}_{(a/8,1/2,-a^{-1}/8)}.$$

Class 4B. cycle shape $1^4 2^2 4^4$, order 4, level 4. The input Jacobi forms are

$$\phi_1 := \phi_{4B}(\tau, z) = 2\zeta^{\pm 1} + (2\zeta^{\pm 3} - 2\zeta^{\pm 1})q^2 + O(q^3);$$

$\phi_2 = \phi_{2A}$ and $\phi_4 = \phi_{1A}$.

Principal part:

$$6\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)}.$$

Class 4C. cycle shape 4^6, order 4, level 16. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{4C}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 8\zeta^{\pm 1} - 8)q + (2\zeta^{\pm 3} - 8\zeta^{\pm 2} + 14\zeta^{\pm 1} - 16)q^2 + O(q^3);\end{aligned}$$

$\phi_2 = \phi_{2B}$; $\phi_4 = \phi_8 = \phi_{16} = \phi_{1A}$.

Principal part:

$$\begin{aligned}&2\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}e_{(0,1/2,0)} + 2q^{-1/4}e_{(1/4,1/2,3/4)} + 2q^{-1/4}e_{(1/2,1/2,1/2)} + 2q^{-1/4}e_{(3/4,1/2,1/4)}\\ &\quad - 4\sum_{a\in(\mathbb{Z}/16\mathbb{Z})^\times} q^{-1/16}\mathfrak{e}_{(a/16,1/2,-3a^{-1}/16)}.\end{aligned}$$

Class 5A. cycle shape $1^4 5^4$, order 5, level 5. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{5A}(\tau, z)\\ &= 2\zeta^{\pm 1} + (2\zeta^{\pm 1} - 4)q + (2\zeta^{\pm 3} - 4\zeta^{\pm 2} + 4\zeta^{\pm 1} - 4)q^2 + O(q^3)\end{aligned}$$

and $\phi_5 = \phi_{1A}$.

Principal part:

$$4\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} - 2\sum_{a\in(\mathbb{Z}/5\mathbb{Z})^\times} q^{-1/20}\mathfrak{e}_{(a/5,1/2,-a^{-1}/5)}.$$

Class 6A. cycle shape $1^2 2^2 3^2 6^2$, order 6, level 6. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{6A}(\tau, z)\\ &= 2\zeta^{\pm 1} - 2 + (-2\zeta^{\pm 2} + 6\zeta^{\pm 1} - 8)q + (2\zeta^{\pm 3} - 8\zeta^{\pm 2} + 16\zeta^{\pm 1} - 20)q^2 + O(q^3)\end{aligned}$$

and $\phi_2 = \phi_{3A}$; $\phi_3 = \phi_{2A}$; $\phi_6 = \phi_{1A}$.

Principal part:

$$4\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} - 2q^{-1/12}\mathfrak{e}_{(1/6,1/2,5/6)} - 2q^{-1/12}\mathfrak{e}_{(5/6,1/2,1/6)}.$$

Class 6B. cycle shape 6^4, order 6, level 36. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{6B}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 12\zeta^{\pm 1} - 16)q + (2\zeta^{\pm 3} - 16\zeta^{\pm 2} + 40\zeta^{\pm 1} - 52)q^2 + O(q^3)\end{aligned}$$

and $\phi_2 = \phi_4 = \phi_{3B}$; $\phi_3 = \phi_9 = \phi_{2B}$; $\phi_6 = \phi_{12} = \phi_{18} = \phi_{36} = \phi_{1A}$.

Principal part:

$$\begin{aligned}&0\mathfrak{e}_{(0,0,0)} + 2\sum_{a\in\mathbb{Z}/6\mathbb{Z}} q^{-1/4}\mathfrak{e}_{(a/6,1/2,-a/6)} - 2\sum_{a\in(\mathbb{Z}/36\mathbb{Z})^\times} q^{-1/9}\mathfrak{e}_{(a/36,1/2,-5a^{-1}/36)}\\ &\quad - 2\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(a/9,1/2,4a^{-1}/9)} - 4\sum_{a\in(\mathbb{Z}/18\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(a/18,1/2,-a^{-1}/9)}\\ &\quad - 4\sum_{a\in(\mathbb{Z}/18\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(-a^{-1}/9,1/2,a/18)}.\end{aligned}$$

Classes 7A/7B. cycle shape $1^3 7^3$, order 7, level 7. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{7AB}(\tau, z)\\ &= 2\zeta^{\pm 1} - 1 + (-\zeta^{\pm 2} + 5\zeta^{\pm 1} - 8)q + (2\zeta^{\pm 3} - 8\zeta^{\pm 2} + 17\zeta^{\pm 1} - 22)q^2 + O(q^3)\end{aligned}$$

and $\phi_7 = \phi_{1A}$.

Principal part:

$$2\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} - \sum_{a\in(\mathbb{Z}/7\mathbb{Z})^\times} q^{-3/28}\mathfrak{e}_{(a/7,1/2,-a^{-1}/7)}.$$

Class 8A. cycle shape $1^2 2^1 4^1 8^2$, order 8, level 8. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{8A}(\tau, z)\\ &= 2\zeta^{\pm 1} - 2 + (-2\zeta^{\pm 2} + 8\zeta^{\pm 1} - 12)q + (2\zeta^{\pm 3} - 12\zeta^{\pm 2} + 30\zeta^{\pm 1} - 40)q^2 + O(q^3)\end{aligned}$$

and $\phi_2 = \phi_{4B}$; $\phi_4 = \phi_{2A}$; $\phi_8 = \phi_{1A}$.

Principal part:

$$2\mathfrak{e}_{(0,0,0)} + 2\mathfrak{e}_{(0,1/2,0)} - \sum_{a\in(\mathbb{Z}/8\mathbb{Z})^\times} q^{-1/8}\mathfrak{e}_{(a/8,1/2,-a/8)}.$$

Class 10A. cycle shape $2^2 10^2$, order 10, level 20. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{10A}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 10\zeta^{\pm 1} - 12)q + (2\zeta^{\pm 3} - 12\zeta^{\pm 2} + 28\zeta^{\pm 1} - 36)q^2 + O(q^3)\end{aligned}$$

and $\phi_2 = \phi_4 = \phi_{5A}$; $\phi_5 = \phi_{2B}$; $\phi_{10} = \phi_{20} = \phi_{1A}$.

Principal part:

$$\begin{aligned}&0\mathfrak{e}_{(0,0,0)} + 2\mathfrak{e}_{(0,1/2,0)} + 2\mathfrak{e}_{(1/2,1/2,1/2)} - 2\sum_{a\in(\mathbb{Z}/20\mathbb{Z})^\times} q^{-1/10}\mathfrak{e}_{(a/20,1/2,-3a^{-1}/20)}\\ &\quad - 2\sum_{a\in(\mathbb{Z}/10\mathbb{Z})^\times} q^{-1/20}\mathfrak{e}_{(a/10,1/2,-a^{-1}/10)} - 2\sum_{a\in(\mathbb{Z}/5\mathbb{Z})^\times} q^{-1/20}\mathfrak{e}_{(a/5,1/2,a^{-1}/5)}.\end{aligned}$$

Class 11A. cycle shape $1^2 11^2$, order 11, level 11. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{11A}(\tau, z)\\ &= 2\zeta^{\pm 1} - 2 + (-2\zeta^{\pm 2} + 4\zeta^{\pm 1} - 4)q + (2\zeta^{\pm 3} - 4\zeta^{\pm 2} + 8\zeta^{\pm 1} - 12)q^2 + O(q^3)\end{aligned}$$

and $\phi_{11} = \phi_{1A}$.

Principal part:

$$0\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} - 2\sum_{a\in(\mathbb{Z}/11\mathbb{Z})^\times} q^{-3/44}\mathfrak{e}_{(a/11,1/2,-2a^{-1}/11)}.$$

Class 12A. cycle shape $2^1 4^1 6^1 12^1$, order 12, level 24. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{12A}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 10\zeta^{\pm 1} - 12)q + (2\zeta^{\pm 3} - 12\zeta^{\pm 2} + 32\zeta^{\pm 1} - 44)q^2 + O(q^3)\end{aligned}$$

and $\phi_2 = \phi_{6A}$; $\phi_3 = \phi_{4A}$; $\phi_4 = \phi_8 = \phi_{3A}$; $\phi_6 = \phi_{2A}$; $\phi_{12} = \phi_{24} = \phi_{1A}$.

Principal part:

$$
\begin{aligned}
&0\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} + 2q^{-1/4}\mathfrak{e}_{(1/2,1/2,1/2)} \\
&\quad - \sum_{a\in(\mathbb{Z}/24\mathbb{Z})^\times} q^{-1/8}\mathfrak{e}_{(a/24,1/2,-a^{-1}/8)} - \sum_{a\in(\mathbb{Z}/24\mathbb{Z})^\times} q^{-1/8}\mathfrak{e}_{(-a^{-1}/8,1/2,a/24)} \\
&\quad - 2\sum_{a\in(\mathbb{Z}/12\mathbb{Z})^\times} q^{-1/12}\mathfrak{e}_{(a/12,1/2,-a^{-1}/12)} - 2\sum_{a\in(\mathbb{Z}/24\mathbb{Z})^\times} q^{-1/24}\mathfrak{e}_{(a/24,1/2,-5a^{-1}/24)}.
\end{aligned}
$$

Class 12B. cycle shape 12^2, order 12, level 144. The input Jacobi forms are

$$
\begin{aligned}
\phi_1 &:= \phi_{12B}(\tau,z) \\
&= 2\zeta^{\pm1} - 4 + (-4\zeta^{\pm2} + 8\zeta^{\pm1} - 8)q + (2\zeta^{\pm3} - 8\zeta^{\pm2} + 20\zeta^{\pm1} - 28)q^2 + O(q^3)
\end{aligned}
$$

and $\phi_2 = \phi_{6B}$; $\phi_3 = \phi_9 = \phi_{4C}$;$\phi_4 = \phi_8 = \phi_{16} = \phi_{3B}$; $\phi_6 = \phi_{18} = \phi_{2B}$; $\phi_{12} = \phi_{24} = \phi_{36} = \phi_{48} = \phi_{72} = \phi_{144} = \phi_{1A}$.

Principal part:

$$
\begin{aligned}
&(-2)\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} + 2q^{-1/4}\mathfrak{e}_{(1/2,1/2,1/2)} + 2\sum_{a\in(\mathbb{Z}/3\mathbb{Z})^\times} q^{-1/4}\mathfrak{e}_{(a/3,1/2,-a/3)} \\
&\quad + 2q^{-1/4}\sum_{a\in(\mathbb{Z}/4\mathbb{Z})^\times} q^{-1/4}\mathfrak{e}_{(a/4,1/2,-a/4)} + 2\sum_{a\in(\mathbb{Z}/6\mathbb{Z})^\times} q^{-1/4}\mathfrak{e}_{(a/6,1/2,-a/6)} \\
&\quad + 2\sum_{a\in(\mathbb{Z}/12\mathbb{Z})^\times} q^{-1/4}\mathfrak{e}_{(a/12,1/2,-a/12)} - \sum_{a\in(\mathbb{Z}/72\mathbb{Z})^\times} q^{-1/9}\mathfrak{e}_{(a/72,1/2,-5a^{-1}/72)} \\
&\quad - \sum_{a\in(\mathbb{Z}/72\mathbb{Z})^\times} q^{-1/9}\mathfrak{e}_{(a/72,1/2,31a^{-1}/72)} - 2\sum_{a\in(\mathbb{Z}/144\mathbb{Z})^\times} q^{-13/144}\mathfrak{e}_{(a/144,1/2,-23a^{-1}/144)} \\
&\quad - 2\sum_{a\in(\mathbb{Z}/48\mathbb{Z})^\times} q^{-1/16}\mathfrak{e}_{(a/48,1/2,13a^{-1}/48)} - 2\sum_{a\in(\mathbb{Z}/48\mathbb{Z})^\times} q^{-1/16}\mathfrak{e}_{(a/48,1/2,-3a^{-1}/48)} \\
&\quad - 2\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(a/9,1/2,a^{-1}/9)} - 2\sum_{a\in(\mathbb{Z}/18\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(a/18,1/2,-5a^{-1}/18)} \\
&\quad - 2\sum_{\substack{a,c\in\mathbb{Z}/36\mathbb{Z}\\ \gcd(a,c,36)=1\\ a\cdot c\in\{7,16,34\}}} q^{-1/36}\mathfrak{e}_{(a/36,1/2,c/36)} - 2\sum_{a\in(\mathbb{Z}/144\mathbb{Z})^\times} q^{-1/144}\mathfrak{e}_{(a/144,1/2,-35a^{-1}/144)}.
\end{aligned}
$$

Classes 14A/14B. cycle shape $1^1 2^1 7^1 14^1$, order 14, level 14. The input Jacobi forms are

$$
\begin{aligned}
\phi_1 &:= \phi_{14AB}(\tau,z) \\
&= 2\zeta^{\pm1} - 3 + (-3\zeta^{\pm2} + 7\zeta^{\pm1} - 8)q + (2\zeta^{\pm3} - 8\zeta^{\pm2} + 19\zeta^{\pm1} - 26)q^2 + O(q^3)
\end{aligned}
$$

and $\phi_2 = \phi_{7AB}$; $\phi_7 = \phi_{2A}$; $\phi_{14} = \phi_{1A}$.

Principal part:

$$
\begin{aligned}
&0\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} - \sum_{a\in(\mathbb{Z}/14\mathbb{Z})^\times} q^{-3/28}\mathfrak{e}_{(a/14,1/2,-a^{-1}/7)} \\
&\quad - \sum_{a\in(\mathbb{Z}/14\mathbb{Z})^\times} q^{-3/28}\mathfrak{e}_{(-a^{-1}/7,1/2,a/14)} - 2\sum_{a\in(\mathbb{Z}/14\mathbb{Z})^\times} q^{-1/28}\mathfrak{e}_{(a/14,1/2,-3a^{-1}/14)}.
\end{aligned}
$$

Classes 15A/15B. cycle shape $1^1 3^1 5^1 15^1$, order 15, level 15. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{15AB}(\tau, z)\\ &= 2\zeta^{\pm 1} - 3 + (-3\zeta^{\pm 2} + 8\zeta^{\pm 1} - 10)q + (2\zeta^{\pm 3} - 10\zeta^{\pm 2} + 25\zeta^{\pm 1} - 34)q^2 + O(q^3)\end{aligned}$$

and $\phi_3 = \phi_{5A}$; $\phi_5 = \phi_{3A}$; $\phi_{15} = \phi_{1A}$.

Principal part:

$$\begin{aligned}&0\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} - \sum_{a\in(\mathbb{Z}/15\mathbb{Z})^\times} q^{-7/60}\mathfrak{e}_{(a/15,1/2,-2a^{-1}/15)}\\ &\quad - \sum_{a\in(\mathbb{Z}/15\mathbb{Z})^\times} q^{-1/20}\mathfrak{e}_{(a/15,1/2,-a^{-1}/5)} - \sum_{a\in(\mathbb{Z}/15\mathbb{Z})^\times} q^{-1/20}\mathfrak{e}_{(-a^{-1}/5,1/2,a/15)}.\end{aligned}$$

Classes 21A/21B. cycle shape $3^1 21^1$, order 21, level 63. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{21AB}(\tau, z)\\ &= 2\zeta^{\pm 1} - 4 + (-4\zeta^{\pm 2} + 11\zeta^{\pm 1} - 14)q + (2\zeta^{\pm 3} - 14\zeta^{\pm 2} + 35\zeta^{\pm 1} - 46)q^2 + O(q^3)\end{aligned}$$

and $\phi_3 = \phi_9 = \phi_{7AB}$; $\phi_7 = \phi_{3B}$; $\phi_{21} = \phi_{63} = \phi_{1A}$.

Principal part:

$$\begin{aligned}&(-2)\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)} + 2q^{-1/4}\mathfrak{e}_{(1/3,1/2,2/3)} + 2q^{-1/4}\mathfrak{e}_{(2/3,1/2,1/3)}\\ &\quad - \sum_{a\in(\mathbb{Z}/63\mathbb{Z})^\times} q^{-31/252}\mathfrak{e}_{(a/63,1/2,-8a^{-1}/63)} - \sum_{a\in(\mathbb{Z}/21\mathbb{Z})^\times} q^{-3/28}\mathfrak{e}_{(a/21,1/2,-a^{-1}/21)}\\ &\quad - \sum_{a\in(\mathbb{Z}/63\mathbb{Z})^\times} q^{-19/252}\mathfrak{e}_{(a/63,1/2,-11a^{-1}/63)}\\ &\quad - \sum_{a\in(\mathbb{Z}/63\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(a/63,1/2,-7a^{-1}/9)} - \sum_{a\in(\mathbb{Z}/63\mathbb{Z})^\times} q^{-1/36}\mathfrak{e}_{(-7a^{-1}/9,1/2,a/63)}.\end{aligned}$$

Classes 23A/23B. cycle shape $1^1 23^1$, order 23, level 23. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= \phi_{23AB}(\tau, z)\\ &= 2\zeta^{\pm 1} - 3 + (-3\zeta^{\pm 2} + 10\zeta^{\pm 1} - 14)q + (2\zeta^{\pm 3} - 14\zeta^{\pm 2} + 32\zeta^{\pm 1} - 40)q^2 + O(q^3)\end{aligned}$$

and $\phi_{23} = \phi_{1A}$.

Principal part:

$$\begin{aligned}&(-2)\mathfrak{e}_{(0,0,0)} + 2q^{-1/4}\mathfrak{e}_{(0,1/2,0)}\\ &\quad - \sum_{a\in(\mathbb{Z}/23\mathbb{Z})^\times} q^{-11/92}\mathfrak{e}_{(a/23,1/2,-3a^{-1}/23)} - \sum_{a\in(\mathbb{Z}/23\mathbb{Z})^\times} q^{-7/92}\mathfrak{e}_{(a/23,1/2,-4a^{-1}/23)}.\end{aligned}$$

Appendix B: Data for Theorem 1.4

In this appendix we describe the correction terms $\mathbf{B}(\hat{F}_g)$ appearing in Theorem 3.4 by listing the family of Jacobi forms $(\phi_d)_{d|N_g} = \hat{\mathbb{J}}((\phi_d)_{d|N_g})$ and the principal parts of $\hat{F}_g$. These terms appear only when g has cycle shape 3B or 4C.

Class 3B. cycle shape 3^8, order 3, level 9. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= 18\frac{\eta(\tau)^3\eta(9\tau)^3}{\eta(3\tau)^2}\phi_{-2,1}(\tau,z)\\ &= (18\zeta^{\pm1}-36)q+(-36\zeta^{\pm2}+90\zeta^{\pm1}-108)q^2+O(q^3)\end{aligned}$$

and $\phi_3=0$, $\phi_9=0$.

Principal part:

$$\begin{aligned}&0\mathfrak{e}_{(0,0,0)}-2q^{-1/4}\mathfrak{e}_{(1/3,1/2,2/3)}-2q^{-1/4}\mathfrak{e}_{(2/3,1/2,1/3)}\\ &\quad+2q^{-1/4}\mathfrak{e}_{(1/3,1/2,1/3)}+2q^{-1/4}\mathfrak{e}_{(2/3,1/2,2/3)}\\ &\quad-2q^{-5/36}\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times}\mathfrak{e}_{(a/9,1/2,-a^{-1}/9)}\\ &\quad+6q^{-1/36}\sum_{a\in(\mathbb{Z}/9\mathbb{Z})^\times}\mathfrak{e}_{(a/9,1/2,-2a^{-1}/9)}.\end{aligned}$$

Class 4C. cycle shape 4^6, order 4, level 16. The input Jacobi forms are

$$\begin{aligned}\phi_1 &:= 16\frac{\eta(2\tau)^4\eta(8\tau)^4}{\eta(4\tau)^4}\phi_{-2,1}(\tau,z)\\ &= (16\zeta^{\pm1}-32)q+(-32\zeta^{\pm2}+128\zeta^{\pm1}-192)q^2+O(q^3)\end{aligned}$$

and $\phi_2=0$, $\phi_4=0$, $\phi_8=0$, $\phi_{16}=0$.

Principal part:

$$\begin{aligned}&0\mathfrak{e}_{(0,0,0)}-2q^{-1/4}\mathfrak{e}_{(1/4,1/2,3/4)}-2q^{-1/4}\mathfrak{e}_{(3/4,1/2,1/4)}\\ &\quad+2q^{-1/4}\mathfrak{e}_{(1/4,1/2,1/4)}+2q^{-1/4}\mathfrak{e}_{(3/4,1/2,3/4)}\\ &\quad-q^{3/16}\sum_{a\in(\mathbb{Z}/16\mathbb{Z})^\times}\mathfrak{e}_{(a/16,1/2,-a^{-1}/16)}\\ &\quad+4q^{-1/16}\sum_{a\in(\mathbb{Z}/16\mathbb{Z})^\times}\mathfrak{e}_{(a/16,1/2,-3a^{-1}/16)}.\end{aligned}$$

References

[1] Hiroki Aoki and Tomoyoshi Ibukiyama. Simple graded rings of Siegel modular forms, differential operators and Borcherds products. *Internat. J. Math.*, 16(3):249–279, 2005.

[2] Richard E. Borcherds. The monster Lie algebra. *Adv. Math.*, 83(1):30–47, 1990.

[3] Richard E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.*, 109(2):405–444, 1992.

[4] Richard E. Borcherds. Automorphic forms on $\mathrm{O}_{s+2,2}(\mathbf{R})$ and infinite products. *Invent. Math.*, 120(1):161–213, 1995.

[5] Richard E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998.

[6] Miranda C. N. Cheng. $K3$ surfaces, $\mathcal{N}=4$ dyons and the Mathieu group M_{24}. *Commun. Number Theory Phys.*, 4(4):623–657, 2010.

[7] Miranda C. N. Cheng and John F. R. Duncan. The largest Mathieu group and (mock) automorphic forms. In *String-Math 2011*, volume 85 of *Proc. Sympos. Pure Math.*, pages 53–82. Amer. Math. Soc., Providence, RI, 2012.

[8] Miranda C. N. Cheng and John F. R. Duncan. On Rademacher sums, the largest Mathieu group and the holographic modularity of moonshine. *Commun. Number Theory Phys.*, 6(3):697–758, 2012.

[9] Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral moonshine. *Commun. Number Theory Phys.*, 8(2):101–242, 2014.

[10] Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral moonshine and the Niemeier lattices. *Res. Math. Sci.*, 1:Art. 3, 81, 2014.
[11] F. Cléry and V. Gritsenko. Siegel modular forms of genus 2 with the simplest divisor. *Proc. Lond. Math. Soc. (3)*, 102(6):1024–1052, 2011.
[12] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bull. London Math. Soc.*, 11(3):308–339, 1979.
[13] Atish Dabholkar and Suresh Nampuri. Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift. *J. High Energy Phys.*, (11):077, 20, 2007.
[14] Justin R. David, Dileep P. Jatkar, and Ashoke Sen. Dyon spectrum in generic $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifolds. *J. High Energy Phys.*, (1):016, 31, 2007.
[15] Robbert Dijkgraaf, Gregory Moore, Erik Verlinde, and Herman Verlinde. Elliptic genera of symmetric products and second quantized strings. *Comm. Math. Phys.*, 185(1):197–209, 1997.
[16] Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde. Counting dyons in $N = 4$ string theory. *Nuclear Phys. B*, 484(3):543–561, 1997.
[17] Tohru Eguchi and Kazuhiro Hikami. Note on twisted elliptic genus of $K3$ surface. *Phys. Lett. B*, 694(4-5): 446–455, 2011.
[18] Tohru Eguchi and Kazuhiro Hikami. Twisted elliptic genus for $K3$ and Borcherds product. *Lett. Math. Phys.*, 102(2):203–222, 2012.
[19] Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the $K3$ surface and the Mathieu group M_{24}. *Exp. Math.*, 20(1):91–96, 2011.
[20] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.
[21] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu twining characters for $K3$. *J. High Energy Phys.*, (9):058, 20, 2010.
[22] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu Moonshine in the elliptic genus of $K3$. *J. High Energy Phys.*, (10):062, 24, 2010.
[23] Terry Gannon. Much ado about Mathieu. *Adv. Math.*, 301:322–358, 2016.
[24] Suresh Govindarajan. Unravelling Mathieu moonshine. *Nuclear Phys. B*, 864(3):823–839, 2012.
[25] Suresh Govindarajan and K. Gopala Krishna. Generalized Kac-Moody algebras from CHL dyons. *J. High Energy Phys.*, (4):032, 39, 2009.
[26] Suresh Govindarajan and K. Gopala Krishna. BKM Lie superalgebras from dyon spectra in $\mathbb{Z}_N$ CHL orbifolds for composite N. *J. High Energy Phys.*, (5):014, 39, 2010.
[27] Suresh Govindarajan and Sutapa Samanta. Mathieu moonshine and Siegel modular forms. *J. High Energy Phys.*, (3):Paper No. 050, 33, 2021.
[28] Valeri A. Gritsenko and Viacheslav V. Nikulin. Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras. *Amer. J. Math.*, 119(1):181–224, 1997.
[29] Valeri A. Gritsenko and Viacheslav V. Nikulin. Automorphic forms and Lorentzian Kac-Moody algebras. II. *Internat. J. Math.*, 9(2):201–275, 1998.
[30] Adrian Hauffe-Waschbüsch and Aloys Krieg. Congruence subgroups and orthogonal groups. *Linear Algebra Appl.*, 618:22–36, 2021.
[31] Jun-ichi Igusa. On Siegel modular forms of genus two. *Amer. J. Math.*, 84:175–200, 1962.
[32] Dileep P. Jatkar and Ashoke Sen. Dyon spectrum in CHL models. *J. High Energy Phys.*, (4):018, 32, 2006.
[33] Daniel Persson and Roberto Volpato. Second-quantized Mathieu moonshine. *Commun. Number Theory Phys.*, 8(3):403–509, 2014.
[34] Martin Raum. M_{24}-twisted product expansions are Siegel modular forms. *Commun. Number Theory Phys.*, 7 (3):469–495, 2013.
[35] Nils R. Scheithauer. Generalized Kac-Moody algebras, automorphic forms and Conway's group. I. *Adv. Math.*, 183(2):240–270, 2004.
[36] Nils R. Scheithauer. On the classification of automorphic products and generalized Kac-Moody algebras. *Invent. Math.*, 164(3):641–678, 2006.
[37] Nils R. Scheithauer. The Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ and some applications. *Int. Math. Res. Not. IMRN*, (8):1488–1545, 2009.
[38] Ashoke Sen. Discrete information from CHL black holes. *J. High Energy Phys.*, (11):138, 21, 2010.
[39] Ashoke Sen. A twist in the dyon partition function. *J. High Energy Phys.*, (5):028, 29, 2010.
[40] Haowu Wang and Brandon Williams. The fake monster algebra and singular Borcherds products. Preprint, 2022. URL `arXiv:2207.14518`.
[41] Brandon Williams. Weilrep. Accessed July 2022. URL `https://github.com/btw-47/weilrep`.
[42] Brandon Williams. Higher pullbacks of modular forms on orthogonal groups. *Forum Math.*, 33(3):631–652, 2021.

Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Korea
Email address: haowu.wangmath@gmail.com

Lehrstuhl A für Mathematik, RWTH Aachen, 52056 Aachen, Germany
Email address: brandon.williams@matha.rwth-aachen.de

arXiv:1503.01472v2 [math.RT] 26 Nov 2015

Proof of the Umbral Moonshine Conjecture

John F. R. Duncan, Michael J. Griffin and Ken Ono

2015 November 25

Abstract

The Umbral Moonshine Conjectures assert that there are infinite-dimensional graded modules, for prescribed finite groups, whose McKay-Thompson series are certain distinguished mock modular forms. Gannon has proved this for the special case involving the largest sporadic simple Mathieu group. Here we establish the existence of the umbral moonshine modules in the remaining 22 cases.

* *MSC2010:* 11F22, 11F37.

*The authors thank the NSF for its support. The first author also thanks the Simons Foundation (#316779), and the third author thanks the A. G. Candler Fund. The authors thank Miranda Cheng, Jeff Harvey, Michael Somos and the anonymous referee for helpful comments and corrections.

1 Introduction and Statement of Results

Monstrous moonshine relates distinguished modular functions to the representation theory of the Monster, $\mathbb{M}$, the largest sporadic simple group. This theory was inspired by the famous observations of McKay and Thompson in the late 1970s [18,51] that

$$196884 = 1 + 196883,$$
$$21493760 = 1 + 196883 + 21296876.$$

The left hand sides here are familiar as coefficients of Klein's modular function (note $q := e^{2\pi i\tau}$),

$$J(\tau) = \sum_{n=-1}^{\infty} c(n)q^n := j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots.$$

The sums on the right hand sides involve the first three numbers arising as dimensions of irreducible representations of $\mathbb{M}$,

$$1,\ 196883,\ 21296876,\ 842609326,\ \dots,\ 258823477531055064045234375.$$

Thompson conjectured that there is a graded infinite-dimensional $\mathbb{M}$-module

$$V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural,$$

satisfying $\dim(V_n^\natural) = c(n)$. For $g \in \mathbb{M}$, he also suggested [50] to consider the graded-trace functions

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|V_n^\natural)q^n,$$

now known as the *McKay-Thompson series*, that arise from the conjectured $\mathbb{M}$-module $V^\natural$. Using the character table for $\mathbb{M}$, it was observed [18,50] that the first few coefficients of each $T_g(\tau)$ coincide with those of a generator for the function field of a discrete group $\Gamma_g < SL_2(\mathbb{R})$, leading Conway and Norton [18] to their famous *Monstrous Moonshine Conjecture*: This is the claim that for each $g \in \mathbb{M}$ there is a specific *genus zero* group Γ_g such that $T_g(\tau)$ is the unique normalized *hauptmodul* for Γ_g, i.e., the unique Γ_g-invariant holomorphic function on $\mathbb{H}$ which satisfies $T_g(\tau) = q^{-1} + O(q)$ as $\Im(\tau) \to \infty$, and remains bounded near any cusp not equivalent to the infinite one.

In a series of ground-breaking works, Borcherds introduced vertex algebras [2], and generalized Kac–Moody Lie algebras [3,4], and used these notions to prove [5] the Monstrous Moonshine Conjecture of Conway and Norton. He confirmed the conjecture for the module $V^\natural$ constructed by Frenkel, Lepowsky, and Meurman [30–32] in the early 1980s. These results provide much more than the predictions of monstrous moonshine. The $\mathbb{M}$-module $V^\natural$ is a vertex operator algebra, one whose automorphism group is precisely $\mathbb{M}$. The construction of Frenkel, Lepowsky and Meurman can be regarded as one of the first examples of an *orbifold conformal field theory*. (Cf. [23].) Here the orbifold in question is the quotient $\left(\mathbb{R}^{24}/\Lambda_{24}\right)/(\mathbb{Z}/2\mathbb{Z})$, of the 24-dimensional torus $\Lambda_{24} \otimes_{\mathbb{Z}} \mathbb{R}/\Lambda_{24} \simeq \mathbb{R}^{24}/\Lambda_{24}$ by the Kummer involution $x \mapsto -x$, where Λ_{24} denotes the Leech lattice.

We refer to [24,32,35,36] for more on monstrous moonshine.

In 2010, Eguchi, Ooguri, and Tachikawa reignited moonshine with their observation [28] that dimensions of some representations of M_{24}, the largest sporadic simple Mathieu group (cf. e.g. [20, 21]), are multiplicities of superconformal algebra characters in the K3 elliptic genus. This observation suggested a manifestation of moonshine for M_{24}: Namely, there should be an infinite-dimensional graded M_{24}-module whose McKay-Thompson series are holomorphic parts of *harmonic Maass forms*, the so-called *mock modular forms*. (See [45, 54, 55] for introductory accounts of the theory of mock modular forms.)

Following the work of Cheng [10], Eguchi and Hikami [27], and Gaberdiel, Hohenegger, and Volpato [33, 34], Gannon established the existence of this infinite-dimensional graded M_{24}-module in [37].

It is natural to seek a general mathematical and physical setting for these results. Here we consider the mathematical setting, which develops from the close relationship between the monster group $\mathbb{M}$ and the Leech lattice Λ_{24}. Recall (cf. e.g. [20]) that the Leech lattice is even, unimodular, and positive-definite of rank 24. It turns out that M_{24} is closely related to another such lattice. Such observations led Cheng, Duncan and Harvey to further instances of moonshine within the setting of even unimodular positive-definite lattices of rank 24. In this way they arrived at the *Umbral Moonshine Conjectures* (cf. §5 of [15], §6 of [16], and §2 of [17]), predicting the existence of 22 further, graded infinite-dimensional modules, relating certain finite groups to distinguished mock modular forms.

To explain this prediction in more detail we recall Niemeier's result [43] that there are 24 (up to isomorphism) even unimodular positive-definite lattices of rank 24. The Leech lattice is the unique one with no root vectors (i.e. lattice vectors with norm-square 2), while the other 23 have root systems with full rank, 24. These *Niemeier root systems* are unions of simple simply-laced root systems with the same Coxeter numbers, and are given explicitly as

$$\begin{gathered} A_1^{24},\ A_2^{12},\ A_3^8,\ A_4^6,\ A_6^4,\ A_{12}^2, \\ A_5^4D_4,\ A_7^2D_5^2,\ A_8^3,\ A_9^2D_6,\ A_{11}D_7E_6,\ A_{15}D_9,\ A_{17}E_7,\ A_{24}, \\ D_4^6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2,\ D_{12}^2,\ D_{16}E_8,\ D_{24},\ E_6^4,\ E_8^3, \end{gathered} \tag{1.1}$$

in terms of the standard ADE notation. (Cf. e.g. [20] or [39] for more on root systems.)

For each Niemeier root system X let N^X denote the corresponding unimodular lattice, let W^X denote the (normal) subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in roots, and define the *umbral group* of X by setting

$$G^X := \mathrm{Aut}(N^X)/W^X. \tag{1.2}$$

(See §A.1 for explicit descriptions of the groups G^X.)

Let m^X denote the Coxeter number of any simple component of X. An association of distinguished $2m^X$-vector-valued mock modular forms $H_g^X(\tau) = (H_{g,r}^X(\tau))$ to elements $g \in G^X$ is described and analyzed in [15–17].

For $X = A_1^{24}$ we have $G^X \simeq M_{24}$ and $m^X = 2$, and the functions $H_{g,1}^X(\tau)$ are precisely the mock modular forms assigned to elements $g \in M_{24}$ in the works [10, 27, 33, 34] mentioned above. Generalizing the M_{24} moonshine initiated by Eguchi, Ooguri and Tachikawa, we have the following conjecture of Cheng, Duncan and Harvey (cf. §2 of [17] or §9.3 of [24]).

Conjecture (Umbral Moonshine Modules)**.** Let X be a Niemeier root system X and set $m := m^X$. There is a naturally defined bi-graded infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \in I^X} \bigoplus_{\substack{D \in \mathbb{Z},\ D \leq 0, \\ D = r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m} \tag{1.3}$$

such that the vector-valued mock modular form $H^X_g = (H^X_{g,r})$ is a McKay-Thompson series for $\check{K}^X$ related[1] to the graded trace of g on $\check{K}^X$ by

$$H^X_{g,r}(\tau) = -2q^{-1/4m}\delta_{r,1} + \sum_{\substack{D \in \mathbb{Z},\ D \leq 0, \\ D = r^2 \pmod{4m}}} \mathrm{tr}(g|\check{K}^X_{r,-D/4m})q^{-D/4m} \tag{1.4}$$

for $r \in I^X$.

In (1.3) and (1.4) the set $I^X \subset \mathbb{Z}/2m\mathbb{Z}$ is defined in the following way. If X has an A-type component then $I^X := \{1, 2, 3, \ldots, m-1\}$. If X has no A-type component but does have a D-type component then $m = 2 \mod 4$, and $I^X := \{1, 3, 5, \ldots, m/2\}$. The remaining cases are $X = E_6^4$ and $X = E_8^3$. In the former of these, $I^X := \{1, 4, 5\}$, and in the latter case $I^X := \{1, 7\}$.

Remark. The functions $H^X_g(\tau)$ are defined explicitly in §B.3. An alternative description in terms of Rademacher sums is given in §B.4.

Here we prove the following theorem.

Theorem 1.1. *The umbral moonshine modules exist.*

Two remarks.

1) Theorem 1.1 for $X = A_1^{24}$ is the main result of Gannon's work [37].

2) The vector-valued mock modular forms $H^X = (H^X_{g,r})$ have "minimal" *principal parts*. This minimality is analogous to the fact that the original McKay-Thompson series $T_g(\tau)$ for the Monster are hauptmoduln, and plays an important role in our proof.

Example. Many of Ramanujan's mock theta functions [46] are components of the vector-valued umbral McKay-Thompson series $H^X_g = (H^X_{g,r})$. For example, consider the root system $X = A_2^{12}$, whose umbral group is a double cover $2.M_{12}$ of the sporadic simple Mathieu group M_{12}. In terms

[1] In the statement of Conjecture 6.1 of [16] the function $H^X_{g,r}$ in (1.4) is replaced with $3H^X_{g,r}$ in the case that $X = A_8^3$. This is now known to be an error, arising from a misspecification of some of the functions H^X_g for $X = A_8^3$. Our treatment of the case $X = A_8^3$ in this work reflects the corrected specification of the corresponding H^X_g which is described and discussed in detail in [17].

of Ramanujan's 3rd order mock theta functions

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$
$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$
$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$
$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2},$$
$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})},$$

we have that

$$H^X_{2B,1}(\tau) = H^X_{2C,1}(\tau) = H^X_{4C,1}(\tau) = -2q^{-\frac{1}{12}} \cdot f(q^2),$$
$$H^X_{6C,1}(\tau) = H^X_{6D,1}(\tau) = -2q^{-\frac{1}{12}} \cdot \chi(q^2),$$
$$H^X_{8C,1}(\tau) = H^X_{8D,1}(\tau) = -2q^{-\frac{1}{12}} \cdot \phi(-q^2),$$
$$H^X_{2B,2}(\tau) = -H^X_{2C,2}(\tau) = -4q^{\frac{2}{3}} \cdot \omega(-q),$$
$$H^X_{6C,2}(\tau) = -H^X_{6D,2}(\tau) = 2q^{\frac{2}{3}} \cdot \rho(-q).$$

See §5.4 of [16] for more coincidences between umbral McKay-Thompson series and mock theta functions identified by Ramanujan almost a hundred years ago.

Our proof of Theorem 1.1 involves the explicit determination of each G^X-module $\check{K}^X$ by computing the multiplicity of each irreducible component for each homogeneous subspace. It guarantees the existence and uniqueness of a $\check{K}^X$ which is compatible with the representation theory of G^X and the Fourier expansions of the vector-valued mock modular forms $H^X_g(\tau) = (H^X_{g,r}(\tau))$.

At first glance our methods do not appear to shed light on any deeper algebraic properties of the $\check{K}^X$, such as might correspond to the vertex operator algebra structure on $V^\natural$, or the monster Lie algebra introduced by Borcherds in [5]. However, we do determine, and utilize, specific recursion relations for the coefficients of the umbral McKay-Thompson series which are analogous to the replicability properties of monstrous moonshine formulated by Conway and Norton in §8 of [18] (cf. also [1]). More specifically, we use recent work [41] of Imamoğlu, Raum and Richter, as generalized [42] by Mertens, to obtain such recursions. These results are based on the process of *holomorphic projection*.

Theorem 1.2. *For each $g \in G^X$ and $0 < r < m$, the mock modular form $H^X_{g,r}(\tau)$ is replicable in the mock modular sense.*

A key step in Borcherds' proof [5] of the monstrous moonshine conjecture is the reformulation of replicability in Lie theoretic terms. We may speculate that the *mock modular replicability* utilized

in this work will ultimately admit an analogous algebraic interpretation. Such a result remains an important goal for future work.

In the statement of Theorem 1.2, replicable means that there are explicit recursion relations for the coefficients of the vector-valued mock modular form in question. For example, we recall the recurrence formula for Ramanujan's third order mock theta function $f(q) = \sum_{n=0}^{\infty} c_f(n)q^n$ that was obtained recently by Imamoğlu, Raum and Richter [41]. If $n \in \mathbb{Q}$, then let

$$\sigma_1(n) := \begin{cases} \sum_{d|n} d & \text{if } n \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{sgn}^+(n) := \begin{cases} \mathrm{sgn}(n) & \text{if } n \neq 0, \\ 1 & \text{if } n = 0, \end{cases}$$

and then define

$$d(N, \widetilde{N}, t, \widetilde{t}) := \mathrm{sgn}^+(N) \cdot \mathrm{sgn}^+(\widetilde{N}) \cdot \left(|N + t| - |\widetilde{N} + \widetilde{t}| \right).$$

Then for positive integers n, we have that

$$\sum_{\substack{m \in \mathbb{Z} \\ 3m^2+m \leq 2n}} \left(m + \frac{1}{6}\right) c_f\left(n - \frac{3}{2}m^2 - \frac{1}{2}m\right) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma\left(\frac{n}{2}\right) - 2 \sum_{\substack{a,b \in \mathbb{Z} \\ 2n = ab}} d\left(N, \widetilde{N}, \frac{1}{6}, \frac{1}{6}\right),$$

where $N := \frac{1}{6}(-3a + b - 1)$ and $\widetilde{N} := \frac{1}{6}(3a + b - 1)$, and the sum is over integers a, b for which $N, \widetilde{N} \in \mathbb{Z}$. This is easily seen to be a recurrence relation for the coefficients $c_f(n)$. The replicability formulas for all of the $H^X_{g,r}(\tau)$ are similar (although some of these relations are slightly more complicated and involve the coefficients of weight 2 cusp forms).

It is important to emphasize that, despite the progress which is represented by our main results, Theorems 1.1 and 1.2, the following important question remains open in general.

Question. *Is there a "natural" construction of $\check{K}^X$? Is $\check{K}^X$ equipped with a deeper algebra structure as in the case of the monster module $V^\natural$ of Frenkel, Lepowsky and Meurman?*

We remark that this question has been answered positively, recently, in one special case: A vertex operator algebra structure underlying the umbral moonshine module $\check{K}^X$ for $X = E_8^3$ has been described explicitly in [25]. See also [14, 26], where the problem of constructing algebraic structures that illuminate the umbral moonshine observations is addressed from a different point of view.

The proof of Theorem 1.1 is not difficult. It is essentially a collection of tedious calculations. We use the theory of mock modular forms and the character table for each G^X (cf. §A.2) to solve for the multiplicities of the irreducible G^X-module constituents of each homogeneous subspace in the alleged G^X-module $\check{K}^X$. To prove Theorem 1.1 it suffices to prove that these multiplicities are non-negative integers. To prove Theorem 1.2 we apply recent work [42] of Mertens on the holomorphic projection of weight $\frac{1}{2}$ mock modular forms, which generalizes earlier work [41] of Imamoğlu, Raum and Richter.

In §2 we recall the facts about mock modular forms that we require, and we prove Theorem 1.2. We prove Theorem 1.1 in §3. The appendices furnish all the data that our method requires. In particular, the umbral groups G^X are described in detail in §A, and explicit definitions for the mock modular forms $H_g^X(\tau)$ are given in §B.

2 Harmonic Maass forms and Mock modular forms

Here we recall some very basic facts about harmonic Maass forms as developed by Bruinier and Funke [9] (see also [45]).

We begin by briefly recalling the definition of a *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier ν (a generalization of the notion of a Nebentypus). If $\tau = x + iy$ with x and y real, we define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{2.1}$$

Suppose Γ is a subgroup of finite index in $SL_2(\mathbb{Z})$ and $k \in \frac{1}{2}\mathbb{Z}$. Then a function $F(\tau)$ which is real-analytic on the upper half of the complex plane is a *harmonic Maass form* of weight k on Γ with multiplier ν if:

(a) The function $F(\tau)$ satisfies the weight k modular transformation,

$$F(\tau)|_k\gamma = \nu(\gamma)F(\tau)$$

for every matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, where $F(\tau)|_k\gamma := F(\gamma\tau)(c\tau + d)^{-k}$, and if $k \in \mathbb{Z} + \frac{1}{2}$, the square root is taken to be the principal branch.

(b) We have that $\Delta_k F(\tau) = 0$,

(c) There is a polynomial $P_F(q^{-1})$ and a constant $c > 0$ such that $F(\tau) - P_F(e^{-2\pi i\tau}) = O(e^{-cy})$ as $\tau \to i\infty$. Analogous conditions are required at each cusp of Γ.

We denote the $\mathbb{C}$-vector space of harmonic Maass forms of a given weight k, group Γ and multiplier ν by $H_k(\Gamma, \nu)$. If no multiplier is specified, we will take

$$\nu_0(\gamma) := \left(\left(\frac{c}{d}\right) \sqrt{\left(\frac{-1}{d}\right)}^{-1} \right)^{2k},$$

where $\left(\frac{*}{d}\right)$ is the Kronecker symbol.

2.1 Main properties

The Fourier expansion of a harmonic Maass form F (see Proposition 3.2 of [9]) splits into two components. As before, we let $q := e^{2\pi i\tau}$.

Lemma 2.1. *If $F(\tau)$ is a harmonic Maass form of weight $2-k$ for Γ where $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, then*

$$F(\tau) = F^+(\tau) + F^-(\tau),$$

where F^+ is the holomorphic part of F, given by

$$F^+(\tau) = \sum_{n \gg -\infty} c_F^+(n) q^n$$

where the sum admits only finitely many non-zero terms with $n < 0$, and F^- is the nonholomorphic part, given by

$$F^-(\tau) = \sum_{n<0} c_F^-(n) \Gamma(k-1, 4\pi y|n|) q^n.$$

Here $\Gamma(s,z)$ is the upper incomplete gamma function.

The holomorphic part of a harmonic Maass form is called a *mock modular form.* We denote the space of harmonic Maass forms of weight $2-k$ for Γ and multiplier ν by $H_k(\Gamma,\nu)$. Similarly, we denote the corresponding subspace of holomorphic modular forms by $M_k(\Gamma,\nu)$, and the space of cusp forms by $S_k(\Gamma,\nu)$. The differential operator $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \overline{\tau}}}$ (see [9]) defines a surjective map

$$\xi_{2-k} : H_{2-k}(\Gamma,\nu) \to S_k(\Gamma,\overline{\nu})$$

onto the space of weight k cusp forms for the same group but conjugate multiplier. The *shadow* of a Maass form $f(\tau) \in H_{2-k}(\Gamma,\nu)$ is the cusp form $g(\tau) \in S_k(\Gamma,\overline{\nu})$ (defined, for now, only up to scale) such that $\xi_{2-k} f(\tau) = \frac{g}{||g||}$, where $|| \bullet ||$ denotes the usual Petersson norm.

2.2 Holomorphic projection of weight $\frac{1}{2}$ mock modular forms

As noted above, the modular transformations of a weight $\frac{1}{2}$ harmonic Maass form may be simplified by multiplying by its shadow to obtain a weight 2 nonholomorphic modular form. One can use the theory of holomorphic projections to obtain explicit identities relating these nonholomorphic modular forms to classical quasimodular forms. In this way, we may essentially reduce many questions about the coefficients of weight $\frac{1}{2}$ mock modular forms to questions about weight 2 holomorphic modular forms. The following theorem is a special case of a more general theorem due to Mertens (cf. Theorem 6.3 of [42]). See also [41].

Theorem 2.2 (Mertens)**.** *Suppose $g(\tau)$ and $h(\tau)$ are both theta functions of weight $\frac{3}{2}$ contained in $S_{\frac{3}{2}}(\Gamma,\nu_g)$ and $S_{\frac{3}{2}}(\Gamma,\nu_h)$ respectively, with Fourier expansions*

$$g(\tau) := \sum_{i=1}^{s} \sum_{n\in\mathbb{Z}} n\chi_i(n) q^{n^2},$$

$$h(\tau) := \sum_{j=1}^{t} \sum_{n\in\mathbb{Z}} n\psi_j(n) q^{n^2},$$

where each χ_i and ψ_i is a Dirichlet character. Moreover, suppose $h(\tau)$ is the shadow of a weight $\frac{1}{2}$ harmonic Maass form $f(\tau) \in H_{\frac{1}{2}}(\Gamma, \overline{\nu_h})$. Define the function

$$D^{f,g}(\tau) := 2 \sum_{r=1}^{\infty} \sum_{\chi_i, \psi_j} \sum_{\substack{m,n \in \mathbb{Z}^+ \\ m^2 - n^2 = r}} \chi_i(m) \overline{\psi_j(n)} (m-n) q^r.$$

If $f(\tau)g(\tau)$ has no singularity at any cusp, then $f^+(\tau)g(\tau) + D^{f,g}(\tau)$ is a weight 2 quasimodular form. In other words, it lies in the space $\mathbb{C}E_2(\tau) \oplus M_2(\Gamma, \nu_g \overline{\nu_h})$, where $E_2(\tau)$ is the quasimodular Eisenstein series $E_2(\tau) := 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1-q^n}$.

Two Remarks.
1) These identities give recurrence relations for the weight $\frac{1}{2}$ mock modular form f^+ in terms of the weight 2 quasimodular form which equals $f^+(\tau)g(\tau) + D^{f,g}(\tau)$. The example after Theorem 1.2 for Ramanujan's third order mock theta function f is an explicit example of such a relation.

2) Theorem 2.2 extends to vector-valued mock modular forms in a natural way.

Proof of Theorem 1.2. Fix a Niemeier lattice and its root system X, and let $M = m^X$ denote its Coxeter number. Each $H^X_{g,r}(\tau)$ is the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass form $\widehat{H}^X_{g,r}(\tau)$. To simplify the exposition in the following section, we will emphasize the case that the root system X is of pure A-type. If the root system X is of pure A-type, the shadow function $S^X_{g,r}(\tau)$ is given by $\hat{\chi}^{X_A}_{g,r} S_{M,r}(\tau)$ (see §B.2), where

$$S_{M,r}(\tau) = \sum_{\substack{n \in \mathbb{Z} \\ n \equiv r \pmod{2M}}} n \, q^{\frac{n^2}{4M}},$$

and $\hat{\chi}^{X_A}_{g,r} = \chi^{X_A}_g$ or $\bar{\chi}^{X_A}_g$ depending on the parity of r is the twisted Euler character given in the appropriate table in §A.3, a character of G^X. (If X is not of pure A-type, then the shadow function $S^X_{g,r}(\tau)$ is a linear combination of similar functions as described in §B.2.)

Given X and g, the symbol $n_g | h_g$ given in the corresponding table in §A.3 defines the modularity for the vector-valued function $(\widehat{H}^X_{g,r}(\tau))$. In particular, if the shadow $(S^X_{g,r}(\tau))$ is nonzero, and if for $\gamma \in \Gamma_0(n_g)$ we have that

$$(S^X_{g,r}(\tau))|_{3/2}\gamma = \sigma_{g,\gamma}(S^X_{g,r}(\tau)),$$

then

$$(\widehat{H}^X_{g,r}(\tau))|_{1/2}\gamma = \overline{\sigma_{g,\gamma}}(\widehat{H}^X_{g,r}(\tau)).$$

Here, for $\gamma \in \Gamma_0(n_g)$, we have $\sigma_{g,\gamma} = \nu_g(\gamma)\sigma_{e,\gamma}$ where $\nu_g(\gamma)$ is a multiplier which is trivial on $\Gamma_0(n_g h_g)$. This identity holds even in the case that the shadow $S^X_{g,r}$ vanishes.

The vector-valued function $(H^X_{g,r}(\tau))$ has poles only at the infinite cusp of $\Gamma_0(n_g)$, and only at the component $H^X_{g,r}(\tau)$ where $r = 1$ if X has pure A-type, or at components where $r^2 \equiv 1 \pmod{4M}$ otherwise. These poles may only have order $\frac{1}{4M}$. This implies that the function $(\widehat{H}^X_{g,r}(\tau) S^X_{g,r}(\tau))$ has no pole at any cusp, and is therefore a candidate for an application of Theorem 2.2.

The modular transformation of $S_{M,r}(\tau)$ implies that

$$(\sigma_{e,S})^2 = (\sigma_{e,T})^{4M} = \mathbf{I}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\mathbf{I}$ is the identity matrix. Therefore $S^X_{M,r}(\tau)$, viewed as a scalar-valued modular function, is modular on $\Gamma(4M)$, and so $(\widehat{H}^X_{g,r}(\tau)S^X_{g,r}(\tau))$ is a weight 2 nonholomorphic scalar-valued modular form for the group $\Gamma(4M)\cap\Gamma_0(n_g)$ with trivial multiplier.

Applying Theorem 2.2, we obtain a function $F^X_{g,r}(\tau)$—call it the holomorphic projection of $\widehat{H^X_{g,r}}(\tau)S^X_{e,r}(\tau)$—which is a weight 2 quasimodular form on $\Gamma(4M)\cap\Gamma_0(n_g)$. In the case that $S^X_{g,r}(\tau)$ is zero, we substitute $S^X_{e,r}(\tau)$ in its place to obtain a function $\widetilde{F}^X_{g,r}(\tau) = H^X_{g,r}(\tau)S^X_{e,r}(\tau)$ which is a weight 2 holomorphic scalar-valued modular form for the group $\Gamma(4M)\cap\Gamma_0(n_g)$ with multiplier ν_g (alternatively, modular for the group $\Gamma(4M)\cap\Gamma_0(n_gh_g)$ with trivial multiplier).

The function $F^X_{g,r}(\tau)$ may be determined explicitly as the sum of Eisenstein series and cusp forms on $\Gamma(4M)\cap\Gamma_0(n_gh_g)$ using the standard arguments from the theory of holomorphic modular forms (i.e. the "first few" coefficients determine such a form). Therefore, we have the identity

$$F^X_{g,r}(\tau) = H^X_{g,r}(\tau)\cdot S^X_{g,r}(\tau) + D^X_{g,r}(\tau), \tag{2.2}$$

where the function $D^X_{g,r}(\tau)$ is the correction term arising in Theorem 2.2. If X has pure A-type, then

$$D^X_{g,r}(\tau) = (\hat{\chi}^{X_A}_{g,r})^2 \sum_{N=1}^{\infty} \sum_{\substack{m,n\in\mathbb{Z}_+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n)q^{\frac{N}{4M}}, \tag{2.3}$$

where

$$\phi_r(\ell) = \begin{cases} \pm 1 & \text{if } \ell \equiv \pm r \pmod{2M} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} A^X_{g,r}(n)q^{n-\frac{D}{4M}}$ where $0 < D < 4M$ and $D \equiv r^2 \pmod{4M}$, and $F^X_{g,r}(\tau) = \sum_{N=0}^{\infty} B^X_{g,r}(n)q^n$. Then by Theorem 2.2, we find that

$$B^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv r \pmod{2M}}} m\cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) + (\hat{\chi}^{X_A}_{g,r})^2 \sum_{\substack{m,n\in\mathbb{Z}^+ \\ m^2-n^2=N}} \phi_r(m)\phi_r(n)(m-n). \tag{2.4}$$

The function $F^X_{g,r}(\tau)$ may be found in the following manner. Using the explicit prescriptions for $H^X_{g,r}(\tau)$ given in §B.3 and (2.2) above, we may calculate the first several coefficients of each component. The Eisenstein component is determined by the constant terms at cusps. Since $D^X_{g,r}(\tau)$ (and the corresponding correction terms at other cusps) has no constant term, these are the same as the constant terms of $\widehat{H^X_{g,r}}(\tau)S^X_{g,r}(\tau)$, which are determined by the poles of $\widehat{H^X_{g,r}}$. Call this Eisenstein

component $E^X_{g,r}(\tau)$. The cuspidal component can be found by matching the initial coefficients of $F^X_{g,r}(\tau) - E^X_{g,r}(\tau)$.

Once the coefficients $B^X_{g,r}(n)$ are known, equation (2.4) provides a recursion relation which may be used to calculate the coefficients of $H^X_{g,r}(\tau)$. If the shadows $S^X_{g,r}(\tau)$ are zero, then we may apply a similar procedure in order to determine $\widetilde{F}^X_{g,r}(\tau)$. For example, suppose $\widetilde{F}^X_{g,r}(\tau) = \sum_{N=0}^{\infty} \widetilde{B}^X_{g,r}(n)q^n$, and X has pure A-type. Then we find that the coefficients $\widetilde{B}^X_{g,r}(N)$ satisfy

$$\widetilde{B}^X_{g,r}(N) = \hat{\chi}^{X_A}_{g,r} \sum_{\substack{m\in\mathbb{Z} \\ m\equiv r \pmod{2M}}} m \cdot A^X_{g,r}\left(N + \frac{D-m^2}{4M}\right) \tag{2.5}$$

Proceeding in this way we obtain the claimed results. □

3 Proof of Theorem 1.1

Here we prove Theorem 1.1. The idea is as follows. For each Niemeier root system X we begin with the vector-valued mock modular forms $(H^X_g(\tau))$ for $g \in G^X$. We use their q-expansions to solve for the q-series whose coefficients are the alleged multiplicities of the irreducible components of the alleged infinite-dimensional G^X-module

$$\check{K}^X = \bigoplus_{r \pmod{2m}} \bigoplus_{\substack{D\in\mathbb{Z},\, D\leq 0, \\ D=r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m}.$$

These q-series turn out to be mock modular forms. The proof requires that we establish that these mock modular forms have non-negative integer coefficients.

Proof of Theorem 1.1. As in the previous section, we fix a root system X and set $M := m^X$, and we emphasize the case when X is of pure A-type.

The umbral moonshine conjecture asserts that

$$H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} \sum_{\chi} m^X_{\chi,r}(n)\chi(g)q^{n-\frac{r^2}{4M}} \tag{3.1}$$

where the second sum is over the irreducible characters of G^X. Here we have rewritten the traces of the graded components $\check{K}^X_{r,n-r^2/4M}$ in 1.4 in terms of the values of the irreducible characters of G^X, where the $m^X_{\chi,r}(n)$ are the corresponding multiplicities. Naturally, if such a $\check{K}^X$ exists, these multiplicities must be non-negative integers for $n > 0$. Similarly, if the mock modular forms $H^X_{g,r}(\tau)$ can be expressed as in 3.1 with $m^X_{\chi,r}(n)$ non-negative integers, then we may construct the umbral moonshine module $\check{K}^X$ explicitly with $\check{K}^X_{r,n-r^2/4M}$ defined as the direct sum of irreducible components with the given multiplicities $m^X_{\chi,r}(n)$.

Let

$$H^X_{\chi,r}(\tau) := \frac{1}{|G^X|} \sum_g \overline{\chi(g)} H^X_{g,r}(\tau). \tag{3.2}$$

It turns out that the coefficients of $H^X_{\chi,r}(\tau)$ are precisely the multiplicities $m^X_{\chi,r}(n)$ required so that 3.1 holds: if

$$H^X_{\chi,r}(\tau) = \sum_{n=0}^{\infty} m^X_{\chi,r}(n) q^{n-\frac{r^2}{4M}}, \tag{3.3}$$

then

$$H^X_{g,r}(\tau) = \sum_{n=0}^{\infty} \sum_{\chi} m^X_{\chi,r}(n)\chi(g) q^{n-\frac{r^2}{4M}}.$$

Thus the umbral moonshine conjecture is true if and only if the Fourier coefficients of $H^X_{\chi,r}(\tau)$ are non-negative integers.

To see this fact, we recall the orthogonality of characters. For irreducible characters χ_i and χ_j,

$$\frac{1}{|G^X|} \sum_{g\in G^X} \overline{\chi_i(g)}\chi_j(g) = \begin{cases} 1 & \text{if } \chi_i = \chi_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

We also have the relation for g and $h \in G^X$,

$$\sum_{\chi} \overline{\chi_i(g)}\chi_i(h) = \begin{cases} |C_{G^X}(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Here $|C_{G^X}(g)|$ is the order of the centralizer of g in G^X. Since the order of the centralizer times the order of the conjugacy class of an element is the order of the group, (3.2) and (3.5) together imply the relation

$$H^X_{g,r}(\tau) = \sum_{\chi} \chi(g) H^X_{\chi,r}(\tau),$$

which in turn implies 3.3.

We have reduced the theorem to proving that the coefficients of certain weight $\frac{1}{2}$ mock modular forms are all non-negative integers. For holomorphic modular forms we may answer questions of this type by making use of Sturm's theorem [49] (see also Theorem 2.58 of [44]). This theorem provides a bound B associated to a space of modular forms such that if the first B coefficients of a modular form $f(\tau)$ are integral, then all of the coefficients of $f(\tau)$ are integral. This bound reduces many questions about the Fourier coefficients of modular forms to finite calculations.

Sturm's theorem relies on the finite dimensionality of certain spaces of modular forms, and so it can not be applied directly to spaces of mock modular forms. However, by making use of holomorphic projection we can adapt Sturm's theorem to this setting.

Let $\widehat{H^X_{\chi,r}}(\tau)$ be defined as above. Recall that the transformation matrix for the vector-valued function $\widehat{H^X_{g,r}}(\tau))$ is $\overline{\sigma_{g,\gamma}}$, the conjugate of the transformation matrix for $(S^X_{e,r}(\tau))$ when $\gamma \in \Gamma_0(n_g h_g)$, and $\sigma_{g,\gamma}$ is the identity for $\gamma \in \Gamma(4M)$. Therefore if

$$N^X_{\chi} := \operatorname{lcm}\{n_g h_g \mid g \in G, \chi(g) \neq 0\},$$

then the scalar-valued functions $\widehat{H^X_{\chi,r}}(\tau)$ are modular on $\Gamma(4M) \cap \Gamma_0(N^X_{\chi})$.

Let

$$A_{\chi,r}(\tau) := H^X_{\chi,r}(\tau) S^X_{e,1}(\tau),$$

and let $\tilde{A}_{\chi,r}(\tau)$ be the holomorphic projection of $A_{\chi,r}(\tau)$. Suppose that $H^X_{\chi,r}(\tau)$ has integral coefficients up to some bound B. Formulas for the shadow functions (cf. §B.2) show that the leading coefficient of $S^X_{e,1}(\tau)$ is 1 and has integral coefficients. This implies that the function

$$A_{\chi,r}(\tau) := H^X_{\chi,r}(\tau) S^X_{e,1}(\tau)$$

also has integral coefficients up to the bound B. The shadow of $H^X_{\chi,r}(\tau)$ is given by

$$S^X_{\chi,r}(\tau) := \frac{1}{|G^X|} \sum_g \overline{\chi(g)} S^X_{g,r}(\tau).$$

If X is pure A-type, then $S^X_{g,r}(\tau) = \chi^{X_A}_{g,r} S_{M,r}(\tau) = (\chi'(g) + \chi''(g)) S_{M,r}(\tau)$ for some irreducible characters χ' and χ'', according to §A.3 and §B.2. Therefore,

$$S^X_{\chi,r}(\tau) = \begin{cases} S_{M,r}(\tau) & \text{if } \chi = \chi' \text{ or } \chi'', \\ 0 & \text{otherwise.} \end{cases}$$

When X is not of pure A-type the shadow is some sum of such functions, but in every case has integer coefficients, and so, applying Theorem 2.2 to $A_{\chi,r}(\tau)$, we find that $\tilde{A}_{\chi,r}(\tau)$ also has integer coefficients up to the bound B. In particular, since $\tilde{A}_{\chi,r}(\tau)$ is modular on $\Gamma(4M) \cap \Gamma_0(N^X_\chi)$, then if B is at least the Sturm bound for this group we have that every coefficient of $\tilde{A}_{\chi,r}(\tau)$ is integral. Since the leading coefficient of $S^X_{e,1}(\tau)$ is 1, we may reverse this argument and obtain that every coefficient of $H^X_{\chi,r}(\tau)$ is integral. Therefore, in order to check that $H^X_{\chi,r}(\tau)$ has only integer coefficients, it suffices to check up to the Sturm bound for $\Gamma(4M) \cap \Gamma_0(N_\chi)$. These calculations were carried out using the `sage` mathematical software [47].

The calculations and argument given above shows that the multiplicities $m^X_{\chi,r}(n)$ are all integers. To complete the proof, it suffices to check that they are also are non-negative. The proof of this claim follows easily by modifying step-by-step the argument in Gannon's proof of non-negativity in the M_{24} case [37] (i.e. $X = A_1^{24}$). Here we describe how this is done.

Expressions for the alleged McKay-Thompson series $H^X_{g,r}(\tau)$ in terms of Rademacher sums and unary theta functions are given in §B.4. Exact formulas are known for all the coefficients of Rademacher sums because they are defined by averaging the special function $\mathrm{r}^{[\alpha]}_{1/2}(\gamma, \tau)$ (see (B.114)) over cosets of a specific modular group modulo Γ_∞, the subgroup of translations. Therefore, Rademacher sums are standard Maass-Poincaré series, and as a result we have formulas for each of their coefficients as convergent infinite sums of Kloosterman-type sums weighted by values of the $I_{1/2}$ modified Bessel function. (For example, see [8] or [53] for the general theory, and [12] for the specific case that $X = A_1^{24}$.) More importantly, this means also that the generating function for the multiplicities $m^X_{\chi,r}(n)$ is a weight $\frac{1}{2}$ harmonic Maass form, which in turn means that exact formulas (modulo the unary theta functions) are also available in similar terms. For positive integers n, this then means that (cf. Theorem 1.1 of [8])

$$m^X_{\chi,r}(n) = \sum_\rho \sum_{m<0} \frac{a^X_\rho(m)}{n^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{K^X_\rho(m,n,c)}{c} \cdot \mathbb{I}^X \left(\frac{4\pi\sqrt{|nm|}}{c} \right), \tag{3.6}$$

where the sums are over the cusps ρ of the group $\Gamma_0(N^X_g)$, and finitely many explicit negative rational numbers m. The constants $a^X_\rho(m)$ are essentially the coefficients which describe the generating function in terms of Maass-Poincaré series. Here $\mathbb{I}$ is a suitable normalization and change of variable for the standard $I_{1/2}$ modified Bessel-function.

The Kloosterman-type sums $K_\rho^X(m,n,c)$ are well known to be related to Salié-type sums (for example see Proposition 5 of [40]). These Salié-type sums are of the form

$$S_\rho^X(m,n,c) = \sum_{\substack{x \pmod c \\ x^2 \equiv -D(m,n) \pmod c}} \epsilon_\rho^X(m,n) \cdot e\left(\frac{\beta^X x}{c}\right),$$

where $\epsilon_\rho^X(m,n)$ is a root of unity, $-D(m,n)$ is a discriminant of a positive definite binary quadratic form, and β^X is a nonzero positive rational number.

These Salié sums may then be estimated using the equidistribution of CM points with discriminant $-D(m,n)$. This process was first introduced by Hooley [38], and it was first applied to the coefficients of weight $\frac{1}{2}$ mock modular forms by Bringmann and Ono [7]. Gannon explains how to make effective the estimates for sums of this shape in §4 of [37], thereby reducing the proof of the M_{24} case of umbral moonshine to a finite calculation. In particular, in equations (4.6-4.10) of [37] Gannon shows how to bound coefficients of the form (3.6) in terms of the Selberg–Kloosterman zeta function, which is bounded in turn in his proof of Theorem 3 of [37]. We follow Gannon's proof *mutatis mutandis*. We find, for most root systems, that the coefficients of each multiplicity generating function are positive beyond the 390th coefficient. In the worst case, for the root system $X = E_8^3$, we find that the coefficients are positive beyond the 1780th coefficient. Moreover, the coefficients exhibit subexponential growth. A finite computer calculation in `sage` has verified the non-negativity of the finitely many remaining coefficients. □

Remark. It turns out that the estimates required for proving non-negativity are the worst for the $X = E_8^3$ case which required 1780 coefficients.

A The Umbral Groups

In this section we present the facts about the umbral groups that we have used in establishing the main results of this paper. We recall (from [16]) their construction in terms of Niemeier root systems in §A.1, and we reproduce their character tables (appearing also in [16]) in §A.2. Note that we use the abbreviations $a_n := \sqrt{-n}$ and $b_n := (-1+\sqrt{-n})/2$ in the tables of §A.2.

The root system description of the umbral groups (cf. §A.1) gives rise to certain characters called *twisted Euler characters* which we recall (from [16]) in §A.3. The data appearing in §A.3 plays an important role in §B.2, where we use it to describe the shadows S_g^X of the umbral McKay-Thompson series H_g^X explicitly.

A.1 Construction

As mentioned in §1, there are exactly 24 self-dual even positive-definite lattices of rank 24 up to isomorphism, according to the classification of Niemeier [43] (cf. also [19,52]). Such a lattice L is determined up to isomorphism by its *root system* $L_2 := \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$. The unique example without roots is the Leech lattice. We refer to the remaining 23 as the *Niemeier lattices*, and we call a root system X a *Niemeier root system* if it occurs as the root system of a Niemeier lattice.

The simple components of Niemeier root systems are root systems of ADE type, and it turns out that the simple components of a Niemeier root system X all have the same Coxeter number. Define m^X to be the Coxeter number of any simple component of X, and call this the *Coxeter number* of X.

For X a Niemeier root system write N^X for the corresponding Niemeier lattice. The *umbral group* attached to X is defined by setting

$$G^X := \mathrm{Aut}(N^X)/W^X \tag{A.1}$$

where W^X is the normal subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in root vectors.

Observe that G^X acts as permutations on the simple components of X. In general this action is not faithful, so define $\overline{G}^X$ to be the quotient of G^X by its kernel. It turns out that the level of the mock modular form H_g^X attached to $g \in G^X$ is given by the order, denoted n_g, of the image of g in $\bar{G}^X$. (Cf. §A.3 for the values n_g.)

The Niemeier root systems and their corresponding umbral groups are described in Table 1. The root systems are given in terms of their simple components of ADE type. Here $D_{10}E_7^2$, for example, means the direct sum of one copy of the D_{10} root system and two copies of the E_7 root system. The symbol ℓ is called the *lambency* of X, and the Coxeter number m^X appears as the first summand of ℓ.

In the descriptions of the umbral groups G^X, and their permutation group quotients $\bar{G}^X$, we write M_{24} and M_{12} for the sporadic simple groups of Mathieu which act quintuply transitively on 24 and 12 points, respectively. (Cf. e.g. [21].) We write $GL_n(q)$ for the general linear group of a vector space of dimension n over a field with q elements, and $SL_n(q)$ is the subgroup of linear transformations with determinant 1, &c. The symbols $AGL_3(2)$ denote the *affine general linear group*, obtained by adjoining translations to $GL_3(2)$. We write Dih_n for the dihedral group of order $2n$, and Sym_n denotes the symmetric group on n symbols. We use n as a shorthand for a cyclic group of order n.

Table 1: The Umbral Groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4 D_4$	A_6^4	$A_7^2 D_5^2$
ℓ	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2 D_6$	$A_{11} D_7 E_6$	A_{12}^2	$A_{15} D_9$	$A_{17} E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10} E_7^2$	D_{12}^2	$D_{16} E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

We also use the notational convention of writing $A.B$ to denote the middle term in a short exact sequence $1 \to A \to A.B \to B \to 1$. This introduces some ambiguity which is nonetheless easily navigated in practice. For example, $2.M_{12}$ is the unique (up to isomorphism) double cover of M_{12} which is not $2 \times M_{12}$. The group $AGL_3(2)$ naturally embeds in $GL_4(2)$, which in turn admits a unique (up to isomorphism) double cover $2.GL_4(2)$ which is not a direct product. The group we denote $2.AGL_3(2)$ is the preimage of $AGL_3(2) < GL_4(2)$ in $2.GL_4(2)$ under the natural projection.

A.2 Character Tables

Table 2: Character table of $G^X \simeq M_{24}$, $X = A_1^{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	0	0	-1	-1	0	0
χ_3	$\circ$	45	-3	5	0	3	-3	1	1	0	0	-1	b_7	$\overline{b_7}$	-1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	-1	-1
χ_4	$\circ$	45	-3	5	0	3	-3	1	1	0	0	-1	$\overline{b_7}$	b_7	-1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	-1	-1
χ_5	$\circ$	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	$\circ$	231	7	-9	-3	0	-1	-1	3	1	1	0	0	0	-1	1	0	-1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	$+$	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	$+$	253	13	-11	10	1	-3	1	1	3	-2	1	1	1	-1	-1	0	0	1	-1	-1	0	0	1	1	0	0
χ_9	$+$	483	35	3	6	0	3	3	3	-2	2	0	0	0	-1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	$\circ$	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	$\circ$	770	-14	10	5	-7	2	-2	-2	0	1	1	0	0	0	0	0	-1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	$\circ$	990	-18	-10	0	3	6	2	-2	0	0	-1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	$\circ$	990	-18	-10	0	3	6	2	-2	0	0	-1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	$+$	1035	27	35	0	6	3	-1	3	0	0	2	-1	-1	1	0	1	0	0	-1	-1	0	0	-1	-1	0	0
χ_{15}	$\circ$	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2b_7$	$2\overline{b_7}$	-1	0	1	0	-1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	$\circ$	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\overline{b_7}$	$2b_7$	-1	0	1	0	-1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	$+$	1265	49	-15	5	8	-7	1	-3	0	1	0	-2	-2	1	0	0	-1	0	0	0	0	0	1	1	0	0
χ_{18}	$+$	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	0	-1	0	0	1	1	0	0	0	0
χ_{19}	$+$	2024	8	24	-1	8	8	0	0	-1	-1	0	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	$+$	2277	21	-19	0	6	-3	1	-3	-3	0	2	2	2	-1	1	0	0	0	0	0	0	0	-1	-1	0	0
χ_{21}	$+$	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	0	0	-1	-1	0	0	1	1	0	0
χ_{22}	$+$	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	0	0	1	1	0	0	-1	-1	1	1
χ_{23}	$+$	5313	49	9	-15	0	1	-3	-3	3	1	0	0	0	-1	-1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	$+$	5544	-56	24	9	0	-8	0	0	-1	1	0	0	0	0	-1	0	1	0	0	0	-1	-1	0	0	1	1
χ_{25}	$+$	5796	-28	36	-9	0	-4	4	0	1	-1	0	0	0	0	1	-1	-1	0	0	0	1	1	0	0	0	0
χ_{26}	$+$	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

Table 3: Character table of $G^X \simeq 2.M_{12}$, $X = A_2^{12}$

$[g]$	FS	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^2]$		1A	1A	2A	1A	1A	3A	3A	3B	3B	2B	2B	5A	5A	6B	3A	3A	4B	4B	4C	4C	10A	10A	11B	11B	11A	11A
$[g^3]$		1A	2A	4A	2B	2C	1A	2A	1A	2A	4B	4C	5A	10A	4A	2B	2C	8A	8B	8C	8D	20A	20B	11A	22A	11B	22B
$[g^5]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	1A	2A	12A	6C	6D	8B	8A	8D	8C	4A	4A	11A	22A	11B	22B
$[g^{11}]$		1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8A	8B	8C	8D	20B	20A	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	11	11	-1	3	3	2	2	-1	-1	-1	3	1	1	-1	0	0	-1	-1	1	1	-1	-1	0	0	0	0
χ_3	+	11	11	-1	3	3	2	2	-1	-1	3	-1	1	1	-1	0	0	1	1	-1	-1	-1	-1	0	0	0	0
χ_4	∘	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	b_{11}	b_{11}	$\overline{b_{11}}$	$\overline{b_{11}}$
χ_5	∘	16	16	4	0	0	-2	-2	1	1	0	0	1	1	1	0	0	0	0	0	0	-1	-1	$\overline{b_{11}}$	$\overline{b_{11}}$	b_{11}	b_{11}
χ_6	+	45	45	5	-3	-3	0	0	3	3	1	1	0	0	-1	0	0	-1	-1	-1	-1	0	0	1	1	1	1
χ_7	+	54	54	6	6	6	0	0	0	0	2	2	-1	-1	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1
χ_8	+	55	55	-5	7	7	1	1	1	1	-1	-1	0	0	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
χ_9	+	55	55	-5	-1	-1	1	1	1	1	3	-1	0	0	1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
χ_{10}	+	55	55	-5	-1	-1	1	1	1	1	-1	3	0	0	1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
χ_{11}	+	66	66	6	2	2	3	3	0	0	-2	-2	1	1	0	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{12}	+	99	99	-1	3	3	0	0	3	3	-1	-1	-1	-1	-1	0	0	1	1	1	1	-1	-1	0	0	0	0
χ_{13}	+	120	120	0	-8	-8	3	3	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	-1	-1	-1
χ_{14}	+	144	144	4	0	0	0	0	-3	-3	0	0	-1	-1	1	0	0	0	0	0	0	-1	-1	1	1	1	1
χ_{15}	+	176	176	-4	0	0	-4	-4	-1	-1	0	0	1	1	-1	0	0	0	0	0	0	1	1	0	0	0	0
χ_{16}	∘	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	a_2	$\overline{a_2}$	a_2	$\overline{a_2}$	0	0	-1	1	-1	1
χ_{17}	∘	10	-10	0	-2	2	1	-1	-2	2	0	0	0	0	0	1	-1	$\overline{a_2}$	a_2	$\overline{a_2}$	a_2	0	0	-1	1	-1	1
χ_{18}	+	12	-12	0	4	-4	3	-3	0	0	0	0	2	-2	0	1	-1	0	0	0	0	0	0	1	-1	1	-1
χ_{19}	−	32	-32	0	0	0	-4	4	2	-2	0	0	2	-2	0	0	0	0	0	0	0	0	0	-1	1	-1	1
χ_{20}	∘	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	a_5	$\overline{a_5}$	0	0	0	0
χ_{21}	∘	44	-44	0	4	-4	-1	1	2	-2	0	0	-1	1	0	1	-1	0	0	0	0	$\overline{a_5}$	a_5	0	0	0	0
χ_{22}	∘	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	a_2	$\overline{a_2}$	$\overline{a_2}$	a_2	0	0	0	0	0	0
χ_{23}	∘	110	-110	0	-6	6	2	-2	2	-2	0	0	0	0	0	0	0	$\overline{a_2}$	a_2	a_2	$\overline{a_2}$	0	0	0	0	0	0
χ_{24}	+	120	-120	0	8	-8	3	-3	0	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	-1	1	-1	1
χ_{25}	∘	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-b_{11}$	b_{11}	$-\overline{b_{11}}$	$\overline{b_{11}}$
χ_{26}	∘	160	-160	0	0	0	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\overline{b_{11}}$	$\overline{b_{11}}$	$-b_{11}$	b_{11}

Table 4: Character table of $G^X \simeq 2.AGL_3(2)$, $X = A_3^8$

$[g]$	FS	1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	7A	14A	7B	14B
$[g^2]$		1A	1A	1A	2A	2B	1A	3A	3A	3A	3A	4A	2C	7A	7A	7B	7B
$[g^3]$		1A	2A	2B	4A	4B	2C	1A	2A	2B	2B	8A	4C	7B	14B	7A	14A
$[g^7]$		1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	1A	2A	1A	2A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	b_7	b_7	$\overline{b_7}$	$\overline{b_7}$
χ_3	∘	3	3	3	−1	−1	−1	0	0	0	0	1	1	$\overline{b_7}$	$\overline{b_7}$	b_7	b_7
χ_4	+	6	6	6	2	2	2	0	0	0	0	0	0	−1	−1	−1	−1
χ_5	+	7	7	7	−1	−1	−1	1	1	1	1	−1	−1	0	0	0	0
χ_6	+	8	8	8	0	0	0	−1	−1	−1	−1	0	0	1	1	1	1
χ_7	+	7	7	−1	3	−1	−1	1	1	−1	−1	1	−1	0	0	0	0
χ_8	+	7	7	−1	−1	−1	3	1	1	−1	−1	−1	1	0	0	0	0
χ_9	+	14	14	−2	2	−2	2	−1	−1	1	1	0	0	0	0	0	0
χ_{10}	+	21	21	−3	1	1	−3	0	0	0	0	−1	1	0	0	0	0
χ_{11}	+	21	21	−3	−3	1	1	0	0	0	0	1	−1	0	0	0	0
χ_{12}	+	8	−8	0	0	0	0	2	−2	0	0	0	0	1	−1	1	−1
χ_{13}	∘	8	−8	0	0	0	0	−1	1	a_3	$\overline{a_3}$	0	0	1	−1	1	−1
χ_{14}	∘	8	−8	0	0	0	0	−1	1	$\overline{a_3}$	a_3	0	0	1	−1	1	−1
χ_{15}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	$\overline{b_7}$	$-\overline{b_7}$	b_7	$-b_7$
χ_{16}	∘	24	−24	0	0	0	0	0	0	0	0	0	0	b_7	$-b_7$	$\overline{b_7}$	$-\overline{b_7}$

Table 5: Character table of $G^X \simeq GL_2(5)/2$, $X = A_4^6$

$[g]$	FS	1A	2A	2B	2C	3A	6A	5A	10A	4A	4B	4C	4D	12A	12B
$[g^2]$		1A	1A	1A	1A	3A	3A	5A	5A	2A	2A	2C	2C	6A	6A
$[g^3]$		1A	2A	2B	2C	1A	2A	5A	10A	4B	4A	4D	4C	4B	4A
$[g^5]$		1A	2A	2B	2C	3A	6A	1A	2A	4A	4B	4C	4D	12A	12B
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
χ_3	+	4	4	0	0	1	1	−1	−1	2	2	0	0	−1	−1
χ_4	+	4	4	0	0	1	1	−1	−1	−2	−2	0	0	1	1
χ_5	+	5	5	1	1	−1	−1	0	0	1	1	−1	−1	1	1
χ_6	+	5	5	1	1	−1	−1	0	0	−1	−1	1	1	−1	−1
χ_7	+	6	6	−2	−2	0	0	1	1	0	0	0	0	0	0
χ_8	∘	1	−1	1	−1	1	−1	1	−1	a_1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$
χ_9	∘	1	−1	1	−1	1	−1	1	−1	$-a_1$	a_1	$-a_1$	a_1	$-a_1$	a_1
χ_{10}	∘	4	−4	0	0	1	−1	−1	1	$2a_1$	$-2a_1$	0	0	$-a_1$	a_1
χ_{11}	∘	4	−4	0	0	1	−1	−1	1	$-2a_1$	$2a_1$	0	0	a_1	$-a_1$
χ_{12}	∘	5	−5	1	−1	−1	1	0	0	a_1	$-a_1$	$-a_1$	a_1	a_1	$-a_1$
χ_{13}	∘	5	−5	1	−1	−1	1	0	0	$-a_1$	a_1	a_1	$-a_1$	$-a_1$	a_1
χ_{14}	+	6	−6	−2	2	0	0	1	−1	0	0	0	0	0	0

Table 6: Character table of $G^X \simeq GL_2(3)$, $X \in \{A_5^4 D_4, E_6^4\}$

$[g]$	FS	1A	2A	2B	4A	3A	6A	8A	8B
$[g^2]$		1A	1A	1A	2A	3A	3A	4A	4A
$[g^3]$		1A	2A	2B	4A	1A	2A	8A	8B
χ_1	$+$	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	-1	1	1	1	-1	-1
χ_3	$+$	2	2	0	2	-1	-1	0	0
χ_4	$+$	3	3	-1	-1	0	0	1	1
χ_5	$+$	3	3	1	-1	0	0	-1	-1
χ_6	$\circ$	2	-2	0	0	-1	1	a_2	$\overline{a_2}$
χ_7	$\circ$	2	-2	0	0	-1	1	$\overline{a_2}$	a_2
χ_8	$+$	4	-4	0	0	1	-1	0	0

Table 7: Character table of $G^X \simeq 3.Sym_6$, $X = D_4^6$

$[g]$	FS	1A	3A	2A	6A	3B	3C	4A	12A	5A	15A	15B	2B	2C	4B	6B	6C
$[g^2]$		1A	3A	1A	3A	3B	3C	2A	6A	5A	15A	15B	1A	1A	2A	3B	3C
$[g^3]$		1A	1A	2A	2A	1A	1A	4A	4A	5A	5A	5A	2B	2C	4B	2B	2C
$[g^5]$		1A	3A	2A	6A	3B	3C	4A	12A	1A	3A	3A	2B	2C	4B	6B	6C
χ_1	$+$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	$+$	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
χ_3	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	3	-1	1	0	-1
χ_4	$+$	5	5	1	1	2	-1	-1	-1	0	0	0	-3	1	-1	0	1
χ_5	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	-1	3	1	-1	0
χ_6	$+$	5	5	1	1	-1	2	-1	-1	0	0	0	1	-3	-1	1	0
χ_7	$+$	16	16	0	0	-2	-2	0	0	1	1	1	0	0	0	0	0
χ_8	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	3	3	-1	0	0
χ_9	$+$	9	9	1	1	0	0	1	1	-1	-1	-1	-3	-3	1	0	0
χ_{10}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	2	-2	0	-1	1
χ_{11}	$+$	10	10	-2	-2	1	1	0	0	0	0	0	-2	2	0	1	-1
χ_{12}	$\circ$	6	-3	-2	1	0	0	2	-1	1	b_{15}	$\overline{b_{15}}$	0	0	0	0	0
χ_{13}	$\circ$	6	-3	-2	1	0	0	2	-1	1	$\overline{b_{15}}$	b_{15}	0	0	0	0	0
χ_{14}	$+$	12	-6	4	-2	0	0	0	0	2	-1	-1	0	0	0	0	0
χ_{15}	$+$	18	-9	2	-1	0	0	2	-1	-2	1	1	0	0	0	0	0
χ_{16}	$+$	30	-15	-2	1	0	0	-2	1	0	0	0	0	0	0	0	0

Table 8: Character table of $G^X \simeq SL_2(3)$, $X = A_6^4$

$[g]$	FS	1A	2A	4A	3A	6A	3B	6B
$[g^2]$		1A	1A	2A	3B	3A	3A	3B
$[g^3]$		1A	2A	4A	1A	2A	1A	2A
χ_1	$+$	1	1	1	1	1	1	1
χ_2	$\circ$	1	1	1	b_3	$\overline{b_3}$	$\overline{b_3}$	b_3
χ_3	$\circ$	1	1	1	$\overline{b_3}$	b_3	b_3	$\overline{b_3}$
χ_4	$+$	3	3	-1	0	0	0	0
χ_5	$-$	2	-2	0	-1	1	-1	1
χ_6	$\circ$	2	-2	0	$-\overline{b_3}$	b_3	$-b_3$	$\overline{b_3}$
χ_7	$\circ$	2	-2	0	$-b_3$	$\overline{b_3}$	$-\overline{b_3}$	b_3

Table 9: Character table of $G^X \simeq Dih_4$, $X = A_7^2 D_5^2$

$[g]$	FS	1A	2A	2B	2C	4A
$[g^2]$		1A	1A	1A	1A	2A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1
χ_3	$+$	1	1	-1	1	-1
χ_4	$+$	1	1	1	-1	-1
χ_5	$+$	2	-2	0	0	0

Table 10: Character table of $G^X \simeq Dih_6$, $X = A_8^3$

$[g]$	FS	1A	2A	2B	2C	3A	6A
$[g^2]$		1A	1A	1A	1A	3A	3A
$[g^3]$		1A	2A	2B	2C	1A	2A
χ_1	$+$	1	1	1	1	1	1
χ_2	$+$	1	1	-1	-1	1	1
χ_3	$+$	2	2	0	0	-1	-1
χ_4	$+$	1	-1	-1	1	1	-1
χ_5	$+$	1	-1	1	-1	1	-1
χ_6	$+$	2	-2	0	0	-1	1

Table 11: Character table of $G^X \simeq 4$, for $X \in \{A_9^2 D_6, A_{12}^2\}$

$[g]$	FS	1A	2A	4A	4B
$[g^2]$		1A	1A	2A	2A
χ_1	$+$	1	1	1	1
χ_2	$+$	1	1	-1	-1
χ_3	$\circ$	1	-1	a_1	$\overline{a_1}$
χ_4	$\circ$	1	-1	$\overline{a_1}$	a_1

Table 12: Character table of $G^X \simeq PGL_2(3) \simeq Sym_4$, $X = D_6^4$

$[g]$	FS	1A	2A	3A	2B	4A
$[g^2]$		1A	1A	3A	1A	2A
$[g^3]$		1A	2A	1A	2B	4A
χ_1	$+$	1	1	1	1	1
χ_2	$+$	1	1	1	-1	-1
χ_3	$+$	2	2	-1	0	0
χ_4	$+$	3	-1	0	1	-1
χ_5	$+$	3	-1	0	-1	1

Table 13: Character table of $G^X \simeq 2$, for $X \in \{A_{11} D_7 E_6, A_{15} D_9, A_{17} E_7, A_{24}, D_{10} E_7^2, D_{12}^2\}$

$[g]$	FS	1A	2A
$[g^2]$		1A	1A
χ_1	$+$	1	1
χ_2	$+$	1	-1

Table 14: Character table of $G^X \simeq Sym_3$, $X \in \{D_8^3, E_8^3\}$

$[g]$	FS	1A	2A	3A
$[g^2]$		1A	1A	3A
$[g^3]$		1A	2A	1A
χ_1	$+$	1	1	1
χ_2	$+$	1	-1	1
χ_3	$+$	2	0	-1

A.3 Twisted Euler Characters

In this section we reproduce certain characters—the *twisted Euler characters*—which are attached to each group G^X, via its action on the root system X. (Their construction is described in detail in §2.4 of [16].)

To interpret the tables, write X_A for the (possibly empty) union of type A components of X, and interpret X_D and X_E similarly, so that if $m = m^X$ Then $X = A^d_{m-1}$ for some d, and $X = X_A \cup X_D \cup X_E$, for example. Then $g \mapsto \bar{\chi}_g^{X_A}$ denotes the character of the permutation representation attached to the action of $\bar{G}^X$ on the simple components of X_A. The characters $g \mapsto \bar{\chi}_g^{X_D}$ and $g \mapsto \bar{\chi}_g^{X_E}$ are defined similarly. The characters $\chi_g^{X_A}$, $\chi_g^{X_D}$, $\chi_g^{X_E}$ and $\check{\chi}_g^{X_D}$ incorporate outer automorphisms of simple root systems induced by the action G^X on X. We refer to §2.4 of [16] for full details of the construction. For the purposes of this work, it suffices to have the explicit descriptions in the tables in this section. The twisted Euler characters presented here will be used to specify the umbral shadow functions in §B.2.

The twisted Euler character tables also attach integers n_g and h_g to each $g \in G^X$. By definition, n_g is the order of the image of $g \in G^X$ in $\bar{G}^X$ (cf. §A.1). The integer h_g may be defined by setting $h_g := N_g/n_g$ where N_g is the product of the shortest and longest cycle lengths appearing in the cycle shape attached to g by the action of G^X on a (suitable) set of simple roots for X.

Table 15: Twisted Euler characters at $\ell = 2$, $X = A_1^{24}$

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
$n_g\|h_g$	1|1	2|1	2|2	3|1	3|3	4|2	4|1	4|4	5|1	6|1	6|6
$\bar{\chi}_g^{X_A}$	24	8	0	6	0	0	4	0	4	2	0

$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB
$n_g\|h_g$	7|1	8|1	10|2	11|1	12|2	12|12	14|1	15|1	21|3	23|1
$\bar{\chi}_g^{X_A}$	3	2	0	2	0	0	1	1	0	1

Table 16: Twisted Euler characters at $\ell = 3$, $X = A_2^{12}$

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
$n_g\|h_g$	1|1	1|4	2|8	2|1	2|2	3|1	3|4	3|3	3|12	4|2	4|1	5|1	5|4	6|24	6|1	6|2	8|4	8|1	10|8	11|1	11|4
$\bar{\chi}_g^{X_A}$	12	12	0	4	4	3	3	0	0	0	4	2	2	0	1	1	0	2	0	1	1
$\chi_g^{X_A}$	12	−12	0	4	−4	3	−3	0	0	0	0	2	−2	0	1	−1	0	0	0	1	−1

Table 17: Twisted Euler characters at $\ell = 4$, $X = A_3^8$

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$n_g\|h_g$	1|1	1|2	2|2	2|4	4|4	2|1	3|1	3|2	6|2	4|8	4|1	7|1	7|2
$\bar{\chi}_g^{X_A}$	8	8	0	0	0	4	2	2	0	0	2	1	1
$\chi_g^{X_A}$	8	−8	0	0	0	0	2	−2	0	0	0	1	−1

Table 18: Twisted Euler characters at $\ell = 5$, $X = A_4^6$

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$n_g\|h_g$	1|1	1|4	2|2	2|1	3|3	3|12	5|1	5|4	2|8	4|1	6|24
$\bar{\chi}_g^{X_A}$	6	6	2	2	0	0	1	1	0	2	0
$\chi_g^{X_A}$	6	−6	−2	2	0	0	1	−1	0	0	0

Table 19: Twisted Euler characters at $\ell = 6$, $X = A_5^4 D_4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g\|h_g$	1|1	1|2	2|1	2|2	3|1	3|2	4|2
$\bar{\chi}_g^{X_A}$	4	4	2	0	1	1	0
$\chi_g^{X_A}$	4	−4	0	0	1	−1	0
$\bar{\chi}_g^{X_D}$	1	1	1	1	1	1	1
$\chi_g^{X_D}$	1	1	−1	1	1	1	−1
$\check{\chi}_g^{X_D}$	2	2	0	2	−1	−1	0

Table 20: Twisted Euler characters at $\ell = 6 + 3$, $X = D_4^6$

$[g]$	1A	3A	2A	6A	3B	6C	4A	12A	5A	15AB	2B	2C	4B	6B	6C
$n_g\|h_g$	1|1	1|3	2|1	2|3	3|1	3|3	4|2	4|6	5|1	5|3	2|1	2|2	4|1	6|1	6|6
$\bar{\chi}_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	4	0	2	1	0
$\chi_g^{X_D}$	6	6	2	2	3	0	0	0	1	1	−4	0	−2	−1	0
$\check{\chi}_g^{X_D}$	12	−6	4	−2	0	0	0	0	2	−1	0	0	0	0	0

Table 21: Twisted Euler characters at $\ell = 7$, $X = A_6^4$

$[g]$	1A	2A	4A	3AB	6AB
$n_g \vert h_g$	1|1	1|4	2|8	3|1	3|4
$\bar{\chi}_g^{X_A}$	4	4	0	1	1
$\chi_g^{X_A}$	4	-4	0	1	-1

Table 22: Twisted Euler characters at $\ell = 8$, $X = A_7^2 D_5^2$

$[g]$	1A	2A	2B	2C	4A
$n_g \vert h_g$	1|1	1|2	2|1	2|1	2|4
$\bar{\chi}_g^{X_A}$	2	2	0	2	0
$\chi_g^{X_A}$	2	-2	0	0	0
$\bar{\chi}_g^{X_D}$	2	2	2	0	0
$\chi_g^{X_D}$	2	-2	0	0	0

Table 23: Twisted Euler characters at $\ell = 9$, $X = A_8^3$

$[g]$	1A	2A	2B	2C	3A	6A
$n_g \vert h_g$	1|1	1|4	2|1	2|2	3|3	3|12
$\bar{\chi}_g^{X_A}$	3	3	1	1	0	0
$\chi_g^{X_A}$	3	-3	1	-1	0	0

Table 24: Twisted Euler characters at $\ell = 10$, $X = A_9^2 D_6$

$[g]$	1A	2A	4AB
$n_g \vert h_g$	1|1	1|2	2|2
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0
$\bar{\chi}_g^{X_D}$	1	1	1
$\chi_g^{X_D}$	1	1	-1

Table 25: Twisted Euler characters at $\ell = 10 + 5$, $X = D_6^4$

$[g]$	1A	2A	3A	2B	4A
$n_g \vert h_g$	$1\vert 1$	$2\vert 2$	$3\vert 1$	$2\vert 1$	$4\vert 4$
$\bar{\chi}_g^{X_D}$	4	0	1	2	0
$\chi_g^{X_D}$	4	0	1	-2	0

Table 26: Twisted Euler characters at $\ell = 12$, $X = A_{11}D_7E_6$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\chi}_g^{X_E}$	1	1
$\chi_g^{X_E}$	1	-1

Table 27: Twisted Euler characters at $\ell = 12 + 4$, $X = E_6^4$

$[g]$	1A	2A	2B	4A	3A	6A	8AB
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$	$2\vert 1$	$2\vert 4$	$3\vert 1$	$3\vert 2$	$4\vert 8$
$\bar{\chi}_g^{X_E}$	4	4	2	0	1	1	0
$\chi_g^{X_E}$	4	-4	0	0	1	-1	0

Table 28: Twisted Euler characters at $\ell = 13$, $X = A_{12}^2$

$[g]$	1A	2A	4AB
$n_g \vert h_g$	$1\vert 1$	$1\vert 4$	$2\vert 8$
$\bar{\chi}_g^{X_A}$	2	2	0
$\chi_g^{X_A}$	2	-2	0

Table 29: Twisted Euler characters at $\ell = 14 + 7$, $X = D_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$	$3\vert 3$
$\bar{\chi}_g^{X_D}$	3	1	0
$\chi_g^{X_D}$	3	1	0

Table 30: Twisted Euler characters at $\ell = 16$, $X = A_{15}D_9$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1

Table 31: Twisted Euler characters at $\ell = 18$, $X = A_{17}E_7$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 2$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1
$\bar{\chi}_g^{X_E}$	1	1

Table 32: Twisted Euler characters at $\ell = 18 + 9$, $X = D_{10}E_7^2$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$
$\bar{\chi}_g^{X_D}$	1	1
$\chi_g^{X_D}$	1	-1
$\bar{\chi}_g^{X_E}$	2	0

Table 33: Twisted Euler characters at $\ell = 22 + 11$, $X = D_{12}^2$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$2\vert 2$
$\bar{\chi}_g^{X_D}$	2	0
$\chi_g^{X_D}$	2	0

Table 34: Twisted Euler characters at $\ell = 25$, $X = A_{24}$

$[g]$	1A	2A
$n_g \vert h_g$	$1\vert 1$	$1\vert 4$
$\bar{\chi}_g^{X_A}$	1	1
$\chi_g^{X_A}$	1	-1

Table 35: Twisted Euler characters at $\ell = 30 + 6, 10, 15$, $X = E_8^3$

$[g]$	1A	2A	3A
$n_g \vert h_g$	$1\vert 1$	$2\vert 1$	$3\vert 3$
$\bar{\chi}_g^{X_E}$	3	1	0

B The Umbral McKay-Thompson Series

In this section we describe the umbral McKay-Thompson series in complete detail. In particular, we present explicit formulas for all the McKay-Thompson series attached to elements of the umbral groups by umbral moonshine in §B.3. Most of these expressions appeared first in [15,16], but some appear for the first time in this work.

In order to facilitate explicit formulations we recall certain standard functions in §B.1. We then, using the twisted Euler characters of §A.3, explicitly describe the shadow functions of umbral moonshine in §B.2. The umbral McKay–Thompson series defined in §B.3 may also be described in terms of Rademacher sums, according to the results of [17]. We present this description in §B.4.

B.1 Special Functions

Throughout this section we assume $q := e^{2\pi i\tau}$, and $u := e^{2\pi iz}$, where $\tau, z \in \mathbb{C}$ with $\mathrm{Im}\ \tau > 0$. The Dedekind eta function is $\eta(\tau) := q^{1/24}\prod_{n>0}(1-q^n)$, where . Write $\Lambda_M(\tau)$ for the function

$$\Lambda_M(\tau) := Mq\frac{\mathrm{d}}{\mathrm{d}q}\left(\log\frac{\eta(M\tau)}{\eta(\tau)}\right) = \frac{M(M-1)}{24} + M\sum_{k>0}\sum_{d|k} d\left(q^k - Mq^{Mk}\right),$$

which is a modular form of weight two for $\Gamma_0(N)$ if $M|N$.

Define the Jacobi theta function $\theta_1(\tau, z)$ by setting

$$\theta_1(\tau, z) := iq^{1/8}u^{-1/2}\sum_{n\in\mathbb{Z}}(-1)^n u^n q^{n(n-1)/2}. \tag{B.1}$$

According to the Jacobi triple product identity we have

$$\theta_1(\tau, z) = -iq^{1/8}u^{1/2}\prod_{n>0}(1-u^{-1}q^{n-1})(1-uq^n)(1-q^n). \tag{B.2}$$

The other Jacobi theta functions are

$$\begin{aligned}
\theta_2(\tau, z) &:= q^{1/8}u^{1/2}\prod_{n>0}(1+u^{-1}q^{n-1})(1+uq^n)(1-q^n),\\
\theta_3(\tau, z) &:= \prod_{n>0}(1+u^{-1}q^{n-1/2})(1+uq^{n-1/2})(1-q^n),\\
\theta_4(\tau, z) &:= \prod_{n>0}(1-u^{-1}q^{n-1/2})(1-uq^{n-1/2})(1-q^n).
\end{aligned} \tag{B.3}$$

Define $\Psi_{1,1}$ and $\Psi_{1,-1/2}$ by setting

$$\begin{aligned}
\Psi_{1,1}(\tau, z) &:= -i\frac{\theta_1(\tau, 2z)\eta(\tau)^3}{\theta_1(\tau, z)^2},\\
\Psi_{1,-1/2}(\tau, z) &:= -i\frac{\eta(\tau)^3}{\theta_1(\tau, z)}.
\end{aligned} \tag{B.4}$$

These are meromorphic Jacobi forms of weight one, with indexes 1 and $-1/2$, respectively. Here, the term meromorphic refers to the presence of simple poles in the functions $z \mapsto \Psi_{1,*}(\tau, z)$, for fixed $\tau \in \mathbb{H}$, at lattice points $z \in \mathbb{Z}\tau + \mathbb{Z}$. (Cf. §8 of [22].)

From §5 of [29] we recall the *index m theta functions*, for $m \in \mathbb{Z}$, defined by setting

$$\theta_{m,r}(\tau, z) := \sum_{k\in\mathbb{Z}} u^{2mk+r} q^{(2mk+r)^2/4m}, \tag{B.5}$$

where $r \in \mathbb{Z}$. Evidently, $\theta_{m,r}$ only depends on $r \mod 2m$. We set $S_{m,r}(\tau) := \frac{1}{2\pi i}\, \partial_z \theta_{m,r}(\tau, z)|_{z=0}$, so that

$$S_{m,r}(\tau) = \sum_{k\in\mathbb{Z}} (2mk + r) q^{(2mk+r)^2/4m}. \tag{B.6}$$

For a m a positive integer define

$$\mu_{m,0}(\tau, z) = \sum_{k\in\mathbb{Z}} u^{2km} q^{mk^2} \frac{uq^k + 1}{uq^k - 1} = \frac{u+1}{u-1} + O(q), \tag{B.7}$$

and observe that we recover $\Psi_{1,1}$ upon specializing (B.7) to $m = 1$. Observe also that

$$\mu_{m,0}(\tau, z + 1/2) = \sum_{k\in\mathbb{Z}} u^{2km} q^{mk^2} \frac{uq^k - 1}{uq^k + 1} = \frac{u-1}{u+1} + O(q). \tag{B.8}$$

Define the even and odd parts of $\mu_{m,0}$ by setting

$$\mu^k_{m,0}(\tau, z) := \frac{1}{2}(\mu_{m,0}(\tau, z) + (-1)^k \mu_{m,0}(\tau, z + 1/2)) \tag{B.9}$$

for $k \mod 2$.

For $m, r \in \mathbb{Z} + \frac{1}{2}$ with $m > 0$ define *half-integral index theta functions*

$$\theta_{m,r}(\tau, z) := \sum_{k\in\mathbb{Z}} e(mk + r/2) u^{2mk+r} q^{(2mk+r)^2/4m}, \tag{B.10}$$

and define also $S_{m,r}(\tau) := \frac{1}{2\pi i}\partial_z \theta_{m,r}(\tau, z)|_{z=0}$, so that

$$S_{m,r}(\tau) = \sum_{k\in\mathbb{Z}} e(mk + r/2)(2mk + r) q^{(2mk+r)^2/4m}. \tag{B.11}$$

As in the integral index case, $\theta_{m,r}$ depends only on $r \mod 2m$. We recover $-\theta_1$ upon specializing $\theta_{m,r}$ to $m = r = 1/2$.

For $m \in \mathbb{Z} + 1/2$, $m > 0$, define

$$\mu_{m,0}(\tau, z) := i \sum_{k\in\mathbb{Z}} (-1)^k u^{2mk+1/2} q^{mk^2+k/2} \frac{1}{1 - uq^k} = \frac{-iu^{1/2}}{y - 1} + O(q). \tag{B.12}$$

Given $\alpha \in \mathbb{Q}$ write $[\alpha]$ for the operator on q-series (in rational, possibility negative powers of q) that eliminates exponents not contained in $\mathbb{Z} + \alpha$, so that if $f = \sum_{\beta\in\mathbb{Q}} c(\beta) q^\beta$ then

$$[\alpha] f := \sum_{n\in\mathbb{Z}} c(n + \alpha) q^{n+\alpha} \tag{B.13}$$

B.2 Shadows

Let X be a Niemeier root system and let $m = m^X$ be the Coxeter number of X. For $g \in G^X$ we define the associated *shadow function* $S_g^X = (S_{g,r}^X)$ by setting

$$S_g^X := S_g^{X_A} + S_g^{X_D} + S_g^{X_E} \tag{B.14}$$

where the $S_g^{X_A}$, &c., are defined in the following way, in terms of the twisted Euler characters $\chi_g^{X_A}$, &c. given in §A.3, and the unary theta series $S_{m,r}$ (cf. (B.6)).

Note that if $m = m^X$ then $S_{g,r}^X = S_{g,r+2m}^X = -S_{g,-r}^X$ for all $g \in G^X$, so we need specify the $S_{g,r}^{X_A}$, &c., only for $0 < r < m$.

If $X_A = \emptyset$ then $S_g^{X_A} := 0$. Otherwise, we define $S_{g,r}^{X_A}$ for $0 < r < m$ by setting

$$S_{g,r}^{X_A} := \begin{cases} \chi_g^{X_A} S_{m,r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_A} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.15}$$

If $X_D = \emptyset$ then $S_g^{X_D} := 0$. If $X_D \neq \emptyset$ then m is even and $m \geq 6$. If $m = 6$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{6,r} + \chi_g^{X_D} S_{6,6-r} & \text{if } r = 1, 5 \mod 6, \\ \check{\chi}_g^{X_D} S_{6,r} & \text{if } r = 3 \mod 6. \end{cases} \tag{B.16}$$

If $m > 6$ and $m = 2 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} 0 & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} + \chi_g^{X_D} S_{m,m-r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.17}$$

If $m > 6$ and $m = 0 \mod 4$ then set

$$S_{g,r}^{X_D} := \begin{cases} \chi_g^{X_D} S_{m,m-r} & \text{if } r = 0 \mod 2, \\ \bar{\chi}_g^{X_D} S_{m,r} & \text{if } r = 1 \mod 2. \end{cases} \tag{B.18}$$

If $X_E = \emptyset$ then $S_g^{X_E} := 0$. Otherwise, m is 12 or 18 or 30. In case $m = 12$ define $S_{g,r}^{X_E}$ for $0 < r < 12$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{12,1} + S_{12,7}) & \text{if } r \in \{1, 7\}, \\ \bar{\chi}_g^{X_E}(S_{12,5} + S_{12,11}) & \text{if } r \in \{5, 11\}, \\ \chi_g^{X_E}(S_{12,4} + S_{12,8}) & \text{if } r \in \{4, 8\}, \\ 0 & \text{else.} \end{cases} \tag{B.19}$$

In case $m = 18$ define $S_{g,r}^{X_E}$ for $0 < r < 18$ by setting

$$S_{g,r}^{X_E} = \begin{cases} \bar{\chi}_g^{X_E}(S_{18,r} + S_{18,18-r}) & \text{if } r \in \{1, 5, 7, 11, 13, 17\}, \\ \bar{\chi}_g^{X_E} S_{18,9} & \text{if } r \in \{3, 15\}, \\ \bar{\chi}_g^{X_E}(S_{18,3} + S_{18,9} + S_{18,15}) & \text{if } r = 9, \\ 0 & \text{else.} \end{cases} \tag{B.20}$$

In case $m = 30$ define $S^{X_E}_{g,r}$ for $0 < r < 30$ by setting

$$S^{X_E}_{g,r} = \begin{cases} \bar{\chi}^{X_E}_g(S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29}) & \text{if } r \in \{1, 11, 19, 29\}, \\ \bar{\chi}^{X_E}_g(S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23}) & \text{if } r \in \{7, 13, 17, 23\}, \\ 0 & \text{else.} \end{cases} \tag{B.21}$$

B.3 Explicit Prescriptions

Here we give explicit expressions for all the umbral McKay-Thompson series H^X_g. Most of these appeared first in [15,16]. The expressions in §§B.3.3, B.3.4, B.3.7, B.3.14 are taken from [26]. The expressions in §§B.3.11, B.3.15, B.3.19, B.3.23 are taken from [14]. The expressions for H^X_g with $X = E^3_8$ appeared first in [25]. The expression for $H^{(6+3)}_{2B,1}$ in §B.3.6, and the expressions for $H^{(12+4)}_{4A,r}$ and $H^{(12+4)}_{8AB,r}$ in §B.3.13, appear here for the first time.

The labels for conjugacy classes in G^X are as in §A.2.

B.3.1 $\ell = 2$, $X = A^{24}_1$

We have $G^{(2)} = G^X \simeq M_{24}$ and $m^X = 2$. So for $g \in M_{24}$, the associated umbral McKay-Thompson series $H^{(2)}_g = (H^{(2)}_{g,r})$ is a 4-vector-valued function, with components indexed by $r \in \mathbb{Z}/4\mathbb{Z}$, satisfying $H^{(2)}_{g,r} = -H^{(2)}_{g,-r}$, and in particular, $H^{(2)}_{g,r} = 0$ for $r = 0 \mod 2$. So it suffices to specify the $H^{(2)}_{g,1}$ explicitly.

Define $H^{(2)}_g = (H^{(2)}_{g,r})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\varphi^{(2)}_1(\tau, z) = -24\mu_{2,0}(\tau, z) + \sum_{r \mod 4} H^{(2)}_{e,r}(\tau)\theta_{2,r}(\tau, z), \tag{B.22}$$

where

$$\varphi^{(2)}_1(\tau, z) := 4\left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right). \tag{B.23}$$

More generally, for $g \in G^{(2)}$ define

$$H^{(2)}_{g,1}(\tau) := \frac{\bar{\chi}^{(2)}_g}{24} H^{(2)}_{e,1}(\tau) - F^{(2)}_g(\tau)\frac{1}{S_{2,1}(\tau)}, \tag{B.24}$$

where $\bar{\chi}^{(2)}_g$ and $F^{(2)}_g$ are as specified in Table 36. Note that $\bar{\chi}^{(2)}_g = \bar{\chi}^{X_A}_g$, the latter appearing in Table 15. Also, $S_{2,1}(\tau) = \eta(\tau)^3$.

The functions $f_{23,a}$ and $f_{23,b}$ in Table 36 are cusp forms of weight two for $\Gamma_0(23)$, defined by

$$\begin{aligned} f_{23,a}(\tau) &:= \frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 3\eta(\tau)^2\eta(23\tau)^2 + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta(2\tau)^2\eta(46\tau)^2, \\ f_{23,b}(\tau) &:= \eta(\tau)^2\eta(23\tau)^2. \end{aligned} \tag{B.25}$$

Note that the definition of $F^{(2)}_g$ appearing here for $g \in 23A \cup 23B$ corrects errors in [11,12].

Table 36: Character Values and Weight Two Forms for $\ell = 2$, $X = A_1^{24}$

$[g]$	$\bar{\chi}_g^{(2)}$	$F_g^{(2)}(\tau)$
$1A$	24	0
$2A$	8	$16\Lambda_2(\tau)$
$2B$	0	$2\eta(\tau)^8\eta(2\tau)^{-4}$
$3A$	6	$6\Lambda_3(\tau)$
$3B$	0	$2\eta(\tau)^6\eta(3\tau)^{-2}$
$4A$	0	$2\eta(2\tau)^8\eta(4\tau)^{-4}$
$4B$	4	$4(-\Lambda_2(\tau)+\Lambda_4(\tau))$
$4C$	0	$2\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^{-2}$
$5A$	4	$2\Lambda_5(\tau)$
$6A$	2	$2(-\Lambda_2(\tau)-\Lambda_3(\tau)+\Lambda_6(\tau))$
$6B$	0	$2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^{-2}$
$7AB$	3	$\Lambda_7(\tau)$
$8A$	2	$-\Lambda_4(\tau)+\Lambda_8(\tau)$
$10A$	0	$2\eta(\tau)^3\eta(2\tau)\eta(5\tau)\eta(10\tau)^{-1}$
$11A$	2	$2(\Lambda_{11}(\tau)-11\eta(\tau)^2\eta(11\tau)^2)/5$
$12A$	0	$2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3\eta(2\tau)^{-1}\eta(3\tau)^{-1}\eta(12\tau)^{-2}$
$12B$	0	$2\eta(\tau)^4\eta(4\tau)\eta(6\tau)\eta(2\tau)^{-1}\eta(12\tau)^{-1}$
$14AB$	1	$(-\Lambda_2(\tau)-\Lambda_7(\tau)+\Lambda_{14}(\tau)-14\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau))/3$
$15AB$	1	$(-\Lambda_3(\tau)-\Lambda_5(\tau)+\Lambda_{15}(\tau)-15\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau))/4$
$21AB$	0	$(7\eta(\tau)^3\eta(7\tau)^3\eta(3\tau)^{-1}\eta(21\tau)^{-1}-\eta(\tau)^6\eta(3\tau)^{-2})/3$
$23AB$	1	$(\Lambda_{23}(\tau)-23f_{23,a}(\tau)-69f_{23,b}(\tau))/11$

B.3.2 $\ell = 3$, $X = A_2^{12}$

We have $G^{(3)} = G^X \simeq 2.M_{12}$ and $m^X = 3$. So for $g \in 2.M_{12}$, the associated umbral McKay-Thompson series $H_g^{(3)} = (H_{g,r}^{(3)})$ is a 6-vector-valued function, with components indexed by $r \in \mathbb{Z}/6\mathbb{Z}$, satisfying $H_{g,r}^{(3)} = -H_{g,-r}^{(3)}$, and in particular, $H_{g,r}^{(3)} = 0$ for $r = 0 \mod 3$. So it suffices to specify the $H_{g,1}^{(3)}$ and $H_{g,2}^{(3)}$ explicitly.

Define $H_g^{(3)} = (H_{g,r}^{(3)})$ for $g = e$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi_1^{(3)}(\tau,z) = -12\mu_{3,0}(\tau,z) + \sum_{r \mod 6} H_{e,r}^{(3)}(\tau)\theta_{3,r}(\tau,z), \tag{B.26}$$

where

$$\varphi_1^{(3)}(\tau,z) := 2\left(\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}\right). \tag{B.27}$$

More generally, for $g \in G^{(3)}$ define

$$H^{(3)}_{g,1}(\tau) := \frac{\bar{\chi}^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g + F^{(3)}_{zg}\right)\frac{1}{S_{3,1}(\tau)}, \tag{B.28}$$

$$H^{(3)}_{g,2}(\tau) := \frac{\chi^{(3)}_g}{12} H^{(3)}_{e,1}(\tau) + \frac{1}{2}\left(F^{(3)}_g - F^{(3)}_{zg}\right)\frac{1}{S_{3,2}(\tau)}, \tag{B.29}$$

where $\chi^{(3)}_g$ and $F^{(3)}_g$ are as specified in Table 37, and z is the non-trivial central element of $G^{(3)}$. The action of $g \mapsto zg$ on conjugacy classes can be read off Table 37, for the horizontal lines indicate the sets $[g] \cup [zg]$.

Note the eta product identities, $S_{3,1}(\tau) = \eta(2\tau)^5/\eta(4\tau)^2$, and $S_{3,2}(\tau) = 2\eta(\tau)^2\eta(4\tau)^2/\eta(2\tau)$. Note also that $\bar{\chi}^{(3)}_g = \bar{\chi}^{X_A}_g$ and $\chi^{(3)}_g = \chi^{X_A}_g$, the latter appearing in Table 16.

Table 37: Character Values and Weight Two Forms for $\ell = 3$, $X = A_2^{12}$

$[g]$	$\bar{\chi}^{(3)}_g$	$\chi^{(3)}_g$	$F^{(3)}_g(\tau)$
1A	12	12	0
2A	12	−12	0
4A	0	0	$-2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
2B	4	4	$-16\Lambda_2(\tau)$
2C	4	−4	$16\Lambda_2(\tau) - \frac{16}{3}\Lambda_4(\tau)$
3A	3	3	$-6\Lambda_3(\tau)$
6A	3	−3	$-9\Lambda_2(\tau) - 2\Lambda_3(\tau) + 3\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau)$
3B	0	0	$8\Lambda_3(\tau) - 2\Lambda_9(\tau) + 2\,\eta^6(\tau)/\eta^2(3\tau)$
6B	0	0	$-2\eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)$
4B	0	0	$-2\eta(2\tau)^8/\eta(4\tau)^4$
4C	4	0	$-8\Lambda_4(\tau)/3$
5A	2	2	$-2\Lambda_5(\tau)$
10A	2	−2	$\sum_{d\mid 20} c_{10A}(d)\Lambda_d(\tau) + \frac{20}{3}\eta(2\tau)^2\eta(10\tau)^2$
12A	0	0	$-2\eta(\tau)\eta(2\tau)^5\eta(3\tau)/\eta(4\tau)^2\eta(6\tau)$
6C	1	1	$2(\Lambda_2(\tau) + \Lambda_3(\tau) - \Lambda_6(\tau))$
6D	1	−1	$-5\Lambda_2(\tau) - 2\Lambda_3(\tau) + \frac{5}{3}\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau)$
8AB	0	0	$-2\eta(2\tau)^4\eta(4\tau)^2/\eta(8\tau)^2$
8CD	2	0	$-2\Lambda_2(\tau) + \frac{5}{3}\Lambda_4(\tau) - \Lambda_8(\tau)$
20AB	0	0	$-2\eta(2\tau)^7\eta(5\tau)/\eta(\tau)\eta(4\tau)^2\eta(10\tau)$
11AB	1	1	$-\frac{2}{5}\Lambda_{11}(\tau) - \frac{33}{5}\eta(\tau)^2\eta(11\tau)^2$
22AB	1	−1	$\sum_{d\mid 44} c_g(d)\Lambda_d(\tau) - \frac{11}{5}\sum_{d\mid 4} c'_g(d)\eta(d\tau)^2\eta(11d\tau)^2 + \frac{22}{3}f_{44}(\tau)$

The function f_{44} is the unique new cusp form of weight 2 for $\Gamma_0(44)$, normalized so that $f_{44}(\tau) =$

$q + O(q^3)$ as $\Im(\tau) \to \infty$. The coefficients $c_g(d)$ and $c'_g(d)$ for $g \in 10A \cup 22A \cup 22B$ are given by

$$c_{10A}(2) = -5,\ c_{10A}(4) = -\frac{5}{3},\ c_{10A}(5) = -\frac{2}{3},\ c_{10A}(10) = 1,\ c_{10A}(20) = -\frac{1}{3}, \tag{B.30}$$

$$c_{22AB}(2) = -\frac{11}{5},\ c_{22AB}(4) = \frac{11}{5},\ c_{22AB}(11) = -\frac{2}{15},\ c_{22AB}(22) = \frac{1}{5},\ c_{22AB}(44) = -\frac{1}{15}, \tag{B.31}$$

$$c'_{22AB}(1) = 1,\ c'_{22AB}(2) = 4,\ c'_{22AB}(4) = 8. \tag{B.32}$$

B.3.3 $\ell = 4$, $X = A_3^8$

We have $m^X = 4$, so the umbral McKay-Thompson series $H_g^{(4)} = (H_{g,r}^{(4)})$ associated to $g \in G^{(4)}$ is an 8-vector-valued function, with components indexed by $r \in \mathbb{Z}/8\mathbb{Z}$.

Define $H_g^{(4)} = (H_{g,r}^{(4)})$ for $g \in G^{(4)}$, $g \notin 4C$, by requiring that

$$\psi_g^{(4)}(\tau, z) = -\chi_g^{(4)} \mu_{4,0}^0(\tau, z) - \bar{\chi}_g^{(4)} \mu_{4,0}^1(\tau, z) + \sum_{r \mod 8} H_{g,r}^{(4)}(\tau) \theta_{4,r}(\tau, z), \tag{B.33}$$

where $\chi_g^{(4)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(4)} := \bar{\chi}_g^{X_A}$ (cf. Table 17), and the $\psi_g^{(4)}$ are meromorphic Jacobi forms of weight 1 and index 4 given explicitly in Table 38.

Table 38: Character Values and Meromorphic Jacobi Forms for $\ell = 4$, $X = A_3^8$

$[g]$	$\chi_g^{(4)}$	$\bar{\chi}_g^{(4)}$	$\psi_g^{(4)}(\tau, z)$
1A	8	8	$2i\theta_1(\tau, 2z)^3 \theta_1(\tau, z)^{-4} \eta(\tau)^3$
2A	-8	8	$2i\theta_1(\tau, 2z)^3 \theta_2(\tau, z)^{-4} \eta(\tau)^3$
2B	0	0	$-2i\theta_1(\tau, 2z)^3 \theta_1(\tau, z)^{-2} \theta_2(\tau, z)^{-2} \eta(\tau)^3$
4A	0	0	$-2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)^2 \theta_2(2\tau, 2z)^{-2} \eta(2\tau)^2 \eta(\tau)^{-1}$
4B	0	0	$-2i\theta_1(2\tau, 2z)\theta_3(2\tau, 2z)^2 \theta_4(2\tau, 2z)\eta(2\tau)^2 \eta(\tau)^{-2} \eta(4\tau)^{-2}$
2C	0	4	$2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)^2 \theta_1(\tau, z)^{-2} \theta_2(\tau, z)^{-2} \eta(\tau)^3$
3A	2	2	$2i\theta_1(3\tau, 6z)\theta_1(\tau, z)^{-1} \theta_1(3\tau, 3z)^{-1} \eta(\tau)^3$
6A	-2	2	$-2i\theta_1(3\tau, 6z)\theta_2(\tau, z)^{-1} \theta_2(3\tau, 3z)^{-1} \eta(\tau)^3$
6BC	0	0	cf. (B.34)
8A	0	0	$-2i\theta_1(\tau, 2z)\theta_2(2\tau, 4z)\theta_2(4\tau, 4z)^{-1} \eta(\tau)\eta(4\tau)\eta(2\tau)^{-1}$
4C	0	2	$2i\theta_1(\tau, 2z)\theta_2(2\tau, 4z)\theta_1(2\tau, 2z)^{-2} \eta(2\tau)^7 \eta(\tau)^{-3} \eta(4\tau)^{-2}$
7AB	1	1	cf. (B.34)
14AB	-1	1	cf. (B.34)

$$\begin{aligned}
\psi^{(4)}_{6BC} &:= \left(\theta_1(\tau,z+\tfrac{1}{3})\theta_1(\tau,z+\tfrac{1}{6})-\theta_1(\tau,z-\tfrac{1}{3})\theta_1(\tau,z-\tfrac{1}{6})\right)\frac{-i\theta_1(3\tau,6z)}{\theta_1(3\tau,3z)\theta_2(3\tau,3z)}\eta(3\tau)\\
\psi^{(4)}_{7AB} &:= \left(\prod_{j=1}^{3}\theta_1(\tau,2z+\tfrac{j^2}{7})\theta_1(\tau,z-\tfrac{j^2}{7})+\prod_{j=1}^{3}\theta_1(\tau,2z-\tfrac{j^2}{7})\theta_1(\tau,z+\tfrac{j^2}{7})\right)\frac{-i}{\theta_1(7\tau,7z)}\frac{\eta(7\tau)}{\eta(\tau)^4}\\
\psi^{(4)}_{14AB} &:= \left(\prod_{j=1}^{3}\theta_1(\tau,2z+\tfrac{j^2}{7})\theta_2(\tau,z-\tfrac{j^2}{7})+\prod_{j=1}^{3}\theta_1(\tau,2z-\tfrac{j^2}{7})\theta_2(\tau,z+\tfrac{j^2}{7})\right)\frac{i}{\theta_2(7\tau,7z)}\frac{\eta(7\tau)}{\eta(\tau)^4}
\end{aligned} \tag{B.34}$$

For use later on, note that $\psi^{(4)}_{1A} = -2\Psi_{1,1}\varphi^{(4)}_1$, where

$$\varphi^{(4)}_1(\tau,z) := \frac{\theta_1(\tau,2z)^2}{\theta_1(\tau,z)^2}. \tag{B.35}$$

B.3.4 $\ell = 5$, $X = A_4^6$

We have $m^X = 5$, so the umbral McKay-Thompson series $H^{(5)}_g = (H^{(5)}_{g,r})$ associated to $g \in G^{(5)}$ is a 10-vector-valued function, with components indexed by $r \in \mathbb{Z}/10\mathbb{Z}$.

Define $H^{(5)}_g = (H^{(5)}_{g,r})$ for $g \in G^{(5)}$, $g \notin 5A \cup 10A$, by requiring that

$$\psi^{(5)}_g(\tau,z) = -\chi^{(5)}_g\mu^0_{5,0}(\tau,z) - \bar{\chi}^{(5)}_g\mu^1_{5,0}(\tau,z) + \sum_{r \mod 10} H^{(5)}_{g,r}(\tau)\theta_{5,r}(\tau,z), \tag{B.36}$$

where $\chi^{(5)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(5)}_g := \bar{\chi}^{X_A}_g$ (cf. Table 18), and the $\psi^{(5)}_g$ are meromorphic Jacobi forms of weight 1 and index 5 given explicitly in Table 39.

Table 39: Character Values and Meromorphic Jacobi Forms for $\ell = 5$, $X = A_4^6$

$[g]$	$\chi^{(5)}_g$	$\bar{\chi}^{(5)}_g$	$\psi^{(5)}_g(\tau,z)$
1A	6	6	$2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(\tau,z)^{-3}\eta(\tau)^3$
2A	-6	6	$-2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_2(\tau,z)^{-3}\eta(\tau)^3$
2B	-2	2	$-2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(\tau,z)^{-1}\theta_2(\tau,z)^{-2}\eta(\tau)^3$
2C	2	2	$2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_1(\tau,z)^{-2}\theta_2(\tau,z)^{-1}\eta(\tau)^3$
3A	0	0	$-2i\theta_1(\tau,2z)\theta_1(\tau,3z)\theta_1(3\tau,3z)^{-1}\eta(3\tau)$
6A	0	0	$-2i\theta_1(\tau,2z)\theta_2(\tau,3z)\theta_2(3\tau,3z)^{-1}\eta(3\tau)$
4AB	0	0	cf. (B.37)
4CD	0	2	cf. (B.37)
12AB	0	0	cf. (B.37)

$$\begin{aligned}
\psi^{(5)}_{4AB}(\tau,z) &:= -i\theta_2(\tau,2z)\frac{\theta_1(\tau,z+\frac{1}{4})\theta_1(\tau,3z+\frac{1}{4})-\theta_1(\tau,z-\frac{1}{4})\theta_1(\tau,3z-\frac{1}{4})}{\theta_2(2\tau,2z)^2}\frac{\eta(2\tau)^2}{\eta(\tau)}\\
\psi^{(5)}_{4CD}(\tau,z) &:= -i\theta_2(\tau,2z)\frac{\theta_1(\tau,z+\frac{1}{4})\theta_1(\tau,3z-\frac{1}{4})+\theta_1(\tau,z-\frac{1}{4})\theta_1(\tau,3z+\frac{1}{4})}{\theta_1(2\tau,2z)\theta_2(2\tau,2z)}\frac{\eta(2\tau)^2}{\eta(\tau)}\\
\psi^{(5)}_{12AB}(\tau,z) &:= i\frac{\theta_2(\tau,2z)}{\theta_2(6\tau,6z)}\left(\theta_1(\tau,z+\tfrac{1}{12})\theta_1(\tau,z+\tfrac{1}{4})\theta_1(\tau,z+\tfrac{5}{12})\theta_1(\tau,3z-\tfrac{1}{4})\right.\\
&\qquad\left.-\,\theta_1(\tau,z-\tfrac{1}{12})\theta_1(\tau,z-\tfrac{1}{4})\theta_1(\tau,z-\tfrac{5}{12})\theta_1(\tau,3z+\tfrac{1}{4})\right)\frac{\eta(6\tau)}{\eta(\tau)^3}
\end{aligned} \tag{B.37}$$

For $g\in 5A$ use the formulas of §B.3.20 to define

$$H^{(5)}_{5A,r}(\tau) := H^{(25)}_{1A,r}(\tau/5) - H^{(25)}_{1A,10-r}(\tau/5) + H^{(25)}_{1A,10+r}(\tau/5) - H^{(25)}_{1A,20-r}(\tau/5) + H^{(25)}_{1A,20+r}(\tau/5). \tag{B.38}$$

For $g\in 10A$ set $H^{(5)}_{10A,r}(\tau) := -(-1)^r H^{(5)}_{5A,r}(\tau)$.

For use later on we note that $\psi^{(5)}_{1A} = -2\Psi_{1,1}\varphi^{(5)}_1$, where

$$\varphi^{(5)}_1(\tau,z) := \frac{\theta_1(\tau,3z)}{\theta_1(\tau,z)}. \tag{B.39}$$

B.3.5 $\ell = 6$, $X = A_5^4 D_4$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6)}_g = (H^{(6)}_{g,r})$ associated to $g\in G^{(6)}$ is a 12-vector-valued function with components indexed by $r\in\mathbb{Z}/12\mathbb{Z}$. We have $H^{(6)}_{g,r} = -H^{(6)}_{g,-r}$, so it suffices to specify the $H^{(6)}_{g,r}$ for $r\in\{1,2,3,4,5\}$.

To define $H^{(6)}_g = (H^{(6)}_{g,r})$ for $g=e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau,z)\varphi^{(6)}_1(\tau,z) = -24\mu_{6,0}(\tau,z) + \sum_{r \mod 12} h_r(\tau)\theta_{6,r}(\tau,z), \tag{B.40}$$

where

$$\varphi^{(6)}_1(\tau,z) := \varphi^{(2)}_1(\tau,z)\varphi^{(5)}_1(\tau,z) - \varphi^{(3)}_1(\tau,z)\varphi^{(4)}_1(\tau,z). \tag{B.41}$$

(Cf. (B.23), (B.27), (B.35), (B.39).) Now define the $H^{(6)}_{1A,r}$ by setting

$$\begin{aligned}
H^{(6)}_{1A,1}(\tau) &:= \frac{1}{24}\left(5h_1(\tau)+h_5(\tau)\right),\\
H^{(6)}_{1A,2}(\tau) &:= \frac{1}{6}h_2(\tau),\\
H^{(6)}_{1A,3}(\tau) &:= \frac{1}{4}h_3(\tau),\\
H^{(6)}_{1A,4}(\tau) &:= \frac{1}{6}h_4(\tau),\\
H^{(6)}_{1A,5}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+5h_5(\tau)\right).
\end{aligned} \tag{B.42}$$

Define $H^{(6)}_{2A,r}$ by requiring

$$H^{(6)}_{2A,r}(\tau) := -(-1)^r H^{(6)}_{1A,r}(\tau). \tag{B.43}$$

For the remaining g, recall (B.13). The $H^{(6)}_{g,r}$ for $g \notin 1A \cup 2A$ are defined as follows for $r = 2$ and $r = 4$, noting that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$.

$$\begin{aligned} H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4C,r}(\tau/2) \\ H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{4B,r}(\tau/2) \\ H^{(6)}_{3A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6C,r}(\tau/2) \\ H^{(6)}_{6A,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{6D,r}(\tau/2) \\ H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{r^2}{24}]H^{(3)}_{8CD,r}(\tau/2) \end{aligned} \tag{B.44}$$

For the $H^{(6)}_{g,3}$ we define

$$\begin{aligned} H^{(6)}_{2B,3}(\tau), H^{(6)}_{4A,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{6A,1}(\tau/3), \\ H^{(6)}_{3A,3}(\tau), H^{(6)}_{6A,3}(\tau) &:= 0, \\ H^{(6)}_{8AB,3}(\tau) &:= -[-\tfrac{9}{24}]H^{(2)}_{12A,1}(\tau/3). \end{aligned} \tag{B.45}$$

Noting that $H^{(2)}_{g,5} = H^{(2)}_{g,1}$ and $H^{(3)}_{g,5} = -H^{(3)}_{g,1}$, the $H^{(6)}_{g,1}$ and $H^{(6)}_{g,5}$ are defined for $o(g) \neq 0 \mod 3$ by setting

$$\begin{aligned} H^{(6)}_{2B,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4C,r}(\tau/2)\right) \\ H^{(6)}_{4A,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{6A,r}(\tau/3) + H^{(3)}_{4B,r}(\tau/2)\right) \\ H^{(6)}_{8AB,r}(\tau) &:= [-\tfrac{1}{24}]\frac{1}{2}\left(H^{(2)}_{12A,r}(\tau/3) + H^{(3)}_{8CD,r}(\tau/2)\right) \end{aligned} \tag{B.46}$$

It remains to specify the $H^{(6)}_{g,r}$ when $g \in 3A \cup 6A$ and r is 1 or 5. These cases are determined by using the formulas of §B.3.17 to set

$$\begin{aligned} H^{(6)}_{3A,1}(\tau), H^{(6)}_{6A,1}(\tau) &:= H^{(18)}_{1A,1}(3\tau) - H^{(18)}_{1A,11}(3\tau) + H^{(18)}_{1A,13}(3\tau), \\ H^{(6)}_{3A,5}(\tau), H^{(6)}_{6A,5}(\tau) &:= H^{(18)}_{1A,5}(3\tau) - H^{(18)}_{1A,7}(3\tau) + H^{(18)}_{1A,17}(3\tau). \end{aligned} \tag{B.47}$$

B.3.6 $\ell = 6 + 3$, $X = D_4^6$

We have $m^X = 6$, so the umbral McKay-Thompson series $H^{(6+3)}_g = (H^{(6+3)}_{g,r})$ associated to $g \in G^{(6+3)}$ is a 12-vector-valued function with components indexed by $r \in \mathbb{Z}/12\mathbb{Z}$. In addition to the identity $H^{(6+3)}_{g,r} = -H^{(6+3)}_{g,-r}$, we have $H^{(6+3)}_{g,r} = 0$ for $r = 0 \mod 2$. Thus it suffices to specify the $H^{(6+3)}_{g,r}$ for $r \in \{1, 3, 5\}$.

Recall (B.13). For $r = 1$, define

$$\begin{aligned}
H^{(6+3)}_{1A,1}(\tau), H^{(6+3)}_{3A,1}(\tau) &:= H^{(6)}_{1A,1}(\tau) + H^{(6)}_{1A,5}(\tau),\\
H^{(6+3)}_{2A,1}(\tau), H^{(6+3)}_{6A,1}(\tau) &:= H^{(6)}_{2B,1}(\tau) + H^{(6)}_{2B,5}(\tau),\\
H^{(6+3)}_{3B,1}(\tau) &:= H^{(6)}_{3A,1}(\tau) + H^{(6)}_{3A,5}(\tau),\\
H^{(6+3)}_{3C,1}(\tau) &:= -2\frac{\eta(\tau)^2}{\eta(3\tau)},\\
H^{(6+3)}_{4A,1}(\tau), H^{(6+3)}_{12A,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) + H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{5A,1}(\tau), H^{(6+3)}_{15A,1}(\tau) &:= [-\tfrac{1}{24}]H^{(2)}_{15AB,1}(\tau/3),\\
H^{(6+3)}_{2C,1}(\tau) &:= H^{(6)}_{4A,1}(\tau) - H^{(6)}_{4A,5}(\tau),\\
H^{(6+3)}_{4B,1}(\tau) &:= H^{(6)}_{8AB,1}(\tau) - H^{(6)}_{8AB,5}(\tau),\\
H^{(6+3)}_{6B,1}(\tau) &:= H^{(6)}_{6A,1}(\tau) - H^{(6)}_{6A,5}(\tau),\\
H^{(6+3)}_{6C,1}(\tau) &:= -2\frac{\eta(2\tau)\,\eta(3\tau)}{\eta(6\tau)}.
\end{aligned} \tag{B.48}$$

Then define $H^{(6+3)}_{2B,1}$ by setting

$$H^{(6+3)}_{2B,1}(\tau) := 2H^{(6+3)}_{4B,1}(\tau) + 2\frac{\eta(\tau)^3}{\eta(2\tau)^2}. \tag{B.49}$$

For $r = 3$ set

$$\begin{aligned}
H^{(6+3)}_{1A,3}(\tau) &:= 2H^{(6)}_{1A,3}(\tau),\\
H^{(6+3)}_{3A,3}(\tau) &:= -H^{(6)}_{1A,3}(\tau),\\
H^{(6+3)}_{2A,3}(\tau) &:= 2H^{(6)}_{2B,3}(\tau),\\
H^{(6+3)}_{6A,3}(\tau) &:= -H^{(6)}_{2B,3}(\tau),\\
H^{(6+3)}_{4A,3}(\tau) &:= 2H^{(6)}_{8AB,3}(\tau),\\
H^{(6+3)}_{12A,3}(\tau) &:= -H^{(6)}_{8AB,3}(\tau),\\
H^{(6+3)}_{5A,3}(\tau) &:= -2[-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau),\\
H^{(6+3)}_{15A,3}(\tau) &:= [-\tfrac{9}{24}]H^{(2)}_{15AB,1}(\tau),
\end{aligned} \tag{B.50}$$

and

$$H^{(6+3)}_{3B,3}(\tau), H^{(6+3)}_{3C,3}(\tau), H^{(6+3)}_{2B,3}(\tau), H^{(6+3)}_{2C,3}(\tau), H^{(6+3)}_{4B,3}(\tau), H^{(6+3)}_{6B,3}(\tau), H^{(6+3)}_{6C,3}(\tau) := 0. \tag{B.51}$$

For $r = 5$ define $H^{(6+3)}_{g,5}(\tau) := H^{(6+3)}_{g,1}(\tau)$ for $[g] \in \{1A, 3A, 2A, 6A, 3B, 3C, 4A, 12A, 5A, 15AB\}$, and set $H^{(6+3)}_{g,5}(\tau) := -H^{(6+3)}_{g,1}(\tau)$ for the remaining cases, $[g] \in \{2B, 2C, 4B, 6B, 6C\}$.

B.3.7 $\ell = 7$, $X = A_6^4$

We have $m^X = 7$, so the umbral McKay-Thompson series $H_g^{(7)} = (H_{g,r}^{(7)})$ associated to $g \in G^{(7)} = G^X \simeq SL_2(3)$ is a 14-vector-valued function, with components indexed by $r \in \mathbb{Z}/14\mathbb{Z}$.

Define $H_g^{(7)} = (H_{g,r}^{(7)})$ for $g \in G^{(7)}$ by requiring that

$$\psi_g^{(7)}(\tau, z) = -\chi_g^{(7)}\mu_{7,0}^0(\tau, z) - \bar{\chi}_g^{(7)}\mu_{7,0}^1(\tau, z) + \sum_{r \mod 14} H_{g,r}^{(7)}(\tau)\theta_{7,r}(\tau, z), \tag{B.52}$$

where $\chi_g^{(7)} := \chi_g^{X_A}$ and $\bar{\chi}_g^{(7)} := \bar{\chi}_g^{X_A}$ (cf. Table 21), and the $\psi_g^{(7)}$ are meromorphic Jacobi forms of weight 1 and index 7 given explicitly in Table 40.

Table 40: Character Values and Meromorphic Jacobi Forms for $\ell = 7$, $X = A_6^4$

$[g]$	$\chi_g^{(7)}$	$\bar{\chi}_g^{(7)}$	$\psi_g^{(7)}(\tau, z)$
1A	4	4	$2i\theta_1(\tau, 4z)\theta_1(\tau, z)^{-2}\eta(\tau)^3$
2A	-4	4	$-2i\theta_1(\tau, 4z)\theta_2(\tau, z)^{-2}\eta(\tau)^3$
4A	0	0	$-2i\theta_1(\tau, 4z)\theta_2(2\tau, 2z)^{-1}\eta(2\tau)\eta(\tau)$
3A	1	1	cf. (B.53)
6A	-1	1	cf. (B.53)

$$\begin{aligned}
\psi_{3A}^{(7)}(\tau, z) &:= -i\frac{\theta_1(\tau, 4z + \frac{1}{3})\theta_1(\tau, z - \frac{1}{3}) + \theta_1(\tau, 4z - \frac{1}{3})\theta_1(\tau, z + \frac{1}{3})}{\theta_1(3\tau, 3z)}\eta(3\tau) \\
\psi_{6A}^{(7)}(\tau, z) &:= -i\frac{\theta_1(\tau, 4z + \frac{1}{3})\theta_1(\tau, z - \frac{1}{6}) - \theta_1(\tau, 4z - \frac{1}{3})\theta_1(\tau, z + \frac{1}{6})}{\theta_2(3\tau, 3z)}\eta(3\tau)
\end{aligned} \tag{B.53}$$

For use later on we note that $\psi_{1A}^{(7)} = -2\Psi_{1,1}\varphi_1^{(7)}$, where

$$\varphi_1^{(7)}(\tau, z) := \frac{\theta_1(\tau, 4z)}{\theta_1(\tau, 2z)}. \tag{B.54}$$

B.3.8 $\ell = 8$, $X = A_7^2 D_5^2$

We have $m^X = 8$, so the umbral McKay-Thompson series $H_g^{(8)} = (H_{g,r}^{(8)})$ associated to $g \in G^{(8)}$ is a 16-vector-valued function with components indexed by $r \in \mathbb{Z}/16\mathbb{Z}$. We have $H_{g,r}^{(8)} = -H_{g,-r}^{(8)}$, so it suffices to specify the $H_{g,r}^{(8)}$ for $r \in \{1, 2, 3, 4, 5, 6, 7\}$.

To define $H_g^{(8)} = (H_{g,r}^{(8)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\left(\varphi_1^{(8)}(\tau, z) + \frac{1}{2}\varphi_2^{(8)}(\tau, z)\right) = -24\mu_{8,0}(\tau, z) + \sum_{r \mod 16} h_r(\tau)\theta_{8,r}(\tau, z), \tag{B.55}$$

where

$$\begin{aligned}
\varphi_1^{(8)}(\tau, z) &:= \varphi_1^{(3)}(\tau, z)\varphi_1^{(6)}(\tau, z) - 5\varphi_1^{(4)}(\tau, z)\varphi_1^{(5)}(\tau, z), \\
\varphi_2^{(8)}(\tau, z) &:= \varphi_1^{(4)}(\tau, z)\varphi_1^{(5)}(\tau, z) - \varphi_1^{(8)}(\tau, z).
\end{aligned} \tag{B.56}$$

(Cf. (B.27), (B.35), (B.39), (B.41).) Now define the $H^{(8)}_{1A,r}$ by setting

$$H^{(8)}_{1A,r}(\tau) := \frac{1}{6}h_r(\tau), \tag{B.57}$$

for $r \in \{1,3,4,5,7\}$, and

$$H^{(8)}_{1A,2}(\tau), H^{(8)}_{1A,6}(\tau) := \frac{1}{12}\left(h_2(\tau) + h_6(\tau)\right). \tag{B.58}$$

Define $H^{(8)}_{2A,r}$ for $1 \leq r \leq 7$ by requiring

$$H^{(8)}_{2A,r}(\tau) := -(-1)^r H^{(8)}_{1A,r}(\tau). \tag{B.59}$$

For the remaining g, recall (B.13). The $H^{(8)}_{g,r}$ for $g \in 2B \cup 2C \cup 4A$ are defined as follows for $r \in \{1,3,5,7\}$, noting that $H^{(4)}_{g,7} = H^{(4)}_{g,-1} = -H^{(4)}_{g,1}$, &c.

$$\begin{aligned} H^{(8)}_{2BC,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4C,r}(\tau/2) \\ H^{(8)}_{4A,r}(\tau) &:= [-\tfrac{r^2}{32}]H^{(4)}_{4B,r}(\tau/2) \end{aligned} \tag{B.60}$$

The $H^{(8)}_{2BC,r}$ and $H^{(8)}_{4A,r}$ vanish for $r = 0 \mod 2$.

B.3.9 $\ell = 9$, $X = A_8^3$

We have $m^X = 9$, so for $g \in G^{(9)}$ the associated umbral McKay-Thompson series $H^{(9)}_g = (H^{(9)}_{g,r})$ is a 18-vector-valued function, with components indexed by $r \in \mathbb{Z}/18\mathbb{Z}$, satisfying $H^{(9)}_{g,r} = -H^{(9)}_{g,-r}$, and in particular, $H^{(9)}_{g,r} = 0$ for $r = 0 \mod 9$. So it suffices to specify the $H^{(9)}_{g,r}$ for $r \in \{1,2,3,4,5,6,7,8\}$.

Define $H^{(9)}_g = (H^{(9)}_{g,r})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau,z)\varphi^{(9)}_1(\tau,z) = -3\mu_{9,0}(\tau,z) + \sum_{r \mod 18} H^{(9)}_{e,r}(\tau)\theta_{9,r}(\tau,z), \tag{B.61}$$

where

$$\varphi^{(9)}_1(\tau,z) := \varphi^{(3)}_1(\tau,z)\varphi^{(7)}_1(\tau,z) - \varphi^{(5)}_1(\tau,z)^2. \tag{B.62}$$

(Cf. (B.27), (B.39), (B.54).)

Recall (B.13). The $H^{(9)}_{2B,r}$ are defined for $r \in \{1,2,4,5,7,8\}$ by setting

$$H^{(9)}_{2B,r}(\tau) := [-\tfrac{r^2}{36}]H^{(3)}_{6C,r}(\tau/3), \tag{B.63}$$

where we note that $H^{(3)}_{g,4} = H^{(3)}_{g,-2} = -H^{(3)}_{g,2}$, &c. We determine $H^{(9)}_{2B,3}$ and $H^{(9)}_{2B,6}$ by using §B.3.17 to set

$$H^{(9)}_{2B,r}(\tau) := H^{(18)}_{1A,r}(2\tau) - H^{(18)}_{1A,18-r}(2\tau) \tag{B.64}$$

for $r \in \{3,6\}$.

The $H^{(9)}_{3A,r}$ are defined by the explicit formulas

$$\begin{aligned}
H^{(9)}_{3A,1}(\tau) &:= [-\tfrac{1}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,2}(\tau) &:= [-\tfrac{4}{36}]f^{(9)}_2(\tau/3),\\
H^{(9)}_{3A,3}(\tau) &:= -\theta_{3,3}(\tau,0),\\
H^{(9)}_{3A,4}(\tau) &:= -[-\tfrac{16}{36}]f^{(9)}_2(\tau/3),\\
H^{(9)}_{3A,5}(\tau) &:= -[-\tfrac{25}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,6}(\tau) &:= \theta_{3,0}(\tau,0),\\
H^{(9)}_{3A,7}(\tau) &:= [-\tfrac{13}{36}]f^{(9)}_1(\tau/3),\\
H^{(9)}_{3A,8}(\tau) &:= [-\tfrac{28}{36}]f^{(9)}_2(\tau/3),
\end{aligned} \tag{B.65}$$

where

$$\begin{aligned}
f^{(9)}_1(\tau) &:= -2\frac{\eta(\tau)\eta(12\tau)\eta(18\tau)^2}{\eta(6\tau)\eta(9\tau)\eta(36\tau)},\\
f^{(9)}_2(\tau) &:= \frac{\eta(2\tau)^6\eta(12\tau)\eta(18\tau)^2}{\eta(\tau)\eta(4\tau)^4\eta(6\tau)\eta(9\tau)\eta(36\tau)} - \frac{\eta(\tau)\eta(2\tau)\eta(3\tau)^2}{\eta(4\tau)^2\eta(9\tau)}.
\end{aligned} \tag{B.66}$$

Finally, the $H^{(9)}_{g,r}$ are determined for $g \in 2A \cup 2C \cup 6A$ by setting

$$\begin{aligned}
H^{(9)}_{2A,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{1A,r}(\tau),\\
H^{(9)}_{2C,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{2B,r}(\tau),\\
H^{(9)}_{6A,r}(\tau) &:= (-1)^{r+1}H^{(9)}_{3A,r}(\tau).
\end{aligned} \tag{B.67}$$

B.3.10 $\ell = 10$, $X = A_9^2D_6$

We have $m^X = 10$, so the umbral McKay-Thompson series $H^{(10)}_g = (H^{(10)}_{g,r})$ associated to $g \in G^{(10)}$ is a 20-vector-valued function with components indexed by $r \in \mathbb{Z}/20\mathbb{Z}$. We have $H^{(10)}_{g,r} = -H^{(10)}_{g,-r}$, so it suffices to specify the $H^{(10)}_{g,r}$ for $1 \le r \le 9$.

To define $H^{(10)}_g = (H^{(10)}_{g,r})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau,z)\varphi^{(10)}_1(\tau,z) = -24\mu_{10,0}(\tau,z) + \sum_{r \mod 20} h_r(\tau)\theta_{10,r}(\tau,z), \tag{B.68}$$

where

$$\varphi^{(10)}_1(\tau,z) := 5\varphi^{(4)}_1(\tau,z)\varphi^{(7)}_1(\tau,z) - \varphi^{(5)}_1(\tau,z)\varphi^{(6)}_1(\tau,z). \tag{B.69}$$

(Cf. (B.35), (B.39), (B.41), (B.54).) Now define the $H^{(10)}_{1A,r}$ for r odd by setting

$$\begin{aligned} H^{(10)}_{1A,1}(\tau) &:= \frac{1}{24}\left(3h_1(\tau)+h_9(\tau)\right), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(3h_3(\tau)+h_7(\tau)\right), \\ H^{(10)}_{1A,5}(\tau) &:= \frac{1}{6}h_5(\tau), \\ H^{(10)}_{1A,3}(\tau) &:= \frac{1}{24}\left(h_3(\tau)+3h_7(\tau)\right), \\ H^{(10)}_{1A,9}(\tau) &:= \frac{1}{24}\left(h_1(\tau)+3h_9(\tau)\right). \end{aligned} \tag{B.70}$$

For $r = 0 \mod 2$ set

$$H^{(10)}_{1A,r}(\tau) := \frac{1}{12}h_r(\tau), \tag{B.71}$$

and define $H^{(10)}_{2A,r}$ for $1 \le r \le 9$ by requiring

$$H^{(10)}_{2A,r}(\tau) := -(-1)^r H^{(10)}_{1A,r}(\tau). \tag{B.72}$$

It remains to specify $H^{(10)}_{g,r}$ for $g \in 4A \cup 4B$. For $r = 0 \mod 2$ set

$$H^{(10)}_{4AB,r}(\tau) := 0. \tag{B.73}$$

For r odd, recall (B.13), and define

$$H^{(10)}_{4A,r}(\tau) := [-\tfrac{r^2}{40}]\frac{1}{2}\left(H^{(2)}_{10A,r}(\tau/5) + H^{(5)}_{4CD,r}(\tau/2)\right). \tag{B.74}$$

B.3.11 $\ell = 10+5$, $X = D_6^4$

We have $m^X = 10$, so the umbral McKay-Thompson series $H^{(10+5)}_g = (H^{(10+5)}_{g,r})$ associated to $g \in G^{(10+5)}$ is a 20-vector-valued function with components indexed by $r \in \mathbb{Z}/20\mathbb{Z}$. We have $H^{(10+5)}_{g,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(10+5)}_{g,r}$ for r odd. Observing that $H^{(10+5)}_{g,r} = -H^{(10+5)}_{g,-r}$ we may determine $H^{(10+5)}_g$ by requiring that

$$\psi^{(5/2)}_g(\tau,z) = -2\chi^{(5/2)}_g i\mu_{5/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2\\ r \mod 5}} e(-r/2)H^{(10+5)}_{g,2r}(\tau)\theta_{5/2,r}(\tau,z), \tag{B.75}$$

where $\chi^{(5/2)}_g := \bar{\chi}^{X_D}_g$ as in Table 25, and the $\psi^{(5/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 5/2 defined as follows.

Table 41: Character Values and Meromorphic Jacobi Forms for $\ell = 10 + 5$, $X = D_6^4$

$[g]$	$\bar{\chi}_g^{(5/2)}$	$\psi_g^{(5/2)}(\tau, z)$
1A	4	$2i\theta_1(\tau, 2z)^2\theta_1(\tau, z)^{-3}\eta(\tau)^3$
2A	0	$-2i\theta_1(\tau, 2z)^2\theta_1(\tau, z)^{-1}\theta_2(\tau, z)^{-2}\eta(\tau)^3$
3A	1	$2i\theta_1(3\tau, 6z)\theta_1(\tau, 2z)^{-1}\theta_1(3\tau, 3z)^{-1}\eta(\tau)^3$
2B	2	$2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)\theta_1(\tau, z)^{-2}\theta_2(\tau, z)^{-1}\eta(\tau)^3$
4A	0	$-2i\theta_1(\tau, 2z)\theta_2(\tau, 2z)\theta_2(2\tau, 2z)^{-1}\eta(\tau)\eta(2\tau)$

B.3.12 $\ell = 12$, $X = A_{11}D_7E_6$

We have $m^X = 12$, so the umbral McKay-Thompson series $H_g^{(12)} = (H_{g,r}^{(12)})$ associated to $g \in G^{(12)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. We have $H_{g,r}^{(12)} = -H_{g,-r}^{(12)}$, so it suffices to specify the $H_{g,r}^{(12)}$ for $1 \leq r \leq 11$.

To define $H_e^{(12)} = (H_{e,r}^{(12)})$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-2\Psi_{1,1}(\tau, z)\left(\varphi_1^{(12)}(\tau, z) + \varphi_2^{(12)}(\tau, z)\right) = -24\mu_{12,0}(\tau, z) + \sum_{r \mod 24} h_r(\tau)\theta_{12,r}(\tau, z), \tag{B.76}$$

where

$$\begin{aligned} \varphi_1^{(12)}(\tau, z) &:= 3\varphi_1^{(3)}(\tau, z)\varphi_1^{(10)}(\tau, z) - 8\varphi_1^{(4)}(\tau, z)\varphi_1^{(9)}(\tau, z) + \varphi_1^{(5)}(\tau, z)\varphi_1^{(8)}(\tau, z), \\ \varphi_2^{(12)}(\tau, z) &:= 4\varphi_1^{(4)}(\tau, z)\varphi_1^{(9)}(\tau, z) - \varphi_1^{(5)}(\tau, z)\varphi_1^{(8)}(\tau, z) - \varphi_1^{(12)}(\tau, z). \end{aligned} \tag{B.77}$$

(Cf. (B.27), (B.35), (B.39), (B.54), (B.56), (B.62), (B.69).) Now define the $H_{1A,r}^{(12)}$ for $r \neq 0 \mod 3$ by setting

$$\begin{aligned} H_{1A,1}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_1(\tau) + h_7(\tau)\right), \\ H_{1A,2}^{(12)}(\tau), H_{1A,10}^{(12)}(\tau) &:= \frac{1}{24}\left(h_2(\tau) + h_{10}(\tau)\right), \\ H_{1A,4}^{(12)}(\tau), H_{1A,8}^{(12)}(\tau) &:= \frac{1}{12}\left(h_4(\tau) + h_8(\tau)\right), \\ H_{1A,5}^{(12)}(\tau) &:= \frac{1}{24}\left(3h_5(\tau) + h_{11}(\tau)\right), \\ H_{1A,7}^{(12)}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 3h_7(\tau)\right), \\ H_{1A,11}^{(12)}(\tau) &:= \frac{1}{24}\left(h_5(\tau) + 3h_{11}(\tau)\right). \end{aligned} \tag{B.78}$$

For $r = 0 \mod 3$ set

$$H_{1A,r}^{(12)}(\tau) := \frac{1}{12}h_r(\tau), \tag{B.79}$$

and define $H_{2A,r}^{(12)}$ by requiring

$$H_{2A,r}^{(12)}(\tau) := -(-1)^r H_{1A,r}^{(12)}(\tau). \tag{B.80}$$

B.3.13 $\ell = 12+4$, $X = E_6^4$

We have $m^X = 12$, so the umbral McKay-Thompson series $H_g^{(12+4)} = (H_{g,r}^{(12+4)})$ associated to $g \in G^{(12+4)}$ is a 24-vector-valued function with components indexed by $r \in \mathbb{Z}/24\mathbb{Z}$. In addition to the identity $H_{g,r}^{(12+4)} = -H_{g,-r}^{(12+4)}$, we have $H_{g,r}^{(12+4)} = 0$ for $r \in \{2,3,6,9,10\}$, $H_{g,1}^{(12+4)} = H_{g,7}^{(12+4)}$, $H_{g,4}^{(12+4)} = H_{g,8}^{(12+4)}$, and $H_{g,5}^{(12+4)} = H_{g,11}^{(12+4)}$. Thus it suffices to specify the $H_{g,1}^{(12+4)}$, $H_{g,4}^{(12+4)}$ and $H_{g,5}^{(12+4)}$.

Recall (B.13). Also, set $S_1^{E_6}(\tau) := S_{12,1}(\tau) + S_{12,7}(\tau)$, and $S_5^{E_6}(\tau) := S_{12,5}(\tau) + S_{12,11}(\tau)$. For $r = 1$ define

$$\begin{aligned}
H_{1A,1}^{(12+4)}(\tau) &:= H_{1A,1}^{(12)}(\tau) + H_{1A,7}^{(12)}(\tau),\\
H_{2B,1}^{(12+4)}(\tau) &:= [-\tfrac{1}{48}]\left(H_{8AB,1}^{(6)}(\tau/2) - H_{8AB,5}^{(6)}(\tau/2)\right),\\
H_{4A,1}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(-2\frac{\eta(2\tau)^8}{\eta(\tau)^4}S_1^{E_6}(\tau) + 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4}S_5^{E_6}(\tau)\right),\\
H_{3A,1}^{(12+4)}(\tau) &:= [-\tfrac{1}{48}]\left(H_{3A,1}^{(6)}(\tau/2) - H_{3A,5}^{(6)}(\tau/2)\right),\\
H_{8AB,1}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(-2F_{8AB,1}^{(12+4)}(\tau)S_1^{E_6}(\tau) + 12F_{8AB,5}^{(12+4)}(\tau/2)S_5^{E_6}(\tau)\right).
\end{aligned} \tag{B.81}$$

In the expression for $g \in 8AB$, we write $F_{8AB,1}^{(12+4)}$ for the unique modular form of weight 2 for $\Gamma_0(32)$ such that

$$F_{8AB,1}^{(12+4)}(\tau) = 1 + 12q + 4q^2 - 24q^5 - 16q^6 - 8q^8 + O(q^9), \tag{B.82}$$

and we write $F_{8AB,5}^{(12+4)}$ for the unique modular form of weight 2 for $\Gamma_0(64)$ such that

$$F_{8AB,5}^{(12+4)}(\tau) = 3q + 4q^3 + 6q^5 - 8q^7 - 9q^9 + 12q^{11} - 18q^{13} - 24q^{15} + O(q^{17}). \tag{B.83}$$

For $r = 4$ define

$$\begin{aligned}
H_{1A,4}^{(12+4)}(\tau) &:= H_{1A,4}^{(12)}(\tau) + H_{1A,8}^{(12)}(\tau),\\
H_{3A,4}^{(12+4)}(\tau) &:= H_{3A,2}^{(6)}(\tau/2) + H_{3A,4}^{(6)}(\tau/2),
\end{aligned} \tag{B.84}$$

and set $H_{g,4}^{(12+4)}(\tau) := 0$ for $g \in 2B \cup 4A \cup 8AB$.

For $r = 5$ define

$$\begin{aligned}
H_{1A,5}^{(12+4)}(\tau) &:= H_{1A,5}^{(12)}(\tau) + H_{1A,11}^{(12)}(\tau),\\
H_{2B,5}^{(12+4)}(\tau) &:= [-\tfrac{25}{48}]\left(H_{8AB,5}^{(6)}(\tau/2) - H_{8AB,1}^{(6)}(\tau/2)\right),\\
H_{4A,5}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(2\frac{\eta(2\tau)^8}{\eta(\tau)^4}S_5^{E_6}(\tau) - 8\frac{\eta(\tau)^4\eta(4\tau)^4}{\eta(2\tau)^4}S_1^{E_6}(\tau)\right),\\
H_{3A,5}^{(12+4)}(\tau) &:= [-\tfrac{25}{48}]\left(H_{3A,5}^{(6)}(\tau/2) - H_{3A,1}^{(6)}(\tau/2)\right),\\
H_{8AB,5}^{(12+4)}(\tau) &:= \frac{1}{S_1^{E_6}(\tau)^2 - S_5^{E_6}(\tau)^2}\left(2F_{8AB,1}^{(12+4)}(\tau)S_5^{E_6}(\tau) - 12F_{8AB,5}^{(12+4)}(\tau/2)S_1^{E_6}(\tau)\right).
\end{aligned} \tag{B.85}$$

Finally, define $H^{(12+4)}_{g,r}$ for $g \in 2A \cup 6A$ by setting

$$\begin{aligned} H^{(12+4)}_{2A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{1A,r}(\tau), \\ H^{(12+4)}_{6A,r}(\tau) &:= -(-1)^r H^{(12+4)}_{3A,r}(\tau). \end{aligned} \tag{B.86}$$

B.3.14 $\ell = 13$, $X = A_{12}^2$

We have $m^X = 13$, so the umbral McKay-Thompson series $H^{(13)}_g = (H^{(13)}_{g,r})$ associated to $g \in G^{(13)} = G^X \simeq \mathbb{Z}/4\mathbb{Z}$ is a 26-vector-valued function, with components indexed by $r \in \mathbb{Z}/26\mathbb{Z}$.

Define $H^{(13)}_g = (H^{(13)}_{g,r})$ for $g \in G^{(13)}$ by requiring that

$$\psi^{(13)}_g(\tau, z) = -\chi^{(13)}_g \mu^0_{13,0}(\tau, z) - \bar{\chi}^{(13)}_g \mu^1_{13,0}(\tau, z) + \sum_{r \mod 26} H^{(13)}_{g,r}(\tau)\theta_{13,r}(\tau, z), \tag{B.87}$$

where $\chi^{(13)}_g := \chi^{X_A}_g$ and $\bar{\chi}^{(13)}_g := \bar{\chi}^{X_A}_g$ (cf. Table 28), and the $\psi^{(13)}_g$ are meromorphic Jacobi forms of weight 1 and index 13 given explicitly in Table 42.

Table 42: Character Values and Meromorphic Jacobi Forms for $\ell = 13$, $X = A_{12}^2$

$[g]$	$\chi^{(13)}_g$	$\bar{\chi}^{(13)}_g$	$\psi^{(13)}_g(\tau, z)$
1A	2	2	$2i\theta_1(\tau, 6z)\theta_1(\tau, z)^{-1}\theta_1(\tau, 3z)^{-1}\eta(\tau)^3$
2A	-2	2	$-2i\theta_1(\tau, 6z)\theta_2(\tau, z)^{-1}\theta_2(\tau, 3z)^{-1}\eta(\tau)^3$
4A	0	0	cf. (B.88)

$$\psi^{(13)}_{4AB}(\tau, z) := -i\theta_2(\tau, 6z)\frac{\theta_1(\tau, z + \frac{1}{4})\theta_1(\tau, 3z + \frac{1}{4}) - \theta_1(\tau, z - \frac{1}{4})\theta_1(\tau, 3z - \frac{1}{4})}{\theta_2(2\tau, 2z)\theta_2(2\tau, 6z)}\frac{\eta(2\tau)^2}{\eta(\tau)} \tag{B.88}$$

For use later on we note that $\psi^{(13)}_{1A} = -2\Psi_{1,1}\varphi^{(13)}_1$, where

$$\varphi^{(13)}_1(\tau, z) := \frac{\theta_1(\tau, z)\theta_1(\tau, 6z)}{\theta_1(\tau, 2z)\theta_1(\tau, 3z)}. \tag{B.89}$$

B.3.15 $\ell = 14 + 7$, $X = D_8^3$

We have $m^X = 14$, so the umbral McKay-Thompson series $H^{(14+7)}_g = (H^{(14+7)}_{g,r})$ associated to $g \in G^{(14+7)}$ is a 28-vector-valued function with components indexed by $r \in \mathbb{Z}/28\mathbb{Z}$. We have $H^{(14+7)}_{g,r} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H^{(14+7)}_{g,r}$ for r odd. Observing that $H^{(14+7)}_{g,r} = -H^{(14+7)}_{g,-r}$ we may determine $H^{(14+7)}_g$ by requiring that

$$\psi^{(7/2)}_g(\tau, z) = -2\bar{\chi}^{(7/2)}_g i\mu_{7/2,0}(\tau, z) + \sum_{\substack{r \in \mathbb{Z}+1/2 \\ r \mod 7}} e(-r/2)H^{(14+7)}_{g,2r}(\tau)\theta_{7/2,r}(\tau, z), \tag{B.90}$$

where $\bar{\chi}^{(7/2)}_g := \bar{\chi}^{X_D}_g$ is the number of fixed points of $g \in G^{(14+7)} \simeq S_3$ in the defining permutation representation on 3 points. The $\psi^{(7/2)}_g$ are the meromorphic Jacobi forms of weight 1 and index 7/2 defined in Table 43.

Table 43: Character Values and Meromorphic Jacobi Forms for $\ell = 14 + 7$, $X = D_8^3$

$[g]$	$\bar{\chi}_g^{(7/2)}$	$\psi_g^{(7/2)}(\tau, z)$
1A	3	$2i\theta_1(\tau, 3z)\theta_1(\tau, z)^{-2}\eta(\tau)^3$
2A	1	$2i\theta_2(\tau, 3z)\theta_1(\tau, z)^{-1}\theta_2(\tau, z)^{-1}\eta(\tau)^3$
3A	0	$-2i\theta_1(\tau, z)\theta_1(\tau, 3z)\theta_1(3\tau, 3z)^{-1}\eta(3\tau)$

B.3.16 $\ell = 16$, $X = A_{15}D_9$

We have $m^X = 16$, so the umbral McKay-Thompson series $H_g^{(16)} = (H_{g,r}^{(16)})$ associated to $g \in G^{(16)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 32-vector-valued function with components indexed by $r \in \mathbb{Z}/32\mathbb{Z}$. We have $H_{g,r}^{(16)} = -H_{g,-r}^{(16)}$, so it suffices to specify the $H_{g,r}^{(16)}$ for $1 \leq r \leq 15$.

To define $H_g^{(16)} = (H_{g,r}^{(16)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-6\Psi_{1,1}(\tau, z)\left(\varphi_1^{(16)}(\tau, z) + \frac{1}{2}\varphi_2^{(16)}(\tau, z)\right) = -24\mu_{16,0}(\tau, z) + \sum_{r \mod 32} h_r(\tau)\theta_{16,r}(\tau, z), \tag{B.91}$$

where

$$\begin{aligned} \varphi_1^{(16)}(\tau, z) &:= 8\varphi_1^{(4)}(\tau, z)\varphi_1^{(13)}(\tau, z) - \varphi_1^{(5)}(\tau, z)\varphi_1^{(12)}(\tau, z) + \varphi_1^{(7)}(\tau, z)\varphi_1^{(10)}(\tau, z), \\ \varphi_2^{(16)}(\tau, z) &:= 12\varphi_1^{(4)}(\tau, z)\varphi_1^{(13)}(\tau, z) - \varphi_1^{(5)}(\tau, z)\varphi_1^{(12)}(\tau, z) - 3\varphi_1^{(16)}(\tau, z). \end{aligned} \tag{B.92}$$

(Cf. (B.35), (B.39), (B.54), (B.69), (B.77), (B.89).) Now define the $H_{1A,r}^{(16)}$ by setting

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{12}h_r(\tau) \tag{B.93}$$

for r odd. For r even, $2 \leq r \leq 14$, use

$$H_{1A,r}^{(16)}(\tau) := \frac{1}{24}\left(h_r(\tau) + h_{16-r}(\tau)\right). \tag{B.94}$$

Define $H_{2A,r}^{(16)}$ by requiring

$$H_{2A,r}^{(16)}(\tau) := -(-1)^r H_{1A,r}^{(16)}(\tau). \tag{B.95}$$

B.3.17 $\ell = 18$, $X = A_{17}E_7$

We have $m^X = 18$, so the umbral McKay-Thompson series $H_g^{(18)} = (H_{g,r}^{(18)})$ associated to $g \in G^{(18)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H_{g,r}^{(18)} = -H_{g,-r}^{(18)}$, so it suffices to specify the $H_{g,r}^{(18)}$ for $1 \leq r \leq 17$.

To define $H_g^{(18)} = (H_{g,r}^{(18)})$ for $g = e$, first define $h(\tau) = (h_r(\tau))$ by requiring that

$$-24\Psi_{1,1}(\tau, z)\phi^{(18)}(\tau, z) = -24\mu_{18,0}(\tau, z) + \sum_{r \mod 36} h_r(\tau)\theta_{18,r}(\tau, z), \tag{B.96}$$

where

$$\phi^{(18)} := \frac{1}{12}\left(\varphi_1^{(18)} + \frac{1}{3}\varphi_3^{(18)} + 4\frac{\theta_1^{12}}{\eta^{12}}\left(\varphi_1^{(12)} + 2\varphi_2^{(12)} + \frac{1}{3}\varphi_3^{(12)}\right)\right). \tag{B.97}$$

For the definition of $\phi^{(18)}$ we require

$$\begin{aligned}
\varphi_2^{(9)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(6)}(\tau,z) - 4\varphi_1^{(5)}(\tau,z)^2 - 4\varphi_1^{(9)}(\tau,z),\\
\varphi_1^{(11)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(7)}(\tau,z) + 2\varphi_1^{(3)}(\tau,z)\varphi_1^{(9)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(8)}(\tau,z),\\
\varphi_3^{(12)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(9)}(\tau,z),\\
\varphi_1^{(14)}(\tau,z) &:= 3\varphi_1^{(5)}(\tau,z)\varphi_1^{(10)}(\tau,z) + \varphi_1^{(3)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(11)}(\tau,z),\\
\varphi_1^{(15)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) + 6\varphi_1^{(3)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z),\\
\varphi_2^{(15)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(12)}(\tau,z) - 2\varphi_1^{(5)}(\tau,z)\varphi_1^{(11)}(\tau,z) - 2\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(18)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(14)}(\tau,z) + 3\varphi_1^{(3)}(\tau,z)\varphi_1^{(16)}(\tau,z) - 4\varphi_1^{(4)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_3^{(18)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_2^{(15)}(\tau,z),
\end{aligned} \tag{B.98}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Now define the $H_{1A,r}^{(18)}$ by setting

$$H_{1A,r}^{(18)}(\tau) := \frac{1}{24}h_r(\tau) \tag{B.99}$$

for r even. For r odd, use

$$\begin{aligned}
H_{1A,1}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_1(\tau) + h_{17}(\tau)\right),\\
H_{1A,3}^{(18)}(\tau) &:= \frac{1}{24}\left(h_3(\tau) + h_9(\tau)\right),\\
H_{1A,5}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_5(\tau) + h_{13}(\tau)\right),\\
H_{1A,7}^{(18)}(\tau) &:= \frac{1}{24}\left(2h_7(\tau) + h_{11}(\tau)\right),\\
H_{1A,9}^{(18)}(\tau) &:= \frac{1}{24}\left(h_3(\tau) + 2h_9(\tau) + h_{15}(\tau)\right),\\
H_{1A,11}^{(18)}(\tau) &:= \frac{1}{24}\left(h_7(\tau) + 2h_{11}(\tau)\right),\\
H_{1A,13}^{(18)}(\tau) &:= \frac{1}{24}\left(h_5(\tau) + 2h_{13}(\tau)\right),\\
H_{1A,15}^{(18)}(\tau) &:= \frac{1}{24}\left(h_{15}(\tau) + h_9(\tau)\right),\\
H_{1A,17}^{(18)}(\tau) &:= \frac{1}{24}\left(h_1(\tau) + 2h_{17}(\tau)\right).
\end{aligned} \tag{B.100}$$

Define $H_{2A,r}^{(18)}$ in the usual way for root systems with a type A component, by requiring

$$H_{2A,r}^{(18)}(\tau) := -(-1)^r H_{1A,r}^{(18)}(\tau). \tag{B.101}$$

B.3.18 $\ell = 18 + 9$, $X = D_{10}E_7^2$

We have $m^X = 18$, so the umbral McKay-Thompson series $H_g^{(18+9)} = (H_{g,r}^{(18+9)})$ associated to $g \in G^{(18+9)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 36-vector-valued function with components indexed by $r \in \mathbb{Z}/36\mathbb{Z}$. We have $H_{g,r}^{(18+9)} = -H_{g,-r}^{(18+9)}$, $H_{g,r}^{(18+9)} = H_{g,18-r}^{(18+9)}$ for $1 \leq r \leq 17$, and $H_{g,r}^{(18+9)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{g,r}^{(18+9)}$ for $r \in \{1, 3, 5, 7, 9\}$.

Define

$$\begin{aligned} H_{1A,r}^{(18+9)}(\tau) &:= H_{1A,r}^{(18)}(\tau) + H_{1A,18-r}^{(18)}(\tau), \\ H_{2A,r}^{(18+9)}(\tau) &:= H_{1A,r}^{(18)}(\tau) - H_{1A,18-r}^{(18)}(\tau), \end{aligned} \tag{B.102}$$

for $r \in \{1, 3, 5, 7, 9\}$.

B.3.19 $\ell = 22 + 11$, $X = D_{12}^2$

We have $m^X = 22$, so the umbral McKay-Thompson series $H_g^{(22+11)} = (H_{g,r}^{(22+11)})$ associated to $g \in G^{(22+11)} \simeq \mathbb{Z}/2\mathbb{Z}$ is a 44-vector-valued function with components indexed by $r \in \mathbb{Z}/44\mathbb{Z}$. We have $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ and $H_{g,r}^{(22+11)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{g,r}^{(22+11)}$ for r odd. Observing that $H_{g,r}^{(22+11)} = -H_{g,-r}^{(22+11)}$ we may determine $H_g^{(22+11)}$ by requiring that

$$\psi_g^{(11/2)}(\tau, z) = -2\bar{\chi}_g^{(11/2)} i\mu_{11/2,0}(\tau, z) + \sum_{\substack{r \in \mathbb{Z}+1/2 \\ r \mod 11}} e(-r/2) H_{g,2r}^{(22+11)}(\tau)\theta_{11/2,r}(\tau, z), \tag{B.103}$$

where $\bar{\chi}_{1A}^{(11/2)} := 2$, $\bar{\chi}_{2A}^{(11/2)} := 0$, and the $\psi_g^{(11/2)}$ are the meromorphic Jacobi forms of weight 1 and index 11/2 defined as follows.

$$\begin{aligned} \psi_{1A}^{(11/2)}(\tau, z) &:= 2i \frac{\theta_1(\tau, 4z)}{\theta_1(\tau, z)\theta_1(\tau, 2z)} \eta(\tau)^3 \\ \psi_{2A}^{(11/2)}(\tau, z) &:= -2i \frac{\theta_1(\tau, 4z)}{\theta_2(\tau, z)\theta_2(\tau, 2z)} \eta(\tau)^3 \end{aligned} \tag{B.104}$$

B.3.20 $\ell = 25$, $X = A_{24}$

We have $m^X = 25$, so for $g \in G^{(25)} \simeq \mathbb{Z}/2\mathbb{Z}$, the associated umbral McKay-Thompson series $H_g^{(25)} = (H_{g,r}^{(25)})$ is a 50-vector-valued function, with components indexed by $r \in \mathbb{Z}/50\mathbb{Z}$, satisfying $H_{g,r}^{(25)} = -H_{g,-r}^{(25)}$, and in particular, $H_{g,r}^{(25)} = 0$ for $r = 0 \mod 25$. So it suffices to specify the $H_{g,r}^{(25)}$ for $1 \leq r \leq 24$.

Define $H_g^{(25)} = (H_{g,r}^{(25)})$ for $g = e$ by requiring that

$$-\Psi_{1,1}(\tau, z)\varphi_1^{(25)}(\tau, z) = -\mu_{25,0}(\tau, z) + \sum_{r \mod 50} H_{e,r}^{(25)}(\tau)\theta_{25,r}(\tau, z), \tag{B.105}$$

where

$$\varphi_1^{(25)}(\tau, z) := \frac{1}{2}\varphi_1^{(5)}(\tau, z)\varphi_1^{(21)}(\tau, z) - \varphi_1^{(7)}(\tau, z)\varphi_1^{(19)}(\tau, z) + \frac{1}{2}\varphi_1^{(13)}(\tau, z)^2. \tag{B.106}$$

For the definition of $\varphi_1^{(25)}$ we require

$$\begin{aligned}
\varphi_1^{(17)}(\tau,z) &:= 4\varphi_1^{(5)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(9)}(\tau,z)^2,\\
\varphi_1^{(19)}(\tau,z) &:= \varphi_1^{(4)}(\tau,z)\varphi_1^{(16)}(\tau,z) + 2\varphi_1^{(7)}(\tau,z)\varphi_1^{(13)}(\tau,z) - \varphi_1^{(5)}(\tau,z)\varphi_1^{(15)}(\tau,z),\\
\varphi_1^{(21)}(\tau,z) &:= \varphi_1^{(5)}(\tau,z)\varphi_1^{(17)}(\tau,z) - 2\varphi_1^{(9)}(\tau,z)\varphi_1^{(13)}(\tau,z),
\end{aligned} \tag{B.107}$$

in addition to the other $\varphi_k^{(m)}$ that have appeared already. Define $H_{2A,r}^{(25)}$ in the usual way for root systems with a type A component, by requiring

$$H_{2A,r}^{(18)}(\tau) := -(-1)^r H_{1A,r}^{(18)}(\tau). \tag{B.108}$$

B.3.21 $\ell = 30+15$, $X = D_{16}E_8$

We have $m^X = 30$, so the umbral McKay-Thompson series $H_g^{(30+15)} = (H_{g,r}^{(30+15)})$ associated to $g \in G^{(30+15)} = \{e\}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have $H_{e,r}^{(30+15)} = -H_{e,-r}^{(30+15)}$, $H_{e,r}^{(30+15)} = H_{e,30-r}^{(30+15)}$ for $1 \le r \le 29$, and $H_{e,r}^{(30+15)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{e,r}^{(30+15)}$ for $r \in \{1,3,5,7,9,11,13,15\}$.

Define

$$\begin{aligned}
H_{1A,1}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} + [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,3}^{(30+15)}(\tau) &:= [-\tfrac{9}{120}]H_{3A,3}^{(10+5)}(\tau/3),\\
H_{1A,5}^{(30+15)}(\tau) &:= [-\tfrac{25}{120}]H_{3A,5}^{(10+5)}(\tau/3),\\
H_{1A,7}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} + [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,11}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,1}^{(30+6,10,15)} - [-\tfrac{1}{120}]H_{3A,1}^{(10+5)}(\tau/3)\right),\\
H_{1A,13}^{(30+15)}(\tau) &:= \frac{1}{2}\left(H_{1A,7}^{(30+6,10,15)} - [-\tfrac{49}{120}]H_{3A,3}^{(10+5)}(\tau/3)\right),\\
H_{1A,15}^{(30+15)}(\tau) &:= -[-\tfrac{105}{120}]H_{3A,5}^{(10+5)}(\tau/3).
\end{aligned} \tag{B.109}$$

B.3.22 $\ell = 30+6,10,15$, $X = E_8^3$

We have $m^X = 30$, and $G^{(30+6,10,15)} = G^X \simeq S_3$. The umbral McKay-Thompson series $H^{(30+6,10,15)}$ is a 60-vector-valued function with components indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. We have

$$H_{g,r}^{(30+6,10,15)}(\tau) = \begin{cases} \pm H_{g,1}^{(30+6,10,15)} & \text{if } r = \pm1,\pm11,\pm19,\pm29 \mod 60,\\ \pm H_{g,7}^{(30+6,10,15)} & \text{if } r = \pm7,\pm13,\pm17,\pm27 \mod 60,\\ 0 & \text{else,} \end{cases} \tag{B.110}$$

so it suffices to specify the $H_{g,r}^{(30+6,10,15)}$ for $r = 1$ and $r = 7$. These functions may be defined as follows.

$$\begin{aligned}
H_{1A,1}^{(30+6,10,15)} &:= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+(k+l+m)/2+3/40}\\
H_{2A,1}^{(30+6,10,15)} &:= -2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0} - \sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+(2k+m)/2+3/40}\\
H_{3A,1}^{(30+6,10,15)} &:= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+3k/2+3/40}\\
H_{1A,7}^{(30+6,10,15)} &= -2\frac{1}{\eta(\tau)^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+3(k+l+m)/2+27/40}\\
H_{2A,7}^{(30+6,10,15)} &= 2\frac{1}{\eta(2\tau)}\left(\sum_{k,m\geq 0} - \sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+3(2k+m)/2+27/40}\\
H_{3A,7}^{(30+6,10,15)} &= -2\frac{\eta(\tau)}{\eta(3\tau)}\sum_{k\in\mathbb{Z}}(-1)^k q^{15k^2/2+9k/2+27/40}
\end{aligned} \tag{B.111}$$

B.3.23 $\ell = 46 + 23$, $X = D_{24}$

We have $m^X = 22$, and $G^{(46+23)} = \{e\}$. The umbral McKay-Thompson series $H_e^{(46+23)} = (H_{e,r}^{(46+23)})$ is a 92-vector-valued function with components indexed by $r \in \mathbb{Z}/92\mathbb{Z}$. We have $H_{e,r}^{(46+23)} = -H_{e,-r}^{(46+23)}$ and $H_{e,r}^{(46+23)} = 0$ for $r = 0 \mod 2$, so it suffices to specify the $H_{e,r}^{(46+23)}$ for r odd. Observing that $H_{e,r}^{(46+23)} = -H_{e,-r}^{(46+23)}$ we may determine $H_e^{(46+23)}$ by requiring that

$$\psi_e^{(23/2)}(\tau,z) = -2i\mu_{23/2,0}(\tau,z) + \sum_{\substack{r\in\mathbb{Z}+1/2\\ r \mod 23}} e(-r/2)H_{g,2r}^{(46+23)}(\tau)\theta_{23/2,r}(\tau,z), \tag{B.112}$$

where $\psi_e^{(23/2)}$ is the meromorphic Jacobi forms of weight 1 and index 23/2 defined by setting

$$\psi_e^{(23/2)}(\tau,z) := 2i\frac{\theta_1(\tau,6z)}{\theta_1(\tau,2z)\theta_1(\tau,3z)}\eta(\tau)^3. \tag{B.113}$$

B.4 Rademacher Sums

Let Γ_∞ denote the subgroup of upper-triangular matrices in $SL_2(\mathbb{Z})$. Given $\alpha \in \mathbb{R}$ and $\gamma \in SL_2(\mathbb{Z})$, define $\mathrm{r}_{1/2}^{[\alpha]}(\gamma,\tau) := 1$ if $\gamma \in \Gamma_\infty$. Otherwise, set

$$\mathrm{r}_{1/2}^{[\alpha]}(\gamma,\tau) := e(-\alpha(\gamma\tau-\gamma\infty))\sum_{k\geq 0}\frac{(2\pi i\alpha(\gamma\tau-\gamma\infty))^{n+1/2}}{\Gamma(n+3/2)}, \tag{B.114}$$

where $e(x) := e^{2\pi i x}$. Let n be a positive integer, and suppose that ν is a multiplier system for vector-valued modular forms of weight $1/2$ on $\Gamma = \Gamma_0(n)$. Assume that $\nu = (\nu_{ij})$ satisfies $\nu_{11}(T) = e^{\pi i/2m}$, for some basis $\{\mathfrak{e}_i\}$, for some positive integer m, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. To this data, attach the Rademacher sum

$$R_{\Gamma,\nu}(\tau) := \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} \nu(\gamma) e\left(-\frac{\gamma\tau}{4m}\right) \mathfrak{e}_1 \, \mathrm{j}(\gamma,\tau)^{1/2} \, \mathrm{r}_{1/2}^{[-1/4m]}(\gamma,\tau), \tag{B.115}$$

where $\Gamma_{K,K^2} := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \mid 0 \le c < K, \, |d| < K^2 \right\}$, and $\mathrm{j}(\gamma,\tau) := (c\tau + d)^{-1}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If the expression (B.115) converges then it defines a mock modular form of weight $1/2$ for Γ whose shadow is given by an explicitly identifiable Poincaré series. We refer to [13] for a review of this, and to [53] for a more general and detailed discussion.

Convergence of (B.115) can be shown by rewriting the Fourier expansion as in [53, Theorem 2] in terms of a sum of Kloostermann sums weighted by Bessel functions. This expression converges at $w = 1/2$ by the analysis discussed at the end of §3, following the method of Hooley as adapted by Gannon. That analysis requires not only establishing that the expressions converge, but also explicitly bounding the rates of convergence.

For the special case that $X = A_8^3$ we require 8-vector-valued functions $\check{t}_g^{(9)} = (\check{t}_{g,r}^{(9)})$ for $g \in G^X$ with order 3 or 6. For such g, define $\check{t}_{g,r}^{(9)}$, for $0 < r < 9$, by setting

$$\check{t}_{3A,r}^{(9)}(\tau) := \begin{cases} 0, & \text{if } r \neq 0 \mod 3, \\ -\theta_{3,3}(\tau,0), & \text{if } r = 3, \\ \theta_{3,0}(\tau,0), & \text{if } r = 6, \end{cases} \tag{B.116}$$

in the case that g has order 3, and

$$\check{t}_{6A,r}^{(9)}(\tau) := \begin{cases} 0, & \text{if } r \neq 0 \mod 3, \\ -\theta_{3,3}(\tau,0), & \text{if } r = 3, \\ -\theta_{3,0}(\tau,0), & \text{if } r = 6, \end{cases} \tag{B.117}$$

when $o(g) = 6$. Here $\theta_{m,r}(\tau,z)$ is as defined in (B.5).

The following result is proved in [17], using an analysis of representations of the metaplectic double cover of $SL_2(\mathbb{Z})$.

Theorem B.1 ([17])**.** Let X be a Niemeier root system and let $g \in G^X$. Assume that the Rademacher sum $R^X_{\Gamma_0(n_g),\check{\nu}_g^X}$ converges. If $X \neq A_8^3$, or if $X = A_8^3$ and $g \in G^X$ does not satisfy $o(g) = 0 \mod 3$, then we have

$$\check{H}_g^X(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}_g^X}. \tag{B.118}$$

If $X = A_8^3$ and $g \in G^X$ satisfies $o(g) = 0 \mod 3$ then

$$\check{H}_{g,r}^X(\tau) = -2R^X_{\Gamma_0(n_g),\check{\nu}_g^X}(\tau) + \check{t}_g^{(9)}(\tau). \tag{B.119}$$

The $X = A_1^{24}$ case of Theorem B.1 was proven first in [12], via different methods.

References

[1] D. Alexander, C. Cummins, J. McKay, and C. Simons, *Completely replicable functions*, Groups, combinatorics & geometry (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 165, Cambridge Univ. Press, Cambridge, 1992, 87–98. MR 1200252 (94g:11029)

[2] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proceedings of the National Academy of Sciences, U.S.A. **83** (1986), no. 10, 3068–3071.

[3] ———, *Generalized Kac-Moody algebras*, J. Algebra **115** (1988), no. 2, 501–512. MR 943273 (89g:17004)

[4] ———, *Central extensions of generalized Kac-Moody algebras*, J. Algebra **140** (1991), no. 2, 330–335. MR 1120425 (92g:17031)

[5] ———, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), no. 2, 405–444.

[6] ———, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562. MR 1625724 (99c:11049)

[7] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), no. 2, 243–266. MR 2231957 (2007e:11127)

[8] ———, *Coefficients of harmonic maass forms*, Partitions, q-Series, and Modular Forms (Krishnaswami Alladi and Frank Garvan, eds.), Developments in Mathematics, vol. 23, Springer New York, 2012, pp. 23–38.

[9] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90. MR 2097357 (2005m:11089)

[10] M. C. Cheng, *$K3$ surfaces, $N = 4$ dyons and the Mathieu group M_{24}*, Commun. Number Theory Phys. **4** (2010), no. 4, 623–657. MR 2793423 (2012e:11076)

[11] M. Cheng and J. Duncan, *The largest Mathieu group and (mock) automorphic forms*, String-Math 2011, Proc. Sympos. Pure Math., vol. 85, Amer. Math. Soc., Providence, RI, 2012, pp. 53–82. MR 2985326

[12] ———, *On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine*, Commun. Number Theory Phys. **6** (2012), no. 3.

[13] ———, *Rademacher Sums and Rademacher Series*, Conformal Field Theory, Automorphic Forms and Related Topics, 2014, pp.143-182, Contributions in Mathematical and Computational Sciences.

[14] ———, *Meromorphic Jacobi Forms of Half-Integral Index and Umbral Moonshine Modules*, in preparation.

[15] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, *Umbral Moonshine*, Commun. Number Theory Phys. **8** (2014), no. 2.

[16] ———, *Umbral Moonshine and the Niemeier Lattices*, Research in the Mathematical Sciences **1** (2014), no. 3.

[17] ———, *Weight One Jacobi Forms and Umbral Moonshine*, in preparation.

[18] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), no. 3, 308–339. MR 554399 (81j:20028)

[19] J. H. Conway and N. J. A. Sloane, *On the enumeration of lattices of determinant one*, J. Number Theory **15** (1982), no. 1, 83–94. MR 666350 (84b:10047)

[20] ———, *Sphere packings, lattices and groups*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1999, With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. MR 1662447 (2000b:11077)

[21] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.*, Oxford: Clarendon Press, 1985.

[22] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms*, arXiv:1208.4074.

[23] L. Dixon, P. Ginsparg and J. Harvey, *Beauty and the beast: superconformal symmetry in a Monster module*, Comm. Math. Phys. **119** (1998), no. 2, 221–241. MR 968697 (90b:81119)

[24] J. Duncan, M. Griffin, and K. Ono, *Moonshine*, Research in the Mathematical Sciences (2015) 2:11.

[25] J. Duncan and J. Harvey, *The Umbral Mooonshine Module for the Unique Unimodular Niemeier Root System*, arXiv:1412.8191.

[26] J. Duncan and A. O'Desky, *Super Vertex Algebras, Meromorphic Jacobi Forms, and Umbral Moonshine*, in preparation.

[27] T. Eguchi and K. Hikami, *Note on Twisted Elliptic Genus of K3 Surface*, Phys.Lett. **B694** (2011), 446–455.

[28] T. Eguchi, H. Ooguri, and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group* M_{24}, Exper.Math. **20** (2011), 91–96.

[29] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Birkhäuser, 1985.

[30] I. Frenkel, J. Lepowsky, and A. Meurman, *A natural representation of the Fischer-Griess Monster with the modular function J as character*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), no. 10, Phys. Sci., 3256–3260. MR MR747596 (85e:20018)

[31] ———, *A moonshine module for the Monster*, Vertex operators in mathematics and physics (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., vol. 3, Springer, New York, 1985, pp. 231–273. MR 86m:20024

[32] ———, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press Inc., Boston, MA, 1988. MR 90h:17026

[33] M. Gaberdiel, S. Hohenegger, and R. Volpato, *Mathieu Moonshine in the elliptic genus of K3*, JHEP **1010** (2010), 062.

[34] ———, *Mathieu twining characters for K3*, JHEP **1009** (2010), 058, 19 pages.

[35] T. Gannon, *Monstrous moonshine: the first twenty-five years*, Bull. London Math. Soc. **38** (2006), no. 1, 1–33. MR 2201600 (2006k:11071)

[36] ———, *Moonshine beyond the Monster*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2006, The bridge connecting algebra, modular forms and physics. MR 2257727 (2008a:17032)

[37] ———, *Much ado about Mathieu*, arXiv:1211.5531.

[38] C. Hooley, *On the number of divisors of a quadratic polynomial*, Acta Math. **110** (1963), 97-114. MR 0153648

[39] J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972, Graduate Texts in Mathematics, Vol. 9. MR 0323842 (48 #2197)

[40] W. Kohnen, *Fourier coefficients of half-integral weight*, Math. Ann. **271** (1985), 237–268. MR 0783554

[41] Ö. Imamoğlu, M. Raum, and O. Richer, *Holomorphic projections and Ramanujan's mock theta functions*, Proc. Natl. Acad. Sci., USA, **111** (2014), 3961–3967.

[42] M. Mertens, *Eichler-Selberg type identities for mixed mock modular forms*, arXiv:1404.5491.

[43] H. V. Niemeier, *Definite quadratische Formen der Dimension* 24 *und Diskriminante* 1, J. Number Theory **5** (1973), 142–178. MR 0316384 (47 #4931)

[44] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conf. Ser. No. 102, Amer. Math. Soc., 2004, Providence MR 2020489 (2005c:11053)

[45] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 347–454. MR 2555930 (2010m:11060)

[46] S. Ramanujan, *The lost notebook and other unpublished papers*, Springer-Verlag, Berlin, 1988, With an introduction by George E. Andrews. MR 947735 (89j:01078)

[47] W. A. Stein et al., *Sage Mathematics Software (Version 6.5)*, The Sage Development Team, 2015, `http://www.sagemath.org`.

[48] J.-P. Serre and H. M. Stark, *Modular forms of weight* 1/2, in Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 27–67. Lecture Notes in Math., Vol. 627. Springer, Berlin, 1977.

[49] J. Sturm, *On the congruence of modular forms*, Springer Lect. Notes Math. **1240** (1984), 275-280. MR 0894516 (88h:11031)

[50] J. G. Thompson, *Finite groups and modular functions*, Bull. London Math. Soc. **11** (1979), no. 3, 347–351. MR 554401 (81j:20029)

[51] ———, *Some numerology between the Fischer-Griess Monster and the elliptic modular function*, Bull. London Math. Soc. **11** (1979), no. 3, 352–353. MR MR554402 (81j:20030)

[52] B. B. Venkov, *On the classification of integral even unimodular* 24*-dimensional quadratic forms*, Trudy Mat. Inst. Steklov. **148** (1978), 65–76, 273, Algebra, number theory and their applications. MR 558941 (81d:10024)

[53] D. Whalen, *Vector-Valued Rademacher Sums and Automorphic Integrals*, arXiv:1406.0571.

[54] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)*, Astérisque (2009), no. 326, Exp. No. 986, vii–viii, 143–164 (2010), Séminaire Bourbaki. Vol. 2007/2008. MR 2605321 (2011h:11049)

[55] S. Zwegers, *Mock Theta Functions*, Ph.D. thesis, Utrecht University, 2002. arXiv:0807.4834.

J. F. R. Duncan, Dept. of Mathematics & Computer Science, Emory University, Atlanta, GA 30322
E-mail address, J. F. R. Duncan: `john.duncan@emory.edu`

M. J. Griffin, Dept. of Mathematics, Princeton University, Princeton, NJ 08544
E-mail address, M. J. Griffin: `mjg4@princeton.edu`

K. Ono, Dept. of Mathematics & Computer Science, Emory University, Atlanta, GA 30322
E-mail address, K. Ono: `ono@mathcs.emory.edu`

Published for SISSA by Springer

Received: May 10, 2019
Revised: December 7, 2019
Accepted: February 2, 2020
Published: February 13, 2020

On Mathieu moonshine and Gromov-Witten invariants

Andreas Banlaki,[a] **Abhishek Chowdhury,**[a,b] **Abhiram Kidambi**[a] **and Maria Schimpf**[a]

[a] *Institut für Theoretische Physik, TU Wien,*
Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria
[b] *IIT Bhubaneshwar,*
SBS Building, Argul, Khordha, 752050, Odisha, India

E-mail: banlaki@hep.itp.tuwien.ac.at, achowdhury@iitbbs.ac.in, abhiram.kidambi@tuwien.ac.at, mschimpf@hep.itp.tuwien.ac.at

Abstract: We provide further evidence that CY_3 manifolds are involved in an intricate way in Mathieu moonshine, i.e., their Gromov-Witten invariants are related to the expansion coefficients of the twined/twisted-twined elliptic genera of $K3$. We use the string duality between CHL orbifolds of heterotic string theory on $K3 \times T^2$ and type IIA string theory on CY_3 manifolds to explicitly show this connection. We then work out two concrete examples where we exactly match the expansion coefficients on both sides of the duality.

Keywords: String Duality, Superstrings and Heterotic Strings, Differential and Algebraic Geometry, Discrete Symmetries

ArXiv ePrint: 1811.11619

Open Access, © The Authors.
Article funded by SCOAP[3].

https://doi.org/10.1007/JHEP02(2020)082

JHEP02(2020)082

Contents

1 Introduction **1**

2 Mathieu moonshine **3**

3 Heterotic compactifications on $K3 \times T^2$ and CHL orbifolds **6**
- 3.1 Instanton embedding and Hilbert space structure 6
- 3.2 The new supersymmetric index 7
  - 3.2.1 CHL orbifolds 8
- 3.3 Threshold corrections and the prepotential 11
- 3.4 Spectrum of heterotic CHL orbifolds 13

4 Dual type IIA compactifications on Calabi-Yau 3-folds **13**

5 Explicit examples **15**

6 Conclusions **17**

A Conventions **18**
- A.1 Jacobi theta functions 18
- A.2 Eisenstein series 19

B Modular properties of Jacobi forms and $\mathcal{E}_N(\tau)$ **19**

C $\mathcal{N} = 4$ superconformal characters **19**

D Explicit expansion of $\mathcal{Z}^{\text{new},2}$ **20**

JHEP02(2020)082

1 Introduction

Moonshine may be labelled as a set of mysterious connections between finite sporadic groups and modular functions. For mathematicians and physicists alike, a major push in this direction came from the seminal paper by Conway and Norton on the discovery of Monstrous Moonshine [1]. It connects two different branches of mathematics that often can be understood using the language of conformal field theory (CFT) and string theory. In mid 1980s, Frenkel, Lepowsky, and Meurman [2] realized that the number of states at a given energy level of a 24 dimensional bosonic string theory compactified on a $\mathbb{Z}_2$ quotient of the Leech lattice [3] (i.e., a chiral CFT) are dimensions of irreducible representations of the Monster group (sometimes dubbed as the "old moonshine"). In 2010, Eguchi, Ooguri and Tachikawa (EOT) [4] discovered a new moonshine phenomenon [5–9] that connects

the elliptic genus of $K3$ to the largest Mathieu group $\mathbb{M}_{24}$. Since the $K3$ manifold plays an important role in mathematics and physics, this new observation has led to a flurry of papers (see [10, 11] for recent review articles). However, the original EOT observation has not yet been explained in a satisfactory manner, but significant progress has been made throughout the years and several new moonshine phenomena have been discovered and others have been better understood, see for example [12–15].

In this paper, we will be interested in strengthening a connection between Mathieu moonshine and Gromov-Witten invariants of CY_3 manifolds. Unlike the original Mathieu moonshine observation which arose by considering the expansion of the $K3$ elliptic genus into $\mathcal{N} = 4$ characters, the elliptic genus of CY_3 is rather simple when expanded in term of the $\mathcal{N} = 2$ characters and thus one does not see anything interesting [16]. A possible alternate route will be to look at other topological invariants such as the Gromov-Witten invariants of the CY_3 to make a connection to moonshine. In particular, we will extend the idea of [17] and establish a connection between the twined and twisted-twined elliptic genera of $K3$ and Gromov-Witten invariants of certain CY_3 manifolds. To do this, we will rely on the string duality between compactifcations to four dimensions of heterotic $E_8 \times E_8$ string theory and type IIA string theory. This allows us, on one hand, to strengthen the connection of CY_3 manifolds to (Mathieu) moonshine and, on the other hand, to also establish a connection between moonshine and algebraic geometry.

As mentioned above, we will be interested in compactifications of the heterotic $E_8 \times E_8$ string theory to four dimensions. In particular we will first focus on compactifications on $K3 \times T^2$. These lead to four dimensional theories with $\mathcal{N} = 2$ spacetime supersymmetry. The action for the $\mathcal{N} = 2$ vector multiplets is controlled by a prepotential $h(S, T, U)$, where S denotes the axio-dilaton, T is the Kähler modulus and U denotes the complex structure modulus of the T^2. The prepotential h is one-loop exact and it has been shown in [17] that the one-loop correction $h^{\text{1-loop}}(T, U)$ has an expansion that is connected to Mathieu moonshine. This can be related back to the EOT observation since the computation of $h^{\text{1-loop}}(T, U)$ involves the new supersymmetric index [18] which, for $K3$ compactifications, is related to the elliptic genus of $K3$. Here we discuss compactifications in which the function $h^{\text{1-loop}}(T, U)$ not only allows an expansion involving the irreducible representations of $\mathbb{M}_{24}$ due to Mathieu moonshine, but is also connected to a bigger part of the $\mathbb{M}_{24}$ character table through the twined and twisted-twined elliptic genera.

In particular, Chaudhuri, Hockney and Lykken (CHL) [19] gave a description for orbifolding the heterotic string compactified on a circle or torus in such a way that supersymmetry is not broken. While looking at such (generalized–)CHL orbifolds, one can find four dimensional $\mathcal{N} = 2$ theories for which the one-loop corrections do not depend on the coefficients appearing in the corresponding moonshine function, but rather on the coefficients of twined functions, i.e., the *McKay-Thompson* series. This has been studied explicitly for the compactifications of the heterotic $E_8 \times E_8$ string theory on $K3 \times T^2$ in [20–22].

Compactifications of the heterotic string theory on $K3 \times T^2$ have played an important role in string-string dualities [23, 24], an example being the aforementioned duality between heterotic $E_8 \times E_8$ string theory and type II string compactifications on certain CY_3 manifolds. These compactifications therefore give rise to the same four dimensional

JHEP02(2020)082

$\mathcal{N} = 2$ theories. In the type IIA string theory, the one-loop threshold corrections are encoded in the Gromov-Witten invariants of the CY_3 manifold, while for type IIB string theory they arise from period integrals of the mirror Calabi-Yau (CY) manifold. Given our heterotic constructions above, we can therefore conclude that via string duality certain CY_3 manifolds are implicated in Mathieu moonshine, i.e., the Gromov-Witten invariants are controlled by (twisted–) characters of the elements of the Mathieu group.[1] In this paper we flesh out this general idea by providing few examples. In future we hope to provide a larger class of CY_3 that are implicitly involved in Mathieu moonshine via string duality.

The paper is organized as follows: in section 2 we briefly review Mathieu moonshine which is then followed by section 3 where we discuss compactifications of the heterotic string theory on $K3 \times T^2$ and extensions to CHL orbifolds. We review the calculation of the new supersymmetric index and show its relation to Mathieu moonshine via the twined and twisted-twined elliptic genera of $K3$. We then discuss the duality to type IIA compactifications on CY_3 manifolds in section 4. Lastly, in section 5, we present explicit examples where we are able to match the Gromov-Witten invariants with the expectation from the twined and twisted-twined elliptic genera on the heterotic side. We summarize our findings in section 6.

2 Mathieu moonshine

In this section, we review some of the relevant details of the Mathieu moonshine, i.e., the connection between the $\mathbb{M}_{24}$ group and the elliptic genus of the $K3$ surface. In general for a superconformal field theory (SCFT) with more than $\mathcal{N} = (2,2)$ supersymmetry and central charges $(c, \bar{c})$ the elliptic genus is defined as [26]

$$\mathcal{Z}^{\text{ell}}(q,y) = Tr_{RR}\left((-1)^{F_L+F_R} y^{J_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right), \tag{2.1}$$

where $y := e^{2\pi i z}, q := e^{2\pi i \tau}$, J_0 is the zero mode of the third component of the affine SU(2) R-symmetry algebra, L_0 is the zero mode of the Virasoro algebra, F_L, F_R are respectively the left and right moving fermion numbers and the trace is taken over the Ramond-Ramond sector. The right moving part is simply the Witten index and hence for a theory with a discrete spectrum $\mathcal{Z}^{\text{ell}}$ is holomorphic.[2] The elliptic genus is an index that counts BPS states of the theory and is independent of moduli of all the theory. From a modular standpoint it is a weak Jacobi form[3] with weight 0 and index $\frac{c}{6}$ w.r.t. the modular group $\text{SL}(2,\mathbb{Z})$. For the special case where the SCFT is an $\mathcal{N} = (4,4)$ non-linear sigma model (NLSM) on $K3$, i.e., for $c = 6$, the elliptic genus is given by [30] (see appendix A for our

[1] Following [25], the generalised *McKay-Thompson* series have the interpretation of being the characters of projective representations of centralizers of the group elements of $\mathbb{M}_{24}$. See (3.14).

[2] For a continuous spectrum holomorphicity is not guaranteed, in fact the elliptic genus behaves like a mock modular form [27]. In many cases certain BPS indices exhibit wall-crossing, i.e., the value of the indices jump as one moves around in the moduli/parameter space of the theory [28, 29].

[3] See appendix B for the defining properties.

JHEP02(2020)082

conventions regarding Jacobi-θ functions)

$$\mathcal{Z}^{\text{ell}}_{K3}(q,y) = 8\sum_{i=2}^{4}\left(\frac{\theta_i(q,y)}{\theta_i(q,1)}\right)^2. \tag{2.2}$$

In [4] it has been noted that when (2.2) is expanded in terms of $\mathcal{N}=4$ superconformal characters, one finds that it has coefficients which are related to the dimensions of irreducible representations of the largest Mathieu group $\mathbb{M}_{24}$. Concretely, defining the (short/long) $\mathcal{N}=4$ superconformal (graded) characters in the Ramond sector as in [31] (see appendix C for our concrete conventions) one finds the expansion

$$\mathcal{Z}^{\text{ell}}_{K3}(q,y) = 24\,\text{ch}^{\mathcal{N}=4}_{6,0,0}(q,y) + \sum_{n=0}^{\infty} A^{(1)}_n \text{ch}^{\mathcal{N}=4}_{6,n,\frac{1}{2}}(q,y), \tag{2.3}$$

where the $24 = 23+1$ and other coefficients $A^{(1)}_n$ may be related to the dimensions of the sum of irreducible representations of $\mathbb{M}_{24}$ as[4,5]

$$\begin{aligned} A^{(1)}_0 &= -2 = -1-1, \\ A^{(1)}_1 &= 90 = 45+\overline{45}, \\ A^{(1)}_2 &= 461 = 231+\overline{231}, \\ A^{(1)}_3 &= 1540 = 770+\overline{770}, \cdots . \end{aligned} \tag{2.4}$$

Since the symplectic automorphisms of $K3$ form a subgroup of $\mathbb{M}_{23}$ [32] and the group of automorphisms of a NLSM on $K3$ is never $\mathbb{M}_{24}$ but in general a subgroup of the Conway group Co_1 [33] the appearance of $\mathbb{M}_{24}$ is not immediately clear.[6] As the elliptic genus only counts BPS states, it is expected that similar to the case of generalized moonshine [42–44] there exists an infinite dimensional graded module[7]

$$\mathcal{H}^{\text{BPS}} = \bigoplus_{n=0}^{\infty} H_n \otimes \mathcal{H}^{\mathcal{N}=4}_n, \tag{2.5}$$

where the sum runs over the contributing irreducible $\mathcal{N}=4$ representations $\mathcal{H}^{\mathcal{N}=4}_n$ and H_n are $\mathbb{M}_{24}$ representations (in general not irreducible) with $\dim(H_n) = |A^{(1)}_n|$ [7]. This would imply that the twined elliptic genera of $K3$ obtained by inserting an element $g \in \mathbb{M}_{24}$ in (2.2), i.e.,

$$\mathcal{Z}^{(\text{ell})}_{K3,g}(q,y) = \text{Tr}_{RR}\left(g\,(-1)^{F_L+F_R} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} e^{2\pi i z J_0}\right) \tag{2.6}$$

[4]Note that the character $\text{ch}_{6,0,\frac{1}{2}}(q,y)$ here is in a massive representation which can be written as a sum of two massless representations, $\lim_{h\searrow\frac{1}{4}} \text{ch}^{\mathcal{N}=4}_{6,h-\frac{1}{4},l}(q,y) = \text{ch}^{\mathcal{N}=4}_{6,0,\frac{1}{2}}(q,y) + 2\text{ch}^{\mathcal{N}=4}_{6,0,0}(q,y)$. We thank the referee for pointing this out.

[5]The correct decomposition of the higher coefficients can be fixed by requiring the twined elliptic genera to behave appropriately, see (2.6) and the following discussion.

[6]Progress towards a complete understanding has been made in [33–37], through the concept of symmetry surfing [38–40] and other directions [41].

[7]The existence of such a module has been proven in [9].

JHEP02(2020)082

form an analogue of the *McKay-Thompson* series of monstrous moonshine, i.e., they admit an expansion similar to (2.3) but now with coefficients $\tilde{A}_n = Tr_{H_n}(g)$ and transform as a Jacobi form of weight 0 and index 1 under the subgroup $\Gamma_0(N)$ of $\mathrm{SL}(2,\mathbb{Z})$. Based on these facts, in [5–8] explicit expressions for all the twined genera have been found. For some cases where $g \in \mathbb{M}_{23}$ admits an interpretation as a symplectic automorphism of $K3$ the corresponding twined genera have been calculated directly in [45] and are in agreement with the above results. The symmetries of NLSM on $K3$ have been classified in [33, 36]. Furthermore, all possible twining genera of NLSM on $K3$ have been conjectured in [36] based on works of [34, 35]. This conjecture has been proven in a "physical" way by demanding absence of unphysical wall-crossings in [37].

In analogy to generalized moonshine, in [25] all the twisted (by g), twined (by h) elliptic genera $\mathcal{Z}^{\mathrm{ell}}_{K3,g,h}$ has been calculated. For every commuting pair of $g, h \in \mathbb{M}_{24}$ they are defined as

$$\mathcal{Z}^{\mathrm{ell}}_{K3,g,h}(q,y) = Tr_{RR,g}\left(h\,(-1)^{F_L} y^{J_0} q^{L_0-\frac{c}{24}} (-1)^{F_R} \bar{q}^{L_0-\frac{\bar{c}}{24}}\right), \tag{2.7}$$

where the trace is now taken over the g-twisted Ramond sector. $\mathcal{Z}^{\mathrm{ell}}_{K3,g,h}$ are expected to fulfil certain properties. Firstly, under modular transformations they transform as

$$\begin{aligned} \mathcal{Z}^{\mathrm{ell}}_{K3,g,h}(\tau, z + l\tau + l') &= e^{-2\pi i m(l^2\tau + 2lz)} \mathcal{Z}^{\mathrm{ell}}_{K3,g,h}\,,\ l, l' \in \mathbb{Z} \\ \mathcal{Z}^{\mathrm{ell}}_{K3,g,h}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= \chi_{g,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) e^{2\pi i \frac{cz^2}{c\tau+d}} \mathcal{Z}^{\mathrm{ell}}_{K3,h^c g^a, h^d g^b}(\tau, z),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \end{aligned} \tag{2.8}$$

for a certain multiplier $\chi_{g,h} : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{U}(1)$. The multipliers are assumed to be constant under conjugation of the pair (g,h) by an element of $k \in \mathbb{M}_{24}$ up to a phase $\xi_{g.h}(k)$ (that depend on certain 2-cocycles c_g) and to have a well defined expansion in terms $\mathcal{N} = 4$ superconformal characters. In particular, $\mathcal{Z}^{\mathrm{ell}}_{K3,e,h}$ agrees with the corresponding twined character. It is postulated in [25] that these properties (in particular $\chi_{g,h}$, $\xi_{g,h}$ and c_g) are all controlled by a 3-cocycle α representing a class in $H^3(\mathbb{M}_{24}, \mathrm{U}(1))$. Moreover, the twisted, twined and twisted-twined genera transform amongst each other following modular transformations as in (2.8). The set of twisted-twined elliptic genera modulo modular transformations is in one to one correspondence with conjugacy classes of abelian subgroups of $\mathbb{M}_{24}$ generated by two elements (g,h).[8] While calculating these genera using the properties mentioned above, many of them vanish identically owing to obstructions, i.e., situations where the properties only allow for vanishing characters. Using the shorthand $\mathcal{Z}^{(\mathrm{ell})}_{K3,g,h} := \phi_{g,h}$, the only un-obstructed twisted-twined elliptic genera (excluding twined genera) are

$$\phi_{2B,4A_2}, \phi_{4B,4A_3}, \phi_{4B,4A_4}, \phi_{2B,8A_{1,2}}, \phi_{3A,3A_3}, \phi_{3A,3B_1}, \tag{2.9}$$

where the last two cases in the above list vanish.

[8] There are 55 such subgroups, 21 of which correspond to cyclic subgroups generated by elements of the form (e,g) and hence are associated to twining genera $\mathcal{Z}^{\mathrm{ell}}_{K3,g}$ already discussed before in [6]. This also includes the twisted and twisted-twined genera obtained from the twined genera by modular transformations. The remaining 34 twisted, twined genera has been calculated in [25] using the properties discussed above.

JHEP02(2020)082

As it will be of importance for us going forward, in the special cases where $(g,h)=(\tilde{g}^r,\tilde{g}^s)$ for $\tilde{g}\in\mathbb{M}_{24}$ where $(r,s)\in\{0,\dots,N-1\}$ and $N=\mathrm{ord}(g)$, which in particular includes the cases where the subgroups generated by g and h are cyclic, compact expressions for $\mathcal{Z}^{(\mathrm{ell})}_{K3,g,h}$ are known [21, 25]. For such cases we define $\mathcal{Z}^{\mathrm{ell}}_{r,s}:=\mathcal{Z}^{\mathrm{ell}}_{\tilde{g}^r,\tilde{g}^s}$ such that

$$\mathcal{Z}^{\mathrm{ell}}_{r,s}(q,y)=\begin{cases}\frac{N}{8}\alpha^{(0,0)}_{\tilde{g}}\mathcal{Z}^{\mathrm{ell}}(q,y) & (r,s)=(0,0)\\ \frac{N}{8}\alpha^{(r,s)}_{\tilde{g}}\mathcal{Z}^{\mathrm{ell}}(q,y)+N\beta^{(r,s)}_{\tilde{g}}(q)\frac{\theta_1^2(q,y)}{\eta^6(q)} & (r,s)\neq(0,0),\end{cases}\tag{2.10}$$

where $\alpha^{(r,s)}_{\tilde{g}}$ are numerical constants and $\beta^{(r,s)}_{\tilde{g}}(q)$ is a weight 2 modular form under $\Gamma_0(N)$. Explicit expressions for $\mathcal{Z}^{\mathrm{ell}}_{r,s}$ can be found in appendix E of [46].

3 Heterotic compactifications on $K3\times T^2$ and CHL orbifolds

In this section we review compactifications of $E_8\times E_8$ heterotic string theory on $K3\times T^2$ and extension to CHL orbifolds thereof. Our discussion is based on some of the results obtained in [20, 21, 46–48]. We will eventually be interested in the prepotential of the vectormoduli and the spectrum of these theories in order to facilitate comparison to potential dual type IIA compactifications.

3.1 Instanton embedding and Hilbert space structure

The fact that $K3$ has a non-vanishing curvature 2-form and as a consequence of the Bianchi identity for the threeform field strength $H^{(3)}$, there is a requirement to embed 24 instantons[9] in $E_8\times E_8$. In general one may embed $(n^{(1)},n^{(2)})$ instantons in some subgroups H_1,H_2 of the two E_8's with $n^{(1)}+n^{(2)}=24$, $n^{(i)}\in\mathbb{N}$, thereby breaking the gauge group to $G_1\times G_2$, where G_i is the commutant of the H_i in E_8.[10] This results in a $\mathcal{N}=(0,4)$ world-sheet supersymmetry and $\mathcal{N}=2$ supersymmetry in four spacetime dimensions. The embedding where one chooses $n_1=24$, $n_2=0$ and $H_1=\mathrm{SU}(2)$, $G_1=E_7$ is called the standard embedding. For the standard embedding, the first E_8 lattice breaks to $D_6\times D_2$ and using the fermionic description of these lattices, one obtains the following structure [21, 47] for the internal Hilbert space

$$\mathcal{H}^{\mathrm{internal}}=\mathcal{H}^{(6,6)}_{D_2K3}\otimes\mathcal{H}^{(6,0)}_{D6}\otimes\mathcal{H}^{(8,0)}_{E_8}\otimes\mathcal{H}^{(2,3)}_{T^2},\tag{3.1}$$

where the upper index labels the left and right central charges. Concretely, $\mathcal{H}^{(6,6)}_{D_2K3}$ consists of 4 left moving bosons on $K3$ together with the 4 fermions coming from the fermionic representation of D_2, and a right-moving $\bar{c}=6$ supersymmetric contribution. $\mathcal{H}^{(6,0)}_{D6}$ contains 12 fermions coming from D_6, $\mathcal{H}^{(8,0)}_{E_8}$ is the Hilbert space of the unbroken E_8 and lastly, $\mathcal{H}^{(2,3)}_{T^2}$ is the Hilbert space of two left moving bosons on T^2 and a supersymmetric $c=3$ theory on T^2. The above case further generalizes to non-standard embeddings where $2(r_1+r_2)$ fermions ($r_{1,2}$ being the rank of the vectorbundle belonging to the instantons embedded in the first/second E_8) from the two E_8's couple to the gauge connection, and the Hilbert space structure generalizes accordingly [49].

[9] 24 is the Euler number of $K3$.

[10] Because of symmetry we can restrict $0\leq n_2\leq 12$.

JHEP02(2020)082

3.2 The new supersymmetric index

We now turn to the new supersymmetric index [18] for heterotic compactification on $K3 \times T^2$. It is the central object which allows us to determine the vector multiplet prepotential as well as the moduli dependence of the gravitational couplings in the effective $\mathcal{N} = 2$ spacetime theory [47, 50–53]. For no Wilson lines on T^2, the new supersymmetric index for the theory on $K3 \times T^2$ takes the general form [48]

$$\begin{aligned}
\mathcal{Z}^{\text{new}}(q,\bar{q}) &= \frac{1}{\eta(q)^2}\mathrm{Tr}_R(F(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{L_0-\frac{\bar{c}}{24}})\Big|_{c=(22,9)} = Z_{K3}(q)\cdot Z_{2,2}(q,\bar{q}),\\
Z_{2,2}(q,\bar{q}) &= \sum_{p\in\Gamma_{2,2}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2} = \sum_{p\in\Gamma_{2,2}} q^{\frac{1}{2}(p_L^2-p_R^2)}e^{-2\pi\tau_2 p_R^2},\\
p_R^2 &= \frac{1}{2T_2U_2}|-m_1U+m_2+n_1T+n_2TU|^2,\\
\frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2+m_1n_1+m_2n_2,
\end{aligned} \tag{3.2}$$

where (T,U) are respectively the Kähler and complex structure moduli of T^2 and the trace is taken over the Ramond sector of the internal $c = (22,9)$ CFT. In particular, one may observe that it factorizes into a holomorphic part coming from $K3$ and the $E_8 \times E_8$ contributions, and into a lattice sum $Z_{2,2}$. The supersymmetric index will in general depend on the topology of the manifold and on the instanton numbers $(n^{(1)}, n^{(2)})$. However, one may move in the hypermultiplet moduli space by (un–) higgsing the hypermultiplets, thereby changing the gauge groups (G_1, G_2) without changing the index. In the above expression (3.2), for the case of $H_1 = H_2 = \mathrm{SU}(2)$, $Z_{K3}(q)$ takes the form

$$Z_{K3}(q) = -2\left[\frac{n^{(1)}}{24}\frac{E_6(q)E_4(q)}{\eta(q)^{24}} + \frac{n^{(2)}}{24}\frac{E_6(q)E_4(q)}{\eta(q)^{24}}\right] = -2\frac{E_6(q)E_4(q)}{\eta(q)^{24}}. \tag{3.3}$$

Since $\tau_2\mathcal{Z}^{\text{new}}$ is a non-holomorphic modular form of weight -2,[11] the form of $Z_{K3}(q)$ as in (3.3) is indeed determined, for arbitrary H_1, H_2, by its modular properties alone up to a multiplicative constant, which can be fixed to 1 by independent arguments [17]. As can be seen from the Hilbert space structure (3.1), $Z_{K3}(q)$ admits an expansion in terms of the characters of $D6$ and the elliptic genus of $K3$ at special values of the elliptic modulus is [47]

$$\begin{aligned}
\eta(q)^{12}Z_{K3}(q) = \frac{1}{2}E_4\Bigg[&\left(\frac{\theta_2(q)}{\eta(q)}\right)^6\mathcal{Z}^{\text{ell}}_{K3}(q,-1) + q^{\frac{1}{4}}\left(\frac{\theta_3(q)}{\eta(q)}\right)^6\mathcal{Z}^{\text{ell}}_{K3}(q,-q^{1/2})\\
&- q^{\frac{1}{4}}\left(\frac{\theta_4(q)}{\eta(q)}\right)^6\mathcal{Z}^{\text{ell}}_{K3}(q,q^{1/2})\Bigg].
\end{aligned} \tag{3.4}$$

Here, the E_4 factor comes from the unbroken E_8, the $D6$ characters enter through the trace over the 12 free fermions coming from the broken E_8. The elliptic genus arises from taking a trace over the remaining 4 fermions, the 4 bosons in the left moving sector together

[11] $\mathcal{Z}^{\text{new}}$ appears in threshold corrections together with modular forms of weight 2.

JHEP02(2020)082

with the supersymmetric right moving sector in the $(R-,R)$, $(NS+,R)$ and $(NS-,R)$ sectors.[12,13] The presence of the $\mathcal{Z}^{\text{ell}}_{K3}$ terms allows us to deduce a connection between $\mathcal{Z}^{\text{new}}$ and the Mathieu group $\mathbb{M}_{24}$, analogous to (2.3) [17]. In particular, one finds

$$-\frac{4E_6(q)}{\eta^{12}}(q) = 24\, g_{h=1/4,l=0}(q) + \sum_{n=0}^{\infty} A_n^{(1)} g_{h=n+1/4,l=1/2}(q) \tag{3.5}$$

where

$$\begin{aligned} g_{h=n+1/4,l}(q) = &\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \text{ch}^{\mathcal{N}=4}_{6,n,l}(q,-1) + q^{\frac{1}{4}}\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 \text{ch}^{\mathcal{N}=4}_{6,n,l}(q,-q^{1/2}) \\ &- q^{\frac{1}{4}}\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 \text{ch}^{\mathcal{N}=4}_{6,n,l}(q,q^{1/2}) \end{aligned} \tag{3.6}$$

and the coefficients $A_n^{(1)}$ as stated in (2.4) are the representation of the Mathieu group. This shows the relation between the new supersymmetric index and the $\mathbb{M}_{24}$ group.

3.2.1 CHL orbifolds

Following the discussion in section 2, the *McKay-Thompson* series is an interesting object to study. To this end, we turn to the study of the new supersymmetric index for the case of CHL orbifolds of heterotic compactification on $K3\times T^2$ following [20, 21, 46]. We consider a NLSM on $K3$ with symmetry group $G < Co_1$ preserving the $\mathcal{N}=(4,4)$ supersymmetry of the model.[14] Such symmetry groups have been classified in [33] (and in [36] for singular $K3s$) where it has been shown that they correspond to groups preserving a 4-plane in the moduli space of the NLSM. We may construct a CHL orbifold by orbifolding by an element $g\in G$ together with a shift by $1/\text{ord}(g)$ along one of the two circles of T^2. The additional shift guarantees that the action is fixed point free and thereby preserves supersymmetry (which in this case is $\mathcal{N}=2$ spacetime supersymmetry) [54].

The concrete orbifolds we will consider are of the type where g is a symplectic automorphism of $K3$, with g being an element of one of the following $\mathbb{M}_{24}$ conjugacy classes[15]

$$1\text{A},\ 2\text{A},\ 3\text{A},\ 4\text{B},\ 5\text{A},\ 6\text{A},\ 7\text{A},\ 7\text{B},\ 8\text{A}. \tag{3.7}$$

At certain points in moduli space where $K3$ is expressible as $T^4/\mathbb{Z}_\nu$, $\nu\in\{2,3,4,6\}$ the new supersymmetric index has been calculated for $K3\times T^2$ heterotic compactification and different instanton embeddings in $E_8\times E_8$ in [48]. CHL orbifolds of such models have been

[12] In our convention, $+$ stands for the insertion of $(-1)^{F_L}$ when taking the trace and $-$ stands for no such insertion. For the right moving part the $(-1)^{F_R}$ insertion is always there so we leave away the $+$.

[13] We have omitted the term proportional to $\left(\frac{\theta_1(q)}{\eta(q)}\right)^6 \mathcal{Z}^{\text{ell}}_{K3}(q,+1)$ coming from the $(R+,R)$ sector which vanishes because $\theta_1(q)=0$. One should however keep in mind that it might contribute to the twisted sectors. We thank the referee for pointing this out.

[14] In general, the $K3$ NLSM is moduli dependent and therefore the symmetry group also depends on which point of the $K3$ moduli space is considered.

[15] There are 26 conjugacy classes in $\mathbb{M}_{24}$ of which 16 have a representative in $\mathbb{M}_{23}$. Of these 16 conjugacy classes, only 9 (as listed in (3.7)) can be realized as a symplectic automorphism of $K3$, i.e., are geometric [55].

JHEP02(2020)082

studied in [20, 21, 46]. For CHL orbifolds of $\mathrm{ord}(g) = N$, the new supersymmetric index has the general form [21]

$$\mathcal{Z}^{\mathrm{new},N} = -\frac{1}{2\eta(q)^{20}} \sum_{a,b=0}^{\nu-1} \sum_{r,s=0}^{N-1} e^{-\frac{2\pi i ab}{\nu^2}} Z_{E_8}^{(a,b)}(q) \times Z_{E_8'}^{(a,b)}(q) \times \frac{1}{2\nu} F(a,r,b,s;q) Z_{2,2}^{(r,s)}(q,\bar{q}), \tag{3.8}$$

where $Z_{E_8}^{(a,b)}(q)$, $Z_{E_8'}^{(a,b)}(q)$ are the partition functions of the shifted E_8 lattice, $Z_{2,2}^{(r,s)}(q,\bar{q})$ is the twisted-twined partition function of T^2 and has the expression

$$Z_{2,2}^{(r,s)}(q,\bar{q}) = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z} \\ n_1\in\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}} e^{2\pi i m_1 s/N}, \tag{3.9}$$

where the momenta are defined as in (3.2) and as before T, U are respectively the Kähler and complex structure moduli of T^2. The twist by an element of order N is reflected in the phase $e^{2\pi i m_1 s/N}$ and the fractional values of n_1. Furthermore,

$$F(a,r,b,s;q) := \mathrm{Tr}_{R,g'^a\tilde{g}^r}\left(g'^b \tilde{g}^s e^{i\pi F_R^{T^4}} q^{L_0} \bar{q}^{\bar{L}_0}\right) \tag{3.10}$$

is the trace over T^4 with 4 left and 4 right moving bosons, and over right moving fermions in the twisted Ramond sector whose fermion number is $F_R^{T^4}$. g' incorporates the $\mathbb{Z}^\nu$ action.[16] For example, in the case of the standard embedding, the orbifolded new supersymmetric index (3.8) may be expanded in terms of the twisted-twined elliptic genus of $K3$ as follows [21]

$$\begin{aligned}\mathcal{Z}^{\mathrm{new},N}(q,\bar{q}) = \frac{1}{N}\sum_{r,s=0}^{N-1} \frac{Z_{2,2}^{(r,s)}(q,\bar{q}) E_4(q)}{\eta(q)^{12}} \times \Bigg[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathcal{Z}^{\mathrm{ell}}_{K3,r,s}(q,-1) \\ + \left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{1/4} \mathcal{Z}^{\mathrm{ell}}_{K3,r,s}(q,-q^{\frac{1}{2}}) - \left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{1/4} \mathcal{Z}^{\mathrm{ell}}_{K3,r,s}(q,q^{\frac{1}{2}}) \Bigg],\end{aligned} \tag{3.11}$$

where we have used the shorthand notation $\mathcal{Z}^{\mathrm{ell}}_{K3,r,s} := \mathcal{Z}^{\mathrm{ell}}_{K3,\tilde{g}^r,\tilde{g}^s}$. The structure of $\mathcal{Z}^{\mathrm{new},N}(q,\bar{q})$ can be understood in the same way as for the un-orbifolded case (3.4). The appearance of the twisted-twined elliptic genera of $K3$ connect these supersymmetric CHL orbifolds to the generalized Mathieu moonshine as discussed in section 2. In particular, for the $(0,N)$ sector one may obtain an expansion similar to (3.5) and (3.6) with $A_i^{(1)}$ now replaced with $A_i^{(N)}$, the coefficients of the corresponding *McKay-Thompson* series [20]. In our case of interest for the examples in section 5, i.e., for $\mathrm{ord}(g) = N = 2$, we have

$$\mathcal{Z}^{\mathrm{ell}}_{K3,0,1}(\tau,z) = 8\, ch^{\mathcal{N}=4}_{6,0,0}(\tau,z) + \sum_{n=0}^{\infty} A_n^{(2)} ch^{\mathcal{N}=4}_{6,n,\frac{1}{2}}(\tau,z), \tag{3.12}$$

where the coefficient 8 is the twisted Euler number of $K3$ and a few values of $A_n^{(2)}$ are given by

$$A_n^{(2)} = -2, -6, 14, -28, 42, -56, 86, -138, \ldots \tag{3.13}$$

[16] For concrete expressions, we refer the reader to [21, 48].

JHEP02(2020)082

They are the coefficients of the *McKay-Thompson* series constructed out of the trace of an element of the $2A$ conjugacy class corresponding to the $\mathbb{Z}_2$ involution of $K3$ embedded in $\mathbb{M}_{24}$. Exactly along the lines of (3.5), the $(0,1)$ sector of $\mathcal{Z}^{\text{new},2}(q,\bar{q})$ has an expansion in terms of $g_{h=1/4,l}(q)$ using (3.6). The $(1,0)$ and $(1,1)$ sectors can be thought of as the modular transformed version of the $(0,1)$ sector. Hence, all the sectors can be related to Mathieu moonshine. In general, it is expected that the twisted-twined elliptic genus admits the following decomposition in terms of the characters of the $\mathcal{N}=4$ superconformal algebra with central charge $c=6$

$$\mathcal{Z}^{\text{ell}}_{K3,r,s}(\tau,z) = \sum_{k=n+\frac{r}{N}\geq 0}^{\infty} \text{Tr}_{\mathcal{H}_{\tilde{g}^r,k}}\left(\rho_{\tilde{g}^r,k}\left(\tilde{g}^s\right)\right) ch^{\mathcal{N}=4}_{6,h=\frac{1}{4}+k,l}(\tau,z). \tag{3.14}$$

Here, $l=\frac{1}{2}$ except when $h=\frac{1}{4}$ for which both $l=\frac{1}{2}$ and $l=0$ are present in the sum. The finite dimensional vector space $\mathcal{H}_{\tilde{g}^r,k}$ is the projective representation of the centralizer $C_{\mathbb{M}_{24}}(\tilde{g})$ which satisfies properties detailed in [25].

In the case of a non-standard embedding it is useful to express $\mathcal{Z}^{\text{new},N}$ in terms of the Eisenstein series. First note that one may write $\mathcal{Z}^{\text{new},N}$ in the general form

$$\mathcal{Z}^{\text{new},N}(q,\bar{q}) = -\,4\sum_{r,s=0}^{N-1} Z^{(r,s)}_{2,2}(q,\bar{q}) f^{(r,s)}(q), \tag{3.15}$$

where $\eta(q)^{24} f^{(r,s)}(q)$ is a modular form of weight 10 under $\Gamma_0(N)$.[17] We start by giving the expression $\mathcal{Z}^{\text{new},N}$ for the standard embedding, making use of (3.15). Here one obtains [46]

$$f^{(r,s)}(q) = \frac{1}{2\eta(q)^{24}} E_4(q)\left[\frac{1}{4}\alpha^{(r,s)}_{\tilde{g}} E_6(q) - \beta^{(r,s)}_{\tilde{g}}(\tau) E_4(q)\right], \tag{3.16}$$

where $\alpha^{(r,s)}_{\tilde{g}}, \beta^{(r,s)}_{\tilde{g}}$ are the same as in (2.10) and explicit expressions can be found in the appendix E of [46]. For the current case of interest, i.e., $N=2$ (order 2 orbifold), we have

$$\begin{gathered}\alpha^{(0,0)}_{\tilde{g}} = 4\,, \qquad \alpha^{(r,s)}_{\tilde{g}} = \frac{4}{3}\,, \qquad (r,s)\neq(0,0) \\ \beta^{(0,0)}_{\tilde{g}}(\tau) = 0\,, \quad \beta^{(0,1)}_{\tilde{g}}(\tau) = -\frac{2}{3}\mathcal{E}_2(\tau)\,, \quad \beta^{(1,0)}_{\tilde{g}}(\tau) = \frac{1}{3}\mathcal{E}_2\left(\frac{\tau}{2}\right)\,, \quad \beta^{(1,1)}_{\tilde{g}}(\tau) = \frac{1}{3}\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\end{gathered} \tag{3.17}$$

where $\mathcal{E}_N$ is a modular form of weight 2 under $\Gamma_0(N)$ defined as

$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau \ln\frac{\eta(\tau)}{\eta(N\tau)}. \tag{3.18}$$

Generalization of the new supersymmetric index to the case of non-standard embedding can be obtained from (3.8). In particular, for a CHL orbifold of order 2, the authors of [21]

[17] The vector space of such forms for $N=2$ is three dimensional and may be generated by E_4E_6, $E_4^2\mathcal{E}_2$ and $\mathcal{E}_2^2E_6$.

JHEP02(2020)082

calculated $\mathcal{Z}^{\text{new},2}$ at the point in $K3$ moduli space where $K3$ is expressible as $T^4/\mathbb{Z}_4$. The concrete expression may again be given with help of (3.15) [21]:[18]

$$
\begin{aligned}
f^{(0,0)}(q) &= \frac{1}{2\eta^{24}(q)} E_4(q) E_6(q) \\
f^{(r,s)}(q) &= \frac{3}{4\eta^{24}(q)} \left(\hat{b} \cdot \left(\frac{3}{2} \beta_{\tilde{g}}^{(r,s)}(\tau) \right)^2 + \left(\frac{2}{3} - \hat{b} \right) E_4(q) \right) \\
&\quad \times \left[\frac{1}{4} \alpha_{\tilde{g}}^{(r,s)} E_6(q) - \beta_{\tilde{g}}^{(r,s)}(q) E_4(q) \right]
\end{aligned}
\tag{3.20}
$$

The constant $\hat{b}$ depends on the 14 different possible instanton embeddings listed in [48] and it takes four different values:

$$
\hat{b} \in \left\{ 0, \frac{4}{9}, \frac{2}{3}, \frac{8}{9} \right\}. \tag{3.21}
$$

The value of $\hat{b}$ also directly determines the difference between the number of hyper and vector multiplets through the relation

$$
N_h - N_v = 144\,\hat{b} - 12. \tag{3.22}
$$

As has been noted in [21], an expansion in terms of the twisted-twined elliptic genera $\mathcal{Z}^{\text{ell}}_{K3,r,s}$ similar to (3.11) also exists in this case. This can be seen by comparing with (3.16) where $E_4(q)$ has been replaced by $\left(\hat{b} \cdot \left(\frac{3}{2} \beta_{\tilde{g}}^{(r,s)}(\tau) \right)^2 + \left(\frac{2}{3} - \hat{b} \right) E_4(q) \right)$ in the twisted-twined sector but the other factor $\left(\frac{1}{4} \alpha_{\tilde{g}}^{(r,s)} E_6(q) - \beta_{\tilde{g}}^{(r,s)}(q) E_4(q) \right)$ which corresponds to the elliptic genus of the 2A orbifold of $K3$ is however unchanged.[19]

3.3 Threshold corrections and the prepotential

Having obtained the new supersymmetric index, we now turn to calculating the one-loop corrections to the prepotential of the vectormoduli for the $d = 4$, $\mathcal{N} = 2$ supersymmetric effective theory arising from the CHL-orbifolded $E_8 \times E_8$ heterotic string theory compactified on $K3 \times T^2$ without Wilson lines [46, 47]. The starting point are the gravitational couplings F_g, $g \in \mathbb{N}$, $g \geq 1$ of the theory where g in the dual type IIA theory will label the

[18] In [45], the relevant $\alpha_{\tilde{g}}^{(r,s)}$ and $\beta_{\tilde{g}}^{(r,s)}$ for CHL orbifolds of the class pA with $p = 2, 3, 5, 7$ has already been computed at the point in $K3$ moduli space where $K3$ is expressible as $T^4/\mathbb{Z}_2$. As an alternative, since $f^{(r,s)}(q)$ has the following transformation property under the $\mathrm{SL}(2,\mathbb{Z})$ generators

$$
\begin{aligned}
f^{(r,s)}(\tau + 1) &= f^{(r,s+r)}(\tau) \\
f^{(r,s)}\left(-\frac{1}{\tau} \right) &= (-i\tau)^{-2} f^{(N-r,s)}(\tau) = (-i\tau)^{-2} f^{(r,N-s)}(\tau),
\end{aligned}
\tag{3.19}
$$

knowing a subset of them is enough to construct all of them. In our case of order 2 orbifolding, knowing $f^{(0,0)}(q)$ and $f^{(0,1)}(q)$ is enough.

[19] Note that it is $\left(\frac{1}{4} \alpha_{\tilde{g}}^{(r,s)} E_6(q) - \beta_{\tilde{g}}^{(r,s)}(q) E_4(q) \right)$ which admits a Mathieu moonshine expansion along the lines discussed after (3.11). We give the explicit expressions in appendix D.

JHEP02(2020)082

genus of the Gromov-Witten invariants. They appear in the one-loop action of the effective theory as terms of the form

$$S^{\text{1-loop}} = \int F_g(T, U, \bar{T}, \bar{U}) F_+^{2g-2} R_+^2, \tag{3.23}$$

where F_+ is the anti-self dual field strength of the graviphoton and R_+ is the self-dual part of the Riemann curvature tensor. In our model the couplings F_g depend on the (complexified) Kähler modulus T and the complex structure U of the torus T^2 (in general they depend on the vectormoduli). Compact expressions for the gravitational couplings F_g can be found in [47, 56]. Though F_g by itself is intricate, one is often interested in the 'holomorphic part' $\bar{F}_g^{\text{hol}}(T, U)$ of $F_g(T, U, \bar{T}, \bar{U})$ which can be obtained by sending $T, U \to \infty$ while holding $\bar{T}, \bar{U}$ fixed and then taking the complex conjugate of the expression [57, 58]. It is $\bar{F}_g^{\text{hol}}$ that predicts certain topological invariants of the dual CY_3, see section 4. In the heterotic frame F_g is given in terms of a one loop integral over the fundamental domain and one can follow the method of orbits [59] to evaluate it (and also the holomorphic counterpart) for genus $g \geq 1$ for several CHL-orbifoldings by conjugacy classes of $\mathbb{M}_{24}$ [46]. In particular, $\bar{F}_g^{\text{hol}}$ enjoys an expansion in polylogarithm functions (Li_{3-2g}) and the expansion holds true for genus $g = 0$. The genus $g = 0$ holomorphic one-loop corrections to the vectormoduli prepotential of the $d = 4$, $\mathcal{N} = 2$ heterotic CHL orbifolds by a $\mathbb{Z}_N$ automorphism acting on $K3$ together with the $\frac{1}{N}$ shift on one of the circles of T^2 is [46][20]

$$\bar{F}_0^{\text{hol},N}(T, U) = \frac{1}{\pi^2} \sum_{s=0}^{N-1} \left(\sum_{(n_1,n_2)>0} e^{-2\pi i n_2 s/N} c^{(n_1/N,s)}(n_1 n_2) \text{Li}_3 \left(e^{2\pi i (n_1 T/N + n_2 U)} \right) + \frac{c^{(0,s)}(0)}{2} \zeta(3) \right) \tag{3.25}$$

where the sum over (n_1, n_2) is over the points

$$\begin{aligned} n_1, n_2 \geq 0, \ (n_1, n_2 \neq (0,0)) \\ n_1 > 0, n_2 < 0 \text{ and } n_1 |n_2| \leq N, \end{aligned} \tag{3.26}$$

$\zeta(n)$ is the Riemann zeta function and $c^{(r,s)}(l)$ are the fourier coefficients of $f^{(r,s)}$ defined in (3.15), i.e.,[21]

$$f^{(r,s)}(q) = \sum_{l \in \frac{\mathbb{Z}}{N}} c^{(r,s)}(l)\, q^l. \tag{3.27}$$

We will use (3.25) to make comparisons to the dual IIA theory.

[20] In [17], the authors have shown that the vectormoduli prepotential (up to a quadratic polynomial in S, T and U) of the heterotic string theory compactified on $K3 \times T^2$ without the Wilson lines is given by

$$F_{\text{vec}} = STU + \frac{1}{3} U^3 + \frac{1}{(2\pi i)^3} c(0) \zeta(3) - \frac{2}{(2\pi i)^3} \sum_{\substack{n_1>0, n_2 \in \mathbb{Z} \\ n_1=0, n_2>0}} c(n_1 n_2) \text{Li}_3 \left(q_T^{n_1} q_U^{n_2} \right) + \mathcal{O}\left(e^{2\pi i S} \right) \tag{3.24}$$

In the weak coupling limit, i.e., when $S^{\text{het.}} \to \text{i}\infty$ the 1-loop contribution is proportional to $\bar{F}_0^{\text{hol},N}(T, U)$ as in (3.25) for $N = 1$. This correspondence also holds for the CHL orbifolds.

[21] Note that as twisted-twined elliptic genera of $K3$ control the coefficients $c^{(r,s)}(l)$, they are related to the (projective–)representations of Mathieu moonshine following the discussion after (3.11).

JHEP02(2020)082

$N_h - N_v$	Gauge group, Shift vector	Twisted sectors	Hypermultiplets	$\hat{b}$
84	$E_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$	g'^0	$(\mathbf{27},\mathbf{2};\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{64})$	
	$\times \mathrm{SO}(14) \times \mathrm{U}(1)$		$+2(\mathbf{1},\mathbf{1};\mathbf{1})$	$\frac{2}{3}$
		$g'^1 + g'^3$	$6(\mathbf{1},\mathbf{1};\mathbf{1}) + 4(\mathbf{1},\mathbf{2};\mathbf{1})$	
	$(2,1,1,0^5;2,0^7)$		$+2(\mathbf{27},\mathbf{1};\mathbf{1}) + 2(\mathbf{1},\mathbf{1};\mathbf{14})$	
		g'^2	$(\mathbf{1},\mathbf{2};\mathbf{14}) + 6(\mathbf{1},\mathbf{2};\mathbf{1})$	
116	$\mathrm{SU}(8) \times \mathrm{SU}(2)$	g'^0	$(\mathbf{28},\mathbf{2};\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{64}) + 2(\mathbf{1},\mathbf{1};\mathbf{1})$	
	$\times \mathrm{SO}(14) \times \mathrm{U}(1)$	$g'^1 + g'^3$	$4(\mathbf{8},\mathbf{1};\mathbf{1}) + 2(\mathbf{8},\mathbf{2};\mathbf{1})$	$\frac{8}{9}$
	$(3,1^5,0^2;2,0^7)$	g'^2	$3(\mathbf{1},\mathbf{2};\mathbf{14}) + 2(\mathbf{1},\mathbf{2};\mathbf{1})$	

Table 1. Completely higgsable models used in the examples in section 5.

3.4 Spectrum of heterotic CHL orbifolds

The spectrum of the CHL-orbifolded heterotic compactifications on $K3 \times T^2$ can be obtained by generalizing the methods of [60]. This has been done in [21] for CHL orbifolds of order 2 for the case when $K3$ is realized as $T^4/\mathbb{Z}_2$ and as $T^4/\mathbb{Z}_4$ for various (non–)standard embeddings. The resulting spectra are listed in tables 6 and 7-9, respectively of [21]. We give the gauge group and matter content of the two examples most important to us in table 1. They arise from compactifications where $K3$ is realized as $T^4/\mathbb{Z}_4$. We denote the generator of the $\mathbb{Z}_4$ action by g'. In table 1, the first column gives the difference in number of the hypermultiplets and the vectormultiplets. The second column states the gauge group and the shift-vector $(\gamma;\tilde{\gamma}) \in E_8 \times E_8$ that encodes the different instanton embeddings. Different instanton embeddings will in general lead to different values for $N_h - N_v$. The third column labels the twisted sectors of the $T^4/\mathbb{Z}_4$ realization of $K3$ from which the hypermultiplets arise. The fourth column gives the hypermultiplet content of the spectrum. In the last column we give the value of $\hat{b}$ according to (3.22).

The gauge group can be higgsed[22] by giving VEV's to the scalars in the hypermultiplets, thereby leaving $N_h - N_v$ unchanged. After maximally higgsing one may still give a VEV to the scalars belonging to the vectormultiplets which breaks the gauge group to U(1) factors. This will reduce N_v while leaving N_h unchanged. To the spectrum thus obtained, one still needs to add the four vectors that come from compactifying the ten dimensional metric $g_{\mu\nu}$ and the B-field $B_{\mu\nu}$ along the two 1-cycles of T^2. This gives the final number of vectorfields and hypermultiplets.

4 Dual type IIA compactifications on Calabi-Yau 3-folds

In this section we briefly review type IIA compactification on CY_3. Compactifying type IIA string theory on a CY_3, X with Hodge numbers $h_{1,1}$ (Kähler structure), $h_{2,1}$ (complex

[22]For a discussion of higgsing in $\mathcal{N} = 2$ gauge theories we refer the reader to, e.g., [61]. The necessary branching rules may be found in [62]. For example, we make use of the 'cascade' gauge group breaking chain say $E_6 \to SO(10) \to SU(5) \to SU(4) \to SU(3) \to SU(2) \to 0$ and $SO(14) \to SO(12) \to SO(10) \to SU(5) \to SU(4) \to SU(3) \to SU(2) \to 0$.

JHEP02(2020)082

structure) leads to a $d = 4$, $\mathcal{N} = 2$ spacetime theory with one gravity multiplet, $h_{1,1}$ vectormultiplets and $h_{2,1} + 1$ hypermultiplets. Due to $\mathcal{N} = 2$ spacetime supersymmetry there is no mixing between vectormultiplet and hypermultiplet moduli, i.e., the moduli space factorizes locally into $M_{\text{vec}} \times M_{\text{hyp}}$. Furthermore, the dilaton is part of a hypermultiplet and the vectormultiplet moduli space is exact at tree level. The vectormoduli are related to Kähler/complex structure moduli in type IIA/B compactifications. For type IIA, the prepotential of the vectormoduli will, in general, receive instanton corrections. In contrast, type IIB string compactifications on a CY_3 leads to a vectormoduli prepotential that is exact at tree level. This enables one to calculate an exact expression for the vectormoduli prepotential on the type IIA side in the form of an instanton expansion by relating it via the mirror map to the dual type IIB theory compactified on the mirror CY_3, $\tilde{X}$. Under the mirror map, the large Kähler structure point is mapped to the large complex structure point where the prepotential can be calculated on the type IIB side. The instanton corrections arise through holomorphic embeddings of the worldsheet into the CY_3, where the worldsheet wraps some 2-cycles in CY_3. For the genus zero case, when the worldsheet is isomorphic to S^2, the expression for the instanton corrections are of the form [63–66]

$$F^{\text{inst}} = \zeta(3)\frac{\chi(X)}{2} + \sum_{d_1,\ldots,d_{h^{1,1}}} n^0_{d_1,\ldots,d_{h^{1,1}}} \text{Li}_3(q_1^{d_1} \ldots q_{h^{1,1}}^{d_{h^{1,1}}}), \tag{4.1}$$

where $\zeta(n)$ is the Riemann zeta function, χ is the Euler character, $q_j := e^{2\pi i y_j}$ where y^j labels the complexified Kähler moduli and $n^0_{d_1,\ldots,d_{h_{1,1}}}$ denote the genus zero Gromov-Witten invariants[23] of multi-degree d_j. These Gromov-Witten invariants count in an appropriate way the number of holomorphic embeddings from a sphere into the CY_3. The first term (which is a constant) arises from the constant embedding i.e. when the sphere is mapped to a point.

The type IIA compactified on CY_3 is dual to the heterotic string compactified on $K3 \times T^2$ as studied in section 3. Calculations on the heterotic side can only be trusted when one considers the limit of weak coupling, i.e., when $S^{\text{het.}} \to \text{i}\infty$ and then compare the prepotential to the dual type IIA side, see (3.24). Under the duality, $S^{\text{het.}}$ will be mapped to one of the Kähler moduli, say y^i, and hence in this limit we have

$$(S^{\text{het}} \to i\infty) \leftrightarrow (q_i = e^{2\pi i y^i} \to 0). \tag{4.2}$$

Comparing (3.25) to (4.1), we see that the two expressions agree[24] if we set

$$F^{\text{inst}} = -2\pi^2 \bar{F}_0^{\text{hol},N} \tag{4.3}$$

and perform the identifications

$$\frac{\chi(X)}{2} = \sum_{s=0}^{N-1} c^{(0,s)}(0), \tag{4.4}$$

$$n^0_{n_1,n_2} = -2\sum_{s=0}^{N-1} e^{-\frac{2\pi i n_2 s}{N}} c^{(r,s)}(n_1 n_2).$$

[23]The are closely related to Gopakumar-Vafa invariants, see [67] for a review.

[24]Since we are considering the situation with no Wilson lines we will in general have to set $h_{1,1} - 3$ moduli to zero corresponding to $q_i = 1$.

JHEP02(2020)082

Therefore, we see that the heterotic prepotential predicts the Euler character[25] and the genus zero Gromov-Witten invariants of the CY_3. In order to find the correct (matching) CY_3's, we need to calculate these invariants. In particular, by restricting to those CY_3's that are realized as a hypersurface in a toric ambient space defined by a four dimensional reflexive polytope, one may calculate the Gromov-Witten invariants using the Mathemathica package *instanton.m* [68].[26] Some results of these computations are discussed in the next section, section 5.

5 Explicit examples

As discussed in the previous sections, we wish to make use of the heterotic-type IIA duality to predict certain CY_3 used in type IIA string compactifications by matching their Gromov-Witten invariants to the coefficients of the expansion of the vectormoduli prepotential on the heterotic side. In this section we give two explicit examples in support of the duality.

On the heterotic side the models we consider are the CHL orbifolds of order 2 whose spectrum is explicitly given in table 1. The prepotential for the order 2 CHL orbifolds can be calculated from (3.25) using (3.20) and (3.21). In order to find CY_3 for which the compactified type IIA theory matches the heterotic side we proceed by looking for CY_3 manifolds which produce the same number of vectormultiplets and hypermultiplets, i.e., with hodge numbers $h_{1,1} = N_v - 1$, $h_{2,1} = N_h - 1$. As explained in subsection 3.4, we maximally Higgs the spectrum of table 1 and add the vector fields coming from compactification of the metric and the antisymmetric B-field thus obtaining N_v number of vectorfields and N_h number of hypermultiplets. For the methods explained in the previous section 4 to work, we have to restrict ourselves to cases where the spectrum can be completely higgsed. Then the vector fields of these theories are the ones coming from the torus compactification of the metric and the antisymmetric B-field. We find complete higgsing is possible for the two cases given in table 1 and this leads to two models with $(N_v, N_h) = (4, 84)$ and $(N_v, N_h) = (4, 116)$. Given the vector and hypermultiplet content we can now proceed to search the database by Kreuzer and Skarke [71] for appropriate CY_3's that are realized as hypersurfaces in toric ambient space defined by a four dimensional reflexive polytope. For $(N_v, N_h) = (4, 84)$ the database contains 4 possible toric CY_3's and for $(N_v, N_h) = (4, 116)$ there are 10. For each of these CY_3's (more precisely for their non-singular resolutions) we calculate the genus zero Gromov-Witten invariants. The instanton corrections to the prepotential on the type IIA side are then obtained from (4.1).

[25] From our discussion above, the Euler character is also determined by the heterotic spectrum.

[26] *instanton.m* calculates the Gromov-Witten invariants in the way outlined above, i.e., by making use of mirror symmetry. It takes the generators of the Mori cone of the toric ambient space as an input which are then used to define appropriate coordinates at the large complex structure point. These generators can be calculated using *mori.x* which is a part of PALP [69]. In general, the toric ambient space is singular and also a (star–)triangulation. Thus a decomposition into four simplices each having its origin as a vertex of the reflexive polytope describing it would resolve the singularities and guarantee a smooth CY_3 manifold. Different triangulations will correspond to different resolutions of the singularities and lead to different Mori cones. *mori.x* can triangulate non-simplicial cones whose secondary fan is at most three dimensional. In other cases, triangulations can be obtained using TOPCOM [70]. However, since the number of possible triangulations increases rapidly with $h_{1,1}$, this is a viable method only when $h_{1,1}$ is small.

JHEP02(2020)082

Finally, we compare the results obtained from the heterotic and type IIA side. The heterotic prepotential (3.25) is determined by the fourier coefficients of $f^{(r,s)}$ as in (3.27). For order 2 CHL orbifolds they are modular forms of weight 10 under $\Gamma_0(2)$ where $(r,s) \in \{(0,0),(0,1),(1,0),(1,1)\}$. Since as mentioned before such forms span a three dimensional vector space, the heterotic prepotential should depend on three parameters only. Assuming string duality holds, it is sufficient to compare a small number of terms in the prepotentials to guarantee the matching of the full expressions. For the two examples stated in table 1 we find CY_3's that give matching prepotentials.

Concretely, for the first example in table 1, we find that the CY_3 is described by the four dimensional reflexive polyhedron with points/vertices given by

$$\begin{aligned} &p_1 = (1,0,0,0), && p_2 = (1,2,0,0), && p_3 = (1,2,4,0), \\ &p_4 = (1,2,0,8), && p_5 = (-7,-6,-4,-8) \end{aligned} \tag{5.1}$$

and has a triangulation which leads to the Mori cone generated by

$$l^1 = (0,1,0,0,1,-2,0),\ l^2 = (0,0,2,0,0,1,1),\ l^3 = (1,0,0,1,0,0,-2). \tag{5.2}$$

After redefining

$$q_1 \to \sqrt{q_T}/q_U, q_2 \to q_U, q_3 \to q_S, \tag{5.3}$$

in (4.1) and using (4.2) we obtain the following non-perturbative corrections to the prepotential (omitting the constant term)

$$\begin{aligned} F^{\text{inst}}(T,U) \propto\ & 320\,\text{Li}_3(\sqrt{q_T}) + 160\,\text{Li}_3(q_T) + 320\,\text{Li}_3(q_T^{3/2}) + 160\,\text{Li}_3(q_T^2) + 320\,\text{Li}_3(q_T^{5/2}) \\ & + 160\,\text{Li}_3(q_T^3) + 320\,\text{Li}_3(q_T^{7/2}) + 4\,\text{Li}_3(\sqrt{q_T}/q_U) + 320\,\text{Li}_3(q_U) \\ & + 9712\,\text{Li}_3(\sqrt{q_T}q_U) + 143872\,\text{Li}_3(q_T q_U) + 1243176\,\text{Li}_3(q_T^{3/2}q_U) \\ & + 8565248\,\text{Li}_3(q_T^2 q_U) + 48219040\,\text{Li}_3(q_T^{5/2}q_U) + 239069184\,\text{Li}_3(q_T^3 q_U) \\ & + 1058238684\,\text{Li}_3(q_T^{7/2}q_U) + 160\,\text{Li}_3(q_U^2) + 143872\,\text{Li}_3(\sqrt{q_T}q_U^2) \\ & + 8493312\,\text{Li}_3(q_T q_U^2) + 239069184\,\text{Li}_3(q_T^{3/2}q_U^2) + 4301347072\ \text{Li}_3(q_T^2 q_U^2) \\ & + 57667865600\,\text{Li}_3(q_T^{5/2}q_U^2) + \dots \end{aligned} \tag{5.4}$$

This matches perfectly with (3.25) using (3.20) and $\hat{b} = \frac{2}{3}$.[27]

Similarly, for the second example in table 1, we find that the CY_3 is described by the reflexive polyhedron

$$\begin{aligned} &p_1 = (1,0,0,0), && p_2 = (1,2,0,0), && p_3 = (1,2,4,0), \\ &p_4 = (1,2,0,4), && p_5 = (-3,-2,0,-4), && p_6 = (-3,-2,-4,0), \\ &p_7 = (-3,-2,0,0), && p_8 = (-3,-2,-4,4), && p_9 = (1,2,4,-4) \end{aligned} \tag{5.5}$$

and has three different triangulations, one of which gives the Mori cone generated by

$$l^1 = (2,1,0,0,0,0,1),\ l^2 = (0,-1,0,1,1,0,-1),\ l^3 = (0,-1,1,0,0,1,-1). \tag{5.6}$$

[27] This can easily be checked with, e.g., Mathematica.

JHEP02(2020)082

As before, we redefine

$$q_1 \to q_U, q_2 \to q_S, q_3 \to \sqrt{q_T}/q_U \tag{5.7}$$

and we find that the first nonpertubative corrrections to the prepotential for this case are (omitting the constant term)

$$\begin{aligned} F^{\text{inst}}(T,U) \propto\ & 256\text{Li}_3(\sqrt{q_T}) + 224\text{Li}_3(q_T) + 256\text{Li}_3(q_T^{3/2}) + 224\text{Li}_3(q_T^2) + 256\,\text{Li}_3(q_T^{5/2}) \\ & + 224\,\text{Li}_3(q_T^3) + 256\,\text{Li}_3(q_T^{7/2}) + 256\,\text{Li}_3(q_U) + 10240\text{Li}_3(\sqrt{q_T}q_U) \\ & + 141312\,\text{Li}_3(q_T q_U) + 1253376\,\text{Li}_3(q_T^{3/2}q_U) + 8529920\,\text{Li}_3(q_T^2 q_U) \\ & + 48328704\,\text{Li}_3(q_T^{5/2}q_U) + 238755840\,\text{Li}_3(q_T^3 q_U) + 1059078144\,\text{Li}_3(q_T^{7/2}q_U) \\ & + 224\,\text{Li}_3(q_U^2) + 141312\,\text{Li}_3(\sqrt{q_T}q_U^2) + 8528640\,\text{Li}_3(q_T q_U^2) \\ & + 238755840\,\text{Li}_3(q_T^{3/2}q_U^2) + 4303479552\,\text{Li}_3(q_T^2 q_U^2) \\ & + 57655783424\,\text{Li}_3(q_T^{5/2}q_U^2) + \ldots \end{aligned} \tag{5.8}$$

This agrees with (3.25) using (3.20) and $\hat{b} = \frac{8}{9}$.

6 Conclusions

Generalized Mathieu moonshine is a fascinating subject connecting the elliptic genus of $K3$ to the $\mathbb{M}_{24}$ sporadic group. In the process of its understanding a host of connections between various topics have been established. In particular, a connection between Mathieu moonshine, Gromov-Witten invariants, and the periods of the holomorphic 3-form of CY_3 has been found in [17, 72].

Here we add to this relation by connecting generalized Mathieu moonshine to Gromov-Witten invariants of CY_3. We find an explicit connection between the twisted-twined elliptic genera of $K3$ and Gromov-Witten invariants of CY_3. This connection is established via the heterotic-type IIA string duality. Based on [21, 46], we study CHL orbifolds of $E_8 \times E_8$ heterotic compactification on $K3 \times T^2$. The vectormoduli prepotential in these theories can be obtained from the new supersymmetric index which admits an expansion in terms of the twisted-twined elliptic genera, which are the central objects of generalized Mathieu moonshine [25]. By heterotic-type IIA string duality, these theories are dual to type IIA string compactifications on some CY_3. The coefficients of the non-perturbative corrections to the vectormoduli prepotential are given by the genus zero Gromov-Witten invariants, which count in an appropriate way the number of different embeddings of the sphere into the CY_3. We find two explicit examples for this duality for the case of CHL orbifolds of order 2. In both examples the spectrum of the heterotic theory can be completely higgsed. In particular we only look for CY_3's realized as hypersurfaces in a toric ambient space.

Building on the knowledge of the heterotic-type IIA duality for the non-orbifolded case [23, 24], it would be interesting to see explicitly how the CHL orbifold action acts on the type IIA side. The two examples of dual CY_3's found here could be a starting point for this. Knowledge thereof would be helpful in constructing further explicit examples, and in the process strengthen the evidence of string-string dualities. Lastly, finding the dual CY_3's for other CHL orbifolds would be an interesting avenue of research and would further deepen the connections established in this paper and in [20, 21, 46].

JHEP02(2020)082

Acknowledgments

We would like to thank Gabriel Cardoso, Aradhita Chattopadhyaya, Justin David, Albrecht Klemm, Johanna Knapp, Suresh Nampuri, Harald Skarke and Timm Wrase for illuminating discussions. We also thank Timm Wrase for comments on the draft. We thank the referee for many helpful comments on the draft. This work is supported by the Austrian Science Fund (FWF): P 28552 and by the OeAD 'Scientific & Technological Cooperation with India' grant Project IN 27/2018 titled 'Mathieu moonshine and $\mathcal{N} = 2$ Heterotic-Type II string duality'. The authors thank the Erwin Schrödinger International Institute for Mathematics and Physics for their hospitality during the workshop 'Moonshine' where part of this work was done. AK thanks the Galileo Galilei Institute for Theoretical Physics, Florence for the hospitality and the INFN for partial support during the beginning of this work.

A Conventions

A.1 Jacobi theta functions

The Jacobi theta functions $\theta_i(q, y),\ i = 1, \cdots, 4$ are defined as

$$\begin{aligned}
\theta_1(q,y) &= -\mathrm{i} \sum_{n+\frac{1}{2}\in\mathbb{Z}} (-1)^{n-\frac{1}{2}}\, y^n q^{\frac{n^2}{2}} \\
&= -\mathrm{i} q^{\frac{1}{8}} \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1-q^n)\left(1-yq^n\right)\left(1-y^{-1}q^n\right) , \qquad \text{(A.1)} \\
\theta_2(q,y) &= \sum_{n+\frac{1}{2}\in\mathbb{Z}} y^n q^{\frac{n^2}{2}} \\
&= q^{\frac{1}{8}} \left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} (1-q^n)\left(1+yq^n\right)\left(1+y^{-1}q^n\right) , \qquad \text{(A.2)} \\
\theta_3(q,y) &= \sum_{n\in\mathbb{Z}} y^n q^{\frac{n^2}{2}} \\
&= \prod_{n=1}^{\infty} (1-q^n)\left(1+yq^{n-\frac{1}{2}}\right)\left(1+y^{-1}q^{n-\frac{1}{2}}\right) , \qquad \text{(A.3)} \\
\theta_4(q,y) &= \sum_{n\in\mathbb{Z}} (-1)^n\, y^n q^{\frac{n^2}{2}} \\
&= \prod_{n=1}^{\infty} (1-q^n)\left(1-yq^{n-\frac{1}{2}}\right)\left(1-y^{-1}q^{n-\frac{1}{2}}\right) , \qquad \text{(A.4)}
\end{aligned}$$

where we have used $q = e^{2\pi\mathrm{i}\tau}$ and $y = e^{2\pi\mathrm{i}z}$. We also use the notation for the truncated Jacobi theta function $\theta_i(q) = \theta_i(q, y = 1),\ i = 1, 2, 3, 4$. The truncated Jacobi theta function can be used to define the Dedekind eta function as

$$\eta(q) = \left(\frac{1}{2}\theta_2(q)\theta_3(q)\theta_4(q)\right)^{\frac{1}{3}} = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \ . \qquad \text{(A.5)}$$

JHEP02(2020)082

A.2 Eisenstein series

The Eisenstein series of weights 4 and 6 can likewise be expressed in terms of the $\theta_i(q)$ functions

$$E_4(q) = \frac{1}{2}\sum_{i=2}^{4}\theta_i(q)^8\,, \tag{A.6}$$

$$E_6(q) = \frac{1}{2}\left(\theta_4(q)^8\left(\theta_2(q)^4+\theta_3(q)^4\right)+\theta_3(q)^8\left(\theta_4(q)^4-\theta_2(q)^4\right)-\theta_2(q)^8\left(\theta_3(q)^4+\theta_4(q)^4\right)\right). \tag{A.7}$$

The Eisenstein series have the following Fourier decompositions

$$E_4(q) = 1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1+240q+2160q^2+\dots\,, \tag{A.8}$$

$$E_6(q) = 1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1-504q-16632q^2+\dots. \tag{A.9}$$

B Modular properties of Jacobi forms and $\mathcal{E}_N(\tau)$

- A weak Jacobi form $\phi(\tau,z)$ of weight w and index m defined on $\mathbb{H}\times\mathbb{C}$ is characterized by the following transformation properties

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = (c\tau+d)^w e^{2\pi i m\frac{cz^2}{c\tau+d}}\phi(\tau,z)\,,\ \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}), \tag{B.1}$$

$$\phi(\tau,z+l\tau+l') = e^{-2\pi i m(l^2\tau+2lz)}\phi(\tau,z),\ l,l'\in\mathbb{Z}\,. \tag{B.2}$$

- $\mathcal{E}_N$ is defined by

$$\mathcal{E}_N(\tau) = \frac{12i}{\pi(N-1)}\partial_\tau\ln\frac{\eta(\tau)}{\eta(N\tau)}. \tag{B.3}$$

  It is a modular form of weight 2 under $\Gamma_0(N)$. It also has the following transformation properties [20]

$$\mathcal{E}_N(\tau+1) = \mathcal{E}_N(\tau)\,,\ \mathcal{E}_N(-1/\tau) = -\tau^2\frac{1}{N}\mathcal{E}_N(\tau/N). \tag{B.4}$$

  Further for $N=2$, the following can be deduced [20]

$$\mathcal{E}_2\left(-\frac{1}{2\tau}\right) = -2\tau^2\mathcal{E}_2(\tau)\,,\ \mathcal{E}_2\left(-\frac{1}{2\tau}+\frac{1}{2}\right) = \tau^2\mathcal{E}_2\left(\frac{\tau+1}{2}\right). \tag{B.5}$$

C $\mathcal{N}=4$ superconformal characters

The (graded) characters of $\mathcal{N}=4$ superconformal algebra with central charge c in the Ramond sector are defined as

$$\mathrm{ch}^{\mathcal{N}=4}_{c,h-\frac{c}{24},\ell}(q,y) = \mathrm{tr}_{\mathcal{H}_{h,\ell}}\left((-1)^F q^{L_0-\frac{c}{24}}e^{4\pi\mathrm{i}zJ_0}\right), \tag{C.1}$$

JHEP02(2020)082

where h and ℓ are respectively the eigenvalues of L_0 (conformal weight) and J_0 (iso-spin), of the highest weight state belonging to the representation $\mathcal{H}_{h,\ell}$. Unitarity requires $h \geq \frac{c}{24}$.

The characters [73] are given by (using conventions from [31])

- Massless representation exist for $h = \frac{c}{24}$, $\ell = 0, \frac{1}{2}, \ldots, \frac{c}{12}$ and are given by

$$\mathrm{ch}^{\mathcal{N}=4}_{c,0,\ell}(q,y) = \frac{\mathrm{i}}{\theta_1(\tau,2z)} \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3} \sum_{\varepsilon=\pm 1} \sum_{m\in\mathbb{Z}} \varepsilon \frac{e^{4\pi\mathrm{i}\varepsilon((\frac{c}{6}+1)m+\ell)(z+\frac{1}{2})}}{(1-y^{-\varepsilon}q^{-m})^2} q^{(\frac{c}{6}+1)m^2+2\ell m}. \tag{C.2}$$

  In particular for $\ell = 0$ this may be written as

$$\mathrm{ch}^{\mathcal{N}=4}_{c,0,0}(q,y) = \frac{-\mathrm{i}}{\theta_1(\tau,2z)} \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3} \sum_{m\in\mathbb{Z}} q^{(\frac{c}{6}+1)m^2} y^{(\frac{c}{3}+2)m} \frac{1+yq^m}{1-yq^m}. \tag{C.3}$$

- Massive representation exist for $h > \frac{c}{24}$, $\ell = \frac{1}{2}, 1, \ldots, \frac{c}{12}$ and are given by

$$\mathrm{ch}^{\mathcal{N}=4}_{c,h-\frac{c}{24},\ell}(q,y) = \mathrm{i} q^{h-\frac{6\ell^2}{c+6}-\frac{c}{24}} \frac{\theta_1(\tau,z)^2}{\theta_1(\tau,2z)\eta(\tau)^3} \left(\vartheta_{\frac{c}{6}+1,2\ell}\left(\tau, z+\frac{1}{2}\right) - \vartheta_{\frac{c}{6}+1,-2\ell}\left(\tau, z+\frac{1}{2}\right)\right), \tag{C.4}$$

  where

$$\vartheta_{P,a}(q,y) = \sum_{n\in\mathbb{Z}} q^{\frac{(2Pn+a)^2}{4P}} y^{2Pn+a}. \tag{C.5}$$

At the unitarity bound $h = \frac{c}{24}$ the massive $\mathcal{N}=4$ characters decompose into massless ones according to

$$\lim_{h\searrow\frac{c}{24}} \mathrm{ch}^{\mathcal{N}=4}_{c,h-\frac{c}{24},l}(q,y) = \mathrm{ch}^{\mathcal{N}=4}_{c,0,l}(q,y) + 2\mathrm{ch}^{\mathcal{N}=4}_{c,0,l-\frac{1}{2}}(q,y) + \mathrm{ch}^{\mathcal{N}=4}_{c,0,l-1}(q,y). \tag{C.6}$$

D Explicit expansion of $\mathcal{Z}^{\mathbf{new},2}$

In this appendix we give the explicit relationships between $f^{(r,s)}$, defined in (3.15),and the twisted-twined elliptic genus of $K3$ for the 2A orbifold appearing in our discussion. To make expressions more readable we only write the parts of $f^{(r,s)}$ that are relevant for this connection.

By comparing (3.15), (3.16) to (3.11) we get the following expansions in terms of the twisted-twined elliptic genera $\mathcal{Z}^{\mathrm{ell}}_{(r,s)}$

$r = s = 0$:

$$\frac{-4}{\eta(q)^{12}} E_6(q) = \left[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathcal{Z}^{\mathrm{ell}}_{K3,0,0}(q,-1) + \left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{1/4} \mathcal{Z}^{\mathrm{ell}}_{K3,0,0}(q,-q^{\frac{1}{2}}) - \left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{1/4} \mathcal{Z}^{\mathrm{ell}}_{K3,0,0}(q,q^{\frac{1}{2}}) \right] \tag{D.1}$$

JHEP02(2020)082

$\boldsymbol{r=0, s=1}$**:**

$$\frac{-4}{\eta(q)^{12}}\frac{1}{3}\left[E_6(q)+2\mathcal{E}_2(\tau)E_4(q)\right]=\left[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathcal{Z}^{\text{ell}}_{K3,0,1}(q,-1)\right. \tag{D.2}$$
$$\left.+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{\text{ell}}_{K3,0,1}(q,-q^{\frac{1}{2}})-\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{\text{ell}}_{K3,0,1}(q,q^{\frac{1}{2}})\right]$$

$\boldsymbol{r=1, s=0}$**:**

$$\frac{-4}{\eta(q)^{12}}\frac{1}{3}\left[E_6(q)-\mathcal{E}_2\left(\frac{\tau}{2}\right)E_4(q)\right]=\left[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathcal{Z}^{\text{ell}}_{K3,1,0}(q,-1)\right. \tag{D.3}$$
$$\left.+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{\text{ell}}_{K3,1,0}(q,-q^{\frac{1}{2}})-\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{\text{ell}}_{K3,1,0}(q,q^{\frac{1}{2}})\right]$$

$\boldsymbol{r=s=1}$**:**

$$\frac{-4}{\eta(q)^{12}}\frac{1}{3}\left[E_6(q)-\mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4(q)\right]=\left[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathcal{Z}^{\text{ell}}_{K3,1,1}(q,-1)\right. \tag{D.4}$$
$$\left.+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{\text{ell}}_{K3,1,1}(q,-q^{\frac{1}{2}})-\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{1/4}\mathcal{Z}^{\text{ell}}_{K3,1,1}(q,q^{\frac{1}{2}})\right]$$

As these are the expressions that show up in $\mathcal{Z}^{\text{new}}$ for both the standard and the non-standard embedding they connect $\mathcal{Z}^{\text{new}}$ to Mathieu moonshine.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] J.H. Conway and S.P. Norton, *Monstrous Moonshine*, *Bull. Lond. Math. Soc.* **11** (1979) 308.

[2] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the monster*, Academic Press (1988) [INSPIRE].

[3] J. Leech, *Notes on sphere packings*, *Can. J. Math.* **19** (1967) 251.

[4] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group M_{24}*, *Exper. Math.* **20** (2011) 91 [arXiv:1004.0956] [INSPIRE].

[5] M.C.N. Cheng, *K3 Surfaces, $N=4$ Dyons and the Mathieu Group M24*, *Commun. Num. Theor. Phys.* **4** (2010) 623 [arXiv:1005.5415] [INSPIRE].

[6] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu twining characters for K3*, *JHEP* **09** (2010) 058 [arXiv:1006.0221] [INSPIRE].

[7] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu Moonshine in the elliptic genus of K3*, *JHEP* **10** (2010) 062 [arXiv:1008.3778] [INSPIRE].

JHEP02(2020)082

[8] T. Eguchi and K. Hikami, *Note on twisted elliptic genus of $K3$ surface*, *Phys. Lett.* **B 694** (2011) 446 [arXiv:1008.4924] [INSPIRE].

[9] T. Gannon, *Much ado about Mathieu*, *Adv. Math.* **301** (2016) 322 [arXiv:1211.5531] [INSPIRE].

[10] J.F.R. Duncan, M.J. Griffin and K. Ono, *Moonshine*, arXiv:1411.6571 [INSPIRE].

[11] S. Kachru, *Elementary introduction to Moonshine*, 2016, arXiv:1605.00697 [INSPIRE].

[12] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, *Umbral Moonshine*, *Commun. Num. Theor. Phys.* **08** (2014) 101 [arXiv:1204.2779] [INSPIRE].

[13] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, *Umbral Moonshine and the Niemeier Lattices*, arXiv:1307.5793 [INSPIRE].

[14] J.A. Harvey and B.C. Rayhaun, *Traces of Singular Moduli and Moonshine for the Thompson Group*, *Commun. Num. Theor. Phys.* **10** (2016) 23 [arXiv:1504.08179] [INSPIRE].

[15] J.F.R. Duncan, M.H. Mertens and K. Ono, *O'nan moonshine and arithmetic*, arXiv:1702.03516.

[16] A. Banlaki, A. Chowdhury, A. Kidambi, M. Schimpf, H. Skarke and T. Wrase, *Calabi-Yau manifolds and sporadic groups*, *JHEP* **02** (2018) 129 [arXiv:1711.09698] [INSPIRE].

[17] M.C.N. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru and T. Wrase, *Mathieu Moonshine and $N = 2$ String Compactifications*, *JHEP* **09** (2013) 030 [arXiv:1306.4981] [INSPIRE].

[18] S. Cecotti, P. Fendley, K.A. Intriligator and C. Vafa, *A New supersymmetric index*, *Nucl. Phys.* **B 386** (1992) 405 [hep-th/9204102] [INSPIRE].

[19] S. Chaudhuri, G. Hockney and J.D. Lykken, *Maximally supersymmetric string theories in $D < 10$*, *Phys. Rev. Lett.* **75** (1995) 2264 [hep-th/9505054] [INSPIRE].

[20] S. Datta, J.R. David and D. Lüst, *Heterotic string on the CHL orbifold of $K3$*, *JHEP* **02** (2016) 056 [arXiv:1510.05425] [INSPIRE].

[21] A. Chattopadhyaya and J.R. David, *$\mathcal{N} = 2$ heterotic string compactifications on orbifolds of $K3 \times T^2$*, *JHEP* **01** (2017) 037 [arXiv:1611.01893] [INSPIRE].

[22] A. Chattopadhyaya and J.R. David, *Dyon degeneracies from Mathieu moonshine symmetry*, *Phys. Rev.* **D 96** (2017) 086020 [arXiv:1704.00434] [INSPIRE].

[23] S. Kachru and C. Vafa, *Exact results for $N = 2$ compactifications of heterotic strings*, *Nucl. Phys.* **B 450** (1995) 69 [hep-th/9505105] [INSPIRE].

[24] S. Ferrara, J.A. Harvey, A. Strominger and C. Vafa, *Second quantized mirror symmetry*, *Phys. Lett.* **B 361** (1995) 59 [hep-th/9505162] [INSPIRE].

[25] M.R. Gaberdiel, D. Persson, H. Ronellenfitsch and R. Volpato, *Generalized Mathieu Moonshine*, *Commun. Num. Theor Phys.* **07** (2013) 145 [arXiv:1211.7074] [INSPIRE].

[26] E. Witten, *Elliptic Genera and Quantum Field Theory*, *Commun. Math. Phys.* **109** (1987) 525 [INSPIRE].

[27] J. Troost, *The non-compact elliptic genus: mock or modular*, *JHEP* **06** (2010) 104 [arXiv:1004.3649] [INSPIRE].

[28] M. Kontsevich and Y. Soibelman, *Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and Mirror Symmetry*, *Lect. Notes Union. Mat. Ital.* **15** (2014) 197 [arXiv:1303.3253] [INSPIRE].

JHEP02(2020)082

[29] S. Cecotti, *Trieste lectures on wall-crossing invariants*, https://people.sissa.it/~cecotti/ictptext.pdf (2010).

[30] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal Algebras and String Compactification on Manifolds with* SU(N) *Holonomy*, *Nucl. Phys.* **B 315** (1989) 193 [INSPIRE].

[31] T. Eguchi and K. Hikami, $N = 4$ *Superconformal Algebra and the Entropy of HyperKähler Manifolds*, *JHEP* **02** (2010) 019 [arXiv:0909.0410] [INSPIRE].

[32] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, *Invent. Math.* **94** (1988) 183.

[33] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Symmetries of* $K3$ σ*-models*, *Commun. Num. Theor. Phys.* **6** (2012) 1 [arXiv:1106.4315] [INSPIRE].

[34] M.C.N. Cheng and S. Harrison, *Umbral Moonshine and* $K3$ *Surfaces*, *Commun. Math. Phys.* **339** (2015) 221 [arXiv:1406.0619] [INSPIRE].

[35] J.F.R. Duncan and S. Mack-Crane, *Derived Equivalences of* $K3$ *Surfaces and Twined Elliptic Genera*, arXiv:1506.06198 [INSPIRE].

[36] M.C.N. Cheng, S.M. Harrison, R. Volpato and M. Zimet, $K3$ *String Theory, Lattices and Moonshine*, arXiv:1612.04404 [INSPIRE].

[37] N.M. Paquette, R. Volpato and M. Zimet, *No More Walls! A Tale of Modularity, Symmetry and Wall Crossing for* 1/4 *BPS Dyons*, *JHEP* **05** (2017) 047 [arXiv:1702.05095] [INSPIRE].

[38] A. Taormina and K. Wendland, *The overarching finite symmetry group of Kummer surfaces in the Mathieu group* M_{24}, *JHEP* **08** (2013) 125 [arXiv:1107.3834] [INSPIRE].

[39] A. Taormina and K. Wendland, *Symmetry-surfing the moduli space of Kummer* $K3$*s*, *Proc. Symp. Pure Math.* **90** (2015) 129 [arXiv:1303.2931] [INSPIRE].

[40] M.R. Gaberdiel, C.A. Keller and H. Paul, *Mathieu Moonshine and Symmetry Surfing*, *J. Phys.* **A 50** (2017) 474002 [arXiv:1609.09302] [INSPIRE].

[41] S. Kachru, N.M. Paquette and R. Volpato, 3D *String Theory and Umbral Moonshine*, *J. Phys.* **A 50** (2017) 404003 [arXiv:1603.07330] [INSPIRE].

[42] S. Carnahan, *Generalized moonshine i: Genus zero functions*, arXiv:0812.3440.

[43] S. Carnahan, *Generalized moonshine II: Borcherds products*, *Duke Math. J.* **161** (2012) 893 [arXiv:0908.4223] [INSPIRE].

[44] S. H.-L. Carnahan, *Generalized Moonshine IV: Monstrous Lie algebras*, arXiv:1208.6254 [INSPIRE].

[45] J.R. David, D.P. Jatkar and A. Sen, *Product representation of Dyon partition function in CHL models*, *JHEP* **06** (2006) 064 [hep-th/0602254] [INSPIRE].

[46] A. Chattopadhyaya and J.R. David, *Gravitational couplings in* $\mathcal{N} = 2$ *string compactifications and Mathieu Moonshine*, *JHEP* **05** (2018) 211 [arXiv:1712.08791] [INSPIRE].

[47] J.A. Harvey and G.W. Moore, *Algebras, BPS states and strings*, *Nucl. Phys.* **B 463** (1996) 315 [hep-th/9510182] [INSPIRE].

[48] S. Stieberger, *(0,2) heterotic gauge couplings and their M-theory origin*, *Nucl. Phys.* **B 541** (1999) 109 [hep-th/9807124] [INSPIRE].

JHEP02(2020)082

[49] S. Harrison, S. Kachru and N.M. Paquette, *Twining Genera of* $(0,4)$ *Supersymmetric* σ*-models on* $K3$, *JHEP* **04** (2014) 048 [arXiv:1309.0510] [INSPIRE].

[50] G. Lopes Cardoso, G. Curio and D. Lüst, *Perturbative couplings and modular forms in* $N = 2$ *string models with a Wilson line*, *Nucl. Phys.* **B 491** (1997) 147 [hep-th/9608154] [INSPIRE].

[51] I. Antoniadis, E. Gava and K.S. Narain, *Moduli corrections to gravitational couplings from string loops*, *Phys. Lett.* **B 283** (1992) 209 [hep-th/9203071] [INSPIRE].

[52] I. Antoniadis, E. Gava and K.S. Narain, *Moduli corrections to gauge and gravitational couplings in four-dimensional superstrings*, *Nucl. Phys.* **B 383** (1992) 93 [hep-th/9204030] [INSPIRE].

[53] S. Cecotti and C. Vafa, *Ising model and* $N = 2$ *supersymmetric theories*, *Commun. Math. Phys.* **157** (1993) 139 [hep-th/9209085] [INSPIRE].

[54] P.S. Aspinwall, *Some relationships between dualities in string theory*, *Nucl. Phys. Proc. Suppl.* **46** (1996) 30 [hep-th/9508154] [INSPIRE].

[55] M.R. Gaberdiel and R. Volpato, *Mathieu Moonshine and Orbifold* $K3$*s*, *Contrib. Math. Comput. Sci.* **8** (2014) 109 [arXiv:1206.5143] [INSPIRE].

[56] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, $N = 2$ *type-II heterotic duality and higher derivative* F *terms*, *Nucl. Phys.* **B 455** (1995) 109 [hep-th/9507115] [INSPIRE].

[57] M. Mariño and G.W. Moore, *Counting higher genus curves in a Calabi-Yau manifold*, *Nucl. Phys.* **B 543** (1999) 592 [hep-th/9808131] [INSPIRE].

[58] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140] [INSPIRE].

[59] I. Florakis and B. Pioline, *On the Rankin-Selberg method for higher genus string amplitudes*, *Commun. Num. Theor. Phys.* **11** (2017) 337 [arXiv:1602.00308] [INSPIRE].

[60] G. Aldazabal, A. Font, L.E. Ibáñez and F. Quevedo, *Chains of* $N = 2$, $D = 4$ *heterotic type-II duals*, *Nucl. Phys.* **B 461** (1996) 85 [hep-th/9510093] [INSPIRE].

[61] C. Nazaroglu, *Jacobi forms of higher index and paramodular groups in* $\mathcal{N} = 2, D = 4$ *compactifications of string theory*, *JHEP* **12** (2013) 074 [arXiv:1309.6404] [INSPIRE].

[62] N. Yamatsu, *Finite-Dimensional Lie Algebras and Their Representations for Unified Model Building*, arXiv:1511.08771 [INSPIRE].

[63] P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys.* **B 359** (1991) 21 [INSPIRE].

[64] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, *Commun. Math. Phys.* **167** (1995) 301 [hep-th/9308122] [INSPIRE].

[65] R. Gopakumar and C. Vafa, *M theory and topological strings. 1.*, hep-th/9809187 [INSPIRE].

[66] R. Gopakumar and C. Vafa, *M theory and topological strings. 2.*, hep-th/9812127 [INSPIRE].

[67] M. Mariño, *Enumerative geometry and knot invariants*, in *70th Meeting between Physicists, Theorist and Mathematicians*, Strasbourg, France, 23–25 May 2002 (2002), [hep-th/0210145] [INSPIRE].

JHEP02(2020)082

[68] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, *Nucl. Phys.* **B 433** (1995) 501 [hep-th/9406055] [INSPIRE].

[69] A.P. Braun, J. Knapp, E. Scheidegger, H. Skarke and N.-O. Walliser, *PALP — a User Manual*, in *Strings, gauge fields and the geometry behind: The legacy of Maximilian Kreuzer*, A. Rebhan, L. Katzarkov, J. Knapp, R. Rashkov and E. Scheidegger eds., pp. 461–550 (2012) [DOI] [arXiv:1205.4147] [INSPIRE].

[70] J. Rambau, *TOPCOM: Triangulations of point configurations and oriented matroids*, in *Mathematical Software — ICMS 2002*, A.M. Cohen, X.-S. Gao and N. Takayama eds., pp. 330–340, World Scientific (2002).

[71] M. Kreuzer and H. Skarke, *Calabi-Yau data*, http://hep.itp.tuwien.ac.at/~kreuzer/CY/.

[72] N.M. Paquette and T. Wrase, *Comments on M_{24} representations and CY_3 geometries*, *JHEP* **11** (2014) 155 [arXiv:1409.1540] [INSPIRE].

[73] T. Eguchi and A. Taormina, *Character Formulas for the $N = 4$ Superconformal Algebra*, *Phys. Lett.* **B 200** (1988) 315 [INSPIRE].

JHEP02(2020)082

MOONSHINE

JOHN F. R. DUNCAN, MICHAEL J. GRIFFIN AND KEN ONO

Abstract. *Monstrous moonshine* relates distinguished modular functions to the representation theory of the Monster $\mathbb{M}$. The celebrated observations that

$$(*) \qquad 1 = 1, \quad 196884 = 1 + 196883, \quad 21493760 = 1 + 196883 + 21296876, \quad \ldots\ldots$$

illustrate the case of $J(\tau) = j(\tau) - 744$, whose coefficients turn out to be sums of the dimensions of the 194 irreducible representations of $\mathbb{M}$. Such formulas are dictated by the structure of the graded monstrous moonshine modules. Recent works in moonshine suggest deep relations between number theory and physics. Number theoretic Kloosterman sums have reappeared in quantum gravity, and mock modular forms have emerged as candidates for the computation of black hole degeneracies. This paper is a survey of past and present research on moonshine. We also compute the quantum dimensions of the monster orbifold, and obtain exact formulas for the multiplicities of the irreducible components of the moonshine modules. These formulas imply that such multiplicities are asymptotically proportional to dimensions. For example, the proportion of 1's in (*) tends to

$$\frac{\dim(\chi_1)}{\sum_{i=1}^{194} \dim(\chi_i)} = \frac{1}{58440767853045028080136021 36} = 1.711\ldots \times 10^{-28}.$$

1. Introduction

This story begins innocently with peculiar numerics, and in its present form exhibits connections to conformal field theory, string theory, quantum gravity, and the arithmetic of mock modular forms. This paper is an introduction to the many facets of this beautiful theory.

We begin with a review of the principal results in the development of monstrous moonshine in §§2-4. We refer to the introduction of [105], and the more recent survey [117], for more on these topics. After describing these classic works, we discuss the interplay between moonshine and Rademacher sums in §5, and related observations which suggest a connection between monstrous moonshine and three-dimensional quantum gravity in

---

2010 *Mathematics Subject Classification.* 11F11, 11F22, 11F37, 11F50, 20C34, 20C35.

The authors are supported by the NSF. The first author also thanks the Simons Foundation (#316779), and the third author also thanks the support of the Asa Griggs Candler Fund. The authors thank John McKay and and John Thompson for answering questions about the history related to the study of distributions within the moonshine module. Thanks are due also to Miranda Cheng, Igor Frenkel, Bob Griess, Daniel Grumiller, Jeff Harvey, Shamit Kachru, Barry Mazur, John McKay, Jean-Pierre Serre and the anonymous referee, for comments, corrections and excellent suggestions.

§6. The remainder of the paper is devoted to more recent works. We describe a generalization of moonshine, the *moonshine tower*, in §7, and in §8 we compute the quantum dimensions of the monster orbifold and discuss the distributions of irreducible representations of the monster arising in monstrous moonshine. We present a survey of the recently discovered *umbral moonshine* phenomenon in §9, and conclude, in §10, with problems for future work.

2. Early Days

The classification of finite simple groups [2] distinguishes twenty-six examples above the others; namely, the *sporadic* simple groups, which are those that belong to none of the naturally occurring infinite families: cyclic groups, alternating groups, or finite groups of Lie type. Distinguished amongst the sporadic simple groups is the *Fischer–Griess monster* $\mathbb{M}$, on account of its size, which is

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \tag{2.1}$$

(cf. [120]). Note that the margin is not small, for the order of the monster is

$$2^{5} \cdot 3^{7} \cdot 5^{3} \cdot 7^{4} \cdot 11 \cdot 13^{2} \cdot 29 \cdot 41 \cdot 59 \cdot 71 \tag{2.2}$$

times that of the next largest sporadic simple group, the baby monster (cf. [170]).

Fischer and Griess independently produced evidence for the monster group in 1973 (cf. [120]). Well before it was proven to exist, Tits gave a lecture on its conjectural properties at the Collège de France in 1975. In particular, he described its order (2.1). Around this time, Ogg had been considering the automorphism groups of certain algebraic curves, and had arrived at the set of primes

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\} \tag{2.3}$$

in a purely geometric way (cf. the Corollaire of [194]). Making what may now be identified as the first observation of monstrous moonshine, Ogg offered a bottle of Jack Daniels[1] for an explanation of this coincidence (cf. Remarque 1 of [194]).

Ogg's observation would ultimately be recognized as reflecting another respect in which the monster is distinguished amongst finite simple groups: as demonstrated by the pioneering construction of Frenkel–Lepowsky–Meurman [103, 104, 105], following the astonishing work of Griess [121, 122], the "most natural" representation of the monster, is infinite-dimensional.

The explanation of this statement takes us back to McKay's famous observation, that

$$196884 = 1 + 196883 \tag{2.4}$$

[1]We refer the reader to [91] for a recent analysis of the Jack Daniels problem.

(cf. [59, 222]), and the generalizations of this observed by Thompson [222], including[2]

$$\begin{aligned} 21493760 &= 1 + 196883 + 21296876, \\ 864299970 &= 2 \times 1 + 2 \times 196883 + 21296876 + 842609326, \\ 20245856256 &= 2 \times 1 + 3 \times 196883 + 2 \times 21296876 + 842609326 + 19360062527. \end{aligned} \tag{2.5}$$

Of course the left hand sides of (2.4) and (2.5) are familiar to number theorists and algebraic geometers, as coefficients in the Fourier coefficients of the *normalized elliptic modular invariant*

$$\begin{aligned} J(\tau) &:= \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} - 744 \\ &= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \dots \end{aligned} \tag{2.6}$$

Here $q := e^{2\pi i \tau}$, and we set $g_2(\tau) := 60G_4(\tau)$ and $g_3(\tau) := 140G_6(\tau)$, where $G_{2k}(\tau)$ denotes the Eisenstein series of weight $2k$,

$$G_{2k}(\tau) := \sum_{(m,n)\neq(0,0)} (m + n\tau)^{-2k}, \tag{2.7}$$

for $k \geq 2$. The functions g_2 and g_3 serve to translate between the two most common parameterizations of a complex elliptic curve: as a quotient $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ for τ in the upper-half plane, $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$, and as the locus of a Weierstrass equation, $y^2 = 4x^3 - g_2 x - g_3$.

The fundamental property of $J(\tau)$, from both the number theoretic and algebro-geometric points of view, is that it is a modular function for $\mathrm{SL}_2(\mathbb{Z})$. In fact, and importantly for the monster's natural infinite-dimensional representation, $J(\tau)$ is a generator for the field of $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic functions on $\mathbb{H}$ that have at most exponential growth as $\Im(\tau) \to \infty$.

The right hand sides of (2.4) and (2.5) are familiar to finite group theorists, as simple sums of dimensions of irreducible representations of the monster $\mathbb{M}$. In fact, the irreducible representations appearing in (2.4) and (2.5) are just the first six, of a total of

[2]As was pointed out to us by J.-P. Serre, the decomposition for 20245856256 that appears in [222] differs from that which develops from the monster's natural, infinite-dimensional representation. We reproduce Serre's decomposition in (2.5).

194, in the character table of $\mathbb{M}$ (cf. [57]), when ordered by size. We have that

$$\begin{aligned}
\chi_1(e) &= 1\\
\chi_2(e) &= 196883\\
\chi_3(e) &= 21296876\\
\chi_4(e) &= 842609326\\
\chi_5(e) &= 18538750076\\
\chi_6(e) &= 19360062527\\
&\vdots\\
\chi_{194}(e) &= 258823477531055064045234375.
\end{aligned} \tag{2.8}$$

Here e denotes the identity element of $\mathbb{M}$, so $\chi_i(e)$ is just the dimension of the irreducible representation of $\mathbb{M}$ with character χ_i.

3. Classical Moonshine

The coincidences (2.4) and (2.5) led Thompson to make the following conjecture [222] which realizes the natural representation of the monster alluded to above.

Conjecture 3.1 (Thompson)**.** There is a naturally defined graded infinite-dimensional monster module, denoted $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$, which satisfies

$$\dim(V_n^\natural) = c(n) \tag{3.1}$$

for $n \geq -1$ (Cf. (2.6)), such that the decompositions into irreducible representations of the monster satisfy (2.4) and (2.5) for $n = 1, 2, 3$ and 4 (and a similar condition for $n = 5$).

At the time that Thompson's conjecture was made, the monster had not yet been proven to exist, but Griess [120], and Conway–Norton [59], had independently conjectured the existence of a faithful representation of dimension 196883, and Fischer–Livingstone–Thorne had constructed the character table of $\mathbb{M}$, by assuming the validity of this claim (cf. [59]) together with conjectural statements (cf. [120]) about the structure of $\mathbb{M}$.

Thompson also suggested [221] to investigate the properties of the graded-trace functions

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|V_n^\natural)q^n, \tag{3.2}$$

for $g \in \mathbb{M}$, now called the *monstrous McKay–Thompson series*, that would arise from the conjectural monster module $V^\natural$. Using the character table constructed by Fischer–Livingstone–Thorne, it was observed [59, 221] that the functions T_g are in many cases directly similar to J in the following respect: the first few coefficients of each T_g coincide

with those of a generator for the function field of a discrete group[3] $\Gamma_g < \mathrm{SL}_2(\mathbb{R})$, commensurable with $\mathrm{SL}_2(\mathbb{Z})$, containing $-I$, and having *width one at infinity*, meaning that the subgroup of upper-triangular matrices in Γ_g coincides with

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}, \tag{3.3}$$

for all $g \in \mathbb{M}$.

This observation was refined and developed by Conway–Norton [59], leading to their famous *monstrous moonshine conjectures*:.

Conjecture 3.2 (Monstrous Moonshine: Conway–Norton)**.** For each $g \in \mathbb{M}$ there is a specific group $\Gamma_g < \mathrm{SL}_2(\mathbb{R})$ such that T_g is the unique *normalized principal modulus*[4] for Γ_g.

This means that each T_g is the unique Γ_g-invariant holomorphic function on $\mathbb{H}$ which satisfies

$$T_g(\tau) = q^{-1} + O(q), \tag{3.4}$$

as $\Im(\tau) \to \infty$, and remains bounded as τ approaches any non-infinite cusp of Γ_g. We refer to this feature of the T_g as the *principal modulus property* of monstrous moonshine.

The hypothesis that T_g is Γ_g-invariant, satisfying (3.4) near the infinite cusp of Γ_g but having no other poles, implies that T_g generates the field of Γ_g-invariant holomorphic functions on $\mathbb{H}$ that have at most exponential growth at cusps, in direct analogy with J. In particular, the natural Riemann surface structure on $\Gamma_g \backslash \mathbb{H}$ (cf. e.g. [212]) must be that of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with finitely many points removed, and for this reason the groups Γ_g are said to have *genus zero*, and the principal modulus property is often referred to as the *genus zero property* of monstrous moonshine.

The reader will note the astonishing predictive power that the principal modulus property of monstrous moonshine bestows: the fact that a normalized principal modulus for a genus zero group Γ_g is unique, means that we can compute the trace of an element $g \in \mathbb{M}$, on any homogeneous subspace of the monster's natural infinite-dimensional representation $V^\natural$, without any information about the monster, as soon as we can guess correctly the discrete group Γ_g. The analysis of Conway–Norton in [59] establishes very strong guidelines for the determination of Γ_g, and once Γ_g has been chosen, the "theory of replicability" (cf. [1, 59, 192]) allows for efficient computation of the coefficients of the normalized principal modulus T_g, given the knowledge of just a few of them (cf. [100], or (3.10)).

It was verified by Atkin–Fong–Smith [213], using results of Thompson [221] (cf. also [208]), that a graded (possibly virtual) infinite-dimensional monster module $V^\natural$, such that the functions T_g of (3.2) are exactly those predicted by Conway–Norton in [59], exists.

[3] The relevant groups Γ_g shall be discussed in detail in Section 8.1.

[4] A principal modulus is also referred to as a *Hauptmodul*.

Theorem 3.3 (Atkin–Fong–Smith)**.** *There exists a (possibly virtual) graded* $\mathbb{M}$*-module* $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *such that if* T_g *is defined by (3.2), then* T_g *is the Fourier expansion of the unique* Γ_g*-invariant holomorphic function on* $\mathbb{H}$ *that satisfies* $T_g(\tau) = q^{-1} + O(q)$ *as* τ *approaches the infinite cusp, and has no poles at any non-infinite cusps of* Γ_g*, where* Γ_g *is the discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *specified by Conway–Norton in [59].*

Thus Thompson's conjecture was confirmed, albeit indirectly. By this point in time, Griess, in an astonishing tour de force, had constructed the monster explicitly, by hand, by realizing it as the automorphism group of a commutative but non-associative algebra of dimension 196884 [121, 122]. (See also [56, 223].) Inspired by Griess' construction, and by the representation theory of affine Lie algebras, which also involves graded infinite-dimensional vector spaces whose graded dimensions enjoy good modular properties (cf. e.g. [149, 150, 151, 154]), Frenkel–Lepowsky–Meurman established Thompson's conjecture in a strong sense.

Theorem 3.4 (Frenkel–Lepowsky–Meurman)**.** *Thompson's Conjecture is true. In particular, the* moonshine module $V^\natural$ *is constructed in [103, 104].*

Frenkel–Lepowsky–Meurman generalized the homogeneous realization of the basic representation of an affine Lie algebra $\hat{\mathfrak{g}}$ due, independently, to Frenkel–Kac [102] and Segal [209], in such a way that *Leech's lattice* Λ [168, 169]—the unique [54] even self-dual positive-definite lattice of rank 24 with no roots—could take on the role played by the root lattice of $\mathfrak{g}$ in the Lie algebra case. In particular, their construction came equipped with rich algebraic structure, furnished by vertex operators, which had appeared first in the physics literature in the late 1960's.

We refer to [102], and also the introduction to [105] for accounts of the role played by vertex operators in physics (up to 1988) along with a detailed description of their application to the representation theory of affine Lie algebras. The first application of vertex operators to affine Lie algebra representations was obtained by Lepowsky–Wilson in [172].

Borcherds described a powerful axiomatic formalism for vertex operators in [15]. In particular, he introduced the notion of a *vertex algebra*, which can be regarded as similar to a commutative associative algebra, except that multiplications depend upon formal variables z_i, and can be singular, in a certain formal sense, along the canonical divisors $\{z_i = 0\}$, $\{z_i = z_j\}$ (cf. [20, 101]).

The appearance of affine Lie algebras above, as a conceptual ingredient for the Frenkel–Lepowsky–Meurman construction of $V^\natural$ hints at an analogy between complex Lie groups and the monster. Borcherds' vertex algebra theory makes this concrete, for Borcherds showed [15] that both in the case of the basic representation of an affine Lie algebra, and in the case of the moonshine module $V^\natural$, the vertex operators defined by Frenkel–Kac, Segal, and Frenkel–Lepowsky–Meurmann, extend naturally to vertex algebra structures.

In all of these examples the *Virasoro algebra*, $\mathcal{V} = \bigoplus_n \mathbb{C}L(n) \oplus \mathbb{C}\mathbf{c}$, being the unique universal central extension of the Lie algebra $\mathbb{C}[t, t^{-1}]\frac{\mathrm{d}}{\mathrm{d}t}$ of polynomial vector fields on

the circle,

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}\mathbf{c}, \quad [L(m), \mathbf{c}] = 0, \tag{3.5}$$

acts naturally on the underlying vector space. (See [155] for a detailed analysis of $\mathcal{V}$. The generator $L(m)$ lies above the vector field $-t^{m+1}\frac{\mathrm{d}}{\mathrm{d}t}$.) This Virasoro structure, which has powerful applications, was axiomatized in [105], with the introduction of the notion of a *vertex operator algebra*. If V is a vertex operator algebra and the central element $\mathbf{c}$ of the Virasoro algebra acts as c times the identity on V, for some $c \in \mathbb{C}$, then V is said to have *central charge* c.

For the basic representation of an affine Lie algebra $\hat{\mathfrak{g}}$, the group of vertex operator algebra automorphisms—i.e. those vertex algebra automorphisms that commute with the Virasoro action—is the adjoint complex Lie group associated to $\mathfrak{g}$. For the moonshine module $V^\natural$, it was shown by Frenkel–Lepowsky–Meurman in [105], that the group of vertex operator algebra automorphisms is precisely the monster.

Theorem 3.5 (Frenkel–Lepowsky–Meurman)**.** *The moonshine module* $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *is a vertex operator algebra of central charge* 24 *whose graded dimension is given by* $J(\tau)$*, and whose automorphism group is* $\mathbb{M}$*.*

Vertex operator algebras are of relevance to physics, for we now recognize them as "chiral halves" of two-dimensional conformal field theories (cf. [106, 107]). From this point of view, the construction of $V^\natural$ by Frenkel–Lepowsky–Meurman constitutes one of the first examples of an orbifold conformal field theory (cf. [73, 74, 75]). In the case of $V^\natural$, the underlying geometric orbifold is the quotient

$$\left(\mathbb{R}^{24}/\Lambda\right)/(\mathbb{Z}/2\mathbb{Z}), \tag{3.6}$$

of the 24-dimensional torus $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}/\Lambda \simeq \mathbb{R}^{24}/\Lambda$ by the Kummer involution $x \mapsto -x$, where Λ denotes the Leech lattice. So in a certain sense, $V^\natural$ furnishes a "24-dimensional" construction of $\mathbb{M}$. We refer to [101, 105, 153, 171] for excellent introductions to vertex algebra, and vertex operator algebra theory.

Affine Lie algebras are special cases of Kac–Moody algebras, first considered by Kac [148] and Moody [185, 186], independently. Roughly speaking, a Kac–Moody algebra is "built" from copies of $\mathfrak{sl}_2$, in such a way that most examples are infinite-dimensional, but much of the finite-dimensional theory carries through (cf. [152]). Borcherds generalized this further, allowing also copies of the three-dimensional Heisenberg Lie algebra to serve as building blocks, and thus arrived [16] at the notion of *generalized Kac–Moody algebra*, or *Borcherds–Kac–Moody (BKM) algebra*, which has subsequently found many applications in mathematics and mathematical physics (cf. [139, 207]).

One of the most powerful such applications occurred in moonshine, when Borcherds introduced a particular example—the *monster Lie algebra* $\mathfrak{m}$—and used it to prove [18] the moonshine conjectures of Conway–Norton. His method entailed using monster-equivariant versions of the denominator identity for $\mathfrak{m}$ to verify that the coefficients

of the McKay–Thompson series T_g, defined by (3.2) according to the Frenkel–Lepowsky–Meurman construction of $V^\natural$, satisfy the replication formulas conjectured by Conway–Norton in [59]. This powerful result reduced the proof of the moonshine conjectures to a small, finite number of identities, that he could easily check by hand.

Theorem 3.6 (Borcherds)**.** *Let $V^\natural$ be the moonshine module vertex operator algebra constructed by Frenkel–Lepowsky–Meurman, whose automorphism group is $\mathbb{M}$. If T_g is defined by (3.2) for $g \in \mathbb{M}$, and if Γ_g is the discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ specified by Conway–Norton in [59], then T_g is the unique normalized principal modulus for Γ_g.*

Recall that an even self-dual lattice of signature (m, n) exists if and only if $m - n = 0 \pmod 8$ (cf. e.g. [63]). Such a lattice is unique up to isomorphism if $mn > 0$, and is typically denoted $II_{m,n}$. In the case that $m = n = 1$ we may take

$$II_{1,1} := \mathbb{Z}e + \mathbb{Z}f, \tag{3.7}$$

where e and f are isotropic, $\langle e, e\rangle = \langle f, f\rangle = 0$, and $\langle e, f\rangle = 1$. Then $me + nf \in II_{1,1}$ has square-length $2mn$. Note that $II_{25,1}$ and $\Lambda \oplus II_{1,1}$ are isomorphic, for Λ the Leech lattice, since both lattices are even and self-dual, with signature $(25, 1)$.

In physical terms the monster Lie algebra $\mathfrak{m}$ is ("about half" of) the space of "physical states" of a bosonic string moving in the quotient

$$\left(\mathbb{R}^{24}/\Lambda \oplus \mathbb{R}^{1,1}/II_{1,1}\right)/(\mathbb{Z}/2\mathbb{Z}) \tag{3.8}$$

of the 26-dimensional torus $II_{25,1} \otimes_{\mathbb{Z}} \mathbb{R}/II_{25,1} \simeq \mathbb{R}^{24}/\Lambda \oplus \mathbb{R}^{1,1}/II_{1,1}$ by the involution $(x, y) \mapsto (-x, y)$ (acting as the Kummer involution on the Leech summand, and trivially on the hyperbolic summand). The monster Lie algebra $\mathfrak{m}$ is constructed in a functorial way from $V^\natural$ (cf. [37]), inherits an action by the monster from $V^\natural$, and admits a monster-invariant grading by $II_{1,1}$.

The denominator identity for a Kac–Moody algebra $\mathfrak{g}$ equates a product indexed by the positive roots of $\mathfrak{g}$ with a sum indexed by the Weyl group of $\mathfrak{g}$. A BKM algebra also admits a denominator identity, which for the case of the monster Lie algebra $\mathfrak{m}$ is the beautiful *Koike–Norton–Zagier formula*

$$p^{-1} \prod_{\substack{m,n\in\mathbb{Z}\\ m>0}} (1 - p^m q^n)^{c(mn)} = J(\sigma) - J(\tau), \tag{3.9}$$

where $\sigma \in \mathbb{H}$ and $p = e^{2\pi i\sigma}$ (and $c(n)$ is the coefficient of q^n in $J(\tau)$, cf. (2.6)). Since the right hand side of (3.9) implies that the left hand side has no terms $p^m q^n$ with $mn \neq 0$, this identity imposes many non-trivial polynomial relations upon the coefficients of $J(\tau)$. Among these is

$$c(4n + 2) = c(2n + 2) + \sum_{k=1}^{n} c(k)c(2n - k + 1), \tag{3.10}$$

which was first found by Mahler [174] by a different method, along with similar expressions for $c(4n)$, $c(4n + 1)$, and $c(4n + 3)$, which are also entailed in (3.9). Taken together

these relations allow us to compute the coefficients of $J(\tau)$ recursively, given just the values

$$\begin{aligned} c(1) &= 196884, \\ c(2) &= 21493760, \\ c(3) &= 864299970, \\ c(5) &= 333202640600. \end{aligned} \tag{3.11}$$

To recover the replication formulas of [59, 192] we require to replace J with T_g, and $c(n) = \dim(V_n^\natural)$ with $\mathrm{tr}(g|V_n^\natural)$ in (3.9), and for this we require a categorification of the denominator identity, whereby the positive integers $c(mn)$ are replaced with $\mathbb{M}$-modules of dimension $c(mn)$.

A categorification of the denominator formula for a finite-dimensional simple complex Lie algebra was obtained by Kostant [166], following an observation of Bott [23]. This was generalized to Kac–Moody algebras by Garland–Lepowsky [119], and generalized further to BKM algebras by Borcherds in [18]. In its most compact form, it is the identity of virtual vector spaces

$$\bigwedge\nolimits_{-1}(\mathfrak{e}) = H(\mathfrak{e}), \tag{3.12}$$

where $\mathfrak{e}$ is the sub Lie algebra of a BKM algebra corresponding to its positive roots (cf. [144, 145, 152]).

In (3.12), which is a version of the *Euler–Poincaré principle*, we understand $\bigwedge_{-1}(\mathfrak{e})$ to be the specialization of the formal series

$$\bigwedge\nolimits_{t}(\mathfrak{e}) := \sum_{k\geq 0} \wedge^k(\mathfrak{e}) t^k \tag{3.13}$$

to $t = -1$, where $\wedge^k(\mathfrak{e})$ is the k-th exterior power of $\mathfrak{e}$, and we write

$$H(\mathfrak{e}) := \sum_{k\geq 0} (-1)^k H_k(\mathfrak{e}) \tag{3.14}$$

for the alternating sum of the Lie algebra homology groups of $\mathfrak{e}$.

In the case of the monster Lie algebra $\mathfrak{m}$, the spaces $\wedge^k(\mathfrak{e})$ and $H_k(\mathfrak{e})$ are graded by $II_{1,1}$, and acted on naturally by the monster. If we use the variables p and q to keep track of the $II_{1,1}$-gradings, then the equality of (3.12) holds in the ring $R(\mathbb{M})[[p,q]][q^{-1}]$ of formal power series in p and q (allowing finitely many negative powers of q), with coefficients in the (integral) representation ring of $\mathbb{M}$. The so-called no-ghost theorem (cf. Theorem 5.1 in [18]) allows to express the $H_k(\mathfrak{e})$ in terms of the homogenous subspaces of $V^\natural$, and the identity (3.12) becomes

$$\bigwedge\nolimits_{-1}\left(\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}} V_{mn}^\natural p^m q^n\right) = \sum_{m\in\mathbb{Z}} V_m^\natural p^{m+1} - \sum_{n\in\mathbb{Z}} V_n^\natural p q^n. \tag{3.15}$$

(It turns out that $H_k(\mathfrak{e}) = 0$ for $k \geq 3$.)

The identity (3.15) returns (3.9) once we replace $V_k^\natural$ everywhere with $\dim(V_k^\natural) = c(k)$, and divide both sides by p. More generally, replacing $V_k^\natural$ with $\mathrm{tr}(g|V_k^\natural)$ for $g \in \mathbb{M}$, we deduce from (3.15) that

$$p^{-1} \exp\left(- \sum_{k>0} \sum_{\substack{m,n\in\mathbb{Z} \\ m>0}} \frac{1}{k} \mathrm{tr}(g^k|V_{mn}^\natural) p^{mk} q^{nk} \right) = T_g(\sigma) - T_g(\tau) \tag{3.16}$$

(cf. [18], and also [147]), which, in turn, implies the replication formulas formulated in [59, 192]. Taking $g = e$ in (3.16) we recover (3.9), so (3.16) furnishes a natural, monster-indexed family of analogues of the identity (3.9).

4. Modularity

Despite the power of the BKM algebra theory developed by Borcherds, and despite some conceptual improvements (cf. [65, 145, 146]) upon Borcherds' original proof of the moonshine conjectures, a conceptual explanation for the principal modulus property of monstrous moonshine is yet to be established. Indeed, there are generalizations and analogs of the notion of replicability which hold for generic modular functions and forms (for example, see [32]), not just those modular functions which are principal moduli.

Zhu explained the modularity of the graded dimension of $V^\natural$ in [238], by proving that this is typical for vertex operator algebras satisfying certain natural (but restrictive) hypotheses, and Dong–Li–Mason extended Zhu's work in [80], obtaining modular invariance results for graded trace functions arising from the action of a finite group of automorphisms.

To prepare for a statement of the results of Zhu and Dong–Li–Mason, we mention that the module theory for vertex operators algebras includes so-called *twisted modules*, associated to finite order automorphisms. If g is a finite order automorphism of V, then V is called *g-rational* in case every g-twisted V-module is a direct sum of simple g-twisted V-modules. Dong–Li–Mason proved [79] that a g-rational vertex operator algebra has finitely many simple g-twisted modules up to isomorphism. So in particular, a rational vertex operator algebra has finitely many simple (untwisted) modules.

Given a module M for the vertex operator algebra V, let us write $\dim_* M$ for its graded dimension,

$$\dim_* M := \sum_n \dim(M_n) q^n. \tag{4.1}$$

When the substitution $q = e^{2\pi i\tau}$ in (4.1) yields a locally uniformly convergent series for $\tau \in \mathbb{H}$ write $Z_M(\tau)$ for the resulting holomorphic function on the upper half plane,

$$Z_M(\tau) := \dim_* M|_{q=e^{2\pi i\tau}}. \tag{4.2}$$

Theorem 4.1 (Zhu, Dong–Li–Mason)**.** *Let V be rational C_2-cofinite vertex operator algebra. Then the graded dimensions $\dim_* M^i$ of its simple modules M^i define holomorphic*

functions $Z_{M^i}(\tau)$ which span a finite-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$*. More generally, if G is a finite subgroup of* $\mathrm{Aut}(V)$ *and V is g-rational for every $g \in G$, then the graded trace functions $\sum_n \mathrm{tr}(\tilde{h}|M_n)q^n$, attached to the triples $(g, \tilde{h}, M)$, where $g, h \in G$ commute, M is a simple h-stable g-twisted module for V, and $\tilde{h}$ is a lift of h to* $\mathrm{GL}(M)$*, span a finite-dimensional representation of* $\mathrm{SL}_2(\mathbb{Z})$*.*

We refer to the Introduction of [80] (see also §2 of [81]) for a discussion of h-stable twisted modules, and the relevant notion of *lift*. Note that any two lifts for h differ only up to multiplication by a non-zero scalar, so $\sum_n \mathrm{tr}(\tilde{h}|M_n)q^n$ is uniquely defined by (g, h, M), up to a non-zero scalar.

In the case of $V^\natural$, there is a unique simple g-twisted module $V_g^\natural = \bigoplus_n (V_g^\natural)_n$ for every $g \in \mathbb{M} = \mathrm{Aut}(V^\natural)$ (cf. Theorem 1.2 of [79]), and $V_g^\natural$ is necessarily h-stable for any $h \in \mathbb{M}$ that commutes with g. Therefore, Theorem 4.1 suggests that the functions

$$T_{(g,\tilde{h})}(\tau) := \sum_n \mathrm{tr}(\tilde{h}|(V_g^\natural)_n)q^n, \tag{4.3}$$

associated to pairs (g, h) of commuting elements of $\mathbb{M}$, may be of interest.

Indeed, this was anticipated a decade earlier by Norton, following observations of Conway–Norton [59] and Queen [199], which associated principal moduli to elements of groups that appear as centralizers of cyclic subgroups in the monster. Norton formulated his *generalized moonshine* conjectures in [191] (cf. also [193], and the Appendix to [182]).

Conjecture 4.2 (Generalized Moonshine: Norton)**.** There is an assignment of holomorphic functions $T_{(g,\tilde{h})} : \mathbb{H} \to \mathbb{C}$ to every pair (g, h) of commuting elements in the monster, such that the following are true:

(1) For every $x \in \mathbb{M}$ we have $T_{(x^{-1}gx, x^{-1}\tilde{h}x)} = T_{(g,\tilde{h})}$.
(2) For every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have that $T_{(g,\tilde{h})\gamma}(\tau)$ is a scalar multiple of $T_{(g,\tilde{h})}(\gamma\tau)$.
(3) The coefficient functions $\tilde{h} \mapsto \mathrm{tr}(\tilde{h}|(V_g^\natural)_n)$, for fixed g and n, define characters of a projective representation of the centralizer of g in $\mathbb{M}$,
(4) We have that $T_{(g,\tilde{h})}$ is either constant or a generator for the function field of a genus zero group $\Gamma_{(g,h)} < \mathrm{SL}_2(\mathbb{R})$.
(5) We have that $T_{(g,\tilde{h})}$ is a scalar multiple of J if and only if $g = h = e$.

Remark. In Conjecture 4.2 (2) above, the right-action of $\mathrm{SL}_2(\mathbb{Z})$ on commuting pairs of elements of the monster is given by

$$(g, h)\gamma := (g^a h^c, g^b h^d) \tag{4.4}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The (slightly ambiguous) $T_{(g,\tilde{h})\gamma}$ denotes the graded trace of a lift of $g^b h^d$ to $\mathrm{GL}(V_{g^a h^c}^\natural)$. Norton's generalized moonshine conjectures reduce to the original Conway–Norton moonshine conjectures of [59] when $g = e$.

Conjecture 4.2 is yet to be proven in full, but has been established for a number of special cases. Theorem 4.1 was used by Dong–Li–Mason in [80], following an observation

of Tuite (cf. [77], and [224, 225, 226] for broader context), to prove Norton's conjecture for the case that g and h generate a cyclic subgroup of $\mathbb{M}$, and this approach, via twisted modules for $V^\natural$, has been extended by Ivanov–Tuite in [142, 143]. Höhn obtained a generalization of Borcherds' method by using a particular twisted module for $V^\natural$ to construct a BKM algebra adapted to the case that g is in the class named $2A$ in [57]—the smaller of the two conjugacy classes of involutions in $\mathbb{M}$—and in so doing established [136] generalized moonshine for the functions $T_{(g,\tilde{h})}$ with $g \in 2A$. So far the most general results in generalized moonshine have been obtained by Carnahan [35, 36, 37]. (See [38] for a recent summary.)

Theorem 4.1 explains why the McKay–Thompson series $T_g(\tau)$ of (3.2), and the $T_{(g,\tilde{h})}(\tau)$ of (4.3) more generally, should be invariant under the actions of (finite index) subgroups of $\mathrm{SL}_2(\mathbb{Z})$, but it does not explain the surprising predictive power of monstrous moonshine. That is, it does not explain why the full invariance groups Γ_g of the T_g should be so large that they admit normalized principal moduli, nor does it explain why the T_g should actually be these normalized principal moduli.

A program to establish a conceptual foundation for the principal modulus property of monstrous moonshine, via *Rademacher sums* and *three-dimensional gravity*, was initiated in [85] by the first author and Frenkel.

5. Rademacher Sums

To explain the conjectural connection between gravity and moonshine, we first recall some history. The roots of the approach of [85] extend back almost a hundred years, to Einstein's theory of general relativity, formulated in 1915, and the introduction of the circle method in analytic number theory, by Hardy–Ramanujan [132]. At the same time that pre-war efforts to quantize Einstein's theory of gravity were gaining steam (see [214] for a review), the circle method was being refined and developed, by Hardy–Littlewood (cf. [131]), and Rademacher [200], among others. (See [227] for a detailed account of what is now known as the *Hardy–Littlewood circle method*.) Despite being contemporaneous, these works were unrelated in science until this century: as we will explain presently, Rademacher's analysis led to a Poincaré series-like expression—the prototypical Rademacher sum—for the elliptic modular invariant $J(\tau)$. It was suggested first in [71] (see also [181]) that this kind of expression might be useful for the computation of partition functions in quantum gravity.

Rademacher "perfected" the circle method introduced by Hardy–Ramanujan, and he obtained an exact convergent series expression for the combinatorial partition function $p(n)$. In 1938 he generalized this work [201] and obtained such exact formulas for the Fourier coefficients of general modular functions. For the elliptic modular invariant $J(\tau) = \sum_n c(n)q^n$ (cf. (2.6)), Rademacher's formula (which was obtained earlier by

Petersson [198], via a different method) may be written as

$$c(n) = 4\pi^2 \sum_{c>0} \sum_{\substack{0<a<c \\ (a,c)=1}} \frac{e^{-2\pi i \frac{a}{c}} e^{2\pi i n \frac{d}{c}}}{c^2} \sum_{k\geq 0} \frac{(4\pi^2)^k}{c^{2k}} \frac{1}{(k+1)!} \frac{n^k}{k!}, \tag{5.1}$$

where d, in each summand, is a multiplicative inverse for a modulo c, and (a, c) is the greatest common divisor of a and c. Having established the formula (5.1), Rademacher sought to reverse the process, and use it to derive the modular invariance of $J(\tau)$. That is, he set out to prove directly that $J_0(\tau + 1) = J_0(-1/\tau) = J_0(\tau)$, when $J_0(\tau)$ is defined by setting $J_0(\tau) = q^{-1} + \sum_{n>0} c(n)q^n$, with $c(n)$ defined by (5.1).

Rademacher achieved this goal in [202], by reorganizing the summation

$$\sum_{n>0} c(n)q^n = 4\pi^2 \sum_{n>0} \sum_{c>0} \sum_{\substack{0<a<c \\ (a,c)=1}} \frac{e^{-2\pi i \frac{a}{c}} e^{2\pi i n (\tau + \frac{d}{c})}}{c^2} \sum_{k\geq 0} \frac{(4\pi^2)^k}{c^{2k}} \frac{1}{(k+1)!} \frac{n^k}{k!} \tag{5.2}$$

into a Poincaré series-like expression for J. More precisely, Rademacher proved that

$$J(\tau) + 12 = e^{-2\pi i \tau} + \lim_{K\to\infty} \sum_{\substack{0<c<K \\ -K^2<d<K^2 \\ (c,d)=1}} e^{-2\pi i \frac{a\tau+b}{c\tau+d}} - e^{-2\pi i \frac{a}{c}}, \tag{5.3}$$

where $a, b \in \mathbb{Z}$ are chosen arbitrarily, in each summand, so that $ad - bc = 1$. We call the right hand side of (5.3) the first *Rademacher sum*.

Rademacher's expression (5.3) for the elliptic modular invariant J is to be compared to the formal sum

$$\sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} e^{-2\pi i m \frac{a\tau+b}{c\tau+d}}, \tag{5.4}$$

for m a positive integer, which we may regard as a (formal) Poincaré series of weight zero for $\mathrm{SL}_2(\mathbb{Z})$. In particular, (5.4) is (formally) invariant for the action of $\mathrm{SL}_2(\mathbb{Z})$, as we see by recognizing the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as representatives for the right coset space $\Gamma_\infty \backslash \Gamma$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and Γ_∞ is defined in (3.3): for a fixed bottom row (c, d) of matrices in $\mathrm{SL}_2(\mathbb{Z})$, any two choices for the top row (a, b) are related by left-multiplication by some element of Γ_∞.

The formal sum (5.4) does not converge for any $\tau \in \mathbb{H}$, so a regularization procedure is required. Rademacher's sum (5.3) achieves this, for $m = 1$, by constraining the order of summation, and subtracting the limit as $\Im(\tau) \to \infty$ of each summand $e^{-2\pi i \frac{a\tau+b}{c\tau+d}}$, whenever this limit makes sense. Rademacher's method has by now been generalized in various ways by a number of authors. The earliest generalizations are due to Knopp [158, 159, 160, 161], and a very general negative weight version of the Rademacher construction was given by Niebur in [189]. We refer to [46] for a detailed review and further references. A nice account of the original approach of Rademacher appears in [157].

We note here that one of the main difficulties in establishing formulas like (5.3) is the demonstration of convergence. When the weight w of the Rademacher sum under consideration lies in the range $0 \leq w \leq 2$, then one requires non-trivial estimates on *sums of Kloosterman sums*, like

$$\sum_{c>0} \sum_{\substack{0<a<c \\ (a,c)=1}} \frac{e^{-2\pi i m \frac{a}{c}} e^{2\pi i n \frac{d}{c}}}{c^2} \tag{5.5}$$

(for the case that $w = 0$ or $w = 2$). The demonstration of convergence generally becomes more delicate as w approaches 1.

In [85] the convergence of a weight zero Rademacher sum $R_\Gamma^{(-m)}(\tau)$ is shown, for m a positive integer and Γ an arbitrary subgroup of $\mathrm{SL}_2(\mathbb{R})$ that is commensurable with $\mathrm{SL}_2(\mathbb{Z})$. Assuming that Γ contains $-I$ and has width one at infinity (cf. (3.3)), we have

$$R_\Gamma^{(-m)}(\tau) = e^{-2\pi i m \tau} + \lim_{K\to\infty} \sum_{(\Gamma_\infty \backslash \Gamma)^\times_{<K}} e^{-2\pi i m \frac{a\tau+b}{c\tau+d}} - e^{-2\pi i m \frac{a}{c}}, \tag{5.6}$$

where the summation, for fixed K, is over non-trivial right cosets of Γ_∞ in Γ (cf. (3.3)), having representatives $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $0 < c < K$ and $|d| < K^2$.

The modular properties of the $R_\Gamma^{(-m)}$ are also considered in [85], and it is at this point that the significance of Rademacher sums in monstrous moonshine appears. To state the relevant result we give the natural generalization (cf. §3.2 of [85]) of the Rademacher–Petersson formula (5.1) for $c(n)$, which is

$$c_\Gamma(-m,n) = 4\pi^2 \lim_{K\to\infty} \sum_{(\Gamma_\infty \backslash \Gamma / \Gamma_\infty)^\times_{<K}} \frac{e^{-2\pi i m \frac{a}{c}} e^{2\pi i n \frac{d}{c}}}{c^2} \sum_{k\geq 0} \frac{(4\pi^2)^k}{c^{2k}} \frac{m^{k+1}}{(k+1)!} \frac{n^k}{k!}, \tag{5.7}$$

where the summation, for fixed K, is over non-trivial double cosets of Γ_∞ in Γ (cf. (3.3)), having representatives $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $0 < c < K$. Note that this formula simplifies for $n = 0$, to

$$c_\Gamma(-m,0) = 4\pi^2 m \lim_{K\to\infty} \sum_{(\Gamma_\infty \backslash \Gamma / \Gamma_\infty)^\times_{<K}} \frac{e^{-2\pi i m \frac{a}{c}}}{c^2}. \tag{5.8}$$

The value $c_\Gamma(-1,0)$ is the *Rademacher constant* attached to Γ. (Cf. §6 of [192] and §5.1 of [85].)

A *normalized Rademacher sum* $T_\Gamma^{(-m)}(\tau)$ is defined in §4.1 of [85] by introducing an extra complex variable and taking a limit. It is shown in §4.4 of [85] that

$$T_\Gamma^{(-m)}(\tau) = R_\Gamma^{(-m)}(\tau) - \frac{1}{2} c_\Gamma(-m,0) \tag{5.9}$$

for any group $\Gamma < \mathrm{SL}_2(\mathbb{R})$ that is commensurable with $\mathrm{SL}_2(\mathbb{Z})$. If Γ has width one at infinity (cf. (3.3)), then also

$$T_\Gamma^{(-m)}(\tau) = q^{-m} + \sum_{n>0} c_\Gamma(-m,n)q^n, \tag{5.10}$$

so in particular, $T_\Gamma^{(-m)}(\tau) = q^{-m} + O(q)$ as $\Im(\tau) \to \infty$. The following theorem by the first author and Frenkel summarizes the central role of Rademacher sums and the principal modulus property.

Theorem 5.1 (Duncan–Frenkel [85]). *Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{R})$ that is commensurable with $\mathrm{SL}_2(\mathbb{Z})$. Then the normalized Rademacher sum $T_\Gamma^{(-m)}$ is Γ-invariant if and only if Γ has genus zero. Furthermore, if Γ has genus zero then $T_\Gamma^{(-1)}$ is the normalized principal modulus for Γ.*

In the case that the normalized Rademacher sum $T_\Gamma^{(-1)}$ is not Γ-invariant, $T_\Gamma^{(-m)}$ is an *abelian integral of the second kind* for Γ, in the sense that it has at most exponential growth at the cusps of Γ, and satisfies $T_\Gamma^{(-m)}(\gamma\tau) = T_\Gamma^{(-m)}(\tau) + \omega(\gamma)$ for $\gamma \in \Gamma$, for some function $\omega : \Gamma \to \mathbb{C}$ (depending on m).

Theorem 5.1 is used as a basis for the formulation of a characterization of the discrete groups Γ_g of monstrous moonshine in terms of Rademacher sums in §6.5 of [85], following earlier work [58] of Conway–McKay–Sebbar. It also facilitates a proof of the following result, which constitutes a uniform construction of the McKay–Thompson series of monstrous moonshine.

Theorem 5.2 (Duncan–Frenkel [85]). *Let $g \in \mathbb{M}$. Then the McKay–Thompson series T_g coincides with the normalized Rademacher sum $T_{\Gamma_g}^{(-1)}$.*

Proof. Theorem 3.6 states that T_g is a normalized principal modulus for Γ_g, and in particular, all the Γ_g have genus zero. Given this, it follows from Theorem 5.1 that $T_{\Gamma_g}^{(-1)}$ is also a normalized principal modulus for Γ_g. A normalized principal modulus is unique if it exists, so we conclude $T_g = T_{\Gamma_g}^{(-1)}$ for all $g \in \mathbb{M}$, as we required to show. □

Perhaps most importantly, Theorem 5.1 is an indication of how the principal modulus property of monstrous moonshine can be explained conceptually. For if we can develop a mathematical theory in which the underlying objects are graded with graded traces that are provably

(1) modular invariant, for subgroups of $\mathrm{SL}_2(\mathbb{R})$ that are commensurable with $\mathrm{SL}_2(\mathbb{Z})$, and
(2) given explicitly by Rademacher sums, such as (5.6),

then these graded trace functions are necessarily normalized principal moduli, according to Theorem 5.1.

We are now led to ask: what kind of mathematical theory can support such results? As we have alluded to above, Rademacher sums have been related to quantum gravity by

articles in the physics literature. Also, a possible connection between the monster and three-dimensional quantum gravity was discussed in [233]. This suggests the possibility that three-dimensional quantum gravity and moonshine are related via Rademacher sums, and was a strong motivation for the work [85]. In the next section we will give a brief review of quantum gravity, since it is an important area of physical inquiry which has played a role in the development of moonshine, but we must first warn the reader: problems have been identified with the existing conjectures that relate the monster to gravity, and the current status of this connection is uncertain.

6. Quantum Gravity

Witten was the first to predict a role for the monster in quantum gravity. In [233] Witten considered *pure quantum gravity* in three dimensions with negative cosmological constant, and presented evidence that the moonshine module $V^\natural$ is a chiral half of the conformal field theory dual to such a quantum gravity theory, at the most negative possible value of the cosmological constant.

To explain some of the content of this statement, note that the action in pure three-dimensional quantum gravity is given explicitly by

$$I_{\mathrm{EG}} := \frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{-g}(R-2\Lambda), \tag{6.1}$$

where G is the *Newton* or *gravitational constant*, R denotes the Ricci scalar, and the *cosmological constant* is the scalar denoted by Λ.

The case that the cosmological constant Λ is negative is distinguished, since then there exist black hole solutions to the action (6.1), as was discovered by Bañados–Teitelboim–Zanelli [9]. These black hole solutions—the *BTZ black holes*—are locally isomorphic to three-dimensional anti-de Sitter space [8], which is a Lorentzian analogue of hyperbolic space, and may be realized explicitly as the universal cover of a hyperboloid

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^3 = -\ell^2 \tag{6.2}$$

in $\mathbb{R}^{2,3}$ (cf. e.g. [175]). The parameter ℓ in (6.2) is called the *radius of curvature*. For a locally anti-de Sitter (AdS) solution to (6.1), the radius of curvature is determined by the cosmological constant, according to

$$\ell^2 = -1/\Lambda. \tag{6.3}$$

In what has become the most cited[5] paper in the history of high energy physics, Maldacena opened the door on a new, and powerful approach to quantum gravity in [176], by presenting evidence for a gauge/gravity duality, in which gauge theories serve as duals to gravity theories in one dimension higher. (See [175] for a recent review.) In the simplest examples, the gauge theories are conformal field theories, and the gravity

[5]Maldacena's groundbreaking paper [176] on the gauge/gravity duality has over 10,000 citations at the time of writing, according to `inspirehep.net`.

theories involve locally AdS spacetimes. The gauge/gravity duality for these cases is now known as the *AdS/CFT correspondence*.

Maldacena's duality furnishes a concrete realization of the *holographic principle*, introduced by 't Hooft [217], and elaborated on by Susskind [216]. For following refinements to Maldacena's proposal due to Gubser–Klebanov–Polyakov [127], and Witten [232], it is expected that gravity theories with $(d+1)$-dimensional locally AdS spacetimes can be understood through the analysis of d-dimensional conformal field theories defined on the boundaries of these AdS spaces. Thus in the case of AdS solutions to three-dimensional quantum gravity, a governing role may be played by two-dimensional conformal theories, which can be accessed mathematically via vertex operator algebras (as we have mentioned in §3).

The conjecture of [233] is that the two-dimensional conformal field theory corresponding to a tensor product of two copies of the moonshine module $V^\natural$ (one "left-moving," the other "right-moving") is the holographic dual to pure three-dimensional quantum gravity with $\ell = 16G$, and therefore

$$\Lambda = -\frac{1}{256G^2}. \tag{6.4}$$

It is also argued that the only physically consistent values of ℓ are $\ell = 16Gm$, for m a positive integer, so that (6.4) is the most negative possible value for Λ, by force of (6.3).

Shortly after this conjecture was formulated, problems with the quantum mechanical interpretation were identified by Maloney–Witten in [179]. Moreover, Gaiotto [115] and Höhn [138] cast doubt on the relevance of the monster to gravity by demonstrating that it cannot act on a holographically dual conformal field theory corresponding to $\ell = 32G$ (i.e. $m = 2$), at least under the hypotheses (namely, an extremality condition, and holomorphic factorization) presented in [233].

Interestingly, the physical problems with the analysis of [233] seem to disappear in the context of *chiral three-dimensional gravity*, which was introduced and discussed in detail by Li–Song–Strominger in [173] (cf. also [178, 215]). This is the gravity theory which motivates much of the discussion in §7 of [85].

In order to define chiral three-dimensional gravity, we first describe *topologically massive gravity*, which was introduced in 1982 by Deser–Jackiw–Templeton [69, 70]. (See also [68].) The action for topologically massive gravity is given by

$$I_{\mathrm{TMG}} := I_{\mathrm{EG}} + I_{\mathrm{CSG}}, \tag{6.5}$$

where I_{EG} is the Einstein–Hilbert action (cf. (6.1)) of pure quantum gravity, and I_{CSG} denotes the *gravitational Chern–Simons term*

$$I_{\mathrm{CSG}} := \frac{1}{32\pi G\mu} \int \mathrm{d}^3x \sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\ \lambda\sigma}\left(\partial_\mu \Gamma^{\sigma}_{\ \rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\ \mu\tau}\Gamma^{\tau}_{\ \nu\rho}\right). \tag{6.6}$$

The $\Gamma^{*}_{\ **}$ are Christoffel symbols, and the parameter μ is called the *Chern–Simons coupling constant*.

Chiral three-dimensional gravity is the special case of topologically massive gravity in which the Chern–Simons coupling constant is set to $\mu = 1/\ell = \sqrt{-\Lambda}$. It is shown in [173] that at this special value of μ, the left-moving central charges of the boundary conformal field theories vanish, and the right-moving central charges are

$$c = \frac{3\ell}{2G} = 24m, \tag{6.7}$$

for m a positive integer, $\ell = 16Gm$.

Much of the analysis of [233] still applies in this setting, and the natural analogue of the conjecture mentioned above states that $V^\natural$ is holographically dual to chiral three-dimensional quantum gravity at $\ell = 16G$, i.e. $m = 1$. However, as argued in detail in [178], the problem of quantizing chiral three-dimensional gravity may be regarded as equivalent to the problem of constructing a sequence of extremal chiral two-dimensional conformal field theories (i.e. vertex operator algebras), one for each central charge $c = 24m$, for m a positive integer. Here, a vertex operator algebra $V = \bigoplus_n V_n$ with central charge $c = 24m$ is called *extremal*, if its graded[6] dimension function satisfies

$$\dim_* V = q^{-m} \frac{1}{\prod_{n>1}(1-q^n)} + O(q) \tag{6.8}$$

(cf. (4.1).) The moonshine module is the natural candidate for $m = 1$ (indeed, it is the only candidate if we assume the uniqueness conjecture of [105]), as the right hand side of (6.8) reduces to $q^{-1} + O(q)$ in this case, but the analysis of [115, 138] also applies here, indicating that the monster cannot act non-trivially on any candidate[7] for $m = 2$. Thus the role of the monster in quantum gravity is still unclear, even in the more physically promising chiral gravity setting.

Nonetheless, the moonshine module $V^\natural$ may still serve as the holographic dual to chiral three-dimensional quantum gravity at $\ell = 16G$, $m = 1$. In this interpretation, the graded dimension, or *genus one partition function* for $V^\natural$—namely, the elliptic modular invariant J—serves as the exact spectrum of physical states of chiral three-dimensional gravity at $\mu = \sqrt{-\Lambda} = 1/16G$, in spacetime asymptotic to the three-dimensional anti-de Sitter space (cf. (6.2)).

Recall that if V is a representation of the Virasoro algebra $\mathcal{V}$ (cf. (3.5)), then $v \in V$ is called a *Virasoro highest weight vector* with *highest weight* $h \in \mathbb{C}$ if $L(m)v = h\delta_{m,0}v$ whenever $m \geq 0$. A *Virasoro descendant* is a vector of the form

$$L(m_1) \cdots L(m_k)v, \tag{6.9}$$

where v is a Virasoro highest weight vector, and $m_1 \leq \cdots \leq m_k \leq -1$.

Assuming that $V^\natural$ is dual to chiral three-dimensional gravity at $m = 1$, the Virasoro highest weight vectors in $V^\natural$ define operators that create black holes, and the Virasoro descendants of a highest weight vector describe black holes embellished by boundary

[6] We regard all vertex operator algebras as graded by $L(0) - \mathbf{c}/24$. Cf. (3.5).

[7] The existence of extremal vertex operator algebras with central charge $c = 24m$ for $m > 1$ remains an open question. We refer to [111, 116, 138, 235] for analyses of this problem.

excitations. In particular, the 196883-dimensional representation of the monster which is contained in the 196884-dimensional homogenous subspace $V_1^\natural < V^\natural$ (cf. (2.4) and (3.1)), represents an 196883-dimensional space of black hole states in the chiral gravity theory.

More generally, the black hole states in the theory are classified, by the monster, into 194 different kinds, according to which monster irreducible representation they belong to.

Question 6.1. *Assuming that the moonshine module $V^\natural$ serves as the holographic dual to chiral three-dimensional quantum gravity at $m = 1$, how are the 194 different kinds of black hole states distributed amongst the homogeneous subspaces $V_n^\natural < V^\natural$. Are some kinds of black holes more or less common than others?*

This question will be answered precisely in §8.

A positive solution to the conjecture that $V^\natural$ is dual to chiral three-dimensional gravity at $m = 1$ may furnish a conceptual explanation for why the graded dimension of $V^\natural$ is the normalized principal modulus for $\mathrm{SL}_2(\mathbb{Z})$. For on the one hand, modular invariance is a consistency requirement of the physical theory—the genus one partition function function is really defined on the moduli space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ of genus one curves, rather than on $\mathbb{H}$—and on the other hand, the genus one partition function of chiral three-dimensional gravity is given by a Rademacher sum, as explained by Manschot–Moore [181], following earlier work [71] by Dijkgraaf–Maldacena–Moore–Verlinde. (Cf. also [178, 179, 180].) So, as we discussed in §5, the genus one partition function must be the normalized principal modulus $J(\tau)$ for $\mathrm{SL}_2(\mathbb{Z})$, according to Theorem 5.1.

In the analysis of [180, 181], the genus one partition function of chiral three-dimensional gravity is a Rademacher sum (5.6), because it is obtained as a sum over three-dimensional hyperbolic structures on a solid torus with genus one boundary, and such structures are naturally parameterized by the coset space $\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})$ (cf. (3.3)), as explained in [177] (see also §5.1 of [71]). The terms $e^{-2\pi i m\frac{a\tau+b}{c\tau+d}}$ in (5.6) are obtained by evaluating $e^{-I_{\mathrm{TMG}}}$, with $\mu = \sqrt{-\Lambda} = 1/16Gm$, on a solution with boundary curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, and the subtraction of $e^{-2\pi i m\frac{a}{c}}$ represents quantum corrections to the classical action.

In [85], the above conjecture is extended so as to encompass the principal modulus property for all elements of the monster, with a view to establishing a conceptual foundation for monstrous moonshine. More specifically, the first main conjecture of [85] states the following.

Conjecture 6.2 (Duncan–Frenkel)**.** There exists a monster-indexed family of *twisted chiral three-dimensional gravity* theories, whose genus one partition functions at

$$\mu = \sqrt{-\Lambda} = 1/16G \tag{6.10}$$

are given by $T^{(-1)}_{\Gamma_g}(-1/\tau)$, where $T^{(-1)}_{\Gamma_g}(\tau)$ is the normalized Rademacher sum attached to Γ_g, satisfying (5.9).

From the point of view of vertex operator algebra theory, $T_g(-1/\tau)$—which coincides with $T^{(-1)}_{\Gamma_g}(-1/\tau)$ according to Theorems 3.6 and 5.1—is the graded dimension of the

unique simple g-twisted $V^\natural$-module $V_g^\natural$ (cf. §4). This non-trivial fact about the functions $T_g(-1/\tau)$ is proven by Carnahan in Theorem 5.1.4 of [37].

Geometrically, the twists of the above conjecture are defined by imposing (generalized) spin structure conditions on solutions to the chiral gravity equations, and allowing orbifold solutions of certain kinds. See §7.1 of [85] for a more complete discussion. The corresponding sums over geometries are then indexed by coset spaces $\Gamma_\infty\backslash\Gamma$, for various groups $\Gamma < \mathrm{SL}_2(\mathbb{R})$, commensurable with $\mathrm{SL}_2(\mathbb{Z})$. According to Theorem 5.1, the genus one partition function corresponding to such a twist, expected to be a Rademacher sum on physical grounds, will only satisfy the basic physical consistency condition of Γ-invariance if Γ is a genus zero group. One may speculate that a finer analysis of physical consistency will lead to the list of conditions given in §6.5 of [85], which characterize the groups Γ_g for $g \in \mathbb{M}$, according to Theorem 6.5.1 of [85]. Thus the discrete groups Γ_g of monstrous moonshine may ultimately be recovered as those defining physically consistent twists of chiral three-dimensional gravity.

On the other hand, it is reasonable to expect that twisted chiral gravity theories are determined by symmetries of the underlying untwisted theory. Conceptually then, but still conjecturally, the monster group appears as the symmetry group of chiral three-dimensional gravity, for which the corresponding twists exist. The principal modulus property of monstrous moonshine may then be explained: as a consequence of Theorem 5.1, together with the statement that the genus one partition function of a twisted theory is $T_\Gamma^{(-1)}(-1/\tau)$, where $T_\Gamma^{(-1)}(\tau)$ is the normalized Rademacher sum attached to the subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ that parameterizes the geometries of the twist.

Before concluding this section we mention two further variations on three-dimensional gravity which may ultimately prove relevant to moonshine. The first of these is *log gravity* which was initiated by work [124] of Grumiller–Johansson, and discussed also in [178]. We refer to [126] for a detailed review. The second is *flat space chiral gravity* which was introduced in [7], and is also reviewed in [126].

For more background on the mathematics and physics of black holes we refer the reader to [125]. We refer to [33, 34] for reviews that focus on the particular role of conformal field theory in understanding quantum gravity.

7. Moonshine Tower

An optimistic view on the relationship between moonshine and gravity is adopted in §7 of [85]. In particular, in §7.2 of [85] the consequences of Conjecture 6.2 for the *second quantization* of chiral three-dimensional gravity are explored. (We warn the reader that the notion of second quantized gravity is very speculative at this stage.)

Motivated in part by the results on second quantized string theory in [72], the existence of a tower of monster modules

$$V^{(-m)} = \bigoplus_{n=-m}^{\infty} V_n^{(-m)}, \tag{7.1}$$

parameterized by positive integer values of m, is predicted in §7.2 of [85]. Moreover, it is suggested that the graded dimension of $V^{(-m)}$ should be given by

$$J^{(-m)} := m\hat{T}(m)J, \tag{7.2}$$

where $\hat{T}(m)$ denotes the *(order m) Hecke operator*, acting on $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic functions on $\mathbb{H}$ according to the rule

$$(\hat{T}(m)f)(\tau) := \frac{1}{m}\sum_{\substack{ad=m\\0\le b<d}} f\left(\frac{a\tau+b}{d}\right). \tag{7.3}$$

Standard calculations (cf. e.g. Chp.VII, §5 of [210]) determine that $m\hat{T}(m)J$ is an $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$, whose Fourier coefficients

$$J^{(-m)}(\tau) = \sum_n c(-m,n)q^n \tag{7.4}$$

are expressed in terms of those of $J(\tau) = \sum_{n=-1}^{\infty} c(n)q^n$, by $c(-m,n) = \delta_{-m,n}$ for $n \le 0$, and

$$c(-m,n) = \sum_{\substack{k>0\\k|(m,n)}} \frac{m}{k} c(mn/k^2), \tag{7.5}$$

for $n > 0$, where (m,n) denotes the greatest common divisor of m and n. In particular, $J^{(-m)}(\tau) = q^{-m} + O(q)$ as $\Im(\tau) \to \infty$. There is only one such $\mathrm{SL}_2(\mathbb{Z})$-invariant holomorphic function on $\mathbb{H}$, so we have

$$J^{(-m)}(\tau) = \sum_{n=-m}^{\infty} \dim(V^{(-m)})q^n = T_\Gamma^{(-m)}(\tau) \tag{7.6}$$

according to (5.10) and Theorem 5.1, when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. So the graded dimension of $V^{(-m)}$ is also a normalized Rademacher sum.

We would like to investigate the higher order analogues of the McKay–Thompson series T_g (cf. (3.2)), encoding the graded traces of monster elements on $V^{(-m)}$, but for this we must first determine the $\mathbb{M}$-module structure on each homogeneous subspace $V_n^{(-m)}$.

A solution to this problem is entailed in Borcherds' proof [18] of the monstrous moonshine conjectures, and the identity (3.15), in particular. To explain this, recall the *Adams operation* ψ^k on virtual G-modules, defined, for $k \ge 0$ and G a finite group, by requiring that

$$\mathrm{tr}(g|\psi^k(V)) = \mathrm{tr}(g^k|V) \tag{7.7}$$

for $g \in G$. (Cf. [6, 162] for more details on Adams operations.) Using the ψ^k we may equip $V^{(-m)}$ with a virtual $\mathbb{M}$-module structure (we will see momentarily that it is actually

an $\mathbb{M}$-module, cf. Proposition 7.2) by defining $V_{-m}^{(-m)} := \mathbb{C}$ to be the one-dimensional trivial $\mathbb{M}$-module, $V_n^{(-m)} := 0$ for $-m < n \leq 0$, and

$$V_n^{(-m)} := \bigoplus_{\substack{k>0 \\ k|(m,n)}} \mathbb{C}^{m/k} \otimes \psi^k(V^\natural_{mn/k^2}) \tag{7.8}$$

for $n > 0$, where $\mathbb{C}^{m/k}$ denotes the trivial $\mathbb{M}$-module of dimension m/k. For convenience later on, we also define $V^{(0)} = V_0^{(0)} := \mathbb{C}$ to be the trivial, one-dimensional $\mathbb{M}$-module, regarded as graded, with grading concentrated in degree $n = 0$.

Evidently ψ^k preserves dimension, so the graded dimension of $V^{(-m)}$ is still given by $J^{(-m)}$, according to (7.5). Define the *order m McKay–Thompson series* $T_g^{(-m)}$, for $m \geq 0$ and $g \in \mathbb{M}$, by setting

$$T_g^{(-m)}(\tau) := q^{-m} + \sum_{n>0} \mathrm{tr}(g|V_n^{(-m)})q^n. \tag{7.9}$$

Then $T_g^{(0)} = 1$ for all $g \in \mathbb{M}$, and $T_g^{(-1)}$ is the original McKay–Thompson series T_g. More generally, we have the following result, which constructs the $T_g^{(-m)}$ uniformly and explicitly as Rademacher sums.

Theorem 7.1. *For $m > 0$ and $g \in \mathbb{M}$ we have $T_g^{(-m)}(\tau) = T_{\Gamma_g}^{(-m)}(\tau)$, where Γ_g is the invariance group of $T_g(\tau)$, and $T_\Gamma^{(-m)}$ denotes the normalized Rademacher sum of order m attached to Γ, as in (5.9). In particular, $T_g^{(-m)}(\tau)$ is a monic integral polynomial of degree m in $T_g(\tau)$.*

Proof. We will use Borcherds' identity (3.15). To begin, note that $T_g^{(-m)}$ is given explicitly in terms of traces on $V^\natural$ by

$$T_g^{(-m)}(\tau) = q^{-m} + \sum_{n>0} \sum_{k|(m,n)} \frac{m}{k} \mathrm{tr}(g|\psi^k(V^\natural_{mn/k^2}))q^n \tag{7.10}$$

according to (7.8) and (7.9). Recall that $R(G)$ denotes the integral representation ring of a finite group G. Extend the ψ^k from $R(G)$ to $R(G)[[p,q]][q^{-1}]$, by setting $\psi^k(Mp^mq^n) = \psi^k(M)p^{km}q^{km}$ for $M \in R(G)$. Then it is a general property of the Adams operations (cf. §5.2 of [147]) that

$$\log \bigwedge\nolimits_{-1}(X) = -\sum_{k>0} \frac{1}{k}\psi^k(X) \tag{7.11}$$

in $R(G)[[p,q]][q^{-1}]\otimes_{\mathbb{Z}}\mathbb{Q}$, for $X\in R(G)[[p,q]][q^{-1}]$. So taking $X=\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}}V^{\natural}_{mn}p^mq^n$ we obtain

$$\begin{aligned}\log\bigwedge\nolimits_{-1}\left(\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}}V^{\natural}_{mn}p^mq^n\right)&=-\sum_{k>0}\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}}\frac{1}{k}\psi^k(V^{\natural}_{mn})p^{km}q^{kn}\\&=-\sum_{\substack{m,n\in\mathbb{Z}\\ m>0}}\sum_{k|(m,n)}\frac{1}{k}\psi^k(V^{\natural}_{mn/k^2})p^mq^n\end{aligned}\tag{7.12}$$

for the logarithm of the left hand side of (3.15). If we now define $V^{(-m)}(q):=\sum_n V^{(-m)}_nq^n$, an element of $R(\mathbb{M})[[q]][q^{-1}]$, then the generating series $\sum_{m>0}p^mV^{(-m)}(q)$ is obtained when we apply $-p\partial_p$ to (7.12), according to the definition (7.8) of the $V^{(-m)}_n$ as elements of $R(\mathbb{M})$. So apply $-p\partial_p\log(\,\cdot\,)$ to both sides of (3.15) to obtain the identity

$$\sum_{m>0}V^{(-m)}(q)p^m=-1-(p\partial_pV^{\natural}(p))\sum_{k\geq 0}V^{\natural}(q)^kV^{\natural}(p)^{-k-1}\tag{7.13}$$

in $R(\mathbb{M})[[p,q]][q^{-1}]$, where $V^{\natural}(q)=V^{(-1)}(q)=q^{-1}+\sum_{n>0}V^{\natural}_nq^n$. The right hand side of (7.13) really is a taylor series in p, for we use $V^{\natural}(p)^{-1}$ as a short hand for $\sum_{k\geq 0}(-1)^kp^{k+1}V^{\natural}_+(p)$, where $V^{\natural}_+(p):=\sum_{n>0}V^{\natural}_np^n$ is the regular part of $V^{\natural}(p)$.

The McKay–Thompson series $T^{(-m)}_g(\tau)$ is just the trace of g on $V^{(-m)}(q)$, so an application of $\mathrm{tr}(g|\,\cdot\,)$ to (7.13) replaces $V^{(-m)}(q)$ with $T^{(-m)}_g(\tau)$, and $V^{\natural}(q)$ with $T_g(\tau)$, etc. and shows that $T^{(-m)}_g$ is indeed a polynomial in T_g, of degree m since the leading term of $T^{(-m)}_g$ is q^{-m} by definition. In particular, $T^{(-m)}_g$ is a modular function for Γ_g, with no poles away from the infinite cusp. Since Γ_g has genus zero, such a function is uniquely determined (up to an additive constant) by the polar terms in its Fourier expansion. The McKay–Thompson series $T^{(-m)}_g$ and the Rademacher sum $T^{(-m)}_{\Gamma_g}$ both satisfy $q^{-m}+O(q)$ as $\Im(\tau)\to\infty$ (cf. (5.10)), and neither have poles away from the infinite cusp, so they must coincide. This completes the proof. □

Remark. The identity obtained by taking the trace of $g\in\mathbb{M}$ on (7.13) may be compactly rewritten

$$\sum_{m\geq 0}T^{(-m)}_g(\tau)p^m=\frac{p\partial_pT_g(\sigma)}{T_g(\tau)-T_g(\sigma)},\tag{7.14}$$

where $p=e^{2\pi i\sigma}$ and $T_g(\sigma)=\sum_m\mathrm{tr}(g|V^{\natural}_m)p^m$. This expression (7.14) is proven for some special cases by a different method in [12].

Recall that the monster group has 194 irreducible ordinary representations, up to equivalence. Let us denote these by M_i, for $1\leq i\leq 194$, where the ordering is as in [57], so that the character of M_i is the function denoted χ_i in [57]. Define $\mathbf{m}_i(-m,n)$ to be

the multiplicity of M_i in $V_n^{(-m)}$, so that

$$V_n^{(-m)} \simeq \bigoplus_{i=1}^{194} M_i^{\oplus \mathbf{m}_i(-m,n)} \tag{7.15}$$

as $\mathbb{M}$-modules, and $c(-m,n) = \sum_{i=1}^{194} \mathbf{m}_i(-m,n)\chi_i(e)$.

A priori, the $\mathbb{M}$-modules $V_n^{(-m)}$ may be virtual, meaning that some of the integers $\mathbf{m}_i(-m,n)$ are negative.

Proposition 7.2. *The $V_n^{(-m)}$ are all (non-virtual) modules for the monster. In particular, the integers $\mathbf{m}_i(-m,n)$ are all non-negative.*

Proof. The claim follows from the modification of Borcherds' proof of Theorem 3.6 presented by Jurisich–Lepowsky–Wilson in [147]. In [147] a certain free Lie sub algebra $\mathfrak{u}^-$ of the monster Lie algebra $\mathfrak{m}$ is identified, for which the identity $\Lambda(\mathfrak{u}^-) = H(\mathfrak{u}^-)$ (or rather, the logarithm of this) yields

$$\sum_{m,n>0} \sum_{k|(m,n)} \frac{1}{k} \psi^k(V^\natural_{mn/k^2}) p^m q^n = \sum_{k>0} \frac{1}{k} \left(\sum_{m,n>0} V^\natural_{m+n-1} p^m q^n \right)^k \tag{7.16}$$

in $R(\mathbb{M})[[p,q]][q^{-1}] \otimes \mathbb{Q}$. (Notice the different range of summation, compared to (3.15).) We apply $p\partial_p$ to (7.16), and recall the definition (7.8) of $V_n^{(-m)}$ to obtain

$$\sum_{m,n>0} V_n^{(-m)} p^m q^n = \sum_{k>0} \left(\sum_{m,n>0} m V^\natural_{m+n-1} p^m q^n \right) \left(\sum_{m,n>0} V^\natural_{m+n-1} p^m q^n \right)^{k-1}. \tag{7.17}$$

The coefficient of $p^m q^n$ in the right hand side of (7.17) is evidently a non-negative integer combination of the $\mathbb{M}$-modules $V_n^\natural$, so the proof of the claim is complete. □

In §8 we will determine the behavior of the multiplicity functions $\mathbf{m}_i(-m,n)$ (cf. (7.15)) as $n \to \infty$. For applications to gravity a slightly different statistic is more relevant. Recall from §6 that it is the Virasoro highest weight vectors—i.e. those $v \in V_n^\natural$ with $L(k)v = 0$ for $k > 0$—that represent black hole states in chiral three-dimensional gravity at $m = 1$. Such vectors generate *highest weight modules* for $\mathcal{V}$, the structure of which has been determined by Feigin–Fuchs in [99]. (See [5] for an alternative treatment.) Specializing to the case that the central element $\mathbf{c}$ (cf. (3.5)) acts as $c = 24m$ times the identity, for some positive integer m, we obtain from the results of [99] that the isomorphism type of an irreducible highest weight module for $\mathcal{V}$ depends only on the $L(0)$-eigenvalue of a generating highest weight vector, v, and if $L(0)v = hv$ for h a non-negative integer, then

$$\dim_* L(h,c) = \begin{cases} q^{-m}(1-q)(q)_\infty^{-1} & \text{if } h = 0, \\ q^{h-m}(q)_\infty^{-1} & \text{if } h > 0, \end{cases} \tag{7.18}$$

(cf. (4.1)) where $L(h,c)$ denotes the irreducible highest weight $\mathcal{V}$-module generated by v, and

$$(q)_\infty := \prod_{n>0}(1-q^n). \tag{7.19}$$

(Note that all the Virasoro modules in this work are graded by $L(0)-\mathbf{c}/24$. See [130] for details of the calculation that returns (7.18) in the case that $m=1$.)

Remark. We may now recognize the leading terms in (6.8) as exactly those of the graded dimension of the Virasoro module $L(0,24m)$.

It is known that $V^\natural$ is a direct sum of highest weight modules for the Virasoro algebra (cf. e.g. [130]). Since the Virasoro and monster actions on $V^\natural$ commute, we have an isomorphism

$$V^\natural \simeq L(0,24)\otimes W^\natural_{-1} \oplus \bigoplus_{n>0} L(n+1,24)\otimes W^\natural_n \tag{7.20}$$

of modules for $\mathcal{V}\times\mathbb{M}$, where $W^\natural_n$ denotes the subspace of $V^\natural_n$ spanned by Virasoro highest weight vectors. To investigate how the black hole states in $V^\natural$ are organized by the representation theory of the monster, we define non-negative integers $\mathbf{n}_i(n)$ by requiring that

$$W^\natural_n \simeq \bigoplus_{i=1}^{194} M_i^{\oplus \mathbf{n}_i(n)}, \tag{7.21}$$

for $n\geq -1$.

Evidently $\mathbf{n}_i(n)\leq \mathbf{m}_i(-1,n)$ for all i and n since $W^\natural_n$ is a subspace of $V^\natural_n$. To determine the precise relationship between the $\mathbf{n}_i(n)$ and $\mathbf{m}_i(-1,n)$, define $U_g(\tau)$ for $g\in\mathbb{M}$ by setting

$$U_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|W^\natural_n)q^n, \tag{7.22}$$

so that $U_g(\tau)=q^{-1}+\sum_{n>0}\sum_{i=1}^{194}\mathbf{n}_i(n)\chi_i(g)q^n$ (cf. (7.21)). Combining (7.18), (7.20) and (7.21), together with the definitions (3.2) of T_g and (7.22) of U_g, we obtain

$$T_g(\tau) = q^{-1}\frac{(1-q)}{(q)_\infty} + \sum_{n>0} q^n \frac{1}{(q)_\infty}\sum_{i=1}^{194}\mathbf{n}_i(n)\chi_i(g), \tag{7.23}$$

or equivalently,

$$U_g(\tau) = (q)_\infty T_g(\tau) + 1 \tag{7.24}$$

for all $g\in\mathbb{M}$. (This computation also appears in [130].)

In §8 we will use (7.24) to determine the asymptotic behavior of the $\mathbf{n}_i(n)$ (cf. Theorem 8.1), and thus the statistics of black hole states, at $\ell=16G$, in the conjectural chiral three-dimensional gravity theory dual to $V^\natural$.

Remark. Note that we may easily construct modules for $\mathcal{V} \times \mathbb{M}$ satisfying the extremal condition (6.8), for each positive integer m, by considering direct sums of the monster modules $V^{(-m)}$ constructed in Proposition 7.2. A very slight generalization of the argument just given will then yield formulas for the graded traces of monster elements on the corresponding Virasoro highest weight spaces. Since it has been shown [115, 138] that such modules cannot admit vertex operator algebra structure, we do not pursue this here.

8. Monstrous Moonshine's Distributions

We now address the problem of determining exact formulas and asymptotic distributions of irreducible components. This work will rely heavily on the modularity of the underlying McKay–Thompson series (i.e. Theorems 3.3 and 7.1).

We prove formulas for the multiplicities $\mathbf{m}_i(-m,n)$ and $\mathbf{n}_i(n)$ which in turn imply the following asymptotics.

Theorem 8.1. *If m is a positive integer and $1 \leq i \leq 194$, then as $n \to +\infty$ we have*

$$\mathbf{m}_i(-m,n) \sim \frac{\dim(\chi_i)|m|^{1/4}}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|mn|}}$$
$$\mathbf{n}_i(n) \sim \frac{\sqrt{12}\ \dim(\chi_i)}{|24n+1|^{1/2}|\mathbb{M}|} \cdot e^{\frac{\pi}{6}\sqrt{23|24n+1|}}$$

These asymptotics immediately imply that the following limits are well-defined

$$\begin{aligned} \delta\left(\mathbf{m}_i(-m)\right) &:= \lim_{n\to+\infty} \frac{\mathbf{m}_i(-m,n)}{\sum_{i=1}^{194} \mathbf{m}_i(-m,n)} \\ \delta\left(\mathbf{n}_i\right) &:= \lim_{n\to+\infty} \frac{\mathbf{n}_i(n)}{\sum_{i=1}^{194} \mathbf{n}_i(n)}. \end{aligned} \tag{8.1}$$

Corollary 8.2. *In particular, we have that*

$$\delta\left(\mathbf{m}_i(-m)\right) = \delta\left(\mathbf{n}_i\right) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$

We illustrate these asymptotics explicitly, for χ_1, χ_2, and χ_{194}, in the case that $m = 1$, in Table 1, where we let $\delta\left(\mathbf{m}_i(-m,n)\right)$ denote the proportion of components corresponding to χ_i in $V_n^{(m)}$.

TABLE 1.

n	$\delta\left(\mathbf{m}_1(-1,n)\right)$	$\delta\left(\mathbf{m}_2(-1,n)\right)$	$\cdots$	$\delta\left(\mathbf{m}_{194}(-1,n)\right)$
-1	1	0	$\cdots$	0
0	0	0	$\cdots$	0
1	1/2	1/2	$\cdots$	0
2	1/3	1/3	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots\times 10^{-4}$	$2.514\ldots\times 10^{-3}$	$\cdots$	$0.00891\ldots$
60	$2.699\ldots\times 10^{-9}$	$2.732\ldots\times 10^{-8}$	$\cdots$	$0.04419\ldots$
80	$4.809\ldots\times 10^{-14}$	$7.537\ldots\times 10^{-13}$	$\cdots$	$0.04428\ldots$
100	$4.427\ldots\times 10^{-18}$	$1.077\ldots\times 10^{-16}$	$\cdots$	$0.04428\ldots$
120	$1.377\ldots\times 10^{-21}$	$5.501\ldots\times 10^{-20}$	$\cdots$	$0.04428\ldots$
140	$1.156\ldots\times 10^{-24}$	$1.260\ldots\times 10^{-22}$	$\cdots$	$0.04428\ldots$
160	$2.621\ldots\times 10^{-27}$	$3.443\ldots\times 10^{-23}$	$\cdots$	$0.04428\ldots$
180	$1.877\ldots\times 10^{-28}$	$3.371\ldots\times 10^{-23}$	$\cdots$	$0.04428\ldots$
200	$1.715\ldots\times 10^{-28}$	$3.369\ldots\times 10^{-23}$	$\cdots$	$0.04428\ldots$
220	$1.711\ldots\times 10^{-28}$	$3.368\ldots\times 10^{-23}$	$\cdots$	$0.04428\ldots$
240	$1.711\ldots\times 10^{-28}$	$3.368\ldots\times 10^{-23}$	$\cdots$	$0.04428\ldots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
∞	$\frac{1}{584407678530450280801360213 6}$	$\frac{196883}{584407678530450280801360213 6}$	$\cdots$	$\frac{2588234775310550640452343 75}{584407678530450280801360213 6}$

The precise values given in the bottom row of Table 1 admit the following decimal approximations:

$$\delta\left(\mathbf{m}_1(-m)\right) = \frac{1}{584407678530450280801360213 6} \approx 1.711\ldots\times 10^{-28}$$

$$\delta\left(\mathbf{m}_2(-m)\right) = \frac{196883}{584407678530450280801360213 6} \approx 3.368\ldots\times 10^{-23} \tag{8.2}$$

$$\delta\left(\mathbf{m}_{194}(-m)\right) = \frac{2588234775310550640452343 75}{584407678530450280801360213 6} \approx 4.428\ldots\times 10^{-2}$$

Before explaining the proof of Theorem 8.1 we describe an application to the computation[8] of quantum dimensions. Suppose that V is a vertex operator algebra and M is a V-module such that the graded-dimension functions Z_V and Z_M are defined (cf. (4.2)).

[8]We are grateful to the referee for suggesting this.

Then the *quantum dimension* of M relative to V is defined by setting

$$\operatorname{qdim}_V M := \lim_{y\to 0} \frac{Z_M(iy)}{Z_V(iy)} \tag{8.3}$$

(for y real and positive), assuming the limit exists. (Cf. §3.1 of [76].)

Dong–Mason initiated a vertex algebraic *quantum Galois theory* in [82]. (Cf. also [78, 83, 128].) In this theory, inclusions of vertex operator algebras take on the role played by inclusions of fields in the classical setting. It is established in [76] that the quantum dimension $\operatorname{qdim}_U V$, for U a sub vertex operator algebra of V, serves as the quantum analogue of the relative dimension $\dim_E F$ of a field F over a subfield E.

Write $V^{\mathbb{M}}$ for the fixed points of the action of $\mathbb{M}$ on $V^\natural$. Then $V^{\mathbb{M}}$ is a sub vertex operator algebra of $V^\natural$, called the *monster orbifold*. Using Theorem 8.1, together with a result from [76], we will compute the quantum dimensions of the monster orbifold. To formulate this precisely, note that, according to the main theorem of [78], we have

$$V^\natural \simeq \bigoplus_{i=1}^{194} V_i^{\mathbb{M}} \otimes M_i, \tag{8.4}$$

as $V^{\mathbb{M}}\times\mathbb{M}$-modules, for some $V^{\mathbb{M}}$-submodules $V_i^{\mathbb{M}}$ in $V^\natural$, where M_i denotes an irreducible module for $\mathbb{M}$ with character χ_i, as in (7.15).

Corollary 8.3. *We have* $\operatorname{qdim}_{V^{\mathbb{M}}} V_i^{\mathbb{M}} = \chi_i(e)$. *In particular, the quantum dimension of* $V_i^{\mathbb{M}}$ *relative to* $V^{\mathbb{M}}$ *exists, for all* $1 \le i \le 194$.

Proof. Recall (cf. (4.2)) that $Z_{V_i^{\mathbb{M}}}(\tau)$ is the function obtained by substituting $q = e^{2\pi i\tau}$ in $\dim_* V_i^{\mathbb{M}} = \sum_n (V_i^{\mathbb{M}})_n q^n$, assuming this series converges. Note that $V^{\mathbb{M}} = V_1^{\mathbb{M}}$. According to Proposition 3.6 of [76], if the limit

$$d_i := \lim_{n\to\infty} \frac{\dim(V_i^{\mathbb{M}})_n}{\dim(V_1^{\mathbb{M}})_n} \tag{8.5}$$

exists, then the quantum dimension of $V_i^{\mathbb{M}}$ relative to $V_1^{\mathbb{M}}$ also exists, and equals d_i. Comparing with (7.15) we see that $\dim(V_i^{\mathbb{M}})_n = \mathbf{m}_i(-1,n)$. Applying Theorem 8.1 we obtain $d_i = \chi_i(e)$, and this completes the proof. □

Note that Corollary 8.3 confirms a special case of Conjecture 6.7 of [76]. It would be interesting to see how generally this method can be applied, to the computation of quantum dimensions of orbifolds V^G, where V is a vertex operator algebra and G is a group of automorphisms of $\operatorname{Aut}(G)$. (See also Problem 10.11.)

8.1. **The modular groups in monstrous moonshine.** To obtain exact formulas, we begin by recalling the modular groups which arise in monstrous moonshine. Suppose $\Gamma_* < \mathrm{GL}_2(\mathbb{R})$ is a discrete group which is commensurable with $\mathrm{SL}_2(\mathbb{Z})$. If Γ_* defines a genus zero quotient of $\mathbb{H}$, then the field of modular functions which are invariant under Γ_* is generated by a single element, the principal modulus (cf. (3.4)). Theorem 3.6 implies that the T_g (defined by (3.2)) are principal moduli for certain groups Γ_g. We can

describe these groups in terms of groups E_g which in turn may be described in terms of the congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod{N} \right\}, \tag{8.6}$$

and the Atkin–Lehner involutions W_e for $\Gamma_0(N)$ given by

$$W_e := \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \tag{8.7}$$

where e is an exact divisor of N (i.e. $e|N$, and $(e, N/e) = 1$), and a, b, c, and d are integers chosen so that W_e has determinant e.

Following Conway–Norton [59] and Conway–McKay–Sebbar [58], we denote the groups E_g by symbols of the form $\Gamma_0(N|h)+e, f, \dots$ (or simply $N|h+e, f, \dots$), where h divides $(N, 24)$, and each of e, f, etc. exactly divide N/h. This symbol represents the group

$$\Gamma_0(N|h) + e, f, \cdots := \begin{pmatrix} 1/h & 0 \\ 0 & 1 \end{pmatrix} \langle \Gamma_0(N/h), W_e, W_f, \dots \rangle \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix},$$

where W_e, W_f, etc. are representative of Atkin–Lehner involutions on $\Gamma_0(N/h)$. We use the notation $\mathcal{W}_g := \{1, e, f, \dots\}$ to denote this list of Atkin–Lehner involutions contained in E_g. We also note that $\Gamma_0(N|h) + e, f, \dots$ contains $\Gamma_0(Nh)$.

The groups E_g are eigengroups for the T_g, so that if $\gamma \in E_g$, then $T_g(\gamma\tau) = \sigma_g(\gamma)T_g$, where σ_g is a multiplicative group homomorphism from E_g to the group of h-th roots of unity. Conway and Norton [59] give the following values for σ_g evaluated on generators of $N|h + e, f, \dots$.

Lemma 8.4 (Conway–Norton)**.** *Assuming the notation above, the following are true:*

(a) $\sigma_g(\gamma) = 1$ *if* $\gamma \in \Gamma_0(Nh)$
(b) $\sigma_g(\gamma) = 1$ *if* γ *is an Atkin–Lehner involution of* $\Gamma_0(Nh)$ *inside* E_g
(c) $\sigma_g(\gamma) = e^{\frac{-2\pi i}{h}}$ *if* $\gamma = \begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix}$
(d) $\sigma_g(\gamma) = e^{-\lambda_g \frac{2\pi i}{h}}$ *if* $\gamma = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}$,

where λ_g *in (d) is* -1 *if* $N/h \in \mathcal{W}_g$, *and* $+1$ *otherwise.*

This information is sufficient to properly describe the modularity of the series $T_g^{(-m)}(\tau)$ on E_g. In section 8.4, we give an explicit procedure for evaluating σ_g. The invariance group Γ_g, denoted by $\Gamma_0(N||h) + e, f, \dots$ (or by the symbol $N||h + e, f, \dots$), is defined as the kernel of σ_g. A complete list of the groups Γ_g can be found in the Appendix (8A) of this paper, or in table 2 of [59].

Theorems 3.3 and 7.1 are summarized by the following uniform statement.

Theorem 8.5. *Let $g \in \mathbb{M}$ and $m \geq 1$. Then $T_g^{(-m)}$ is the unique weakly holomorphic modular form of weight zero for Γ_g that satisfies $T_g^{(-m)} = q^{-m} + O(q)$ as τ approaches the infinite cusp, and has no poles at any cusps inequivalent to the infinite one.*

Remark. A *weakly holomorphic* modular form is a meromorphic modular form whose poles (if any) are supported at cusps.

8.2. **Harmonic Maass forms.** Maass–Poincaré series allow us to obtain formulas for weakly holomorphic modular forms and mock modular forms. We begin by briefly recalling the definition of a *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ and multiplier ν (a generalization of the notion of a Nebentypus). If $\tau = x + iy$ with x and y real, we define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \tag{8.8}$$

and if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, define

$$(\gamma : \tau) := (c\tau + d).$$

Suppose Γ is a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$ and $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$. Then a real analytic function $F(\tau)$ is a *harmonic Maass form* of weight k on Γ with multiplier ν if:

(a) The function $F(\tau)$ satisfies the modular transformation with respect to the weight k slash operation,

$$F(\tau)|_k\gamma := (\gamma : \tau)^{-k} F(\gamma\tau) = \nu(\gamma) F(\tau)$$

for every matrix $\gamma \in \Gamma$, where if $k \in \mathbb{Z} + \frac{1}{2}$, the square root is taken to be the principal branch. In particular, if ν is trivial, then F is invariant under the action of the slash operator.

(b) We have that $\Delta_k F(\tau) = 0$,

(c) The singularities of F (if any) are supported at the cusps of Γ, and for each cusp ρ there is a polynomial $P_{F,\rho}(q^{-1}) \in \mathrm{C}[q^{-1/t_\rho}]$ and a constant $c > 0$ such that $F(\tau) - P_{F,\rho}(e^{-2\pi i\tau}) = O(e^{-cy})$ as $\tau \to \rho$ from inside a fundamental domain. Here t_ρ is the width of the cusp ρ.

Remark. The polynomial $P_{F,\rho}$ above is referred to as the *principal part of F at ρ*. In certain applications, condition (c) of the definition may be relaxed to admit larger classes of harmonic Maass forms. However, for our purposes we will only be interested in those satisfying the given definition, having a holomorphic principal part.

We denote the complex vector space of such functions by $H_k(\Gamma, \nu)$, and note that in order for $H_k(\Gamma, \nu)$ to be nonzero, ν must satisfy $(\gamma : \delta\tau)^k (\delta : \tau)^k \nu(\gamma)\nu(\delta) = (\gamma\delta : \tau)^k \nu(\gamma\delta)$ for every $\gamma, \delta \in \Gamma$.

Let $\mathcal{S}(\Gamma)$ denote some fixed complete set of inequivalent representatives of the cusps of Γ. For each representative $\rho = \frac{\alpha}{\gamma}$ with $(\alpha, \gamma) = 1$, fix a matrix $L_\rho = \begin{pmatrix} -\delta & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

so that $\rho = L_\rho^{-1}\infty$. Following Rankin [206], let t_ρ be the cusp width and let κ_ρ be the cusp parameter, defined as the least nonnegative integer so that $\nu(L_\rho T^{t_\rho} L_\rho^{-1}) = e^{2\pi i\kappa_\rho}$, where $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The stabilizer of ρ in Γ is given by $\Gamma_\rho := \langle \pm T^{t_\rho} \rangle$, so for example $\Gamma_\infty = \langle \pm T \rangle$. Given $F(\tau) \in H_{2-k}(\Gamma, \nu)$, we refer to $F_\rho(\tau) := F(\tau)|_{2-k} L_\rho$ as the expansion of F at the cusp ρ. We note that this expansion depends on the choice of L_ρ. These facts imply that the expansion of F_ρ can be given as a Fourier series of the form

$$F_\rho(\tau) = \sum_n a(n, y) e^{2\pi i x(n+\kappa_\rho)/t_\rho}.$$

More precisely, we have the following. The Fourier expansion of harmonic Maass forms F at a cusp ρ (see Proposition 3.2 of [31]) splits into two components. As before, we let $q := e^{2\pi i\tau}$.

Lemma 8.6. *If $F(\tau)$ is a harmonic Maass form of weight $2-k$ for Γ where $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, and if ρ is a cusp of Γ, then*

$$F_\rho(\tau) = F_\rho^+(\tau) + F_\rho^-(\tau)$$

where F_ρ^+ is the holomorphic part of F_ρ, given by

$$F_\rho^+(\tau) := \sum_{n \gg -\infty} c_{F,\rho}^+(n) q^{(n+\kappa_\rho)/\tau_\rho},$$

and F_ρ^- is the nonholomorphic part, given by

$$F_\rho^-(\tau) + \sum_{n<0} c_{F\rho}^-(n) \Gamma(k-1, 4\pi y|(n+\kappa_\rho)/t_\rho|) q^{(n+\kappa_\rho)/t_\rho}.$$

By inspection, we see that weakly holomorphic modular forms are themselves harmonic Maass forms. In fact, under the given definition, all harmonic Maass forms of positive weight are weakly holomorphic. On the other hand, if the weight is non-positive, then the space of harmonic Maass forms may be larger than the space of weakly holomorphic modular forms. However, as with the weakly holomorphic modular forms, a harmonic Maass form is uniquely defined by its principal parts at all cusps. This is clear if the form is weakly holomorphic. If the nonholomorphic part is non-zero, we have the following lemma which follows directly from the work of Bruinier and Funke [31].

Lemma 8.7. *If $F \in H_{2-k}(\Gamma_0(N))$ has the property that $F^- \neq 0$, then the principal part of F is nonconstant for at least one cusp.*

Sketch of the Proof. Bruinier and Funke define a pairing $\{\bullet, \bullet\}$ on harmonic weak Maass Forms. The particular quantity $\{2iy^k \overline{\frac{\partial}{\partial \overline{\tau}} F}, F\}$ can be described in terms of either a Petersson norm, or in terms of products of coefficients of the principal part of F with coefficients of the nonholomorphic part. Since the Petersson norm is non-zero, at least one coefficient of the principal part is also non-zero. □

Hence, if F and G are harmonic Maass forms of non-positive weight whose principal parts are equal at all cusps, then $F - G$ is holomorphic and vanishes at cusps, and therefore identically 0.

8.3. **Maass–Poincaré series.** The Maass–Poincaré series define a basis for a space of harmonic Maass forms and provide exact formulas for their coefficients. The following construction of the Maass–Poincaré series follows the method and notation of Bringmann and the third author [26] which builds on the early work of Rademacher, followed by more contemporary work of Fay, Niebur, among many others [98, 188, 189]. The Poincaré series we construct in this section are modular for congruence subgroups $\Gamma_0(N)$ which we will then use to construct the McKay–Thompson series. Although we could follow similar methods to construct Poincaré series for the groups Γ_g directly, we restrict our attention to these groups since the congruence subgroups $\Gamma_0(N)$ are more standard.

For $s \in \mathrm{C}$, $w \in \mathbb{R} \setminus \{0\}$, and $k \geq 3/2$, $k \in \frac{1}{2}\mathbb{Z}$, let

$$\mathcal{M}_s(w) := |y|^{\frac{k}{2}-1} M_{\mathrm{sign}(w)(1-k/2), s-\frac{1}{2}}(|w|), \tag{8.9}$$

where $M_{\nu,\mu}(z)$ is the M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) u = 0,$$

and (here and throughout this paper) $\tau = x + iy$. Using this function, let

$$\phi_s(\tau) := \mathcal{M}_s(4\pi y) e^{2\pi i x}. \tag{8.10}$$

Given a positive integer m and a cusp ρ, Maass–Poincaré series provide a form with principal part equal to $q^{(-m+\kappa_\rho)/t_\rho}$ plus a constant at the cusp ρ, and constant at all other cusps, thereby forming a basis for $H_{2-k}(\Gamma, \nu)$.

Suppose $m > 0$ and $L \in \mathrm{SL}_2(\mathbb{Z})$ with $\rho = L^{-1}\infty$. Then we have the Maass–Poincaré series

$$\mathcal{P}_L(\tau, m, \Gamma, 2-k, s, \nu) := \sum_{M \in \Gamma_\rho \backslash \Gamma} \frac{\phi_s\left(\frac{-m+\kappa_\rho}{t_\rho} \cdot L^{-1} M\tau\right)}{(L^{-1} : M\tau)^{2-k} (M : \tau)^{2-k} \nu(M)}. \tag{8.11}$$

It is easy to check that $\phi_s(\tau)$ is an eigenfunction of Δ_{2-k} with eigenvalue

$$s(1-s) + \frac{k^2 - 2k}{4}.$$

The right hand side of (8.11) converges absolutely for $\Re(s) > 1$, however Bringmann and the third author establish conditional convergence when $s \geq 3/4$ [26], giving Theorem 8.8 below. The theorem is stated for the specific case $\Gamma = \Gamma_0(N)$ for some N and $k \geq 3/2$, in which case we modify the notation slightly and define

$$\mathcal{P}_L(\tau, m, N, 2-k, \nu) := \frac{1}{\Gamma(k)} \mathcal{P}_L(\tau, m, \Gamma_0(N), 2-k, \frac{k}{2}, \nu) \tag{8.12}$$

In the statement of the theorem below, K_c is a modified Kloosterman sum given by (8.13)

$$K_c(2-k,L,\nu,m,n) := \sum_{\substack{0\le a<ct_\rho \\ a\equiv -\frac{c\cdot(\alpha,N)}{\alpha\gamma} \pmod{\frac{N}{(\gamma,N)}} \\ ad\equiv 1 \pmod{c}}} \frac{(S:\tau)^{2-k}\exp\left(2\pi i\left(\frac{a\cdot\frac{(-m+\kappa_\rho)}{t_\rho}+d\cdot\frac{(n+\kappa_\infty)}{t_\infty}}{c}\right)\right)}{(L^{-1}:LS\tau)^{2-k}(LS:\tau)^{2-k}\nu(LS)},$$

where $S=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$. If ν is trivial, we omit it from the notation. We also have that $\delta_{L,S}(m)$ is an indicator function for the cusps $\rho = L^{-1}\infty$ and $\mu = S^{-1}\infty$ given by

$$\delta_{L,S}(m) := \begin{cases} \nu(M)^{-1}e^{2\pi i r\frac{-m+\kappa_\rho}{t_\rho}} & \text{if } M = LT^rS^{-1}\in\Gamma_0(N), \\ 0 & \text{if } \mu \not\sim \rho \text{ in } \Gamma_0(N). \end{cases}$$

Using this notation, we have the following theorem which gives exact formulas for the coefficients and principal part of $\mathcal{P}_L(\tau,m,N,2-k,\nu)$, which is a generalization of Theorem 3.2 of [26].

Theorem 8.8. *Suppose that $\frac{3}{2}\le k\in\frac{1}{2}\mathbb{Z}$, and suppose $\rho = L^{-1}\infty$ is a cusp of $\Gamma_0(N)$. If m is a positive integer, then $\mathcal{P}_L(\tau,m,N,2-k,\nu)$ is in $H_{2-k}(\Gamma_0(N),\nu)$. Moreover, the following are true:*

(1) *We have*

$$\mathcal{P}_L^+(\tau,m,N,2-k,\nu) = \delta_{\rho,I}(m)\cdot q^{-m+\kappa_\infty} + \sum_{n\ge 0} a^+(n)q^n.$$

Moreover, if $n>0$, then $a^+(n)$ is given by

$$-i^k 2\pi\left|\frac{-m+\kappa_\rho}{t_\rho(n+\kappa_\infty)}\right|^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ (c,N)=(\gamma,N)}} \frac{K_c(2-k,L,\nu,-m,n)}{c}\cdot I_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{|-m+\kappa_\rho|\,|n+\kappa_\infty|}{t_\rho}}\right),$$

where I_k is the usual I-Bessel function.

(2) *If $S\in\mathrm{SL}_2(\mathbb{Z})$, then there is some $c\in\mathbb{C}$ so that the principal part of $\mathcal{P}_L(\tau,m,N,2-k)$ at the cusp $\mu = S^{-1}\infty$ is given by*

$$\delta_{L,S}(m)q^{\frac{-m+\kappa_\rho}{t_\rho}} + c$$

Remark. We shall use Theorem 8.8 to prove Theorem 8.1. We note that one could prove Theorem 8.1 without referring to the theory of Maass–Poincaré series. One could make use of ordinary weakly holomorphic Poincaré series. However, we have chosen to use Maass–Poincaré series and the theory of harmonic Maass forms because these results are more general, and because of the recent appearance of harmonic Maass forms in the theory of *umbral moonshine* (see §9).

Sketch of the proof. Writing $\rho = \frac{\alpha}{\gamma}$, Bringmann and the third author prove this theorem for the case that $\gamma \mid N$ and $(\alpha, N) = 1$, along with the assumption that μ and ρ are in a fixed complete set of inequivalent cusps, so that $\delta_{\mu,\rho} = 1$ or 0. This general form is useful to us particularly since it works equally well for the cusps ∞ with L taken to be the identity, and for 0 with L taken to be $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Here and for the remainder of the paper we let $\mathcal{S}_N$ denote any complete set of inequivalent cusps of $\Gamma_0(N)$, and for each $\rho \in \mathcal{S}_N$, we fix some L_ρ with $\rho = L_\rho^{-1}\infty$. Rankin notes [206] (Proof of Theorem 4.1.1(iii)) that given some choice of $\mathcal{S}_N$, each right coset of $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ is in $\Gamma_0(N) \cdot L_\rho T^r$ for some unique $\rho \in \mathcal{S}_N$. Moreover, the r in the statement is unique modulo t_ρ, so the function $\delta_{L,S}(m)$ given above is well-defined on all matrices in $\mathrm{SL}_q(\mathbb{Z})$.

In the proof given by Bringmann and the third author, the sum of Kloosterman sums $\displaystyle\sum_{\substack{c>0 \\ (c,N)=(\gamma,N)}} \frac{K_c(2-k, L, \nu, -m, n)}{c} \ldots$ is written as a sum over representatives of the double coset $\Gamma_\rho \backslash L^{-1}\Gamma_0(N)/\Gamma_\infty$ (omitting the identity if present). Following similar arguments, but without the assumptions on α and γ, we find the indices of summation given in (8.13) and Theorem 8.8. As in their case, we find that the principal part of $\mathcal{P}_{L_\rho}(\tau, m, N, 2-k)$ at a cusp μ is constant if $\mu \not\sim \rho$ and is $\delta_{L_\rho, L_\mu} q^{\frac{-m}{t_\rho}} + c$ for some constant if $\mu = \rho$. Therefore, if μ is a cusp with $L_\mu = M^{-1} L_\rho T^r$ for some $M \in \Gamma_0(N,)$, then the principal part of $\mathcal{P}_{L_\rho}(\tau, m, N, 2-k, \nu)$ at μ is clearly $\nu(M)^{-1} e^{2\pi i r \frac{-m+\kappa}{t_\rho}} q^{\frac{-m+\kappa_\rho}{t_\rho}} + c$. □

Since harmonic Maass forms with a nonholomorphic part have a non-constant principal part at some cusp, we have the following theorem.

Theorem 8.9. [26, Theorem 1.1] *Assuming the notation above, if $\frac{3}{2} \leq k \in \frac{1}{2}\mathbb{Z}$, and $F(\tau) \in H_{2-k}(\Gamma_0(N), \nu)$ has principal part $P_\rho(\tau) = \sum_{m \geq 0} a_\rho(-m) q^{\frac{-m+\kappa_\rho}{t_\rho}}$ for each cusp $\rho \in \mathcal{S}_N$, then*

$$F(\tau) = \sum_{\rho \in \mathcal{S}_N} \sum_{m>0} a_\rho(-m) \mathcal{P}_\rho(\tau, m, N, 2-k, \nu) + g(\tau),$$

where $g(\tau)$ is a holomorphic modular form. Moreover, we have that $c = 0$ whenever $k > 2$, and is a constant when $k = 2$.

8.4. **Exact formulas for $T_g^{(-m)}$.** Using Theorems 8.8 and 8.9, we can write exact formulas for the coefficients of the $T_g^{(-m)}$ provided we know its principal parts at all cusps of $\Gamma_0(Nh)$. With this in mind, we now regard T_g as a modular function on $\Gamma_0(Nh)$ with trivial multiplier. The location and orders of the poles were determined by Harada and Lang [129].

Lemma 8.10. [129, Lemma 7, 9] *Suppose the Γ_g is given by the symbol $N||h+e, f...$, and let $L = \begin{pmatrix} -\delta & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then $T_g|_0 L$ has a pole if and only if $\left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}$, for some $e \in \mathcal{W}_g$ (Note, here we allow $e = 1$). The order of the pole is given by $\frac{(h,\gamma)^2}{eh^2}$.*

Harada and Lang prove this lemma by showing that if u is an integer chosen such $\frac{u\gamma - \alpha\cdot(h,\gamma)}{h}$ is integral and divisible by e and $U = \begin{pmatrix} \frac{e\cdot h}{(h,\gamma)} & \frac{u}{h} \\ 0 & \frac{(h,\gamma)}{h} \end{pmatrix}$, then LU is an Atkin–Lehner involution $W_e \in E_g$. Therefore, we have that

$$T_g|_0 L = \sigma_g(LU) T_g\left(\frac{(h,\gamma)^2}{eh^2}\tau - \frac{u\cdot(h,\gamma)}{eh^2}\right). \tag{8.14}$$

Harada and Lang do not compute $\sigma_g(LU)$, however we will need these values in order to apply Theorem 8.9. Using Lemma 8.4, the following procedure allows us to compute $\sigma_g(M)$ for any matrix $M \in E_g$.

Given a matrix $M \in E_g$, we may write M as $M = \begin{pmatrix} ae & \frac{b}{h} \\ cN & de \end{pmatrix}$ with $e \in \mathcal{W}_g$ and $ade - bc\frac{N}{eh} = 1$. We may also write $h = h_e \cdot h_{\overline{e}}$, where $h_{\overline{e}}$ is the largest divisor of h co-prime to e. Since $c\frac{N}{eh}$ is co-prime to both d and e, we may chose integers A, B, and C (mod h) such that:

- $c\frac{N}{eh}A + d$ is co-prime to $h_{\overline{e}}$ but is divisible by h_e,
- $B \equiv -(eaA+b)(c\frac{N}{h}A + ed)^{-1} \pmod{h_{\overline{e}}}$ and $Bc\frac{N}{eh} + b \equiv 0 \pmod{h_e}$,
- $C \equiv -c(c\frac{N}{h}A + ed)^{-1} \pmod{h_{\overline{e}}}$, and $C \equiv 0 \pmod{h_e}$.

A calculation shows that $\widehat{M} := \begin{pmatrix} 1 & \frac{B}{h} \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} 1 & \frac{A}{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ CN & 1 \end{pmatrix} \begin{pmatrix} h_e & 0 \\ 0 & h_e \end{pmatrix}$ is an Atkin–Lehner involution W_E for $\Gamma_0(Nh)$ where $E = e\cdot h_e^2$. By Lemma 8.4, this implies $\sigma_g(\widehat{M}) = 1$, and therefore

$$\sigma_g(M) = exp\left(\frac{2\pi i}{h}(A + B + \lambda_g C)\right).$$

Combined with Lemma 8.10, this leads us to the following proposition.

Proposition 8.11. *Given a matrix $L = \begin{pmatrix} -\delta & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, let u and U be chosen as above, and define*

$$\epsilon_g(L) := \sigma_g(LU) \cdot e^{2\pi i \frac{u\cdot(h,\gamma)}{eh^2}}.$$

Then by (8.14), we have that

$$T_g|_0 L = \epsilon_g(L) q^{-\frac{(h,\gamma)^2}{eh^2}} + O(q).$$

Using this notation, we are equipped to find exact formulas for the $T_g^{(-m)}$.

Theorem 8.12. *Let $g \in \mathbb{M}$, with $\Gamma_g = N|h+e, f, \ldots,$ and let $\mathcal{S}_{Nh}$ and $\mathcal{W}_g$ be as above. if m and n are positive integers, then there is a constant c for which*

$$T_g^{(-m)}(\tau) = c + \sum_{e \in \mathcal{W}_g} \sum_{\substack{\frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho)^m \mathcal{P}^+_{\alpha/\gamma}(\tau, m, Nh, 0).$$

The n-th coefficient of $T_g^{(-m)}(\tau)$ is given by

$$\sum_{e \in \mathcal{W}_g} \sum_{\substack{\rho = \frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho)^m 2\pi \left| \frac{-m}{n} \cdot \frac{(h,\gamma)^2}{eh^2} \right|^{\frac{1}{2}} \times$$

$$\sum_{\substack{c>0 \\ (c,Nh)=(\gamma,Nh)}} \frac{K_c(0, L, -m, n)}{c} \cdot I_1\left(\frac{4\pi}{c} \sqrt{\left| \frac{-mn \cdot (h,\gamma)^2}{eh^2} \right|} \right).$$

Proof. Every modular function is a harmonic Maass form. Therefore, the idea is to exhibit a linear combination of Maass–Poincaré series with exactly the same principal parts at all cusps as $T_g^{(-m)}$. By Lemma 8.7 and Theorem 8.9, this form equals $T_g^{(-m)}$ up to an additive constant. Lemma 8.4 (c) implies that the coefficients $c_g(n)$ of T_g are supported on the arithmetic progression $n \equiv -1 \pmod{h}$. The function $T_g^{(-m)}$ is a polynomial in T_g, and as such must be the sum of powers of T_g each of which is congruent to $m \pmod{h}$. Therefore, if $M \in \Gamma_g$, then $T^{(-m)}|_0 M = \sigma_g(M)^m T^{(-m)}$. Given $L \in \mathrm{SL}_2(\mathbb{Z})$, let U be a matrix as in (8.14) so that $LU \in \Gamma_g$. By applying Proposition 8.11, we find

$$T_g^{(-m)}|_0 L = \sigma_g(LU)^m T_g^{(-m)}|_0 U^{-1} = \epsilon_g(L)^m q^{-m\frac{(h,\gamma)^2}{eh^2}} + O(q).$$

Theorem 8.9, along with the observations that $t_\rho = \frac{(h,\gamma)^2}{eh^2}$ and $\kappa_\rho = 0$ for every $\rho = \frac{\alpha}{\gamma} = L_\rho^{-1}\infty$, implies the first part of the theorem. The formula for the coefficients follows by Theorem 8.8. □

8.5. **Exact formulas for U_g up to a theta function.** Following a similar process to that in the previous section, we may construct a series $\widehat{U}_g(\tau) = q^{-\frac{23}{24}} + O(q^{\frac{1}{24}})$ with principal parts matching those of $\eta(\tau)T_g(\tau)$ at all cusps. Then according to (7.24), the difference $q^{\frac{1}{24}}(U_g - 1) - \widehat{U}_g$ is a weight $\frac{1}{2}$ holomorphic modular form, which by a celebrated result [211] of Serre–Stark, is a finite linear combination of unary theta functions. This will not affect the asymptotics in Theorem 8.1. The functions T_g and $\widehat{U}_g$ differ primarily in their weight, and in that $\widehat{U}_g$ has a non-trivial multiplier $\nu_\eta : M \to \frac{\eta(M\tau)}{(M:\tau)^{1/2}\eta(\tau)}$. They also have slightly different orders of poles, which is accounted for by the fact that the multiplier ν_η implies that $\kappa_\rho = t_\rho/24$ at every cusp ρ for the $\widehat{U}_g$, rather than 0 for the T_g. The proof of the following theorem is the same as that of Theorem 8.12, *mutatis mutandis.*

Theorem 8.13. *Let $g \in \mathbb{M}$, with $\Gamma_g = N|h+e, f, \ldots,$ and let $\mathcal{S}_{Nh}$ and $\mathcal{W}_g$ be as above. If m is a positive integer then*

$$\widehat{U}_g = \sum_{e \in \mathcal{W}_g} \sum_{\substack{\rho = \frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho) \mathcal{P}^+_{L_\rho}(\tau, 1, Nh, 1/2, \nu_\eta).$$

For n a non-negative integer, the coefficient of $q^{n+\frac{1}{24}}$ in $\widehat{U}_g$ is given by

$$\sum_{e \in \mathcal{W}_g} \sum_{\substack{\rho = \frac{\alpha}{\gamma} \in \mathcal{S}_{Nh} \\ \left(\frac{\gamma}{(\gamma,h)}, \frac{N}{h}\right) = \frac{N}{eh}}} \epsilon_g(L_\rho) \frac{1-i}{\sqrt{2}} 2\pi \left| \frac{-\frac{(h,\gamma)^2}{eh^2} + \frac{1}{24}}{n + \frac{1}{24}} \right|^{\frac{1}{4}} \times$$

$$\sum_{\substack{c>0 \\ (c,Nh)=(\gamma,Nh)}} \frac{K_c(\frac{1}{2}, L, \nu_\eta, -1, n)}{c} \cdot I_{\frac{1}{2}} \left(\frac{4\pi}{c} \sqrt{\left| -\frac{(h,\gamma)^2}{eh^2} + \frac{1}{24} \right| \left| n + \frac{1}{24} \right|} \right).$$

This immediately admits the following corollary.

Corollary 8.14. *Given the notation above, there is a weight $\frac{1}{2}$ linear combination of theta functions $h_g(\tau)$ for which the coefficient q^n in $U_g(\tau) - q^{-\frac{1}{24}} h_g(\tau)$ coincides with the coefficient of $q^{n+\frac{1}{24}}$ in $\widehat{U}_g$, given explicitly in Theorem 8.13.*

8.6. **Proof of Theorem 8.1.**

Proof of Theorem 8.1. Following Harada and Lang [129], we begin by defining the functions

$$T^{(-m)}_{\chi_i}(\tau) := \frac{1}{|\mathbb{M}|} \sum_{g \in \mathbb{M}} \chi_i(g) T^{(-m)}_g(\tau). \tag{8.15}$$

The orthogonality of characters imply that for g and $h \in \mathbb{M}$,

$$\sum_{i=1}^{194} \overline{\chi_i(g)} \chi_i(h) = \begin{cases} |C_{\mathbb{M}}(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.16}$$

Here $|C_{\mathbb{M}}(g)|$ is the order of the centralizer of g in $\mathbb{M}$. Since the order of the centralizer times the order of the conjugacy class of an element is the order of the group, (8.16) and (8.15) together imply the inverse relation

$$T^{(-m)}_g(\tau) = \sum_{i=1}^{194} \overline{\chi_i(g)} T^{(-m)}_{\chi_i}(\tau).$$

In particular we have that $T_e^{(-m)}(\tau) = \sum_{i=1}^{194} \dim(\chi_i) T_{\chi_i}^{(-m)}(\tau)$, and therefore we can identify the $\mathbf{m}_i(-m,n)$ as the Fourier coefficients of the $T_{\chi_i}^{(-m)}(\tau) = \sum_{n=-m}^{\infty} \mathbf{m}_i(-m,n)q^n$.

Using Theorem 8.12, we obtain exact formulas for the coefficients of $T_{\chi_i}^{(-m)}(\tau)$. Let $g \in \mathbb{M}$ with $\Gamma_g = N_g || h_g + e_g, f_g, \ldots$. If m and n are positive integers, then the nth coefficient is given exactly by

$$\frac{1}{|\mathbb{M}|} \sum_{g \in \mathbb{M}} \chi_i(g) \sum_{e \in \mathcal{W}_g} \sum_{\substack{\frac{\alpha}{\gamma} \in \mathcal{S}_{N_g h_g} \\ \left(\frac{\gamma}{(\gamma,h_g)}, \frac{N_g}{h_g}\right) = \frac{N_g}{e h_g}}} \epsilon_g(L_\rho)^m 2\pi \left| \frac{-m}{n} \cdot \frac{(h_g,\gamma)^2}{e h_g^2} \right|^{\frac{1}{2}}$$

$$\sum_{\substack{c>0 \\ (c,N_g h_g) = (\gamma, N_g h_g)}} \frac{K_c(2-k, L, \nu, -m, n)}{c} \cdot I_1\left(\frac{4\pi}{c} \sqrt{\left| \frac{-mn \cdot (h_g,\gamma)^2}{e h_g^2} \right|} \right),$$

where $\mathcal{S}_{N_g h_g}$ and $\mathcal{W}_g$ are given as above.

Using the well-known asymptotics for the I-Bessel function

$$I_k(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4k^2-1}{8x} + \ldots \right),$$

we see that the formula for $\mathbf{m}_i(-m,n)$ is dominated by the $c=1$ term which appears only for $g = \mathrm{e}$ (so that $N_{\mathrm{e}} = h_{\mathrm{e}} = 1$). This term yields the asymptotic

$$\mathbf{m}_i(-m,n) \sim \frac{\chi_i(\mathrm{e}) \cdot |m|^{1/4}}{\sqrt{2} n^{3/4} |\mathbb{M}|} \cdot e^{4\pi\sqrt{|mn|}}$$

as in the statement of the theorem.

The asymptotics for $\mathbf{n}_i(n)$ follows similarly, using the formula

$$U_{\chi_i}(\tau) := \frac{1}{|\mathbb{M}|} \sum_{g \in \mathbb{M}} \chi_i(g) U_g^{(-m)}(\tau).$$

We note that the coefficients of the theta functions $h_g(\tau)$ in Corollary 8.14 are bounded by constants and so do not affect the asymptotics. This yields

$$\mathbf{n}_i(n) \sim \frac{\sqrt{12}\ \chi_i(\mathrm{e})}{|24n+1|^{1/2} |\mathbb{M}|} \cdot e^{\frac{\pi}{6}\sqrt{23|24n+1|}}$$

as in the theorem. □

8.7. **Examples of the exact formulas.** We conclude with a few examples illustrating the exact formulas for the McKay–Thompson series. These formulas for the coefficients generally converge rapidly. However the rate of convergence is not uniform and often requires many more terms to converge to a given precision.

Example. We first consider the case that g is the identity element. Then we have $\Gamma_g = \mathrm{SL}_2(\mathbb{Z})$, which has only the cusp infinity. In this case Theorem 8.12 reduces to the well known expansion

$$T_g = J(\tau) - 744 = q^{-1} + \sum_{n\geq 1} \frac{2\pi}{\sqrt{n}} \cdot \sum_{c>0} \frac{K_c(\infty, -m, n)}{c} \cdot I_1\left(\frac{4\pi\sqrt{n}}{c}\right) q^n.$$

Table 2 below contains several approximations made by bounding the size of the c term in the summation.

Table 2.

	$n=1$	$n=5$	$n=10$
$c \leq 25$	196883.661...	333202640598.254...	22567393309593598.047...
≤ 50	196883.881...	333202640599.429...	22567393309593598.660...
≤ 75	196883.840...	333202640599.828...	22567393309593599.369...
≤ 100	196883.958...	333202640599.827...	22567393309593599.681...
∞	196884	333202640600	22567393309593600

Example. The second example we consider is g in the conjugacy class 4B. In this case we have $\Gamma_g = 4||2+2 \supset \Gamma_0(8)$. The function T_g has a pole at each of the four cusps of $\Gamma_0(8)$:

(1) The cusp ∞ has $e=1$, width $t=1$, and coefficient $\epsilon(L_\infty)=1$.
(2) The cusp 0 has $e=2$, width $t=8$, and coefficient $\epsilon(L_0)=1$.
(3) The cusp $1/2$ has $e=2$, width $t=2$, and coefficient $\epsilon(L_{1/2})=i$.
(4) The cusp $1/4$ has $e=1$, width $t=1$, and $\epsilon(L_{1/4})=-1$.

Table 3 below contains several approximations as in Table 2.

Table 3.

	$n=1$	$n=5$	$n=10$
$c \leq 25$	51.975...	4760.372...	0.107...
≤ 50	52.003...	4759.860...	0.117...
≤ 75	52.041...	4760.066...	0.092...
≤ 100	51.894...	4760.049...	0.040...
∞	52	4760	0

9. Umbral Moonshine

In this penultimate section, we review the recently discovered, and rapidly developing field of umbral moonshine.

9.1. **K3 Surfaces.** Eguchi–Ooguri–Tachikawa reignited the field of moonshine with their 2010 observation [94] that dimensions of representations of the largest Mathieu group, M_{24}, occur as multiplicities of superconformal algebra characters in the K3 elliptic genus.

To formulate their observation more precisely, recall that a *complex K3 surface* is a compact connected complex manifold M of dimension 2, with $\Omega_M^2 \simeq \mathcal{O}_M$ and $H^1(M, \mathcal{O}_M) = 0$. (See [10, 11] for introductory accounts.) Following Witten [167, 231], the *elliptic genus* of a complex manifold M of dimension d is defined to be

$$Z_M := \int_M \mathrm{ch}(\mathbb{E}_{q,y})\,\mathrm{td}(M), \tag{9.1}$$

where $\mathrm{td}(M)$ is the Todd class of M, and $\mathrm{ch}(\mathbb{E}_{q,y})$ is the Chern character of the formal power series

$$\mathbb{E}_{q,y} = y^{\frac{d}{2}} \bigotimes_{n=1}^{\infty} \left(\bigwedge\nolimits_{-y^{-1}q^{n-1}} T_M \otimes \bigwedge\nolimits_{-yq^n} T_M^* \otimes \bigvee\nolimits_{q^n} T_M \otimes \bigvee\nolimits_{q^n} T_M^* \right), \tag{9.2}$$

whose coefficients are virtual vector bundles, obtained as sums of tensor products of the exterior and symmetric powers of the holomorphic tangent bundle T_M, and its dual bundle T_M^*. (Cf. also §1 of [123] or Appendix A of [72].) In (9.2) we interpret $\bigvee_t E$ in direct analogy with $\bigwedge_t E$ (cf. (3.13)), replacing exterior powers $\wedge^k E$ with symmetric powers $\vee^k E$.

Since a complex K3 surface M has trivial canonical bundle, and hence vanishing first Chern class, its elliptic genus is a *weak Jacobi form* $Z_M(\tau, z)$ of weight zero and index $\dim(M)/2 = 1$—see [123] or [137] for proofs of this fact—once we set $q = e(\tau)$ and $y = e(z)$. This means[9] that $Z_M(\tau, z)$ is a holomorphic function on $\mathbb{H} \times \mathbb{C}$, satisfying

$$\begin{aligned} Z_M(\tau, z) &= e\left(\frac{-cz^2}{c\tau + d}\right) Z_M\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \\ &= e(\lambda^2\tau + 2\lambda z) Z_M(\tau, z + \lambda\tau + \mu) \end{aligned} \tag{9.3}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$, with a Fourier expansion $Z_M(\tau, z) = \sum_{n,r} c(n, r) q^n y^r$ such that $c(n, r) = 0$ whenever $n < 0$.

It is known that $Z_M(\tau, z)$ specializes to the Euler characteristic $\chi(M) = 24$ when $z = 0$ (cf. e.g. [123]). The space of weak Jacobi forms of weight zero and index one is one-dimensional (cf. [97]), so Z_M is independent of the choice of M. In fact, we have

$$Z_M(\tau, z) = 8\left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2}\right), \tag{9.4}$$

[9] See [67] or [97] for more on Jacobi forms, including the general transformation formula in case of weight different from zero or index different from one.

where the $\theta_i(\tau, z)$ are the usual Jacobi theta functions:

$$\begin{aligned}
\theta_2(\tau, z) &:= \sum_{n\in\mathbb{Z}} y^{n+1/2} q^{(n+1/2)^2/2} = y^{1/2} q^{1/8} \prod_{n>0} (1 + y^{-1} q^{n-1})(1 + yq^n)(1 - q^n) \\
\theta_3(\tau, z) &:= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} = \prod_{n>0} (1 + y^{-1} q^{n-1/2})(1 + yq^{n-1/2})(1 - q^n) \\
\theta_4(\tau, z) &:= \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2} = \prod_{n>0} (1 - y^{-1} q^{n-1/2})(1 - yq^{n-1/2})(1 - q^n).
\end{aligned} \tag{9.5}$$

In Witten's original analysis [167, 231] the elliptic genus Z_M is the graded trace of an integer-valued operator on a Hilbert space arising from a *supersymmetric nonlinear sigma model* on M. In the case that M is a K3 surface—see [3, 4] for analyses of the sigma models associated to K3 surfaces—it is expected that the corresponding Hilbert space admits an unitary action by the (small) $N = 4$ superconformal algebra (cf. [96]). At least this is known for some special cases, so (9.4) can be written as an integer combination of the irreducible unitary $N = 4$ algebra characters. This leads (cf. [92, 95]) to an expression

$$Z_M(\tau, z) = 24\mu(\tau, z) \cdot \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} + H(\tau) \cdot \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3}, \tag{9.6}$$

where $\theta_1(\tau, z)$ is the Jacobi theta function

$$\theta_1(\tau, z) := i \sum_{n\in\mathbb{Z}} (-1)^n y^{n+1/2} q^{(n+1/2)^2/2} = iy^{1/2} q^{1/8} \prod_{n>0} (1 - y^{-1} q^{n-1})(1 - yq^n)(1 - q^n), \tag{9.7}$$

$\mu(\tau, z)$ denotes the Appell–Lerch sum defined by

$$\mu(\tau, z) := \frac{iy^{1/2}}{\theta_1(\tau, z)} \sum_{n\in\mathbb{Z}} (-1)^n \frac{y^n q^{n(n+1)/2}}{1 - yq^n}, \tag{9.8}$$

and $q^{1/8} H(\tau)$ is a power series in q with integer coefficients,

$$H(\tau) = -2q^{-1/8} + 90q^{7/8} + 462q^{15/8} + 1540q^{23/8} + 4554q^{31/8} + 11592q^{39/8} + \dots. \tag{9.9}$$

The surprising observation of [94] is that each coefficient of a non-polar term appearing in (9.9) is twice the dimension of an irreducible representation (cf. [57]) of the sporadic simple group M_{24}, discovered by Émile Mathieu [183, 184] more than 150 years ago. (Generally, the coefficient of a positive power of q in (9.9) is some non-negative integer combination of dimensions of representations of M_{24}.) Thus $H(\tau)$ serves as an analogue of $J(\tau)$ (cf. (2.6)) for M_{24}, and

$$90 = 45 + 45 \tag{9.10}$$

is the Mathieu analogue of McKay's monstrous observation (2.4).

The analogy with monstrous moonshine was quickly taken up, with the determination by Cheng [39], Eguchi–Hikami [93], and Gaberdiel–Hohenegger–Volpato [108, 109], of *Mathieu McKay–Thompson series*

$$H_g(\tau) = -2q^{-1/8} + \sum_{n>0} \mathrm{tr}(g|K_{n-1/8})q^{n-1/8} \tag{9.11}$$

associated to a graded M_{24}-module $K = \bigoplus_{n>0} K_{n-1/8}$, such that $K_{7/8}$ is the sum of the two 45-dimensional irreducible representations of M_{24}, and $K_{15/8}$ the sum of the two 231-dimensional irreducible representations, etc. (See [43] for a detailed review of Mathieu moonshine, and explicit descriptions of the H_g in particular.) We have the following beautiful recent result of Gannon [43].

Theorem 9.1 (Gannon)**.** *There is a graded M_{24}-module $K = \bigoplus_{n>0} K_{n-1/8}$ for which (9.11) is true (given that the H_g are as described in [43]).*

Remark. A concrete construction of K remains unknown.

The observer may ask: how were the H_g determined, if the module K is as yet unknown? To explain this, note that the subscript in M_{24} is a reference to the fact that M_{24} is distinguished amongst permutation groups: it may be characterized as the unique proper subgroup of the alternating group A_{24} that acts quintuply transitively on 24 points (cf. [66]). Write χ_g for the number of fixed points of an element $g \in M_{24}$, in this defining permutation representation.

The first few terms of H_g are determined by the Eguchi–Ooguri–Tachikawa observation on (9.9), for it indicates that the coefficient of $q^{7/8}$ in H_g should be the trace of g on the sum of the two 45-dimensional irreducible representations, and the coefficient of $q^{15/8}$ should be the trace of g on the sum of the two 231-dimensional irreducible representations, etc. To determine the remaining infinitely many terms, modularity may be used: the series H_g, determined in [39, 93, 108, 109], have the property that

$$Z_g(\tau, z) := \chi_g \mu(\tau, z) \cdot \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} + H_g(\tau) \cdot \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \tag{9.12}$$

is a weak Jacobi form of weight zero and index one for $\Gamma_0^J(N) := \Gamma_0(N) \ltimes \mathbb{Z}^2$ (with non-trivial multiplier when $\chi_g = 0$), where $N = o(g)$ is the order of g, and $\Gamma_0(N)$ is as in (8.6).

Thus Mathieu moonshine entails twisted, or *twined* versions (9.12) of the K3 elliptic genus (9.4), but the single variable series $H_g(\tau)$ may also be studied in their own right, as automorphic objects of a particular kind: it turns out that they are *mock modular forms*[10] of weight 1/2, for various groups $\Gamma_0(N)$, with *shadows* $\chi_g \eta(\tau)^3$. This means that

[10]The notion of mock modular form has arisen recently, from Zwegers' foundational work [239] on Ramanujan's mock theta functions [204, 205], and the subsequent contributions [25] and [237]. We refer to [67, 195, 237] for nice introductions to the theory.

the *completed* functions

$$\widehat{H}_g(\tau) := H_g(\tau) + \chi_g \frac{1}{2\sqrt{i}} \int_{-\overline{\tau}}^{\infty} \overline{\eta\left(-\overline{w}\right)^3} \frac{\mathrm{d}w}{\sqrt{w+\tau}} \tag{9.13}$$

are harmonic Maass forms of weight 1/2, with the same multiplier system as $\eta(\tau)^{-3}$ when $\chi_g \neq 0$. (In case $\chi_g = 0$, i.e. when H_g is already a modular form, the multiplier is slightly different. See e.g. [43]. The groups $\Gamma_0(o(g))$ for which $\chi_g \neq 0$ are characterized in [45].) The function $H_g(\tau)$, being the holomorphic part of $\widehat{H}_g(\tau)$, is the *mock modular form*.

In contrast to the twined K3 elliptic genera Z_g, the mock modular forms H_g are distinguished, in a manner directly analogous to the McKay–Thompson series T_g of monstrous moonshine: it is shown in [44] that the H_g admit a uniform description in terms of Rademacher sums, in direct analogy with Theorem 5.2. (We refer to [44] or the review [43] for a precise statement of this result.) Since the coincidence between the monstrous McKay–Thompson series and (normalized) Rademacher sums depends in a crucial way upon the genus zero property of monstrous moonshine, as evidenced by Theorem 5.1, it is natural to identify the Rademacher sum realization of the H_g as the Mathieu moonshine counterpart to the genus zero property of monstrous moonshine.

As we have hinted above, the Rademacher sum property that distinguishes the T_g and H_g does not hold for the weight zero Jacobi forms Z_g (cf. (9.12)). A Poincaré series approach to Jacobi forms is described in [24], using the foundations established in [27, 28], and it is verified there that the Z_g are not all realized in this way. On the other hand, the main result of [24] is the Poincaré series construction of certain Maass–Jacobi forms of weight one, naturally associated to elements of M_{24}. Thus we can expect that Jacobi forms of weight one, rather than the Z_g of (9.12), will play an important role in a comprehensive conceptual explanation of the Mathieu moonshine phenomenon.

Note that some of the functions Z_g admit a geometric interpretation in terms of K3 surfaces. Namely, it has been established in [64] that if $\bar{g}$ is a symplectic automorphism of a K3 surface M then the natural $\bar{g}$-equivariant modification of (9.1) coincides with Z_g, for a suitable element $g \in M_{24}$. However, not all Z_g arise in this way. Please see §9.4 for a fuller discussion of this.

9.2. **Niemeier Lattices.** Vector-valued versions of the Rademacher sums that characterize the H_g were used in [48] to identify Mathieu moonshine as a special case of six directly similar correspondences, between conjugacy classes in certain finite groups and distinguished (vector-valued) mock modular forms of weight 1/2. Since the mock modular forms arising seemed to be characterized by their shadows, this was dubbed *umbral moonshine* in [48].

The conjectures of [48] were greatly expanded in [49], following an observation of Glauberman (cf. the Acknowledgement in [49]), that the finite groups identified in [48] also appear as automorphism groups of codes associated to deep holes in the Leech lattice (cf. [60] or [57]).

To explain the significance of this, recall that an *integral lattice* is a free abelian group L together with a symmetric bilinear form $\langle \cdot , \cdot \rangle : L \times L \to \mathbb{Z}$. A lattice L is called

positive-definite if $\langle\lambda,\lambda\rangle \geq 0$ for all $\lambda \in L$, with equality only when $\lambda = 0$. It is called *even* if $\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$ for all $\lambda \in L$, and *self-dual* if $L = L^*$, for L^* the *dual* of L,

$$L^* := \{\mu \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle\lambda,\mu\rangle \in \mathbb{Z} \Leftarrow \lambda \in L\}\,. \tag{9.14}$$

The even self-dual positive-definite lattices of rank 24 have been classified [190] (see also [61, 228]) by Niemeier: there are 24 in total, up to isomorphism. They are characterized by their root systems—i.e. the configurations of their vectors with square length equal to 2—and the Leech lattice Λ (cf. §3) is the unique such lattice whose root system is empty. We refer to the remaining 23 as the *Niemeier lattices*. The *Niemeier root systems* are the root systems of the Niemeier lattices, and they are described explicitly as

$$A_1^{24},\ A_2^{12},\ A_3^8,\ A_4^6,\ A_6^4,\ A_{12}^2, \tag{9.15}$$

$$\begin{gathered} A_5^4D_4,\ A_7^2D_5^2,\ A_8^3,\ A_9^2D_6,\ A_{11}D_7E_6,\ A_{15}D_9,\ A_{17}E_7,\ A_{24}, \\ D_4^6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2,\ D_{12}^2,\ D_{16}E_8,\ D_{24},\ E_6^4,\ E_8^3, \end{gathered} \tag{9.16}$$

in terms of the irreducible, simply-laced (i.e. ADE type) root systems. (See [63] or [140] for more on root systems.)

In (9.15) and (9.16) we use juxtaposition as a shorthand for direct sum, so that A_1^{24} denotes 24 copies of the A_1 root system, and $A_{11}D_7E_6$ is shorthand for $A_{11} \oplus D_7 \oplus E_6$, etc. The subscripts indicate ranks. The Coxeter numbers of the ADE root systems are given by

$$m(A_n) = n+1,\ m(D_n) = 2n-2,\ m(E_6) = 12,\ m(E_7) = 18,\ m(E_8) = 30, \tag{9.17}$$

and one can check that the Niemeier root systems (9.15), (9.16) are exactly those unions of ADE type root systems for which the total rank is 24, and the Coxeter number is constant across irreducible components.

For X a Niemeier root system and N^X the corresponding Niemeier lattice, define the *outer automorphism group* of N^X by setting

$$\mathrm{Out}(N^X) := \mathrm{Aut}(N^X)/W^X, \tag{9.18}$$

where W^X denotes the subgroup of $\mathrm{Aut}(N^X)$ generated by reflections in root vectors. Applying this construction to the Leech lattice, corresponding to $X = \emptyset$, we obtain the *Conway group*,

$$Co_0 := \mathrm{Aut}(\Lambda), \tag{9.19}$$

so named in light of Conway's detailed description [53, 55] of its structure. A number of the 26 sporadic simple groups appear as subgroups, or quotients of subgroups of Co_0, including the three sporadic simple Conway groups, Co_1, Co_2 and Co_3. The Conway group Co_0 is a double cover of the first, and largest of these,

$$Co_1 \simeq \mathrm{Aut}(\Lambda)/\{\pm\mathrm{Id}\}. \tag{9.20}$$

Note that M_{24} is naturally a subgroup of Co_0, and also Co_1, for if $\{\lambda_i\} \subset \Lambda$ is a set of 24 vectors such that $\langle\lambda_i,\lambda_j\rangle = 8\delta_{ij}$, then the subgroup of Co_0 that stabilizes this set $\{\lambda_i\}$ is

a copy of M_{24}.

$$M_{24} \simeq \{g \in Co_0 \mid \{g(\lambda_i)\} \subset \{\lambda_i\}\}\,. \tag{9.21}$$

According to Conway–Parker–Sloane [60], the Niemeier root systems classify the *deepest holes* in the Leech lattice, being the points in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ at maximal distance from vectors in Λ. Moreover, this correspondence is strong enough that the Niemeier outer automorphism groups $\mathrm{Out}(N^X)$ are also visible inside the Conway group, Co_0. More precisely, if $x \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is a deep hole, with corresponding Niemeier root system X according to [60], then the stabilizer $\mathrm{Aut}(\Lambda, x)$ of x in $\mathrm{Aut}(\Lambda)$ has a normal subgroup C^x such that

$$\mathrm{Aut}(\Lambda, x)/C^x \simeq \mathrm{Out}(N^X). \tag{9.22}$$

The subgroup C^x even encodes a method for constructing N^X, as is explained in detail in [62], for if L^X denotes the sub lattice of N^X generated by roots, then N^X is determined by its image in $(L^X)^*/L^X$ (cf. (9.14)) under the natural map $N^X \to (L^X)^*/L^X$. Write C^X for this subgroup of $(L^X)^*/L^X$, called the *glue code* of X in [62] (see also [61]). Then C^x is isomorphic to C^X, according to [62].

$$1 \to C^X \simeq C^x \to \mathrm{Aut}(\Lambda, x) \to \mathrm{Out}(N^X) \to 1 \tag{9.23}$$

Thus $\mathrm{Out}(N^X)$ acts as automorphisms on the glue code C^X, and Glauberman's observation suggests an extension of the results of [48], whereby distinguished vector-valued mock modular forms $H_g^X = (H_{g,r}^X)$ are associated to elements g in the *umbral groups*

$$G^X := \mathrm{Out}(N^X), \tag{9.24}$$

for each Niemeier root system X. The realization of this suggestion is described in detail in [49]. For $X = A_1^{24}$, the glue code C^X is a copy of the *extended binary Golay code* (cf. [63] or [203]), and G^X is its full automorphism group, M_{24}. Thus, from the Niemeier root system perspective, Mathieu moonshine is the special case of umbral moonshine corresponding to the root system A_1^{24}.

In (9.15) we have separated out the Niemeier root systems of the form A_n^d with $d = 24/n$ even. It is exactly these cases of umbral moonshine that are discussed in [48]. The original umbral moonshine observation of Eguchi–Ooguri–Tachikawa stemmed from consideration of the weight zero, index one weak Jacobi form Z_M (cf. (9.4)), realized as the K3 elliptic genus. The analysis of [48] is, to some extent, similarly motivated, including the attachment of a weight zero, index n weak Jacobi form $Z_g^{(n+1)}(\tau, z)$ to each $g \in G^X$, for each Niemeier root system $X = A_n^d$ with $d = 24/n$ even.

A notion of *extremal Jacobi form* is formulated in [48], motivated by the representation theory of the $N = 4$ superconformal algebra, and it is proven[11] there that the six functions

[11]The main step in the classification given in [48] is a demonstration that the existence of an extremal Jacobi form of index $m-1$ implies the vanishing of $L(f, 1)$ for all new forms f of weight 2 and level m, where $L(f, s)$ is the Dirichlet series naturally attached to f (cf. e.g. §3.6 of [212]). At this point one expects extremal Jacobi forms to be very few in number, on the strength of the Birch–Swinnerton-Dyer conjecture (cf. [14, 230]), for example. This machinery is evidently quite powerful, and we may anticipate further applications to umbral moonshine in the future.

$Z^{(n+1)} := Z_e^{(n+1)}$, for $n \in \{1, 2, 3, 4, 6, 12\}$, exhaust all examples. Thus the cases (9.15) of umbral moonshine considered in [48] are distinguished from the point of view of Jacobi forms of weight zero.

By contrast, there seems to be no natural way to associate weight zero Jacobi forms to the Niemeier root systems not[12] of the *pure A-type*, A_n^d. Rather, the mock modular forms H_g^X described in [49] naturally appear as the *theta-coefficients* of *finite parts* of certain meromorphic Jacobi forms ψ_g^X of weight 1 and index m,

$$\psi_g^X(\tau, z) = \psi_g^{X,P}(\tau, z) + \sum_{r \pmod{2m}} H_{g,r}^X(\tau)\theta_{m,r}(\tau, z), \tag{9.25}$$

where $m = m(X)$ is the Coxeter number of any irreducible component of X (cf. (9.17)).

Here, meromorphic means that we allow poles in the functions $z \mapsto \psi_g^X(\tau, z)$, at torsion points $z \in \mathbb{Q}\tau + \mathbb{Q}$. The Weierstrass $\wp$ function

$$\wp(\tau, z) := \frac{1}{z^2} + \sum_{\substack{\omega \in \mathbb{Z}\tau + \mathbb{Z} \\ \omega \neq 0}} \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \tag{9.26}$$

is a natural example (with weight two and index zero).

The decomposition (9.25) is described in detail in [49], following the general structural results on meromorphic Jacobi forms established in [67, 239]. For now let us just mention that the first summand on the right hand side is the *polar part* of ψ_g^X, defined as in §8.2 of [67], and

$$\theta_{m,r}(\tau, z) := \sum_{k \in \mathbb{Z}} y^{2km+r} q^{(2km+r)^2/4m} \tag{9.27}$$

evidently depends only on r modulo $2m$.

A number of the meromorphic Jacobi forms attached to Niemeier root systems by umbral moonshine also appear amongst the specific examples of [67], where the main application is the computation of quantum degeneracies of black holes in certain string theories. However, whilst some speculations are offered in §5.5 of [48], no direct relationship between umbral moonshine and string theory has been formulated as yet.

We have seen in §9.1 that the mock modular forms attached to M_{24} by Mathieu moonshine (i.e. umbral moonshine for $X = A_1^{24}$) may be characterized as Rademacher sums, and this serves as an umbral analogue of the principal modulus/genus zero property of monstrous moonshine, on the strength of Theorem 5.1. It is natural to ask for an extension of this result to all cases of umbral moonshine.

Conjecture 5.4 of [48] amounts to the prediction that vector-valued generalizations of the Rademacher sums of [44] will recover the H_g^X for $X = A_n^d$ with d even (cf. (9.15)), and Conjecture 3.2 of [50] is an extension of this to all Niemeier root systems X. Thus a positive solution to Conjecture 3.2 of [50] will verify the umbral analogue of the principal

[12]The cases A_8^3 and A_{24} do come with weight zero Jacobi forms attached, which are obtained via a slight weakening of the notion of extremal Jacobi form formulated in [48]. Cf. §4.3 of [49].

modulus property of monstrous moonshine. So far, the Rademacher sum conjecture for umbral moonshine is known to be true only in the case that $X = A_1^{24}$, but a program to analyze the Rademacher sum conjecture for more general cases of umbral moonshine, via the theory of Maass–Jacobi forms (cf. [27, 28]), has been initiated in [24].

A notion of optimal growth was formulated in §6.3 of [49], following the work [67], with a view to extending Conjecture 5.4 of [48]. It is now known that this condition does not uniquely determine the H_g^X for general X (see [50] for a full discussion of this), but all the H_g^X serve as examples. With this in mind, it is interesting to note that many of Ramanujan's mock theta functions [204, 205] appear as components of the umbral McKay–Thompson series H_g^X. (Cf. §4.7 of [48] and §5.4 of [49].)

9.3. **Modules.** As we have explained above, the Rademacher sum property of umbral moonshine is a natural counterpart to the principal modulus, or genus zero property of monstrous moonshine (cf. §3), formulated in a detailed way by Conway–Norton [59].

The natural counterpart to Thompson's conjecture, Conjecture 3.1, verified by the Frenkel–Lepowsky–Meurman construction [103, 104, 105] of the moonshine module $V^\natural$, together with Borcherds' work [18], is the following (cf. §6.1 of [49], and §2 of [50]).

Conjecture 9.2 (Cheng–Duncan–Harvey)**.** For each Niemeier root system X, there is a bi-graded G^X-module

$$\check{K}^X = \bigoplus_{r\in I^X} \bigoplus_{\substack{D\in\mathbb{Z} \\ D=r^2 \pmod{4m}}} \check{K}^X_{r,-D/4m}, \tag{9.28}$$

such that the vector-valued umbral McKay–Thompson series $H_g^X = (H_{g,r}^X)$ is recovered[13] from the graded trace of g on $\check{K}^X$ via

$$H_{g,r}^X(\tau) = -2q^{-1/4m}\delta_{r,1} + \sum_{\substack{D\in\mathbb{Z} \\ D=r^2 \pmod{4m}}} \mathrm{tr}(g|\check{K}^X_{r,-D/4m})q^{-D/4m} \tag{9.29}$$

for $r \in I^X$.

In (9.28) and (9.29), $m = m(X)$ is the Coxeter number of any irreducible component of X, as in (9.25). The $H_{g,r}^X$ satisfy $H_{g,-r}^X = -H_{g,r}^X$, so the umbral McKay–Thompson series H_g^X is determined by its components $H_{g,r}^X$ with $0 < r < m$. If the highest rank irreducible component of X is of type D or E then there are more symmetries amongst the $H_{g,r}^X$, and the definition of the set $I^X \subset \mathbb{Z}/2m\mathbb{Z}$ reflects this: if X has an A-type component then $I^X := \{1, 2, 3, \ldots, m-1\}$. If the highest rank component of X is of type

[13] In the original formulation, Conjecture 6.1 of [49], the function $H_{g,r}^X$ in (9.29) is replaced with $3H_{g,r}^X$ in the case that $X = A_8^3$. It also predicted that $\check{K}^X_{r,-D/4m}$ is a virtual G^X-module in case $X = A_8^3$ and $D = 0$. Recently, a modification of the specification of the H_g^X for $X = A_8^3$ has been discovered, which leads to the simpler, more uniform formulation appearing here. We refer to [50] for a full discussion of this.

D then $m = 2 \mod 4$, and $I^X := \{1, 3, 5, \ldots, m/2\}$. The remaining cases are $X = E_6^4$, in which case $I^X := \{1, 4, 5\}$, and $X = E_8^3$, for which $I^X := \{1, 7\}$.

As mentioned in §9.1, the existence of the module $\check{K}^X$ for $X = A_1^{24}$ has been proven by Gannon [118]. More specifically, Gannon has shown that the coefficients of the non-negative powers of q in H_g^X for $X = A_1^{24}$ are traces of elements of M_{24} on direct sums of irreducible M_{24}-modules. A priori, we might have needed $\mathbb{C}$-linear combinations of such traces in order to recover the $H_{g,r}^X$.

In forthcoming work [86], the authors confirm the validity of Conjecture 9.2.

Theorem 9.3 (Duncan–Griffin–Ono)**.** *Conjecture 9.2 is true.*

Theorem 9.3 serves, to a certain extent, as the umbral counterpart to Borcherds' result, Theorem 3.6. Indeed, the method of [86] may be used to give an alternative proof of the existence of the $\mathbb{M}$-module $V^\natural$, for which the associated graded trace functions are the normalized principal moduli of the genus zero groups Γ_g, identified by Conway–Norton in [59].

Nonetheless, there is still work to be done, for in order to have a direct counterpart to Theorem 3.6 we require concrete constructions of the $\check{K}^X$. In the case of monstrous moonshine, the construction of $V^\natural$ due to Frenkel–Lepowsky–Meurman came equipped with rich algebraic structure, ultimately leading to the notion of vertex operator algebra, and powerful connections to physics. We can expect that a full explanation of the umbral moonshine phenomena will require analogues of this for all the $\check{K}^X$.

Just such an analogue for $X = E_8^3$ has recently been obtained in [87], where a super vertex operator algebra V^X is constructed, together with an action of $G^X \simeq S_3$, such that the components of the vector-valued mock modular forms $H_g^X = (H_{g,r}^X)$ are recovered from traces of elements of G^X on canonically-twisted modules for V^X. The main ingredient in the construction of [87] is an adaptation of the familiar (to specialists) lattice vertex algebra construction (cf. [15, 105]), to cones in indefinite lattices. The choice of cone is in turn inspired by Zwegers' work [240] on a particular pair of the fifth order mock theta functions of Ramanujan.

In [47, 90] a different approach to the module problem is considered, whereby meromorphic Jacobi forms associated to the H_g^X are recovered as graded traces on canonically-twisted modules for certain super vertex algebras. In [90] constructions are given for the ψ_g^X of (9.25), for $X \in \{A_3^8, A_4^6, A_6^4, A_{12}^2\}$. In [47] certain half-integral index analogues of the ψ_g^X are recovered, for $X \in \{D_6^4, D_8^3, D_{12}^2, D_{24}\}$.

As we will explain in more detail in the next section, recent work [41, 52] constructs modules underlying assignments of vector-valued mock modular forms to the sporadic simple groups M_{24}, M_{23} and M_{22}. Here, M_{23} denotes the maximal subgroup of M_{24} composed of elements fixing any given point in the defining permutation representation (cf. §9.1), and M_{22} is obtained similarly from M_{23}, as the subgroup stabilizing a point in its natural permutation representation of degree 23. Although the mock modular forms realized in [41, 52] are not directly related to the H_g^X, it seems likely that the construction

used therein holds important hints for future developments in the module problem for umbral moonshine.

9.4. **Sigma Models.** Recall from §9.1 that $\Omega^2_M \simeq \mathcal{O}_M$ for M a complex K3 surface. An automorphism of M that induces the trivial action on $H^0(M, \Omega^2_M)$ is called *symplectic*. It is a celebrated result of Mukai [187] (cf. also [163]), that the finite groups of symplectic automorphisms of complex K3 surfaces are, up to isomorphism, precisely the subgroups of M_{24} that have at least five orbits in the unique non-trivial permutation representation on 24 points, including at least one fixed point.

Since a symplectic automorphism of a complex K3 surface M induces a supersymmetry preserving automorphism of a sigma model attached to M (cf. §9.1), and since it is the supersymmetry preserving automorphisms of a K3 sigma model that can be used to twine the K3 elliptic genus (9.4), the problem of classifying the supersymmetry preserving automorphism groups of nonlinear K3 sigma models was considered by Gaberdiel–Hohenegger–Volpato in [110].

One might have anticipated that all supersymmetry preserving K3 sigma model automorphism groups would be contained in M_{24}, but this is not the case. Rather, the main result of [110], being a quantum analogue of Mukai's classification of finite symplectic automorphism groups of K3 surfaces, is that the groups of supersymmetry preserving automorphisms of K3 sigma models are, up to isomorphism, precisely the subgroups of $Co_0 = \mathrm{Aut}(\Lambda)$ (cf. (9.19)) that fix a sublattice of Λ with rank at least four.

Note that the results of [110] are obtained subject to certain conjectural assumptions about the structure of the moduli space of K3 sigma models. Nonetheless, it seems fair to conclude that the K3 sigma models do not furnish quite the right theoretical setting for solving the mysteries of umbral moonshine. For not all of the M_{24}-twinings (9.12) of the K3 elliptic genus (9.4) arise as twinings defined by K3 sigma model automorphisms, since, for example, there are elements of M_{24} (cf. (9.21)) that do not fix a rank four lattice in the Leech lattice, Λ. (Cf. also, the last sentence of §9.1.)

That notwithstanding, we can expect to learn useful information about umbral moonshine from further investigation of K3 sigma models. The history of monstrous moonshine provides a useful point of comparison: in advance of his proof of the Conway–Norton conjectures, Borcherds considered a certain BKM algebra (cf. §3) in [17], which was, at the time, called the monster Lie algebra, although it turned out to be only indirectly connected to the monster. The Lie algebra constructed in [17] is now known as the *fake monster Lie algebra* (cf. §2 of [18]), and has found a number of applications outside of moonshine. For example, the denominator function of the fake monster Lie algebra (cf. Example 2 in §10 of [19]) is used to prove facts about families of K3 surfaces in [21, 164, 236].

At the level of vertex operator algebras, the fake monster Lie algebra corresponds to the lattice vertex algebra V_Λ attached to the Leech lattice. This may be regarded as a "fake" moonshine module, for it has exactly the same graded dimension as $V^\natural$, up to the

constant term,

$$\dim_* V_\Lambda = J(\tau) + 24. \tag{9.30}$$

(Cf. (4.1).) There is no action of the monster on V_Λ, although there is an action by a group[14] of the shape $2^{24}.Co_0 = 2^{24}.(2.Co_1)$ (cf. (9.20)), whereas the monster contains a maximal subgroup with the shape $(2.2^{24}).Co_1$. The Frenkel–Lepowsky–Meurman construction of $V^\natural$ takes V_Λ as a main ingredient. (Cf. (3.6).)

It is striking that the Conway group $Co_0 = 2.Co_1$ plays a prominent role in so many of the objects we have discussed: it is visible within the monster, and within the automorphism group of V_Λ. It serves for K3 sigma models as M_{24} does for K3 surfaces, as discussed above, and all of the umbral moonshine groups (9.24) are visible within Co_0, according to (9.22).

Moreover, there is moonshine for the Conway group, in direct analogy with that for the monster, in the sense that there is an assignment of normalized principal moduli T_g^s to elements $g \in Co_0$ which are realized as trace functions on a graded infinite-dimensional Co_0-module. A proof of this statement has recently appeared in [88].

To explain this, take $g \in Co_0$, let $\{\varepsilon_i\}$ be the eigenvalues associated to the action of g on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, and define

$$T_g^s(\tau) := q^{-1/2} \prod_{n>0} \prod_{i=1}^{24} (1 - \varepsilon_i q^{n-1/2}) + \chi_g, \tag{9.31}$$

where

$$\chi_g := \sum_i \varepsilon_i \tag{9.32}$$

is the character value associated to the action of g on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. Then $T_g^s(2\tau) = q^{-1} + O(q)$ is the normalized principal modulus for a genus zero group, according to Conway–Norton [59] and Queen [199]. (See also [165].)

It has been demonstrated in [88] that the functions T_g^s are the graded traces attached to the action of Co_0 on a distinguished[15] super vertex operator algebra, $V^{s\natural} = \bigoplus_{n=-1}^{\infty} V_{n/2}^{s\natural}$.

$$T_g^s(\tau) = \sum_{n=-1}^{\infty} \mathrm{tr}(\mathfrak{z}g | V_{n/2}^{s\natural}) q^{n/2} \tag{9.33}$$

[14]Given groups A and B, say that a group G has the *shape* $A.B$, and write $G = A.B$, if G contains a normal subgroup isomorphic to A such that $G/A \simeq B$. In this setting it is typical to write p^n as a shorthand for an elementary abelian p-group with p^n elements.

[15]The super vertex operator algebra $V^{s\natural}$ admits actions by both Co_0 (cf. (9.19)) and the simple group Co_1 (cf. (9.20)). It's construction as a Co_1-module was sketched first in §15 of [104], described later in §5 of [22], and subsequently studied in detail in [84]. The Co_0-module structure on $V^{s\natural}$ is mentioned in [84], following [22], but it seems that the modular properties of the trace functions associated to the Co_0-action were not considered until [88].

(In (9.33) we write $\mathfrak{z}$ for the super space involution, acting as $(-1)^n\mathrm{Id}$ on $V^{s\natural}_{n/2}$.) Thus the super vertex operator algebra $V^{s\natural}$ solves the Conway moonshine analogue of Thompson's Conjecture 3.1, and $V^{s\natural}$ is the natural analogue of the moonshine module $V^\natural$ for the Conway group, Co_0.

The Conway module $V^{s\natural}$ is closely related to monstrous moonshine, for, in addition to being directly analogous to $V^\natural$, many of the discrete groups $\Gamma_g < \mathrm{SL}_2(\mathbb{R})$, for $g \in \mathbb{M}$, also arise as invariance groups of principal moduli attached to Co_0 via its action on $V^{s\natural}$. (Cf. [88].) On the other hand, $V^{s\natural}$ enjoys a close connection to K3 sigma models, for it is shown in [89] that the data defining a K3 sigma model gives rise to a bi-grading on a distinguished, canonically-twisted[16] $V^{s\natural}$-module

$$V^{s\natural}_{\mathrm{tw}} = \bigoplus_{n,r} (V^{s\natural}_{\mathrm{tw}})_{n,r}, \tag{9.34}$$

such that the associated graded traces of compatible elements of Co_0 are weak Jacobi forms.

More specifically, following §2.1 of [110], we may regard the data of a K3 sigma model as equivalent[17] to a choice of positive-definite 4-space $\Pi < II_{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$ (cf. (3.7)), such that

$$\delta \in \Pi^\perp \cap II_{4,20} \implies \langle \delta, \delta \rangle \neq -2. \tag{9.35}$$

Then the supersymmetry preserving automorphism group of the nonlinear sigma model defined by Π is the group

$$G_\Pi := \mathrm{Aut}(II_{4,20}, \Pi), \tag{9.36}$$

composed of orthogonal transformations of $II_{4,20}$ that extend to the identity on Π, according to §2.2 of [110]. One of the main results of [110] is that G_Π may be identified with a subgroup of Co_0.

The construction of [89] uses $V^{s\natural}_{\mathrm{tw}}$ to attach a graded trace function

$$\phi_g(\tau, z) := -\sum_{n,r} \mathrm{tr}(\mathfrak{z}g|(V^{s\natural}_{\mathrm{tw}})_{n,r}) q^n y^r \tag{9.37}$$

to each pair (g, Π), where $\Pi < II_{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$ satisfies (9.35), and $g \in G_\Pi$, and $\mathfrak{z}$ is a certain naturally defined involution on $V^{s\natural}_{\mathrm{tw}}$ (analogous to the $\mathfrak{z}$ in (9.33)). It is shown in [89] that ϕ_g is a weak Jacobi form of weight zero and index one for $\Gamma^J_0(N)$ (cf. (9.12)), for some N, for all choices of Π and $g \in G_\Pi$. Moreover, ϕ_g is found to coincide with the g-twined K3 elliptic genus associated to the sigma model defined by Π, for all the examples computed in [110, 114, 229]. (These examples account for about half of the conjugacy classes of

[16]Twisted modules for vertex algebras are discussed in §4. In the case of a super vertex algebra there is a canonically defined involution coming from the superspace structure, called the *parity involution*. A module for a super vertex algebra that is twisted with respect to the parity involution is called *canonically-twisted*.

[17]This convention excludes some interesting K3 sigma models, such as those considered in [51]. We refer to [3, 4] for detailed discussions of K3 sigma model moduli.

Co_0 that fix a 4-space in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.) In particular, taking $g = e$ in (9.37) recovers the K3 elliptic genus (9.4), but in the form

$$\phi_e(\tau, z) = -2^{11} \frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} \frac{\Delta(2\tau)}{\Delta(\tau)} + \frac{1}{2} \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} \frac{\Delta(\tau)^2}{\Delta(2\tau)\Delta(\tau/2)} - \frac{1}{2} \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2} \frac{\Delta(\tau/2)}{\Delta(\tau)}, \tag{9.38}$$

where $\Delta(\tau) := \eta(\tau)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$.

Thus $V_{\mathrm{tw}}^{s\natural}$ serves as a kind of universal object for K3 sigma models. This is interesting, for generally it is difficult to construct the Hilbert spaces underlying a K3 sigma model, and therefore difficult to compute the associated twined K3 elliptic genera, for instance, for all but a few special examples.

In [141], Huybrechts has related the positive-definite 4-spaces $\Pi < II_{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying (9.35) to pairs (X, σ), where X is a projective complex K3 surface, and σ is a stability condition on the bounded derived category of coherent sheaves on X. In this way he has obtained an alternative analogue of Mukai's result [187], whereby symplectic automorphisms of K3 surfaces are replaced by symplectic derived autoequivalences. (The results of [89] are formulated in this language.)

A number of the functions Z_g (cf. (9.12)) arising in Mathieu moonshine are realized as ϕ_g for some $g \in Co_0$. So the construction of [89] relates $V^{s\natural}$ to umbral moonshine, but the connection goes deeper, for it is shown in [89] that a natural generalization of the definition (9.37) recovers a number of the Jacobi forms attached to other root systems of the form $X = A_n^d$ (cf. §9.2), beyond the special case $X = A_1^{24}$. It is interesting to compare this to the results of [51] (see also the precursor [134]), which demonstrate a role for K3 surface geometry in all cases of umbral moonshine (i.e., for all the Niemeier root systems), by considering sigma models attached to K3 surfaces admitting du Val singularities. Indeed, many of the Jacobi forms computed in [51] appear also in [89].

From the discussion above we see that the Conway module $V^{s\natural}$ provides evidence for a deep connection between monstrous and umbral moonshine. Further support for the notion that monstrous and umbral moonshine share a common origin is obtained in [196], where the generalized Borcherds products of [30] are used to relate the trace functions of monstrous and umbral moonshine directly.

We elaborate now upon the results of [41, 52], which, as mentioned at the end of §9.3, fall outside of umbral moonshine as formulated in [49], but are nonetheless related. Actually, these works are further applications of the Conway moonshine module $V^{s\natural}$, for in [41] the canonically-twisted $V^{s\natural}$-module $V_{\mathrm{tw}}^{s\natural}$ (cf. (9.34)) is equipped with module structures for the $N = 2$ and $N = 4$ superconformal algebras, which in turn give rise to an assignment of distinguished vector-valued mock modular forms to elements of the sporadic simple Mathieu groups M_{23} and M_{22}, respectively. This work furnishes the first examples of concretely constructed modules for sporadic simple groups, such that the associated graded trace functions define mock modular forms. The methods of [41] are extended in [52], to the case of the $Spin(7)$ algebra (a certain extension of the $N = 1$ superconformal algebra, cf. [13] and references therein) and vector-valued mock modular forms for M_{24} are obtained. Interestingly, Conway's sporadic groups Co_2 and Co_3, and

the sporadic groups of McLaughlin and Higman–Sims (all rather larger than M_{24}, or any of the other umbral groups G^X) also appear in the analysis of [41, 52].

As a further indication of the important role that K3 sigma models will play in illuminating umbral moonshine, we mention the interesting work [113, 218, 219, 220], which seeks to explain the Mathieu moonshine observation by formulating a precise mechanism for combining symmetries of distinct K3 sigma models into a single group. We note in particular, that a fixed-point-free maximal subgroup of M_{24} is constructed in this way in [220].

We conclude this section with references [42, 133, 134, 135, 156, 197, 234] to a number of other occurrences of umbral groups in geometry and physics, all promising connections to Mathieu moonshine, or umbral moonshine more generally. We also note the recent work [40], which analyzes all cases of generalized umbral moonshine, thereby extending the investigation of generalized Mathieu moonshine that was initiated in [112].

10. Open Problems

We conclude the article by identifying some open problems for future research that are suggested by our results in §7 and §8, and the developments described in §9.

Problem 10.1. We have seen that the known connections between monstrous moonshine and physics owe much to the Frenkel–Lepowsky–Meurman construction of the moonshine module $V^\natural$, and its associated vertex operator algebra structure. Just as the vertex operator algebra structure on $V^\natural$ gives a strong solution to Thompson's conjecture, Conjecture 3.1, we can expect concrete constructions of the $\check{K}^X$—whose existence is now guaranteed thanks to Theorem 9.3—to be necessary for the elucidation of the physical origins of umbral moonshine. As we have described in §9.3, progress on this problem has been obtained recently in [47, 87, 90], and the related work [41] may also be useful, in the determination of a general, algebraic solution to the module problem for umbral moonshine.

Problem 10.2. Norton's generalized moonshine conjectures were discussed in §4, and generalized umbral moonshine has been investigated in [40, 112]. A special case of generalized moonshine for the Conway group is established in [88], but the full formulation and proof of generalized Conway moonshine remains open. Given the close connections between $V^{s\natural}$ and umbral moonshine discussed in §9.4, it will be very interesting to determine the precise relationship between the corresponding generalized moonshine theories. We can expect that the elucidation of these structures will be necessary for a full understanding of the role that umbral moonshine plays in physics.

Problem 10.3. As discussed in §9.2, the fact (cf. Theorem 5.2) that the McKay–Thompson series of monstrous moonshine are realized as Rademacher sums admits conjectural analogues for umbral moonshine. (See Conjecture 3.2 of [50] for a precise formulation.) So far this has been established only for $X = A_1^{24}$, corresponding to Mathieu moonshine (cf. [44]), and the general case remains open. As explained in §9.2, a positive

solution to Conjecture 3.2 of [50] will establish an umbral moonshine counterpart to the principal modulus/genus zero property of monstrous moonshine.

Problem 10.4. As discussed above, in §5 and in §9.2, Rademacher sums play a crucial role in both monstrous and umbral moonshine, by serving to demonstrate the distinguished nature of the automorphic functions arising in each setting. In the case of monstrous moonshine, the Rademacher sum property also indicates a potentially powerful connection to physics, via three-dimensional gravity, as explained in §6. Thus it is an interesting problem to formulate umbral moonshine analogues of the conjectures of [85], discussed here in §§6,7.

Problem 10.5. Relatedly, it follows from the results of [85] that the McKay–Thompson series T_g^s (cf. (9.33)), attached to elements g in the Conway group Co_0 via its action on $V^{s\natural}$ (cf. §9.4), are also realized as Rademacher sums. That is, Theorem 5.2 generalizes naturally to Conway moonshine. Thus it is natural to investigate the higher order analogues $V^{s(-m)}$ of the super vertex operator algebra $V^{s\natural}$, and the Conway group analogues of the three-dimensional gravity conjectures of [85]. Some perspectives on this are available in [138, 179, 233].

Problem 10.6. The notion of extremal vertex operator algebra is defined by (6.8). So far the only known example is the moonshine module $V^\natural$. As explained in §6, the construction of a series of extremal vertex operator algebras, with central charges the positive integer multiples of 24, would go a long way towards the construction of a chiral three-dimensional quantum gravity theory. This problem also has a super analogue, cf. [138].

Problem 10.7. The monster modules $V^{(-m)}$, defined in §7, cannot be vertex operator algebras for $m > 1$, for an action of the Virasoro algebra would generate a non-zero vector with non-positive eigenvalue for $L(0) - \mathbf{c}/24$, and this would violate the condition $\sum_n \dim(V_n^{(-m)})q^n = q^{-m} + O(q)$. Nonetheless, we may ask: do the $V^{(-m)}$ admit vertex algebra structure? Or, is there another natural algebraic structure, which characterizes the monster group actions on the $V^{(-m)}$?

Problem 10.8. Relatedly, $\mathcal{V} \times \mathbb{M}$-modules satisfying the extremal condition (6.8) may be easily constructed from the monster modules $V^{(-m)}$, as is mentioned at the conclusion of §7. What is the algebraic significance of these spaces? We know from [115, 138] that they cannot admit vertex operator algebra structure compatible with the given $\mathcal{V} \times \mathbb{M}$ actions. Is there some other kind of algebraic structure which is compatible with this symmetry?

Problem 10.9. The result of Corollary 8.2 implies that the $V_n^{(-m)}$ and $W_n^\natural$ tend to direct sums of copies of the regular representation of $\mathbb{M}$, as $n \to \infty$. This means that if we write each homogeneous subspace of each module, particularly the moonshine module $V^\natural$, as the sum of a free part (free over the group ring of $\mathbb{M}$) and a non-free part, then the non-free part tends to 0 (relative to the free part) as $n \to \infty$. Is there something to be

learnt from an analysis of the non-free parts of $V^{(-m)}$, $W^\natural$? As one can see from Table 8, some irreducible representations of the monster feature more often in the non-free part than others. We thank Bob Griess for posing this question.

Problem 10.10. It is a natural problem to generalize the methods employed in §8, to determine the distributions of irreducible representations of the umbral groups G^X (cf. (9.24)) in the umbral moonshine modules $\check{K}^X$ (cf. (9.28)). Similarly, one may also consider the distributions of the Co_0-modules in $V^{s\natural} = V^{s(-1)}$, and in the $V^{s(-m)}$ more generally. In all of these cases, questions analogous to Problem 10.9 may reward investigation.

Problem 10.11. In Corollary 8.3 we have used our asymptotic results (Theorem 8.1) on multiplicities of monster modules inside $V^\natural$ to compute the quantum dimensions of the monster orbifold, and in so doing confirmed a special case of Conjecture 6.7 of [76]. How generally can this method be applied, to orbifolds V^G, where V is a vertex operator algebra and G is a compact group of automorphisms of V? Note the following strengths of the asymptotic approach: thanks to Proposition 3.6 of [76], we did not need to verify that the monster orbifold of $V^\natural$ is a rational vertex operator algebra, nor did we need to assume the positivity condition of Theorem 6.3 in [76] (which in any case does not hold for $V^\natural$).

Appendix A. Monstrous Groups

The table below contains the symbols $\Gamma_g = N||h+e, f, \ldots$, for each conjugacy class of the monster. Following [58], if $h = 1$, we omit the '$||1$' from the symbol. If $\mathcal{W}_g = \{1\}$, then we write $N||h$, whereas if it contains every exact divisor of N/h, we write $N||h+$.

The naming of the conjugacy classes is as in [57]. We follow the convention of writing $23AB$ as a shorthand for $23A \cup 23B$, since these conjugacy classes are related by inversion in the monster. (There are 22 such pairs. The Monster group has 196 conjugacy classes in total.) Since the monstrous McKay–Thompson series have real coefficients, $T_g = T_{g^{-1}}$ and $\Gamma_g = \Gamma_{g^{-1}}$ for all g in the monster. Note however that $27A$ and $27B$ are not related by inversion, even though $\Gamma_{27A} = \Gamma_{27B}$. To the authors best knowledge, this coincidence has not yet been explained.

1A	1	12C	$12\|\|2+$	21D	$21+21$
2A	$2+$	12D	$12\|\|3+$	22A	$22+$
2B	2	12E	$12+3$	22B	$22+11$
3A	$3+$	12F	$12\|\|2+6$	23AB	$23+$
3B	3	12G	$12\|\|2+2$	24A	$24\|\|2+$
3C	$3\|\|3$	12H	$12+12$	24B	$24+$
4A	$4+$	12I	12	24C	$24+8$
4B	$4\|\|2+$	12J	$12\|\|6$	24D	$24\|\|2+3$
4C	4	13A	$13+$	24E	$24\|\|6+$
4D	$4\|\|2$	13B	13	24F	$24\|\|4+6$
5A	$5+$	14A	$14+$	24G	$24\|\|4+2$
5B	5	14B	$14+7$	24H	$24\|\|2+12$
6A	$6+$	14C	$14+14$	24I	$24+24$
6B	$6+6$	15A	$15+$	24J	$24\|\|12$
6C	$6+3$	15B	$15+5$	25A	$25+$
6D	$6+2$	15C	$15+15$	26A	$26+$
6E	6	15D	$15\|\|3$	26B	$26+26$
6F	$6\|\|3$	16A	$16\|\|2+$	27A	$27+$
7A	$7+$	16B	16	27B	$27+$
7B	7	16C	$16+$	28A	$28\|\|2+$
8A	$8+$	17A	$17+$	28B	$28+$
8B	$8\|\|2+$	18A	$18+2$	28C	$28+7$
8C	$8\|\|4+$	18B	$18+$	28D	$28\|\|2+14$
8D	$8\|\|2$	18C	$18+9$	29A	$29+$
8E	8	18D	18	30A	$30+6, 10, 15$
8F	$8\|\|4$	18E	$18+18$	30B	$30+$
9A	$9+$	19A	$19+$	30C	$30+3, 5, 15$
9B	9	20A	$20+$	30D	$30+5, 6, 30$
10A	$10+$	20B	$20\|\|2+$	30E	$30\|\|3+10$
10B	$10+5$	20C	$20+4$	30F	$30+2, 15, 30$
10C	$10+2$	20D	$20\|\|2+5$	30G	$30+15$
10D	$10+10$	20E	$20\|\|2+10$	31AB	$31+$
10E	10	20F	$20+20$	32A	$32+$
11A	$11+$	21A	$21+$	32B	$32\|\|2+$
12A	$12+$	21B	$21+3$	33A	$33+11$
12B	$12+4$	21C	$21\|\|3+$	33B	$33+$

34A	34+
35A	35+
35B	$35 + 35$
36A	36+
36B	$36 + 4$
36C	$36\|\|2+$
36D	$36 + 36$
38A	38+
39A	39+
39B	$39\|\|3+$
39CD	$39 + 39$
40A	$40\|\|4+$
40B	$40\|\|2+$
40CD	$40\|\|2 + 20$
41A	41+
42A	42+
42B	$42 + 6, 14, 21$
42C	$42\|\|3 + 7$
42D	$42 + 3, 14, 42$
44AB	44+
45A	45+
46AB	$46 + 23$
46CD	46+
47AB	47+
48A	$48\|\|2+$
50A	50+
51A	51+
52A	$52\|\|2+$
52B	$52\|\|2 + 26$
54A	54+
55A	55+
56A	56+
56BC	$56\|\|4 + 14$
57A	$57\|\|3+$
59AB	59+
60A	$60\|\|2+$
60B	60+
60C	$60 + 4, 15, 60$
60D	$60 + 12, 15, 20$
60E	$60\|\|2 + 5, 6, 30$
60F	$60\|\|6 + 10$
62AB	62+
66A	66+
66B	$66 + 6, 11, 66$
68A	$68\|\|2+$
69AB	69+
70A	70+
70B	$70 + 10, 14, 35$
71AB	71+
78A	78+
78BC	$78 + 6, 26, 39$
84A	$84\|\|2+$
84B	$84\|\|2 + 6, 14, 21$
84C	$84\|\|3+$
87AB	87+
88AB	$88\|\|2+$
92AB	92+
93AB	$93\|\|3+$
94AB	94+
95AB	95+
104AB	$104\|\|4+$
105A	105+
110A	110+
119AB	119+

References

[1] D. Alexander, C. Cummins, J. McKay, and C. Simons, *Completely replicable functions*, Groups, combinatorics & geometry (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 165, Cambridge Univ. Press, Cambridge, 1992, pp. 87–98. MR 1200252 (94g:11029)

[2] M. Aschbacher, *The status of the classification of the finite simple groups*, Notices Amer. Math. Soc. **51** (2004), no. 7, 736–740. MR 2072045

[3] P. S. Aspinwall, *$K3$ surfaces and string duality*, Fields, strings and duality (Boulder, CO, 1996), World Sci. Publ., River Edge, NJ, 1997, pp. 421–540. MR 1479699 (98j:81243)

[4] P. S. Aspinwall and D. R. Morrison, *String theory on $K3$ surfaces*, Mirror symmetry, II, AMS/IP Stud. Adv. Math., vol. 1, Amer. Math. Soc., Providence, RI, 1997, pp. 703–716. MR 1416354 (97i:81128)

[5] A. Astashkevich, *On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras*, Comm. Math. Phys. **186** (1997), no. 3, 531–562. MR 1463812 (98h:17028)

[6] M. F. Atiyah, *K-theory*, second ed., Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989, Notes by D. W. Anderson. MR 1043170 (90m:18011)

[7] A. Bagchi, S. Detournay, and D. Grumiller, *Flat-Space Chiral Gravity*, Phys.Rev.Lett. **109** (2012), 151301.

[8] M. Bañados, M. Henneaux, C.Teitelboim, and J. Zanelli, *Geometry of the $2+1$ black hole*, Phys. Rev. D (3) **48** (1993), no. 4, 1506–1525. MR 1236812 (94g:83079)

[9] M. Bañados, C. Teitelboim, and J. Zanelli, *Black hole in three-dimensional spacetime*, Phys. Rev. Lett. **69** (1992), no. 13, 1849–1851. MR MR1181663 (93e:83042)

[10] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, Springer-Verlag, Berlin, 2004. MR 2030225 (2004m:14070)

[11] A. Beauville et al., *Géométrie des surfaces $K3$: modules et périodes*, Société Mathématique de France, Paris, 1985, Papers from the seminar held in Palaiseau, October 1981–January 1982, Astérisque No. 126 (1985). MR 785216 (87h:32052)

[12] L. Beneish and H. Larson, *Traces of singular values of Hauptmoduln*, Int. J. Numb. Th., accepted for publication.

[13] N. Benjamin, S. M. Harrison, S. Kachru, N. M. Paquette, and D. Whalen, *On the elliptic genera of manifolds of Spin(7) holonomy*, ArXiv e-prints (2014).

[14] B. J. Birch and H. P. F. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. Reine Angew. Math. **218** (1965), 79–108. MR 0179168 (31 #3419)

[15] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proceedings of the National Academy of Sciences, U.S.A. **83** (1986), no. 10, 3068–3071.

[16] ———, *Generalized Kac-Moody algebras*, J. Algebra **115** (1988), no. 2, 501–512. MR 943273 (89g:17004)

[17] ———, *The monster Lie algebra*, Adv. Math. **83** (1990), no. 1, 30–47. MR 1069386 (91k:17027)

[18] ———, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109, No.2** (1992), 405–444.

[19] ———, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562. MR 1625724 (99c:11049)

[20] ———, *Quantum vertex algebras*, Taniguchi Conference on Mathematics Nara '98, Adv. Stud. Pure Math., vol. 31, Math. Soc. Japan, Tokyo, 2001, pp. 51–74. MR 1865087 (2002k:17054)

[21] R. Borcherds, L. Katzarkov, T. Pantev, and N. I. Shepherd-Barron, *Families of $K3$ surfaces*, J. Algebraic Geom. **7** (1998), no. 1, 183–193. MR 1620702 (99d:14029)

[22] R. Borcherds and A. J. E. Ryba, *Modular Moonshine. II*, Duke Math. J. **83** (1996), no. 2, 435–459. MR 1390654 (98b:17030)

[23] R. Bott, *Homogeneous vector bundles*, Ann. of Math. (2) **66** (1957), 203–248. MR 0089473 (19,681d)

[24] K. Bringmann, J. Duncan, and L. Rolen, *Maass-Jacobi Poincaré series and Mathieu Moonshine*, ArXiv e-prints (2014).

[25] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), no. 2, 243–266. MR 2231957 (2007e:11127)

[26] ———, *Coefficients of harmonic maass forms*, Partitions, q-Series, and Modular Forms (Krishnaswami Alladi and Frank Garvan, eds.), Developments in Mathematics, vol. 23, Springer New York, 2012, pp. 23–38.

[27] K. Bringmann and O. Richter, *Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms*, Adv. Math. **225** (2010), no. 4, 2298–2315. MR 2680205 (2011h:11052)

[28] ———, *Exact formulas for coefficients of Jacobi forms*, Int. J. Number Theory **7** (2011), no. 3, 825–833. MR 2805582 (2012f:11095)

[29] J. D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, Comm. Math. Phys. **104** (1986), no. 2, 207–226. MR 836000 (87d:83034)

[30] J. H. Bruinier and K. Ono, *Heegner divisors, L-functions and harmonic weak Maass forms*, Ann. of Math. (2) **172** (2010), no. 3, 2135–2181. MR 2726107 (2012c:11101)

[31] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90. MR 2097357 (2005m:11089)

[32] J. H. Bruinier, W. Kohnen, and K. Ono, *The arithmetic of the values of modular functions and the divisors of modular forms*, Compos. Math. **140** (2004), no. 3, 552–566. MR 2041768 (2005h:11083)

[33] S. Carlip, *Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole*, Class.Quant.Grav. **22** (2005), R85–R124.

[34] ———, *Effective Conformal Descriptions of Black Hole Entropy*, Entropy **13** (2011), 1355–1379.

[35] S. Carnahan, *Generalized moonshine I: genus-zero functions*, Algebra Number Theory **4** (2010), no. 6, 649–679. MR 2728485 (2012b:11065)

[36] ———, *Generalized moonshine, II: Borcherds products*, Duke Math. J. **161** (2012), no. 5, 893–950. MR 2904095

[37] ———, *Generalized Moonshine IV: Monstrous Lie algebras*, (2012).

[38] ———, *Monstrous Lie Algebras*, RIMS Kôkyûroku **1872** (2014).

[39] M. Cheng, *$K3$ surfaces, $N = 4$ dyons and the Mathieu group M_{24}*, Commun. Number Theory Phys. **4** (2010), no. 4, 623–657. MR 2793423 (2012e:11076)

[40] M. Cheng, P. de Lange, and D. Whalen, *Generalised Umbral Moonshine*, (2015).

[41] M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, and T. Wrase, *Mock Modular Mathieu Moonshine Modules*, ArXiv e-prints (2014).

[42] M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, and T. Wrase, *Mathieu Moonshine and N=2 String Compactifications*, JHEP **1309** (2013), 030.

[43] M. Cheng and J. Duncan, *The largest Mathieu group and (mock) automorphic forms*, String-Math 2011, Proc. Sympos. Pure Math., vol. 85, Amer. Math. Soc., Providence, RI, 2012, pp. 53–82. MR 2985326

[44] ———, *On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine*, Commun. Number Theory Phys. **6** (2012), no. 3.

[45] ———, *On the Discrete Groups of Mathieu Moonshine*, Perspectives in Representation Theory, Contemporary Mathematics **610** (2014).

[46] ———, *Rademacher sums and rademacher series*, Conformal Field Theory, Automorphic Forms and Related Topics (Winfried Kohnen and Rainer Weissauer, eds.), Contributions in Mathematical and Computational Sciences, vol. 8, Springer Berlin Heidelberg, 2014, pp. 143–182 (English).

[47] ———, *Meromorphic Jacobi Forms of Half-Integral Index and Umbral Moonshine Modules*, (2015).

[48] M. Cheng, J. Duncan, and J. Harvey, *Umbral Moonshine*, Commun. Number Theory Phys. **8** (2014), no. 2.

[49] ———, *Umbral Moonshine and the Niemeier Lattices*, Research in the Mathematical Sciences **1** (2014), no. 3.

[50] ———, *Weight One Jacobi Forms and Umbral Moonshine*, (2015).

[51] M. Cheng and S. Harrison, *Umbral Moonshine and K3 Surfaces*, ArXiv e-prints (2014).

[52] M. C. N. Cheng, S. M. Harrison, S. Kachru, and D. Whalen, *Exceptional Algebra and Sporadic Groups at c=12*, ArXiv e-prints (2015).

[53] J. H. Conway, *A perfect group of order* $8,315,553,613,086,720,000$ *and the sporadic simple groups*, Proc. Nat. Acad. Sci. U.S.A. **61** (1968), 398–400. MR 0237634 (38 #5915)

[54] ———, *A characterisation of Leech's lattice*, Invent. Math. **7** (1969), 137–142. MR 0245518 (39 #6824)

[55] ———, *A group of order* $8,315,553,613,086,720,000$, Bull. London Math. Soc. **1** (1969), 79–88. MR 0248216 (40 #1470)

[56] ———, *A simple construction for the Fischer-Griess monster group*, Invent. Math. **79** (1985), no. 3, 513–540. MR MR782233 (86h:20019)

[57] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.*, Oxford: Clarendon Press, 1985.

[58] J. Conway, J. McKay, and A. Sebbar, *On the discrete groups of Moonshine*, Proc. Amer. Math. Soc. **132** (2004), 2233–2240.

[59] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), no. 3, 308–339. MR 554399 (81j:20028)

[60] J. H. Conway, R. A. Parker, and N. J. A. Sloane, *The covering radius of the Leech lattice*, Proc. Roy. Soc. London Ser. A **380** (1982), no. 1779, 261–290. MR 660415 (84m:10022b)

[61] J. H. Conway and N. J. A. Sloane, *On the enumeration of lattices of determinant one*, J. Number Theory **15** (1982), no. 1, 83–94. MR 666350 (84b:10047)

[62] ———, *Twenty-three constructions for the Leech lattice*, Proc. Roy. Soc. London Ser. A **381** (1982), no. 1781, 275–283. MR 661720 (84m:10020)

[63] ———, *Sphere packings, lattices and groups*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1999, With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. MR 1662447 (2000b:11077)

[64] T. Creutzig and G. Höhn, *Mathieu moonshine and the geometry of K3 surfaces*, Commun. Number Theory Phys. **8** (2014), no. 2, 295–328. MR 3271177

[65] C. J. Cummins and T. Gannon, *Modular equations and the genus zero property of moonshine functions*, Invent. Math. **129** (1997), no. 3, 413–443. MR 1465329 (98k:11046)

[66] R. T. Curtis, *Natural constructions of the Mathieu groups*, Math. Proc. Cambridge Philos. Soc. **106** (1989), no. 3, 423–429. MR 1010366 (90h:20020)

[67] A. Dabholkar, S. Murthy, and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms*, (2012).

[68] S. Deser and R. Jackiw, *Erratum: "Topologically massive gauge theories" [Ann. Physics* **140** *(1982), no. 2, 372–411; MR0665601 (84j:81128)] by Deser, Jackiw and S. Templeton*, Ann. Physics **185** (1988), no. 2, 406. MR 965585 (90a:81077b)

[69] S. Deser, R. Jackiw, and S. Templeton, *Topologically massive gauge theories*, Ann. Physics **140** (1982), no. 2, 372–411. MR 665601 (84j:81128)

[70] S. Deser, R. Jackiw, and S. Templeton, *Three-Dimensional Massive Gauge Theories*, Phys.Rev.Lett. **48** (1982), 975–978.

[71] R. Dijkgraaf, J. Maldacena, G. Moore, and E. Verlinde, *A black hole farey tail*, (2007).

[72] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Commun. Math. Phys. **185** (1997), 197–209.

[73] L. Dixon, P. Ginsparg, and J. Harvey, *Beauty and the beast: superconformal symmetry in a Monster module*, Comm. Math. Phys. **119** (1988), no. 2, 221–241. MR 968697 (90b:81119)

[74] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds*, Nuclear Phys. B **261** (1985), no. 4, 678–686. MR 818423 (87k:81104a)

[75] ———, *Strings on orbifolds. II*, Nuclear Phys. B **274** (1986), no. 2, 285–314. MR 851703 (87k:81104b)

[76] C. Dong, X. Jiao, and F. Xu, *Quantum dimensions and quantum Galois theory*, Trans. Amer. Math. Soc., **365** (2013), no. 12, 6441–6469. MR 3105758

[77] C. Dong, H. Li, and G. Mason, *Some twisted sectors for the Moonshine module*, Moonshine, the Monster, and related topics (South Hadley, MA, 1994), Contemp. Math., vol. 193, Amer. Math. Soc., Providence, RI, 1996, pp. 25–43. MR 1372716 (97b:17021)

[78] ———, *Compact automorphism groups of vertex operator algebras*, Internat. Math. Res. Notices, (1996), no. 18, 913–921. MR 1420556 (98a:17044)

[79] ———, *Twisted representations of vertex operator algebras*, Math. Ann. **310** (1998), no. 3, 571–600. MR 1615132 (99d:17030)

[80] ———, *Modular invariance of trace functions in orbifold theory and generalized Moonshine*, Communications in Mathematical Physics **214** (2000), 1–56.

[81] C. Dong and G. Mason, *Nonabelian orbifolds and the boson-fermion correspondence*, Comm. Math. Phys. **163** (1994), no. 3, 523–559. MR 1284796 (95i:17031)

[82] ———, *On quantum Galois theory*, Duke Math. J. **86** (1997), no. 2, 305–321. MR 1430435 (97k:17042)

[83] ———, *Quantum Galois theory for compact Lie groups*, J. Algebra, **214** (1999), no. 1, 92–102. MR 1684904 (2000g:17043b)

[84] J. F. Duncan, *Super-Moonshine for Conway's largest sporadic group*, Duke Math. J. **139** (2007), no. 2, 255–315.

[85] J. F. R. Duncan and I. B. Frenkel, *Rademacher sums, moonshine and gravity*, Commun. Number Theory Phys. **5** (2011), no. 4, 1–128.

[86] J. F. R. Duncan, M. Griffin and K. Ono, *Proof of the Umbral Moonshine Conjecture*, ArXiv e-prints (2015).

[87] J. F. R. Duncan and J. A. Harvey, *The Umbral Mooonshine Module for the Unique Unimodular Niemeier Root System*, ArXiv e-prints (2014).

[88] J. F. R. Duncan and S. Mack-Crane, *The Moonshine Module for Conway's Group*, ArXiv e-prints (2014).

[89] ———, *Derived Equivalences of K3 Surfaces and Twined Elliptic Genera*, (2015).

[90] J. Duncan and A. O'Desky, *Super Vertex Algebras, Meromorphic Jacobi Forms, and Umbral Moonshine*, (2015).

[91] J. F. R. Duncan and K. Ono, *The Jack Daniels Problem*, ArXiv e-prints (2014).

[92] T. Eguchi and K. Hikami, *Superconformal Algebras and Mock Theta Functions 2. Rademacher Expansion for K3 Surface*, Communications in Number Theory and Physics **3,** (2009), 531–554.

[93] ———, *Note on Twisted Elliptic Genus of K3 Surface*, Phys.Lett. **B694** (2011), 446–455.

[94] T. Eguchi, H. Ooguri, and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group M_{24}*, Exper.Math. **20** (2011), 91–96.

[95] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, *Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy*, Nucl. Phys. **B315** (1989), 193.

[96] T. Eguchi and A. Taormina, *Unitary representations of the $N = 4$ superconformal algebra*, Phys. Lett. B **196** (1987), no. 1, 75–81. MR 910253 (88j:17022)

[97] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Birkhäuser, 1985.

[98] J. Fay, *Fourier coefficients of the resolvent for a Fuchsian group*, J. Reine Angew. Math. **293/294** (1977), 143–203. MR 0506038 (58 #21944)

[99] B. L. Feĭgin and D. B. Fuchs, *Verma modules over the Virasoro algebra*, Topology (Leningrad, 1982), Lecture Notes in Math., vol. 1060, Springer, Berlin, 1984, pp. 230–245. MR 770243 (86g:17004)

[100] D. Ford, J. McKay, and S. Norton, *More on replicable functions*, Comm. Algebra **22** (1994), no. 13, 5175–5193. MR 1291027 (95i:11036)

[101] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, second ed., Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2004. MR 2082709 (2005d:17035)

[102] I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980/81), no. 1, 23–66. MR 84f:17004

[103] I. Frenkel, J. Lepowsky, and A. Meurman, *A natural representation of the Fischer-Griess Monster with the modular function J as character*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), no. 10, Phys. Sci., 3256–3260. MR MR747596 (85e:20018)

[104] ———, *A moonshine module for the Monster*, Vertex operators in mathematics and physics (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., vol. 3, Springer, New York, 1985, pp. 231–273. MR 86m:20024

[105] ———, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press Inc., Boston, MA, 1988. MR 90h:17026

[106] M. Gaberdiel, *An Introduction to conformal field theory*, Rept.Prog.Phys. **63** (2000), 607–667.

[107] ———, *2D conformal field theory and vertex operator algebras*, (2005).

[108] M. Gaberdiel, S. Hohenegger, and R. Volpato, *Mathieu Moonshine in the elliptic genus of K3*, JHEP **1010** (2010), 062.

[109] ———, *Mathieu twining characters for K3*, JHEP **1009** (2010), 058, 19 pages.

[110] ———, *Symmetries of K3 sigma models*, Commun. Number Theory Phys. **6** (2012), no. 1, 1–50. MR 2955931

[111] M. Gaberdiel, C. Keller, and R. Volpato, *Genus two partition functions of chiral conformal field theories*, Commun. Number Theory Phys. **4** (2010), no. 2, 295–363. MR 2725054 (2011h:81208)

[112] M. Gaberdiel, D. Persson, H. Ronellenfitsch, and R. Volpato, *Generalized Mathieu Moonshine*, Commun. Number Theory Phys. **7** (2013), no. 1, 145–223. MR 3108775

[113] M. Gaberdiel, A. Taormina, R. Volpato, and K. Wendland, *A K3 sigma model with $Z_2^8 : M_{20}$ symmetry*, (2013).

[114] M. Gaberdiel and R. Volpato, *Mathieu Moonshine and Orbifold K3s*, (2012).

[115] D. Gaiotto, *Monster symmetry and Extremal CFTs*, (2008).

[116] D. Gaiotto and X. Yin, *Genus two partition functions of extremal conformal field theories*, JHEP **0708** (2007), 029.

[117] T. Gannon, *Monstrous moonshine: the first twenty-five years*, Bull. London Math. Soc. **38** (2006), no. 1, 1–33. MR 2201600 (2006k:11071)

[118] ———, *Much ado about Mathieu*, (2012).

[119] H. Garland and J. Lepowsky, *Lie algebra homology and the Macdonald-Kac formulas*, Invent. Math. **34** (1976), no. 1, 37–76. MR 0414645 (54 #2744)

[120] R. L. Griess, Jr., *The structure of the "monster" simple group*, Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975), Academic Press, New York, 1976, pp. 113–118. MR 0399248 (53 #3099)

[121] ———, *A construction of F_1 as automorphisms of a $196,883$-dimensional algebra*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), no. 2, part 1, 686–691. MR 605419 (82d:20022)

[122] ———, *The friendly giant*, Invent. Math. **69** (1982), no. 1, 1–102. MR 671653 (84m:20024)

[123] V. Gritsenko, *Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms*, Algebra i Analiz **11** (1999), no. 5, 100–125. MR 1734348 (2001i:11051)

[124] D. Grumiller and N. Johansson, *Instability in cosmological topologically massive gravity at the chiral point*, Journal of High Energy Physics, **7** (2008).

[125] D. Grumiller, R. McNees, and J. Salzer, *Black holes and thermodynamics - The first half century*, (2014).

[126] D. Grumiller, W. Riedler, J. Rosseel, and T. Zojer, *Holographic applications of logarithmic conformal field theories*, J.Phys. **A46** (2013), 494002.

[127] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys.Lett. **B428** (1998), 105–114.

[128] A. Hanaki, M. Miyamoto, and D. Tambara, *Quantum Galois theory for finite groups*, Duke Math. J. **97** (1999), no. 3, 541–544. MR 1682988 (2000g:17043a)

[129] K. Harada and M.-L. Lang, *The McKay-Thompson series associated with the irreducible characters of the Monster*, Moonshine, the Monster, and related topics (South Hadley, MA, 1994), Contemp. Math., vol. 193, Amer. Math. Soc., Providence, RI, 1996, pp. 93–111. MR 1372718 (97b:11059)

[130] ———, *Modular forms associated with the Monster module*, The Monster and Lie algebras (Columbus, OH, 1996), Ohio State Univ. Math. Res. Inst. Publ., vol. 7, de Gruyter, Berlin, 1998, pp. 59–83. MR 1650637 (2000f:11046)

[131] G. H. Hardy, *Collected papers of G. H. Hardy (Including Joint papers with J. E. Littlewood and others). Vol. I*, Edited by a committee appointed by the London Mathematical Society, Clarendon Press, Oxford, 1966. MR 0201267 (34 #1151)

[132] G. H. Hardy and S. Ramanujan, *Asymptotic formulæ in combinatory analysis [Proc. London Math. Soc. (2)* **17** *(1918), 75–115]*, Collected papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, pp. 276–309. MR 2280879

[133] S. Harrison, S. Kachru, and N. Paquette, *Twining Genera of (0,4) Supersymmetric Sigma Models on K3*, JHEP **1404** (2014), 048.

[134] J. A. Harvey and S. Murthy, *Moonshine in Fivebrane Spacetimes*, ArXiv e-prints (2013).

[135] J. A. Harvey, S. Murthy, and C. Nazaroglu, *ADE Double Scaled Little String Theories, Mock Modular Forms and Umbral Moonshine*, ArXiv e-prints (2013).

[136] G. Hoehn, *Generalized Moonshine for the Baby Monster*, (2003).

[137] ———, *Komplexe elliptische Geschlechter und S^1-aequivariante Kobordismustheorie (Complex elliptic genera and S^1-equivariant cobordism theory)*, ArXiv Mathematics e-prints (2004).

[138] G. Hoehn, *Self-dual vertex operator superalgebras of large minimal weight*, arXiv:0801.1822v1, January 2008.

[139] S. Hohenegger, *Automorphic products, generalized Kac-Moody algebras and string amplitudes*, J.Phys.Conf.Ser. **346** (2012), 012008.

[140] J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972, Graduate Texts in Mathematics, Vol. 9. MR 0323842 (48 #2197)

[141] D. Huybrechts, *On derived categories of K3 surfaces, symplectic automorphisms and the Conway group*, ArXiv e-prints (2013).

[142] R. Ivanov and M. Tuite, *Rational generalised moonshine from abelian orbifoldings of the moonshine module*, Nuclear Phys. B **635** (2002), no. 3, 435–472. MR 1915258 (2003g:11040)

[143] ———, *Some irrational generalised moonshine from orbifolds*, Nuclear Phys. B **635** (2002), no. 3, 473–491. MR 1915259 (2003f:11057)

[144] E. Jurisich, *An exposition of generalized Kac-Moody algebras*, Lie algebras and their representations (Seoul, 1995), Contemp. Math., vol. 194, Amer. Math. Soc., Providence, RI, 1996, pp. 121–159. MR 1395597 (97e:17035)

[145] ———, *Generalized Kac-Moody Lie algebras, free Lie algebras and the structure of the Monster Lie algebra*, J. Pure Appl. Algebra **126** (1998), no. 1-3, 233–266. MR 1600542 (99b:17032)

[146] ———, *Borcherd's proof of the Conway-Norton conjecture*, Moonshine: the first quarter century and beyond, London Math. Soc. Lecture Note Ser., vol. 372, Cambridge Univ. Press, Cambridge, 2010, pp. 219–235. MR 2681781 (2012a:20026)

[147] E. Jurisich, J. Lepowsky, and R. L. Wilson, *Realizations of the Monster Lie algebra*, Selecta Math. (N.S.) **1** (1995), no. 1, 129–161. MR 1327230 (96e:17059)

[148] V. G. Kac, *Simple irreducible graded Lie algebras of finite growth*, Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 1323–1367. MR 0259961 (41 #4590)

[149] ———, *Infinite-dimensional Lie algebras, and the Dedekind η-function*, Funkcional. Anal. i Priložen. **8** (1974), no. 1, 77–78. MR 0374210 (51 #10410)

[150] ———, *Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function and the very strange formula*, Adv. in Math. **30** (1978), no. 2, 85–136. MR 513845 (83a:17014a)

[151] ———, *An elucidation of: "Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function and the very strange formula". $E_8^{(1)}$ and the cube root of the modular invariant j*, Adv. in Math. **35** (1980), no. 3, 264–273. MR 563927 (83a:17014b)

[152] ———, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990. MR 1104219 (92k:17038)

[153] ———, *Vertex algebras for beginners*, second ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998. MR 1651389 (99f:17033)

[154] V. G. Kac and D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), no. 2, 125–264. MR 750341 (86a:17007)

[155] V. G. Kac and A. K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Series in Mathematical Physics, vol. 2, World Scientific Publishing Co., Inc., Teaneck, NJ, 1987. MR 1021978 (90k:17013)

[156] S. Katz, A. Klemm, and R. Pandharipande, *On the motivic stable pairs invariants of K3 surfaces*, ArXiv e-prints (2014).

[157] M. I. Knopp, *Rademacher on $J(\tau)$, Poincaré series of nonpositive weights and the Eichler cohomology*, Notices Amer. Math. Soc. **37** (1990), no. 4, 385–393. MR MR1050700 (91e:11047)

[158] ———, *Construction of a class of modular functions and forms*, Pacific J. Math. **11** (1961), 275–293. MR MR0122994 (23 #A325)

[159] ———, *Construction of a class of modular functions and forms. II*, Pacific J. Math. **11** (1961), 661–678. MR MR0130229 (24 #A96)

[160] ———, *Construction of automorphic forms on H-groups and supplementary Fourier series*, Trans. Amer. Math. Soc. **103** (1962), 168–188. MR MR0136735 (25 #197)

[161] ———, *On abelian integrals of the second kind and modular functions*, Amer. J. Math. **84** (1962), 615–628. MR MR0146379 (26 #3901)

[162] D. Knutson, *λ-rings and the representation theory of the symmetric group*, Lecture Notes in Mathematics, Vol. 308, Springer-Verlag, Berlin-New York, 1973. MR 0364425 (51 #679)

[163] S. Kondō, *Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of $K3$ surfaces*, Duke Math. J. **92** (1998), no. 3, 593–603, With an appendix by Shigeru Mukai. MR 1620514 (99i:14042)

[164] ———, *On the Kodaira dimension of the moduli space of $K3$ surfaces. II*, Compositio Math. **116** (1999), no. 2, 111–117. MR 1686793 (2000j:14055)

[165] T. Kondo, *The automorphism group of Leech lattice and elliptic modular functions*, J. Math. Soc. Japan **37** (1985), no. 2, 337–362. MR 780666 (86k:11033)

[166] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387. MR 0142696 (26 #265)

[167] P. S. Landweber (ed.), *Elliptic curves and modular forms in algebraic topology*, Lecture Notes in Mathematics, vol. 1326, Springer-Verlag, Berlin, 1988. MR 970278 (91a:57021)

[168] J. Leech, *Some sphere packings in higher space*, Canad. J. Math. **16** (1964), 657–682. MR 0167901 (29 #5166)

[169] ———, *Notes on sphere packings*, Canad. J. Math. **19** (1967), 251–267. MR 0209983 (35 #878)

[170] J. Leon and C. Sims, *The existence and uniqueness of a simple group generated by* $\{3,4\}$*-transpositions*, Bull. Amer. Math. Soc. **83** (1977), no. 5, 1039–1040. MR 0444765 (56 #3113)

[171] J. Lepowsky and H. Li, *Introduction to vertex operator algebras and their representations*, Progress in Mathematics, vol. 227, Birkhäuser Boston Inc., Boston, MA, 2004. MR 2023933 (2004k:17050)

[172] J. Lepowsky and R. L. Wilson, *Construction of the affine Lie algebra* $A_1^{(1)}$, Comm. Math. Phys. **62** (1978), no. 1, 43–53. MR 0573075 (58 #28089)

[173] W. Li, W. Song, and A. Strominger, *Chiral gravity in three dimensions*, J. High Energy Phys. (2008), no. 4, 082, 15. MR 2425221 (2010a:83101)

[174] K. Mahler, *On a class of non-linear functional equations connected with modular functions*, J. Austral. Math. Soc. Ser. A **22** (1976), no. 1, 65–118. MR 0441867 (56 #258)

[175] J. Maldacena, *The gauge gravity duality*, Proceedings of Xth Quark Confinement and the Hadron Spectrum (Confinement X). 8-12 October, 2012. TUM Camous Garching, Munich, Germany.

[176] ———, *The large N limit of superconformal field theories and supergravity*, Adv.Theor.Math.Phys. **2** (1998), 231–252.

[177] J. Maldacena and A. Strominger, *AdS(3) black holes and a stringy exclusion principle*, JHEP **9812** (1998), 005.

[178] A. Maloney, W. Song, and A. Strominger, *Chiral Gravity, Log Gravity and Extremal CFT*, Phys.Rev. **D81** (2010), 064007.

[179] A. Maloney and E. Witten, *Quantum gravity partition functions in three dimensions*, J. High Energy Phys. (2010), no. 2, 029, 58. MR 2672754 (2011m:83022)

[180] J. Manschot, AdS_3 *partition functions reconstructed*, J. High Energy Phys. (2007), no. 10, 103, 7. MR MR2357875

[181] J. Manschot and G. W. Moore, *A Modern Fareytail*, Commun.Num.Theor.Phys. **4** (2007), 103–159.

[182] G. Mason, *Finite groups and modular functions*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, With an appendix by S. P. Norton, pp. 181–210. MR 933359 (89c:11066)

[183] É. Mathieu, *Mémoire sur l'étude des fonctions de plusiers quantités, sur la manière de les former et sur les substitutions qui les laissent invariables*, Journal de Mathématiques Pures et Appliquées **6** (1861), 241–323.

[184] ———, *Sur la fonction cinq fois transitive de 24 quantités*, Journal de Mathématiques Pures et Appliquées **18** (1873), 25–46.

[185] R. V. Moody, *Lie algebras associated with generalized Cartan matrices*, Bull. Amer. Math. Soc. **73** (1967), 217–221. MR 0207783 (34 #7598)

[186] ———, *A new class of Lie algebras*, J. Algebra **10** (1968), 211–230. MR 0229687 (37 #5261)

[187] S. Mukai, *Finite groups of automorphisms of* $K3$ *surfaces and the Mathieu group*, Invent. Math. **94** (1988), no. 1, 183–221. MR 958597 (90b:32053)

[188] D. Niebur, *A class of nonanalytic automorphic functions*, Nagoya Math. J. **52** (1973), 133–145. MR MR0337788 (49 #2557)

[189] ———, *Construction of automorphic forms and integrals*, Trans. Amer. Math. Soc. **191** (1974), 373–385. MR MR0344196 (49 #8936)

[190] H. V. Niemeier, *Definite quadratische Formen der Dimension* 24 *und Diskriminante* 1, J. Number Theory **5** (1973), 142–178. MR 0316384 (47 #4931)

[191] S. P. Norton, *Generalized Moonshine*, Proc. Symp. Pure Math **47** (1987), 208–209.

[192] S. P. Norton, *More on moonshine*, Computational group theory (Durham, 1982), Academic Press, London, 1984, pp. 185–193. MR 760657 (86h:20020)

[193] _____, *From Moonshine to the Monster*, Proceedings on Moonshine and related topics (Montréal, QC, 1999), CRM Proc. Lecture Notes, vol. 30, Amer. Math. Soc., Providence, RI, 2001, pp. 163–171. MR 1877765 (2002k:20025)

[194] A. P. Ogg, *Automorphismes de courbes modulaires*, Séminaire Delange-PisotPoitou (16e année: 1974/75), Théorie des nombres, Fasc. 1, Exp. No. 7, Secrétariat Mathématique, Paris, 1975, p. 8. MR 0417184 (54 #5242)

[195] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 347–454. MR 2555930 (2010m:11060)

[196] K. Ono, L. Rolen, and S. Trebat-Leder, *Classical and Umbral Moonshine: Connections and p-adic Properties*, J. Ramanujan Math. Soc., to appear (ArXiv e-prints (2014)).

[197] N. Paquette and T. Wrase, *Comments on M_{24} representations and CY_3 geometries*, (2014).

[198] H. Petersson, *Über die Entwicklungskoeffizienten der automorphen Formen*, Acta Math. **58** (1932), no. 1, 169–215. MR 1555346

[199] L. Queen, *Modular functions arising from some finite groups*, Math. Comp. **37** (1981), no. 156, 547–580. MR 628715 (83d:20008)

[200] H. Rademacher, *On the Partition Function $p(n)$*, Proc. London Math. Soc. (2) **43** (1937), 241–254.

[201] _____, *The Fourier Coefficients of the Modular Invariant $J(\tau)$*, Amer. J. Math. **60** (1938), no. 2, 501–512. MR MR1507331

[202] _____, *The Fourier Series and the Functional Equation of the Absolute Modular Invariant $J(\tau)$*, Amer. J. Math. **61** (1939), no. 1, 237–248. MR MR1507375

[203] E. Rains and N. J. A. Sloane, *Self-dual codes*, Handbook of coding theory, Vol. I, II, North-Holland, Amsterdam, 1998, pp. 177–294. MR 1667939

[204] S. Ramanujan, *The lost notebook and other unpublished papers*, Springer-Verlag, Berlin, 1988, With an introduction by George E. Andrews. MR 947735 (89j:01078)

[205] _____, *Collected papers of Srinivasa Ramanujan*, AMS Chelsea Publishing, Providence, RI, 2000, Edited by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson, Third printing of the 1927 original, With a new preface and commentary by Bruce C. Berndt. MR 2280843 (2008b:11002)

[206] R. A. Rankin, *Modular forms and functions*, Cambridge University Press, Cambridge-New York-Melbourne, 1977. MR 0498390 (58 #16518)

[207] U. Ray, *Generalized Kac-Moody algebras and some related topics*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 1, 1–42 (electronic). MR 1803076 (2001j:17044)

[208] W. Reynolds, *Thompson's characterization of characters and sets of primes*, J. Algebra **156** (1993), no. 1, 237–243. MR 1213794 (94b:20012)

[209] G. Segal, *Unitary representations of some infinite-dimensional groups*, Comm. Math. Phys. **80** (1981), no. 3, 301–342. MR 626704 (82k:22004)

[210] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7. MR 0344216 (49 #8956)

[211] J.-P. Serre and H. M. Stark, *Modular forms of weight* 1/2, in Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 27–67. Lecture Notes in Math., Vol. 627. Springer, Berlin, 1977.

[212] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1. MR MR0314766 (47 #3318)

[213] S. D. Smith, *On the head characters of the Monster simple group*, Finite groups—coming of age (Montreal, Que., 1982), Contemp. Math., vol. 45, Amer. Math. Soc., Providence, RI, 1985, pp. 303–313. MR 822245 (87h:20037)

[214] J. Stachel, *The early history of quantum gravity (1916–1940)*, Black holes, gravitational radiation and the universe, Fund. Theories Phys., vol. 100, Kluwer Acad. Publ., Dordrecht, 1999, pp. 525–534. MR 1663704

[215] A. Strominger, *A Simple Proof of the Chiral Gravity Conjecture*, (2008).

[216] L. Susskind, *The world as a hologram*, Journal of Mathematical Physics **36** (1995), 6377–6396.

[217] G. 't Hooft, *Dimensional Reduction in Quantum Gravity*, ArXiv General Relativity and Quantum Cosmology e-prints (1993).

[218] A. Taormina and K. Wendland, *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, J. High Energy Phys. (2013), no. 8, 125, front matter+62. MR 3106313

[219] ———, *A twist in the M24 moonshine story*, (2013).

[220] ———, *Symmetry-surfing the moduli space of Kummer K3s*, (2013).

[221] J. G. Thompson, *Finite groups and modular functions*, Bull. London Math. Soc. **11** (1979), no. 3, 347–351. MR MR554401 (81j:20029)

[222] ———, *Some numerology between the Fischer-Griess Monster and the elliptic modular function*, Bull. London Math. Soc. **11** (1979), no. 3, 352–353. MR MR554402 (81j:20030)

[223] J. Tits, *On R. Griess' "friendly giant"*, Invent. Math. **78** (1984), no. 3, 491–499. MR 768989 (86f:20019)

[224] M. P. Tuite, *Monstrous Moonshine from orbifolds*, Commun. Math. Phys. **146** (1992), 277–309.

[225] ———, *On the relationship between monstrous moonshine and the uniqueness of the moonshine module*, Commun. Math. Phys. **166** (1995), 495–532.

[226] ———, *Generalized moonshine from abelian orbifold constructions*, Contemp. Maths. **193** (1995), 353–368.

[227] R. C. Vaughan, *The Hardy-Littlewood method*, second ed., Cambridge Tracts in Mathematics, vol. 125, Cambridge University Press, Cambridge, 1997. MR 1435742 (98a:11133)

[228] B. B. Venkov, *On the classification of integral even unimodular* 24*-dimensional quadratic forms*, Trudy Mat. Inst. Steklov. **148** (1978), 65–76, 273, Algebra, number theory and their applications. MR 558941 (81d:10024)

[229] R. Volpato, *On symmetries of N=(4,4) sigma models on T4*, ArXiv e-prints (2014).

[230] A. Wiles, *The Birch and Swinnerton-Dyer conjecture*, The millennium prize problems, Clay Math. Inst., Cambridge, MA, 2006, pp. 31–41. MR 2238272

[231] E. Witten, *Elliptic genera and quantum field theory*, Comm. Math. Phys. **109** (1987), no. 4, 525–536. MR 885560 (89i:57017)

[232] ———, *Anti-de Sitter space and holography*, Adv.Theor.Math.Phys. **2** (1998), 253–291.

[233] ———, *Three-Dimensional Gravity Revisited*, (2007).

[234] T. Wrase, *Mathieu moonshine in four dimensional $\mathcal{N} = 1$ theories*, JHEP **1404** (2014), 069.

[235] X. Yin, *Partition functions of three-dimensional pure gravity*, Commun. Number Theory Phys. **2** (2008), no. 2, 285–324. MR 2442775 (2009m:83105)

[236] K.-I. Yoshikawa, *Nikulin's $K3$ surfaces, adiabatic limit of equivariant analytic torsion, and the Borcherds Φ-function*, Complex analysis in several variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, Adv. Stud. Pure Math., vol. 42, Math. Soc. Japan, Tokyo, 2004, pp. 339–345. MR 2087068 (2005h:14093)

[237] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann)*, Astérisque (2009), no. 326, Exp. No. 986, vii–viii, 143–164 (2010), Séminaire Bourbaki. Vol. 2007/2008. MR 2605321 (2011h:11049)

[238] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, Journal of the American Mathematical Society **9** (1996), no. 1, 237–302.

[239] S. Zwegers, *Mock Theta Functions*, Ph.D. thesis, Utrecht University, 2002.
[240] ———, *On two fifth order mock theta functions*, Ramanujan J. **20** (2009), no. 2, 207–214. MR 2558702 (2011a:11046)

Department of Mathematics, Applied Mathematics and Statistics, Case Western Reserve University, Cleveland, Ohio 44106
E-mail address: john.duncan@case.edu

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322
E-mail address: mjgrif3@emory.edu

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322
E-mail address: ono@mathcs.emory.edu

arXiv:0908.4223v3 [math.RT] 3 Dec 2014

GENERALIZED MOONSHINE II: BORCHERDS PRODUCTS

SCOTT CARNAHAN

Abstract. The goal of this paper is to construct infinite dimensional Lie algebras by using infinite product identities, and to use these Lie algebras to reduce the generalized moonshine conjecture to a pair of hypotheses about group actions on vertex algebras and Lie algebras. We expect the Lie algebras that we construct to manifest as algebras of physical states in an orbifold conformal field theory (yet to be fully constructed) with symmetries given by the monster simple group.

We introduce vector-valued modular functions attached to families of modular functions of different levels, and we prove infinite product identities for a distinguished class of automorphic functions on a product of two half-planes. We recast this result using the Borcherds-Harvey-Moore singular theta-lift, and show that the vector-valued functions attached to completely replicable modular functions with integer coefficients lift to automorphic functions with infinite product expansions at all cusps.

For each element of the monster simple group, we construct an infinite dimensional Lie algebra, such that its denominator formula is an infinite product expansion of the automorphic function arising from that element's McKay-Thompson series. These Lie algebras have the unusual property that their simple roots and all root multiplicities are known. We show that under certain hypotheses, characters of groups acting on these Lie algebras form functions on the upper half-plane that are either constant or invariant under a genus zero congruence group.

Contents

Introduction 1
1. Main results 4
2. Summary 5
3. Automorphic infinite products 5
3.1. Vector-valued functions 6
3.2. Hecke-monic modular functions 11
3.3. Replicable functions and products 14
3.4. Singular theta-lifts 16
3.5. A monstrous example 22
4. Lie algebras 26
4.1. Denominator formulas 26
4.2. Compatibility 29
4.3. Twisted denominator formulas 31
4.4. Twisted modules - hypotheses 33
4.5. Conditional generalized moonshine 35
5. Open problems 37
References 39

Introduction

In this paper, we are interested in automorphic functions derived from modular functions, infinite dimensional Lie algebras, and the generalized moonshine conjecture. We use the machinery of automorphic infinite products to construct Lie algebras, and apply distinguished structural features

of the Lie algebras to reduce the conjecture to some precisely stated hypotheses motivated by conformal field theory.

Let $\mathfrak{H}$ denote the complex upper half-plane, and let $j : \mathfrak{H} \to \mathbb{C}$ be Klein's modular invariant. It is characterized as the unique holomorphic function on $\mathfrak{H}$ that is invariant under the action of $SL_2(\mathbb{Z})$ by Möbius transformations, with a simple pole of residue one at infinity, and a zero at a third root of unity. The function j is invariant under translation by one, so it admits a Fourier expansion $j(\tau) = q^{-1} + 744 + 196884q + \ldots$, where $q = e^{2\pi i\tau}$, $\tau \in \mathfrak{H}$. We shall consider an alternative normalization by defining $J(\tau) = j(\tau) - 744$. This function is also holomorphic and invariant under $SL_2(\mathbb{Z})$, and it is the unique such function whose Fourier expansion $\sum_n c(n)q^n$ has the form $q^{-1} + O(q)$, i.e., $c(n) = 0$ for $n < -1$, $c(-1) = 1$, and $c(0) = 0$. During the 1980s, Koike, Norton, and Zagier independently proved the following remarkable formula:

$$J(\sigma) - J(\tau) = p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)} \qquad (p = e^{2\pi i\sigma}, |p| < e^{-2\pi}, |q| < e^{-2\pi}).$$

One of the most interesting qualities of this identity is that the left side of the equation is a difference of two single-variable power series, while there is no obvious reason for the same to be true for the infinite product on the right side. The cancellation of all mixed terms $p^i q^j, ij \neq 0$ implies infinitely many equations relating the coefficients $c(n)$. For example, the vanishing of the pq^2 term on the right side implies $c(4) = c(3) + \binom{c(1)}{2}$, i.e., $20245856256 = 864299970 + \binom{196884}{2}$.

The product identity plays an essential role in monstrous moonshine, because it is the Weyl-Kac-Borcherds denominator formula for the monster Lie algebra, which is a rank two generalized Kac-Moody algebra that admits a homogeneous action of the monster simple group $\mathbb{M}$. The left side of the identity describes the action of the Weyl group on the simple roots together with a Weyl vector, and the exponents $c(mn)$ on the right side give the multiplicities of the positive roots, so the denominator formula can be employed to present the Lie algebra in terms of generators and relations. In his proof of the monstrous moonshine conjecture [Borcherds-1992], Borcherds constructed this Lie algebra in two ways: the first construction, via generators and relations, yielded the denominator formula, and the second construction, by applying a certain functor to an object with a known monster action, endowed the Lie algebra with an action of the monster. The object in question is the monster vertex algebra $V^\natural \cong \bigoplus_{n\geq -1} V_n$, constructed by Frenkel, Lepowsky, and Meurman, and the construction of the monster action on this structure requires most of a rather large book [Frenkel-Lepowsky-Meurman-1988] for building the required machinery. To show that the two Lie algebras were isomorphic, Borcherds identified the root spaces of the Lie algebra with homogeneous spaces in $V^\natural$ to show that their root multiplicities matched. Using the twisted denominator formulas arising from this action, together with facts about $V^\natural$ established in [Frenkel-Lepowsky-Meurman-1988], he showed that the characters of automorphisms of $V^\natural$ are equal to the genus zero modular functions listed in [Conway-Norton-1979], and this proved the monstrous moonshine conjecture for the monster vertex algebra.

Even without a priori knowledge of the monster Lie algebra, one might expect that the Koike-Norton-Zagier formula is a sort of decategorification of a combinatorial or representation-theoretic object, since the coefficients of J (and therefore exponents in the identity) are nonnegative integers. Indeed, similar infinite product formulas appeared in the representation theory of Kac-Moody algebras well before moonshine was discovered. However, the monster action makes this case a particularly rich example: for example, the identity $c(4) = c(3) + \binom{c(1)}{2}$ not only indicates the dimensions of homogeneous spaces in the Lie algebra, but gets promoted to an isomorphism $V_4 \cong V_3 \oplus \bigwedge^2 V_1$ of representations of the monster.

One might ask if this phenomenon is singular, or if there are similar identities, perhaps with modular forms of nonzero weight or level greater than one, that carry additional representation-theoretic information. The answer seems to be positive, but limited, in the sense that there may

be only finitely many interesting cases. One can employ work of Borcherds, Harvey, and Moore to produce automorphic infinite products, and many of these are the Weyl denominators for Lie algebras that may appear in nature as algebras of BPS states or infinitesimal symmetries of a string theory [Harvey-Moore-1995]. However, among the known examples, there is rarely a natural way to endow the root spaces with group actions. The known exceptions all have a basis in conformal field theory, and they include the fake monster Lie algebra with an action of automorphisms of a central extension of the Leech lattice [Borcherds-1992], some supersymmetric variants such as [Scheithauer-2000], and a Lie algebra with baby monster action [Hoehn-2003] constructed to solve a special case of Norton's generalized moonshine conjecture [Norton-1987]. This last construction is of principal interest in this paper.

The generalized moonshine conjecture arose from numerical experimentation, but is indicative of some vast structure:

Conjecture. *There exists a map that attaches to each element g of the monster, an infinite dimensional graded representation $V^\natural(g)$ of a central extension of the centralizer $C_{\mathbb{M}}(g)$, satisfying the following properties:*

(1) *For each $h \in C_{\mathbb{M}}(g)$ and any lift $\tilde{h}$ to the chosen central extension, the graded character*

$$Z(g, \tilde{h}, \tau) = \sum_{n \in \mathbb{Q}} \mathrm{Tr}(\tilde{h} | V^\natural(g)_n) e^{2\pi i (n-1)\tau}$$

is either constant or the q-expansion of a congruence Hauptmodul, i.e., a modular function on $\mathfrak{H}$ that is invariant under a group $\Gamma_{g,h}$, and generates the function field of the quotient $\mathfrak{H}/\Gamma_{g,h}$, where $\Gamma_{g,h} \supset \Gamma(N)$ for some $N > 0$.

(2) *If we let $\mathcal{Z}(g, h, \tau)$ denote the set of functions $Z(g, \tilde{h}, \tau)$ as $\tilde{h}$ ranges over lifts of h, then the functions in this set are constant multiples of each other, and the set of these functions is invariant under simultaneous conjugation of the pair (g, h) in $\mathbb{M}$.*

(3) *For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have the equality of sets $\mathcal{Z}(g, h, \frac{a\tau+b}{c\tau+d}) = \mathcal{Z}(g^a h^c, g^b h^d, \tau)$.*

(4) *$\mathcal{Z}(g, h, \tau)$ contains $J(\tau) = q^{-1} + 196884q + \dots$ if and only if $g = h = 1 \in \mathbb{M}$.*

The assignment $(g, h) \mapsto \mathcal{Z}(g, h, \tau)$ of pairs of commuting elements of the monster to sets of functions on $\mathfrak{H}$ has been called a generalized character, since it appears to be related to the characters that show up in complex oriented cohomology theories [Hopkins-Kuhn-Ravenel-2000]. We use the notation $V^\natural$ here, even though the monster vertex algebra had not been constructed when the conjecture was posed, because in hindsight, the conjecture is fundamentally about the object constructed in [Frenkel-Lepowsky-Meurman-1988].

This conjecture is still open, but in some special cases, candidate representations have been explicitly constructed. In those and some additional cases computed by Norton and Queen, it is known what $Z(g, h, \tau)$ must be, assuming the conjecture is true. Furthermore, shortly after the conjecture was published, Dixon, Ginsparg, and Harvey [Dixon-Ginsparg-Harvey-1988] suggested that the representations of centralizers had a physical interpretation as twisted Hilbert spaces for a conformal field theory with monster symmetry. While the conformal field theory in question has yet to be constructed in full, the physical interpretation has a rephrasing in terms of known algebraic objects: Using the notion of twisted module of a vertex algebra introduced in [Frenkel-Lepowsky-Meurman-1988] for the construction of $V^\natural$, one replaces the twisted Hilbert spaces mentioned before with irreducible twisted modules of the monster vertex algebra $V^\natural$, so elements of $Z(g, h, \tau)$ are interpreted as graded characters of lifts of h acting on an irreducible g-twisted module of $V^\natural$.

With the interpretation of the conjecture in terms of twisted modules in hand, much of the effort in proving it has focused on studying the twisted modules, and under the assumption that this interpretation is correct, candidate functions have been identified as characters in the following cases:

- Borcherds's proof of the original moonshine conjecture implies that $\mathcal{Z}(1,h,\tau)$ are congruence Hauptmoduln for all $h \in \mathbb{M}$. In this case, the twisted module is $V^\natural$, where the twisting is by the identity element.
- Dong, Li, and Mason proved a general result about existence and uniqueness of irreducible twisted modules. They then showed, by reducing to Borcherds's theorem, that, if g and h generate a cyclic group, then the resulting elements of $Z(g,h,\tau)$ are congruence Hauptmoduln that obey the $SL_2(\mathbb{Z})$ compatibility [Dong-Li-Mason-2000].
- Hoehn showed that if g lies in conjugacy class 2A, then the elements of $\mathcal{Z}(g,h,\tau)$ are congruence Hauptmoduln for all h in the centralizer of g [Hoehn-2003].

Hoehn's proof employs a nontrivially twisted module, but the overall structure of the proof is quite similar to Borcherds's proof of the moonshine conjecture, involving the construction of a generalized vertex operator algebra along the lines of [Frenkel-Lepowsky-Meurman-1988] and the construction of a large Lie algebra with a finite group action by the use of a functor.

Based on Hoehn's success, it is reasonable to hope that additional cases of the generalized moonshine conjecture can be attacked with the strategy of constructing a Lie algebra with known root multiplicities (arising from a well-behaved product formula) and an action of a large finite group, and then evaluating twisted denominator formulas. In a previous paper [Carnahan-2010], we showed that when the group and Lie algebra satisfied certain conditions, then the characters were automatically genus zero. In particular, this allows us to circumvent the explicit computations at the end of [Hoehn-2003]. More importantly, this result makes it possible to analyze characters without classifying all pairs of commuting elements, and allows us to work with groups whose character tables are unknown. In this paper, we will construct for each $g \in \mathbb{M}$, a Lie algebra W_g whose denominator identity recovers an automorphic infinite product. When g is the identity, we get the monster Lie algebra, and when g is in conjugacy class 2A, we get Hoehn's Lie algebra. These Lie algebras have the property that we know both the simple roots and the multiplicities of all roots, and they are given as coefficients of special functions. Outside the case of affine Kac-Moody algebras, this is only known for a few special cases of infinite dimensional Lie algebras, which includes some constructions by Borcherds [Borcherds-1998a], Gritsenko and Nikulin [Grisenko-Nikulin-1997], [Gritsenko-Nikulin-1998], Hoehn [Hoehn-2003], and Scheithauer [Scheithauer-2008], [Scheithauer-2004], [Scheithauer-2009] (this list may be missing some contributions, and the author apologizes for those left out). The positive subalgebra E_g yields characters $Z(g,h,\tau)$ that behave as Norton conjectured, assuming some hypotheses we shall explain in detail. The most important of these hypotheses is the existence of a suitable group action, and we intend to construct such an action later in this series using vertex algebra manipulations.

1. Main results

Theorem. *(Theorem 3.24 and subsequent corollary)*

If f is a completely replicable modular function of level N with integer coefficients, then there exists a vector-valued modular function F canonically attached to f, such that the automorphic function $f(\sigma)-f(\tau)$ on $\mathfrak{H}\times\mathfrak{H}$ has an infinite product expansion at all cusps with exponents described by coefficients of F. At the cusp $(i\infty,0)$, this product is:

$$p^{-1}\prod_{m>0,n\in\frac{1}{N}\mathbb{Z}}(1-p^m q^n)^{c_{m,Nn}(mn)},$$

where $p=e^{2\pi i\sigma}$, $q=e^{2\pi i\tau}$, and $c_{m,Nn}(mn)$ is the q^{mn} coefficient of the (m,Nn) component of F.

In this paper, we will adopt the convention that the McKay-Thompson series T_g of an element g in the monster is the graded character $Tr(gq^{L_0-1}|V^\natural)$ of g acting on the monster vertex operator algebra $V^\natural$. These series were only given conceptual grounding as characters of an object when $V^\natural$

was constructed in [Frenkel-Lepowsky-Meurman-1988], proving the McKay-Thompson conjecture asserting the existence of a natural monster representation with graded dimension equal to J.

Theorem. *(Theorem 4.2) For each element g of the monster simple group, let $T_g(\tau)$ be its McKay-Thompson series. There exists a unique generalized Kac-Moody Lie algebra whose denominator formula is the product expansion of $T_g(\sigma) - T_g(\tau)$ at the cusp $(i\infty, 0)$.*

The following result requires unproven hypotheses that are introduced near the end of the paper.

Theorem. *(Theorem 4.12) Suppose hypothesis C_g holds for all $g \in \mathbb{M}$ and hypothesis D holds for $V^\natural$ and all commuting pairs (g, h). Then the generalized moonshine conjecture is true (where the $SL_2(\mathbb{Z})$-compatibility is only given up to a nonzero constant).*

2. Summary

In section 3, we construct automorphic functions on the orthogonal group $O_{2,2}$ with product formulas in two different ways. We begin by directly proving the formulas using equivariant Hecke operators. After this, we show that many of these products arise from a Borcherds-Harvey-Moore regularized theta-lift. The automorphic functions have a particularly degenerate form, in the sense that they analytically continue to functions on $\mathfrak{H} \times \mathfrak{H}$ whose invariance group produces the affine plane (minus some cusp lines) as a quotient space; under appropriate coordinate choices, the functions descend to have the form $x - y$. This degenerate form is essential to our study, because it allows us to identify the simple roots of the Lie algebras in the following section.

In section 4, we present an application of this theory to generalized moonshine. We show that the product formulas arising from McKay-Thompson series are the denominator formulas of generalized Kac-Moody algebras. The main obstruction is the presence of exponents that cannot be dimensions of vector spaces, such as negative integers. We show that under certain assumptions concerning the size of irreducible twisted modules of the monster vertex algebra, the root spaces of the Lie algebras can be identified with eigenspaces in the twisted modules. Under an additional assumption concerning group actions on these Lie algebras, we show that the characters of the actions on the twisted modules are weakly Hecke-monic, and reduce the generalized moonshine conjecture to a conjecture concerning spaces of genus one functions.

We conclude in section 5 with some open problems.

Acknowledgements. The author wishes to thank Richard Borcherds, Gerald Hoehn, and Jacob Lurie. Borcherds suggested generalized moonshine as a dissertation project, and gave much useful advice and perspective. Hoehn offered many helpful comments on an earlier version of this paper. Lurie suggested that equivariant Hecke operators could be useful for understanding exponential elliptic cohomology operations, and this led to their use in product formulas. This research was partly supported by NSF grant DMS-0354321.

3. Automorphic infinite products

The theory of automorphic infinite products originates in early work of Euler and Jacobi, but the representation-theoretic significance of the Fourier expansions of functions started with the independent discovery by Kac and Moody that the denominators of affine Kac-Moody algebras give rise to Macdonald identities (see e.g., [Kac-1990]). Borcherds expanded this work to generalized Kac-Moody algebras, and gave several examples, including the fake monster Lie algebra and monstrous Lie superalgebras. Later work [Borcherds-1995b] gave a coherent recipe for generating product formulas for automorphic forms on $O_{2,n-2}$ whose exponents are coefficients of modular forms. Harvey and Moore reinterpreted these products using a regularized theta-lift [Harvey-Moore-1995], and for certain cases with integer multiplicities, they proposed a physical interpretation of the Lie algebras as infinitesimal symmetries of a heterotic string theory. Borcherds then generalized their

construction to work at higher level, using vector-valued modular forms F that transform according to Weil's metaplectic representation (for more explanation of these ideas, see the introduction to [Borcherds-1998a] and references therein).

In this section, we construct automorphic functions on $\mathfrak{H} \times \mathfrak{H}$ with distinguished infinite product expansions at cusps. We employ Borcherds's toolkit of vector-valued functions, but we first find product formulas using equivariant Hecke operators together with an analytic continuation argument. After that, we introduce the Borcherds-Harvey-Moore theta-lift, and show that in many cases, the products we found come from this more general theory.

The regularized theta-lift has logarithmic singularities along certain quadratic divisors that specify linear fractional relations between the coordinates, and by exponentiating, we get a highly invariant function Ψ with zeroes, poles, and possibly branches on the same loci. Furthermore, for each vector-valued function F, the exponentiated lift Ψ has an infinite product expansion at each cusp, with exponents given by coefficients of F. We show that each McKay-Thompson series $T_g(\tau)$ from monstrous moonshine can be assembled into a vector-valued function whose lift is equal to $T_g(\sigma) - T_g(\tau)$. This gives another way to identify the product expansion of Ψ at $(i\infty, 0)$ with $T_g(\sigma) - T_g(-1/\tau)$, and this will yield the Lie algebras we want in the next section.

The product expansions of $T_g(\sigma) - T_g(-1/\tau)$ were given in the author's Ph.D. dissertation, using an exhaustive enumeration of poles of all McKay-Thompson series. We do not reproduce the argument here, because the current arguments using Hecke operators are more conceptual and easier to follow.

3.1. **Vector-valued functions.** Let M be a free abelian group of rank 4, and let $Q : M \to \mathbb{Z}$ be a quadratic form. We let $(u, v) = Q(u + v) - Q(u) - Q(v)$ be the corresponding even bilinear form, and we assume it is nondegenerate with signature $(2, 2)$. We write $e(x) = e^{2\pi i x}$ for the normalized exponential, and $M^\vee := \mathrm{Hom}(M, \mathbb{Z})$ for the dual lattice. The bilinear form induces an inclusion of M into $M^\vee$ as a finite index subgroup, and $M^\vee/M$ inherits a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form which we will call Q.

Definition 3.1. A vector-valued modular function of type ρ_M is a meromorphic function $F : \mathfrak{H} \to \mathbb{C}[M^\vee/M]$, written as $F(\tau) = \sum_{\gamma \in M^\vee/M} F_\gamma(\tau) e_\gamma$, satisfying

$$F_\gamma(\tau + 1) = e(Q(\gamma)) F_\gamma$$
$$F_\gamma(-1/\tau) = |M^\vee/M|^{-1/2} \sum_{\delta \in M^\vee/M} e(-(\gamma, \delta)) F_\delta(\tau)$$

These functions produce finite-dimensional representations of $SL_2(\mathbb{Z})$, known as the Weil representations associated to M. In the following paragraphs, we describe a technique for constructing vector-valued functions suggested by Borcherds. Essentially, one combines modular functions of various levels and applies lemma 2.6 in [Borcherds-1998a].

We define the lattice $II_{1,1}$ to be the group $\mathbb{Z} \times \mathbb{Z}$, equipped with the quadratic form $Q(a, b) = ab$. The associated bilinear form is even and unimodular. For a positive integer $N > 0$, we define $II_{1,1}(N)$ to be the group $\mathbb{Z} \times \mathbb{Z}$ with quadratic form $Q(a, b) = Nab$. If we identify $II_{1,1}(N)$ with the sublattice of $II_{1,1}$ whose underlying group is $\mathbb{Z} \times N\mathbb{Z}$, then the dual lattice is identified with $\frac{1}{N}\mathbb{Z} \times \mathbb{Z}$, and the quotient is isomorphic to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with quadratic form $Q(a, b) = ab/N \in \mathbb{Q}/\mathbb{Z}$.

Lemma 3.2. *The group $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ decomposes into $\sigma_0(N)$ orbits under the right vector action of $SL_2(\mathbb{Z})$ following reduction modulo N, i.e., where $(i, j) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (ai + cj, bi + dj)$, and $\sigma_0(N)$ is the number of positive integers dividing N. There is an explicit bijection between orbits and positive integer divisors of N, given by attaching to each $m|N$, the orbit that contains the element $(0, m)$.*

Proof. For any $(i,j) \in \mathbb{Z}\times\mathbb{Z}$, let $m = (i,j,N)$. Then there exists a (not necessarily unique) matrix A of the following form:

$$A = \begin{pmatrix} * & * \\ i/m & j/m \end{pmatrix} \in SL_2(\mathbb{Z}/(N/m)\mathbb{Z}).$$

If we replace the integers i,j,m with their residue classes mod N, we have $(0,m)\cdot A \equiv (i,j)$. Since the reduction map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/\frac{N}{m}\mathbb{Z})$ is surjective, the orbit of $(0,m)$ under $SL_2(\mathbb{Z})$ is equal to the set of all $(k,\ell) \in \mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}$ satisfying $(k,\ell,N) = m$. This yields the desired bijection. □

For each $m|N$, let $f_{(m)} : \mathfrak{H} \to \mathbb{C}$ be a modular function invariant under $\Gamma_1(N/m)$. We construct a $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}]$-valued function $\hat{F}$ on $\mathfrak{H}$ by setting $\hat{F}_{i,j}(\tau) = f_{((i,j,N))}(A\tau)$, where A is as given in the proof of Lemma 3.2. In particular, $\hat{F}_{0,m} = f_{(m)}$, and for any $A \in SL_2(\mathbb{Z})$, we have $\hat{F}_{(0,m)\cdot A} = f_{(m)}(A\tau)$. This definition yields the following invariance relation: $\hat{F}_{i,j}(\frac{a\tau+b}{c\tau+d}) = \hat{F}_{ai+cj,bi+dj}(\tau)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. We can think of $\hat{F}$ as a way to encode all $SL_2(\mathbb{Z})$ transformations of $f_{(m)}$ as m ranges over divisors of N. We note that this assignment yields a well-defined function, because the ambiguity in our choice of A is given by left translation by elements of $\Gamma_1(N/m)$.

Let $M = II_{1,1}(N)\times II_{1,1}$ and fix the inclusion $M \subset II_{1,1}\times II_{1,1}$ as $\mathbb{Z}\times N\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \subset \mathbb{Z}^4$ with quadratic form $Q(a,b,c,d) = ab+cd$. Then we identify $M^\vee/M = (\frac{1}{N}\mathbb{Z}\times\mathbb{Z})/(\mathbb{Z}\times N\mathbb{Z})$ with $\mathbb{Z}/N\mathbb{Z}\times\mathbb{Z}/N\mathbb{Z}$ with a coordinate switch: $(a,b,0,0)+M \mapsto (b+N\mathbb{Z}, Na+N\mathbb{Z})$, and define F to be the Fourier transform of $\hat{F}$ in the second coordinate, i.e., along the rows of a matrix of functions:

$$F_{i,k} = \frac{1}{N}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} e(-jk/N)\hat{F}_{i,j}$$

Lemma 3.3. *F is a vector-valued function of type ρ_M.*

Proof. We need to check two conditions:

$$\begin{aligned} F_{i,k}(\tau+1) &= \frac{1}{N}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} e(-jk/N)\hat{F}_{i,j}(\tau+1) \\ &= \frac{1}{N}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} e(-jk/N)\hat{F}_{i,j+i}(\tau) \\ &= \frac{1}{N}e(ik/N)\sum_{j\in\mathbb{Z}/N\mathbb{Z}} e(-(j+i)k/N)\hat{F}_{i,j+i}(\tau) \\ &= e(ik/N)F_{i,k}(\tau) \end{aligned}$$

so F behaves correctly under $T : \tau \mapsto \tau+1$.

$$\begin{aligned} F_{i,k}(-1/\tau) &= \frac{1}{N}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} e(-jk/N)\hat{F}_{i,j}(-1/\tau) \\ &= \frac{1}{N}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} e(-jk/N)\hat{F}_{j,-i}(\tau) \\ &= \frac{1}{N}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} e(-jk/N)\sum_{l\in\mathbb{Z}/N\mathbb{Z}} e(-il/N)F_{j,l}(\tau) \\ &= \frac{1}{N}\sum_{j,l\in\mathbb{Z}/N\mathbb{Z}} e(\frac{-jk-il}{N})F_{j,l}(\tau) \end{aligned}$$

so F behaves correctly under $S : \tau \mapsto -1/\tau$. □

Proposition 3.4. *The Fourier transform described above is a linear isomorphism between the space of families $\{f_{(m)}\}_{m|N}$ of modular functions with $f_{(m)}$ invariant under $\Gamma_1(N/m)$ and the space of vector-valued functions of type ρ_M.*

Proof. The map is clearly linear, and by the previous lemma, it takes families $\{f_{(m)}\}_{m|N}$ to vector-valued functions of type ρ_M. Since the Fourier transform is invertible, the kernel of this map is trivial. It suffices to show that the map is surjective.

Let F be a vector-valued function of type ρ_M. We apply the inverse Fourier transform to get a $\mathbb{C}[\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}]$-valued function $\hat{F}$ on $\mathfrak{H}$:

$$\hat{F}_{i,j}(\tau) = \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(jk/N) F_{i,k}(\tau)$$

We wish to show that for each $m|N$, $\hat{F}_{0,m}(\tau)$ is invariant under $\Gamma_1(N/m)$ and that for $m = (i, j, N)$, $\hat{F}_{i,j}(\tau) = \hat{F}_{0,m}(\frac{a\tau+b}{c\tau+d})$ for all $a, b, c, d \in \mathbb{Z}$ satisfying $ad - bc = 1$, $c \equiv i/m \pmod{N}$, and $d \equiv j/m \pmod{N}$. Since the two maps $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$ generate $SL_2(\mathbb{Z})$, it suffices to show that $\hat{F}_{i,j}(\tau + 1) = \hat{F}_{i,j+i}(\tau)$ and $\hat{F}_{i,j}(-1/\tau) = \hat{F}_{j,-i}(\tau)$.

$$\begin{aligned}
\hat{F}_{i,j}(\tau + 1) &= \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(jk/N) F_{i,k}(\tau + 1) \\
&= \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(jk/N) e(ik/N) F_{i,k}(\tau) \\
&= \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(k(j + i)/N) F_{i,k}(\tau) \\
&= \hat{F}_{i,j+i}(\tau)
\end{aligned}$$

so $\hat{F}$ behaves correctly under T.

$$\begin{aligned}
\hat{F}_{i,j}(-1/\tau) &= \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(jk/N) F_{i,k}(-1/\tau) \\
&= \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(jk/N) \frac{1}{N} \sum_{m,n \in \mathbb{Z}/N\mathbb{Z}} e(\frac{-in - km}{N}) F_{m,n}(\tau) \\
&= \sum_{n \in \mathbb{Z}/N\mathbb{Z}} e(-in/N) \frac{1}{N} \sum_{m \in \mathbb{Z}/N\mathbb{Z}} F_{m,n}(\tau) \sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(\frac{k(j - m)}{N}) \\
&= \sum_{n \in \mathbb{Z}/N\mathbb{Z}} e(-in/N) F_{j,n}(\tau) \\
&= \hat{F}_{j,-i}(\tau)
\end{aligned}$$

so $\hat{F}$ behaves correctly under S.

To conclude, we note that:

(1) For all m, $\hat{F}_{0,m}$ is invariant under T, since $\hat{F}_{0,m} = \hat{F}_{0,m+0}$.
(2) For all $m|N$, $\hat{F}_{0,m}$ is invariant under $\Gamma(N/m)$, since $\hat{F}_{0,m}(\frac{(aN/m+1)\tau+(bN/m)}{(cN/m)\tau+(dN/m+1)}) = \hat{F}_{0,m}(\tau)$ for all $a, b, c, d \in \mathbb{Z}$ such that $\begin{pmatrix} aN/m + 1 & bN/m \\ cN/m & dN/m + 1 \end{pmatrix}$ has determinant 1.

Since $\Gamma_1(N/m)$ is generated by $\Gamma(N/m)$ and T, $\hat{F}_{0,m}$ is invariant under $\Gamma_1(N/m)$. □

Corollary 3.5. *A vector-valued function F of type ρ_M is uniquely determined by its restriction to components $F_{0,i}$ for $i \in \mathbb{Z}/N\mathbb{Z}$. In particular, if these components are zero, then F is zero.*

Proof. F is uniquely determined by its inverse Fourier transform $\hat{F}$. $\hat{F}$ is uniquely determined by its restriction to the components $\hat{F}_{0,m}$ for $0 \leq m < N$, and these components are uniquely determined by their Fourier transforms $F_{0,i}$. $\square$

3.1.1. *Moduli interpretation.* Modular functions (or forms) can be viewed as functions (or sections of line bundles) on moduli spaces of structured elliptic curves. We can think of the functions $\hat{F}$ in a similar way, except unlike the usual cases, this moduli space is not in general connected. Since each $\hat{F}$ is determined by a set of functions $\{f_{(m)}\}_{m|N}$ where $f_{(m)} : \mathfrak{H} \to \mathbb{C}$ is invariant under $\Gamma_1(N/m)$, $\hat{F}$ is a function on the disjoint union of modular curves $\coprod_{m|N} Y_1(N/m)$. Each curve $Y_1(N/m)$ is the coarse space for the moduli problem classifying elliptic curves equipped with a point of order exactly N/m, i.e., as an analytic object, $Y_1(N/m)$ receives a universal map from the fibered category over complex analytic spaces whose objects are diagrams $E \underset{e}{\overset{p}{\leftrightarrows}} S$ of complex analytic spaces, with $E \to S$ smooth and proper with one dimensional genus one fibers, and $e, p : S \to E$ sections, such that p has order N/m under the S-group law induced by e, and whose morphisms are fibered diagrams. Using this interpretation, we can think of $\hat{F}$ as a function on the space of elliptic curves $E \overset{e}{\leftrightarrows} S$ equipped with a point $p : S \to E$, such that $[N]p = e$.

We have been viewing the component functions of $\hat{F}$ as functions on the complex upper half-plane, so it will be convenient to consider the above disjoint union of curves as the quotient of a disjoint union of half-planes by an action of $SL_2(\mathbb{Z})$, in a way that specializes to the classical uniformization of $Y(1)$ when $N = 1$. We begin by recalling two geometric objects attached to an arbitrary finite group G, introduced in Section 2 of [Carnahan-2010]:

(1) Consider the category whose objects are diagrams $\begin{tikzcd} P \arrow[r, "a"] & E \arrow[r, "\pi"] & S \arrow[l, bend left, "\tilde{e}"] \end{tikzcd}$ of complex analytic spaces equipped with sheaf isomorphisms $\psi : R^1\pi_*\underline{\mathbb{Z}} \to \underline{\mathbb{Z} \times \mathbb{Z}}$, such that:
- $a : P \to E$ is a G-torsor,
- $\pi : E \to S$ is a smooth proper map whose fibers are genus one curves,
- $\tilde{e} : S \to P$ is a section of $\pi \circ a$,
- The exterior square $\wedge^2\psi : \wedge^2 R^1\pi_*\underline{\mathbb{Z}}(1) \overset{\cup}{\to} R^2\pi_*\underline{\mathbb{Z}}(1) \to \underline{\mathbb{Z}}$ is the negative of the canonical isomorphism,

and whose morphisms are fibered diagrams, where the maps $P \to P'$ over $E \to E'$ are G-equivariant, equipped with the canonical isomorphisms of sheaves under pullback. This stack is represented by the analytic space $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, G) \times \mathfrak{H}$. By dualizing ϕ, we find that the connected component attached to a fixed homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to G$ parametrizes complex analytic families of elliptic curves E with G-torsors P and a preferred basis of fiberwise $H_1(E) \cong \mathbb{Z} \times \mathbb{Z}$, such that the monodromy of P along a basis element in H_1 from the base point is described by applying ϕ. When $N = 1$, we have $P = E$, and the above category reduces to the moduli description of the upper half-plane given in [Deligne-1969] and [Conrad-20??].

(2) Consider the category whose objects are diagrams $P \overset{a}{\to} E \underset{\pi}{\overset{e}{\leftrightarrows}} S$ of complex analytic spaces, such that:
- $a : P \to E$ is a G-torsor,
- $\pi : E \to S$ is a smooth proper map whose fibers are genus one curves,
- $e : S \to E$ is a section of π,

and whose morphisms are fibered diagrams, where the maps $P \to P'$ over $E \to E'$ are G-equivariant. This is commonly called the complex analytic stack of elliptic curves with G-torsors, and we will denote it by $\mathcal{M}_{Ell}^G$.

The second stack is the quotient of the first by a canonical action of $G \times SL_2(\mathbb{Z})$, where elements of G translate among lifts $\tilde{e}$ of e, and elements of $SL_2(\mathbb{Z})$ transform the isomorphism ψ. In terms of coordinates, the points of the first space $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, G) \times \mathfrak{H}$ are given by triples (g, h, τ), where $g, h \in G$ commute, and $\tau \in \mathfrak{H}$. The action of G identifies (g, h, τ) with the simultaneous conjugates $(kgk^{-1}, khk^{-1}, \tau)$ as k ranges over elements of G, since changes of base point conjugate the monodromy of a torsor. The action of $SL_2(\mathbb{Z})$ identifies $(g^a h^c, g^b h^d, \tau)$ with $(g, h, \frac{a\tau+b}{c\tau+d})$ for all $\binom{ab}{cd} \in SL_2(\mathbb{Z})$. We can view a function on $\mathcal{M}_{Ell}^G$ as a function $f(g, h, \tau)$ on $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, G) \times \mathfrak{H}$ that is invariant under this action of $G \times SL_2(\mathbb{Z})$.

We specialize this picture to the case $G = \mathbb{Z}/N\mathbb{Z}$. Functions on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ are functions on $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z}/N\mathbb{Z}) \times \mathfrak{H}$ that are invariant under the action of $\mathbb{Z}/N\mathbb{Z} \times SL_2(\mathbb{Z})$ described in the previous paragraph. In contrast to the general case, the action of $\mathbb{Z}/N\mathbb{Z}$ is trivial, since it is given by conjugation in an abelian group. We then have an identification of the coarse moduli space with a disjoint union of analytic quotients of the upper half plane, and by the analysis of $SL_2(\mathbb{Z})$-orbits in $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ in Lemma 3.2, it is a disjoint union of quotients by $\Gamma_1(N/m)$ as m ranges over divisors of N. In particular, this identification with $\coprod Y_1(N/m)$ allows us to make an identification of function spaces. We summarize the situation as follows:

Lemma 3.6. *For any positive integer N, we have linear isomorphisms between the spaces:*

1. *Families $\{f_{(m)}\}_{m|N}$ of modular functions, where each $f_{(m)}$ is invariant under $\Gamma_1(N/m)$.*
2. *Functions f on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$.*
3. *Functions $\hat{F}_{i,j}(\tau)$ on $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \times \mathfrak{H}$ that obey the relation $\hat{F}_{i,j}(\frac{a\tau+b}{c\tau+d}) = \hat{F}_{ai+cj,bi+dj}(\tau)$ for $\binom{ab}{cd} \in SL_2(\mathbb{Z})$.*
4. *Vector-valued functions of type ρ_M for $M = II_{1,1}(N) \times II_{1,1}$.*

Proof. The isomorphism $3 \leftrightarrow 1$ is given by restriction: $f_{(m)} = \hat{F}_{0,m}$. The fact that this is an isomorphism is a consequence of Lemma 3.2.

The bijection $2 \leftrightarrow 3$ is due to the identification of the coarse moduli space of $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ as an analytic quotient of $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \times \mathfrak{H}$ by the action of $SL_2(\mathbb{Z})$ given above.

The correspondence between $4 \leftrightarrow 1$ was proved in Proposition 3.4. □

3.1.2. *Rationality.* Let f be a function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$, and let F be the corresponding vector-valued form. We give sufficient conditions for F to have rational coefficients.

Lemma 3.7. *If the Fourier coefficients $c_{0,k}(n) \in \mathbb{Q}$ for all $k \in \mathbb{Z}/N\mathbb{Z}$, and $n \in \mathbb{Q}$, then all coefficients of F are rational. The analogous statement holds if we replace $\mathbb{Q}$ with any subring of $\mathbb{C}$ containing $\mathbb{Q}$.*

Proof. From the main theorem of [McGraw-2003], the space of vector-valued functions of type ρ_M has a basis consisting of functions all of whose Fourier expansions have only integer coefficients. If we choose such a basis $\{f^i\}$, then there exist rational numbers a_i such that $F_{0,k} = \sum_i a_i f^i_{0,k}$ for all k. By corollary 3.5, F is uniquely determined by its restriction to $F_{0,k}$, so $F = \sum_i a_i f^i$.

This argument works for any ring containing $\mathbb{Q}$, since we may allow the elements a_i to lie in the larger ring. □

We define $\hat{c}_{i,j}(n)$ to be the q^n coefficient of $f(g^i, g^j, \tau)$, so its q-expansion is $\sum \hat{c}_{i,j}(n) q^n$. We define $c_{i,k}(n) = \frac{1}{N} \sum_{j \in \mathbb{Z}/N\mathbb{Z}} e(-jk/N) \hat{c}_{i,j}(n)$, so these are the coefficients of the vector-valued function F.

Lemma 3.8. *If $c_{i,k}(n) \in \mathbb{Q}$ for all $i, k \in \mathbb{Z}/N\mathbb{Z}$, $n < 0$, and $c_{0,k}(0) \in \mathbb{Q}$ for all k, then all coefficients of F are rational. The analogous statement holds if we replace $\mathbb{Q}$ with any subring of $\mathbb{C}$ containing $\mathbb{Q}$.*

Proof. By the main theorem of [McGraw-2003], there exists a basis $\{f^i\}$ for the space of vector-valued functions of type ρ_M with integer coefficients. Let V be the space of vector-valued functions of type ρ_M such that the expansions of all components are regular, and the constant terms of the $(0,k)$ components are zero. By our hypotheses, there exists a set $\{a_i\}$ of rational numbers such that $F-\sum_i a_i f^i$ lies in V. The inverse Fourier transform gives a bijection between V and the space of families $\{f_{(m)}\}_{m|N}$ of modular functions that are regular at all cusps, and whose expansions at infinity have vanishing constant term. The regularity implies these modular functions are constants, and the vanishing of the constant term implies these functions are zero. V is therefore composed of the constant function zero, and $F=\sum_i a_i f^i$.

As in the previous lemma, this argument works for larger rings, since we can choose a_i appropriately. □

3.2. **Hecke-monic modular functions.** Recall the following definitions from [Carnahan-2010, Sections 1,2].

(1) Given a finite group G, the equivariant Hecke operator nT_n is defined for functions on $\mathcal{M}_{Ell}^G$ by summing over pullbacks of G-torsors over degree n isogenies. When G is cyclic of order N and generated by g, we can write a formula for nT_n in terms of functions on the complex upper half-plane:

$$nT_n f(g^i,g^j,\tau)=\sum_{\substack{ad=n\\0\le b<d}} f(g^{di},g^{aj-bi},\frac{a\tau+b}{d})$$

(2) A function f on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ is Hecke-monic if and only if for all $n>0$ and any connected component $X\subset\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ there exists a monic polynomial $\Phi_{X,n}\in\mathbb{C}[x]$ of degree n such that the restriction of nT_nf to X is equal to the restriction of $\Phi_{X,n}(f)$ to X.

Definition 3.9. A function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ is principally normalized if $f(1,g^i,\tau)$ has a q-expansion of the form $q^{-1}+O(q)$ for all $i\in\mathbb{Z}/N\mathbb{Z}$.

Lemma 3.10. *If f is a principally normalized Hecke-monic function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$, then for each $m|N$, $f(1,g^m,\tau)$ is a normalized Hauptmodul invariant under $\Gamma_0(N/m)$.*

Proof. Since f is Hecke-monic, and $f(1,g^k,\tau)$ has q-expansion of the form $q^{-1}+O(q)$ for all k, Corollary 3.6 in [Carnahan-2010] implies each $f(1,g^k,\tau)$ is a normalized Hauptmodul, i.e., each $f(1,g^k,\tau)$ has the q-expansion of the required form, is invariant under a genus zero group of Möbius transformations, and generates the function field of the upper half-plane quotient.

Let $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N/m)$. Then $f(1,g^m,\frac{a\tau+b}{c\tau+d})=f(1^ag^{cm},1^bg^{dm},\tau)=f(1,g^{dm},\tau)$. We assumed f was principally normalized, so the two functions $f(1,g^m,\frac{a\tau+b}{c\tau+d})$ and $f(1,g^m,\tau)$ both have q-expansions of the form $q^{-1}+O(q)$. By the Hauptmodul property of $f(1,g^m,\tau)$, the expansions must be equal. □

Lemma 3.11. *Let f be a principally normalized Hecke-monic function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$. Then,*

$$\log p(f(1,g,\sigma)-f(1,g,\tau))=-\sum_{m>0}T_mf(1,g,\tau)p^m$$

in a neighborhood of $p=q=0$, where $p=e(\sigma)$ and $q=e(\tau)$.

Proof. Following [Norton-1984], we define numbers $H_{m,n}$ by the bivarial transform (see the next section for details): the power series $q^{-n}+n\sum_m H_{m,n}q^m$ satisfies two properties:

(1) it is n times the coefficient of p^n in $-\log p(f(1,g,\sigma)-f(1,g,\tau))$
(2) it is the unique polynomial in $f(1,g,\tau)$ whose expansion has the form $q^{-n}+O(q)$.

By [Carnahan-2010] Lemma 5.2, $mT_m f(1,g,\tau)$ is also the unique polynomial in $f(1,g,\tau)$ whose expansion has the form $q^{-n}+O(q)$. Therefore, the expansion of $\log p(f(1,g,\sigma)-f(1,g,\tau))$ as a series in p has coefficients given by $-T_m f(1,g,\tau)$.

By the genus zero property, $f(1,g,\tau)$ is injective in a nontrivial neighborhood of zero in the q-disc, so $\log p(f(1,g,\sigma)-f(1,g,\tau))$ is the sum of $\log(1-pq^{-1})$ and a power series that converges absolutely in a nontrivial neighborhood of $p=q=0$ in the unit polydisc. Therefore, the product also converges on the complement of the locus $pq=0$ in this neighborhood, and the formal equality extends to a nontrivial domain. □

Lemma 3.12. *Let f be a principally normalized Hecke-monic function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$, and let F be the corresponding vector-valued form with coefficients $c_{i,k}(n)$, as defined just before Lemma 3.8. Then $f(1,g,\sigma)-f(1,g,\tau)$ has an infinite product expansion of the form:*

$$p^{-1}\prod_{m>0,n\in\mathbb{Z}}\prod_{j\in\mathbb{Z}/N\mathbb{Z}}(1-e(j/N)p^m q^n)^{c_{0,j}(mn)}$$

in a neighborhood of the cusp $(i\infty,i\infty)$ of $\mathfrak{H}\times\mathfrak{H}$.

Proof. We multiply by p and take the logarithm:

$$\begin{aligned}
&\sum_{m>0,n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} c_{0,j}(mn)\log(1-e(j/N)p^m q^n)\\
&= \sum_{m>0,n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}/N\mathbb{Z}} c_{0,j}(mn)\sum_{i=1}^{\infty}\frac{-1}{i}e(ij/N)p^{mi}q^{ni}\\
&= -\sum_{m>0,n\in\mathbb{Z}}\sum_{0<a|(m,n)}\frac{1}{a}\sum_{r\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}} c_{0,Nr}(mn/a^2)e(ar)p^m q^n\\
&= -\sum_{m>0}p^m\sum_{ad=m}\frac{1}{a}\sum_{n\in\mathbb{Z}}\hat{c}_{0,a}(dn)q^{an}\\
&= -\sum_{m>0}p^m\sum_{ad=m}\frac{1}{a}\sum_{0\le b<d}\frac{e(b/d)}{d}\sum_{n\in\mathbb{Z}}\hat{c}_{0,a}(n)q^{an/d}\\
&= -\sum_{m>0}p^m\frac{1}{m}\sum_{\substack{ad=m\\0\le b<d}}\sum_{n\in\mathbb{Z}}\hat{c}_{0,a}(n)e(b/d)q^{an/d}\\
&= -\sum_{m>0}p^m\frac{1}{m}\sum_{\substack{ad=m\\0\le b<d}}\hat{F}_{0,a}\left(\frac{a\tau+b}{d}\right)\\
&= -\sum_{m>0}T_m f(1,g,\tau)p^m
\end{aligned}$$

By Lemma 3.11, this is equal to $\log p(f(1,g,\sigma)-f(1,g,\tau))$, and both converge on a nontrivial neighborhood of $p=q=0$ in the unit polydisc. Therefore, the product is equal to $f(1,g,\sigma)-f(1,g,\tau)$ on a nontrivial neighborhood of $p=q=0$. □

We now wish to consider a product expansion of $f(1,g,\sigma)-f(1,g,\tau)$ in a neighborhood of the cusp $(i\infty,0)$. Here, we take "expansion of $f(1,g,\tau)$ at zero" to mean the q-expansion of $f(1,g,-1/\tau)$. Since $f(1,g,-1/\tau)=f(g,1,\tau)$, we are really seeking an infinite product expansion of $f(1,g,\sigma)-f(g,1,\tau)$ at (∞,∞).

Proposition 3.13. *Let f be a principally normalized Hecke-monic function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$, and let F be the corresponding vector-valued function, with coefficients $c_{i,k}(n)$ as in Lemma 3.12. Then in a*

neighborhood of infinity, $f(1,g,\sigma)-f(g,1,\tau)$ has a product expansion:

$$p^{-1}\prod_{m>0,n\in\frac{1}{N}\mathbb{Z}}(1-p^mq^n)^{c_{m,Nn}(mn)}$$

Proof. We multiply the product by p and take the logarithm to get:

$$\begin{aligned}
&\sum_{m>0,n\in\frac{1}{N}\mathbb{Z}} c_{m,Nn}(mn)\sum_{i>0}\frac{-1}{i}p^{mi}q^{ni}\\
&\quad=-\sum_{m>0,n\in\frac{1}{N}\mathbb{Z}}\sum_{0<a|(m,Nn)}\frac{1}{a}c_{m/a,Nn/a}(mn/a^2)p^mq^n\\
&\quad=-\sum_{m>0}\sum_{ad=m}\frac{1}{a}\sum_{n\in\frac{1}{N}\mathbb{Z}}c_{d,Nn}(dn)p^mq^{an}\\
&\quad=-\sum_{m>0}\sum_{ad=m}\frac{1}{a}\sum_{0\le b<d}\frac{1}{d}\sum_{n\in\frac{1}{N}\mathbb{Z}}\sum_{\substack{r\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}\\ n\in dr+\mathbb{Z}}}e(-br)c_{d,Nr}(n)e(br)q^{an/d}p^m\\
&\quad=-\sum_{m>0}\frac{1}{m}\sum_{\substack{ad=m\\0\le b<d}}\sum_{n\in\frac{1}{N}\mathbb{Z}}\sum_{\substack{r\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}\\ n\in dr+\mathbb{Z}}}e(-br)c_{d,Nr}(n)e\left(n\frac{a\tau+b}{d}\right)p^m\\
&\quad=-\sum_{m>0}\frac{1}{m}\sum_{\substack{ad=m\\0\le b<d}}\sum_{n\in\frac{1}{N}\mathbb{Z}}\hat{c}_{d,-b}(n)e\left(n\frac{a\tau+b}{d}\right)p^m\\
&\quad=-\sum_{m>0}\frac{1}{m}\sum_{\substack{ad=m\\0\le b<d}}f\left(g^d,g^{-b},\frac{a\tau+b}{d}\right)p^m\\
&\quad=-\sum_{m>0}T_mf(g,1,\tau)p^m\\
&\quad=-\sum_{m>0}T_mf(1,g,-1/\tau)p^m
\end{aligned}$$

By Lemma 3.11, this sum agrees with the expansion of $\log p(f(1,g,\sigma)-f(1,g,\tau))$ at $(i\infty,0)$ as elements of the formal power series ring $\mathbb{C}((q^{1/N}))[[p]]$. The function $p(f(1,g,\sigma)-f(1,g,\tau))$ has no zeroes or poles in a neighborhood of the line $\{(\sigma,\tau)\in\mathfrak{H}^*\times\mathfrak{H}|\sigma=i\infty,\tau\in i\mathbb{R}_{>0}\}$, so $\log p(f(1,g,\sigma)-f(1,g,\tau))$ is single-valued and holomorphic in a neighborhood of the same line. To show that the two cusp expansions at $(i\infty,i\infty)$ and $(i\infty,0)$ are analytic continuations of each other, it suffices to show that they converge on neighborhoods whose union contains this line. We note that by a standard theorem of several complex variables (e.g., [Gunning-1990, Theorem B.2]), the domain of convergence is always a complete Reinhardt domain with log-convex base.

Because the function $\tau\mapsto f(1,g,\tau)$ has a simple pole at infinity and is holomorphic on $\mathfrak{H}$, we can use the following injectivity property: for any $y\in\mathbb{R}_{>0}$, there exists σ with sufficiently large imaginary part that any $\tau\in\mathfrak{H}$ satisfying $\Im(\tau)>y$ and $f(1,g,\tau)=f(1,g,\sigma)$ also satisfies $\tau-\sigma\in\mathbb{Z}$. Equivalently, given any y, there exists $\alpha\in\mathbb{R}_{>0}$ such that the power series expansion of $\log p(f(1,g,\sigma)-f(1,g,\tau))$ converges in the region $\{(p,q)\mid |p|<e(i\alpha),|q|<e(iy)\}$, i.e., on a polydisc. Since y is arbitrary, we have convergence on the complement of an arbitrarily small segment of the line $\{(\sigma,\tau)\in\mathfrak{H}^*\times\mathfrak{H}|\sigma=i\infty,\tau\in i\mathbb{R}_{>0}\}$. This reduces the problem to showing that

the power series expansion of $\log p(f(1,g,\sigma)-f(g,1,\tau))$ converges in a nontrivial neighborhood of the cusp $(i\infty, i\infty)$.

Case 1: Suppose $f(g,1,\tau)$ has a pole at infinity, i.e., $f(1,g,\tau)$ has a pole at zero. Corollary 3.6 in [Carnahan-2010] asserts that $f(1,g,\tau)$ has global symmetries, i.e.,, there exists some element $\binom{ab}{cd} \in PSL_2(\mathbb{R})$ that takes zero to infinity, such that $f(1,g,\tau) = f(1,g,\frac{a\tau+b}{c\tau+d})$ for all $\tau \in \mathfrak{H}$. Since zero is taken to infinity, we have $d=0$, and the fact that f is a Hauptmodul implies the transformation must have the form $\tau \mapsto a - b/\tau$ for $b|N$ and $a \in \mathbb{Q}$. Then using Lemma 3.11,

$$\begin{aligned} -\sum_{m>0} T_m f(1,g,-1/\tau)p^m &= -\sum_{m>0} T_m f(1,g,a-\frac{b}{-1/\tau}) \\ &= -\sum_{m>0} T_m f(1,g,a+b\tau) \\ &= \log p(f(1,g,\sigma)-f(1,g,a+b\tau)) \end{aligned}$$

a is real and b is nonzero, so the last line is the sum of $\log(1-e(-a)pq^{-b})$ and a power series that converges on a nontrivial neighborhood of $p=q=0$. This implies $\log p(f(1,g,\sigma)-f(g,1,\tau))$ converges near $(i\infty, i\infty)$.

Case 2: Suppose $f(g,1,\tau)$ is regular at infinity, i.e., $f(1,g,\tau)$ is regular at zero. By continuity, there exists a neighborhood $U \subset \mathfrak{H}\times\mathfrak{H}$, whose closure contains the cusp $(i\infty, 0)$, satisfying $f(1,g,\sigma) \neq f(1,g,\tau)$ for all $(\sigma,\tau) \in U$. The function $\log p(f(1,g,\sigma)-f(1,g,\tau))$ is therefore single-valued and holomorphic on U and has a unique power series expansion that converges absolutely uniformly on some open subset of U containing $(i\infty, 0)$. □

3.3. **Replicable functions and products.** This section is not essential for the main results, but it describes a direct relationship between replicability of series and the form of infinite products.

Definition 3.14. ([Norton-1984]) For any formal Laurent series $f(q) = \sum_{n\in\mathbb{Z}} c(n)q^n \in \mathbb{C}((q))$ of the form $q^{-1}+O(q)$, one has the bivarial transform, which yields numbers $H_{m,n}$ by the formal expansion $\log \frac{f(p)-f(q)}{p^{-1}-q^{-1}} = -\sum_{m,n=1}^{\infty} H_{m,n}p^m q^n$. The series f is called replicable if all of its coefficients are integers and the value of $H_{m,n}$ depends only on mn and (m,n). A holomorphic function on the upper half plane is replicable if it admits a q-expansion that is a replicable series.

Lemma 3.15. *A Laurent series f with integer coefficients is replicable if and only if one has the following formal infinite product expansion*

$$f(p)-f(q) = p^{-1}\prod_{m>0,n\geq -1}(1-p^m q^n)^{c(m,n)}$$

such that the exponents $c(m,n)$ depend only on mn and (m,n).

Proof. The presence of the p^{-1} in the front of the product restricts the form of f to have a simple pole of residue 1 at $q=0$. Taking the formal logarithm of both sides of the product formula yields:

$$\begin{aligned} \log(f(p)-f(q)) &= \log(p^{-1}-q^{-1}) - \sum_{i>0}\sum_{m,n=1}^{\infty}\frac{c(m,n)}{i}p^{mi}q^{ni} \\ &= \log(p^{-1}-q^{-1}) - \sum_{m,n=1}^{\infty} p^m q^n \sum_{t|(m,n)} c(m/t,n/t)/t. \end{aligned}$$

We see that $H_{m,n} = \sum_{t|(m,n)} c(m/t,n/t)/t$, and note that if $c(m,n)$ only depends on mn and (m,n), then the same is true for $H_{m,n}$. We can use Möbius inversion to reverse this process:

$$\begin{aligned}
\sum_{m,n=1}^{\infty} H_{m,n}p^m q^n &= \sum_{i=1}^{\infty}\frac{1}{i}\sum_{m,n=1}^{\infty} c(m,n)p^{mi}q^{ni} \\
\sum_{t=1}^{\infty}\frac{\mu(t)}{t}\sum_{m,n=1}^{\infty} H_{m,n}p^{mt}q^{nt} &= \sum_{i,t=1}^{\infty}\frac{\mu(t)}{it}\sum_{m,n=1}^{\infty} c(m,n)p^{mit}q^{nit} \\
&= \sum_{s=1}^{\infty}\sum_{it=s}\frac{\mu(t)}{s}\sum_{m,n=1}^{\infty} c(m,n)p^{ms}q^{ns} \\
\sum_{m,n=1}^{\infty} p^m q^n \sum_{t|(m,n)}\frac{\mu(t)}{t}H_{m/t,n/t} &= \sum_{m,n=1}^{\infty} p^m q^n \sum_{s|(m,n)}\frac{c(m/s,n/s)}{s}\sum_{t|s}\mu(t) \\
\sum_{t|(m,n)}\frac{\mu(t)}{t}H_{m/t,n/t} &= \sum_{s|(m,n)}\frac{c(m/s,n/s)}{s}\delta_{s,1} \\
&= c(m,n)
\end{aligned}$$

By the same reasoning as before, if each $H_{m,n}$ depends only on mn and (m,n), then the same holds for $c(m,n)$. □

The notion of replicable function was originally introduced in [Conway-Norton-1979], where it was defined in terms of explicit relations between coefficients of the q-expansion. This interpretation was formalized in [Norton-1984] as the existence of "replicate functions", and a proof of equivalence was sketched. Given a formal Laurent series of the form $f(q) = q^{-1}+O(q) = \sum_{n\in\mathbb{Z}} c(n)q^n \in \mathbb{Z}((q))$, and an integer $n > 0$ there is a unique normalized Faber polynomial $\Phi_n(x)$ (depending on f), defined by the property that $\Phi_n(f(q)) = q^{-n} + O(q)$. Since the polynomial in $f(q)$ that is n times the coefficient of p^n in the formal series $-\log p(f(p) - f(q))$ also has this form, we have $\Phi_n(f(q)) = q^{-n} + n\sum_m H_{m,n}q^m$. It is straightforward to show that $f(q)$ is replicable if and only if for any $t \in \mathbb{Z}_{>0}$ there exists a series $f^{(t)}(q) = q^{-1}+O(q)$ such that $\Phi_n(f(q)) = \sum_{ad=n,0\le b<d} f^{(a)}(e(b/d)q^{a/d})$. The series $f^{(t)} = \sum_{n>0} a_n^{(t)}q^n$ is called the t-th replicate of f, and by suitable use of Möbius inversion, one can show that it is unique if it exists, with coefficients satisfying the relations:

$$H_{m,n} = \sum_{t|(m,n)}\frac{1}{t}a^{(t)}_{mn/t^2} \qquad \text{and} \qquad a_k^{(t)} = t\sum_{s|t}\mu(s)H_{kts,t/s}.$$

Definition 3.16. A replicable series f is has order N if $f^{(n+N)} = f^{(n)}$, for all $n > 0$.

In Proposition 5.3 of [Carnahan-2010], we gave an interpretation of replicability of a function as the condition that all equivariant Hecke operators T_n applied to the function are equal to certain polynomials applied to the function. In particular, Lemma 5.2 identifies the series $\Phi_n(f(q))$ with $nT_nf(q)$, and Corollary 5.4 asserts that finite order replicable functions are either Hauptmoduln invariant under some $\Gamma_1(N)$ or have the form q^{-1}, $q^{-1} - q$, or $q^{-1} + q$.

Definition 3.17. A replicable series f is completely replicable if for all $n > 0$, the series $f^{(n)}$ is also replicable.

If f is replicable, then the condition that f is completely replicable is equivalent to the (a priori stronger) condition that $(f^{(s)})^{(t)}$ exists and is equal to $f^{(st)}$ for all $s, t > 0$ [Kozlov-1994]. Theorem 1.3 of [Cummins-Gannon-1997] asserts that completely replicable functions of finite order are either Hauptmoduln, or have finite expansions of the form q^{-1}, $q^{-1} + q$, or $q^{-1} - q$. By Corollary 5.6 in [Carnahan-2010], there is a natural bijection between principally normalized Hecke-monic functions on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$ and completely replicable modular functions of order dividing N.

Lemma 3.18. *If f is a completely replicable series of order N, then the exponents $c(m,n)$ in the formal product expansion of $f(p)-f(q)$ depend only on mn and (m,n,N).*

Proof. Suppose f is completely replicable of order N. From the previous discussion, we know that $H(m,n)=\sum_{t|(m,n)} c(m/t,n/t)/t=\sum_{t|(m,n)}\frac{1}{t}a^{(t)}_{mn/t^2}$. Möbius inversion then yields $c(m,n)=\sum_{st|(m,n)}\frac{\mu(s)}{st}a^{(t)}_{mn/s^2t^2}$. Following the arguments after equation (10.4) of [Borcherds-1992], we note that if b is any function on positive integers such that $b(n)$ depends only on (n,N) for some N, then $\sum_{st=k}\mu(s)b(t)$ is zero unless $k|N$. Since the coefficients $a^{(t)}_n$ depend only on n and (t,N), the sum $\sum_{st=k}\mu(s)a^{(t)}_{mn/s^2t^2}$ vanishes unless $k|N$. In particular, we obtain the simplified equation $c(m,n)=\sum_{st|(m,n,N)}\frac{\mu(s)}{st}a^{(t)}_{mn/s^2t^2}$, which depends only on mn and (m,n,N). □

We suspect that a converse holds as well, i.e., that the conclusion of this lemma gives a characterization of finite order completely replicable functions.

3.4. **Singular theta-lifts.** We rephrase the product formula results of Section 3.2 in terms of Borcherds-Harvey-Moore singular theta-lifts. This section is not strictly necessary for the Lie algebra constructions in this paper, but the formalism allows us to consider product expansions at cusps other than $(i\infty,i\infty)$ and $(i\infty,0)$. These are often Weyl denominators for infinite dimensional Lie superalgebras, but it is not at all clear what physical interpretation they should have.

We introduce some geometry, borrowing heavily from [Borcherds-1998a]. Let $O_{2,2}$ be the automorphism group of the quadratic form $Q(a,b,c,d)\mapsto ab+cd$. We employ the transpose-inverse outer automorphism of SL_2 to construct two commuting embeddings $SL_2\to O_{2,2}$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & d & 0 & -c \\ c & 0 & d & 0 \\ 0 & -b & 0 & a \end{pmatrix} \text{ or } \begin{pmatrix} a & 0 & 0 & b \\ 0 & d & -c & 0 \\ 0 & -b & a & 0 \\ c & 0 & 0 & d \end{pmatrix}$$

This induces an injection $SL_2(\mathbb{Z})\times SL_2(\mathbb{Z})\to \mathrm{Aut}(II_{1,1}\times II_{1,1})$ as an index four subgroup. If we view $M=II_{1,1}(N)\times II_{1,1}$ as a sublattice of $II_{1,1}\times II_{1,1}$ in these coordinates as $\mathbb{Z}\times N\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, the above map pulls back to $\Gamma_0(N)\times\Gamma_0(N)\to\mathrm{Aut}(M)$. The Grassmannian $G(M)\cong O_{2,2}(\mathbb{R})/O_2(\mathbb{R})\times O_2(\mathbb{R})$ is the space of two dimensional positive definite subspaces of $M\otimes\mathbb{R}$. If we choose a point in $G(M)$ and an ordered basis X_M,Y_M of the corresponding positive definite plane, then $X_M+iY_M\in M\otimes\mathbb{C}$. The vector X_M+iY_M has norm zero if and only if X_M and Y_M are orthogonal and of equal norm. There is an action of $\mathbb{C}^\times$ on the space of ordered orthogonal bases whose vectors have equal positive norm, by simultaneous rotations and dilations. The manifold $G(M)$, which is isomorphic as a hermitian symmetric domain to $\mathfrak{H}\times\mathfrak{H}$, can be viewed as an analytic open subset of the quadric surface of all norm zero vectors in $\mathbb{P}(M\otimes\mathbb{C})$. In particular, the subspace of $\mathbb{P}(M\otimes\mathbb{C})$ for which X_M (equivalently, Y_M) has positive norm has two connected components, and we will choose one of them to identify with $G(M)$, placing an orientation on the plane spanned by X_M and Y_M. Each copy of $\Gamma_0(N)$ above acts on M and $M\otimes\mathbb{C}$ by norm-preserving transformations, and the product $\Gamma_0(N)\times\Gamma_0(N)$ in $\mathrm{Aut}(M)$ acts on the norm zero vectors in the projectivization, preserving $G(M)$. The canonical $\mathbb{C}^\times$-torsor over $\mathbb{P}(M\otimes\mathbb{C})$ restricts to a $\mathbb{C}^\times$-torsor P over $G(M)$, and since $G(M)$ is contractible, we can trivialize the associated line bundle. Trivializations are not equivariant under the action of $\Gamma_0(N)\times\Gamma_0(N)$, and we say that a function on $G(M)$ is an automorphic form of weight k if under an inverse trivialization of the the kth tensor power of this line bundle, the function becomes an invariant section. We are concerned with automorphic functions, i.e, weight zero forms.

We choose once and for all an identification $G(M)\cong\mathfrak{H}\times\mathfrak{H}$ by sending the $\mathbb{C}^\times$-span of the norm zero vector $(i,i,1,1)\in M\otimes\mathbb{C}$ to $(i,i)\in\mathfrak{H}\times\mathfrak{H}$. We can specify the rest of the points by

Möbius transformations on each copy of $\mathfrak{H}$, essentially using Iwasawa decomposition: $\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}$ sends i to $x+iy$, so by using the above embeddings $SL_2 \to O_{2,2}$ and rescaling appropriately, $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$ corresponds to the $\mathbb{C}^\times$-span of:

$$(-y_1y_2 + x_1x_2 + ix_1y_2 + ix_2y_1, -1, x_1 + iy_1, x_2 + iy_2) = (z_1z_2, -1, z_1, z_2)$$

Lemma 3.19. *Given a vector $\lambda = (a, b, c, d) \in M \otimes \mathbb{R}$ of negative norm,*

$$\lambda^\perp = \{(z_1, z_2) \in \mathfrak{H} \times \mathfrak{H} | z_1 = \frac{-cz_2 + a}{bz_2 + d}\}$$

Proof. Orthogonality with (a, b, c, d) is equivalent to the condition $-a + bz_1z_2 + cz_2 + dz_1 = 0$. Solving for z_1 yields the answer. □

Remark: The negative norm condition on λ is equivalent to the corresponding matrix $\begin{pmatrix} -c & a \\ b & d \end{pmatrix}$ having positive determinant, so the condition is necessary for the set $\lambda^\perp$ to be nonempty.

Cusps are represented by norm zero vectors in M, and they can be produced by taking limits of Möbius transformations - in particular, the cusp at $(1, 0, 0, 0)$ corresponds to $(i\infty, i\infty)$ under our identification. The orbits of cusps in $\mathfrak{H} \times \mathfrak{H}$ under the action of $\Gamma_0(N) \times \Gamma_0(N)$ naturally map to $\operatorname{Aut} M$ orbits of primitive norm zero vectors in M. Cusp generation can be done explicitly as follows: Any $a/c \in \mathbb{P}^1(\mathbb{Q})$ is the image of $i\infty$ under some $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ in $SL_2(\mathbb{Z})$, so the cusp at $(a_1/c_1, a_2/c_2)$ corresponds to the vector:

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & d_1 & 0 & -c_1 \\ c_1 & 0 & d_1 & 0 \\ 0 & -b_1 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & 0 & b_2 \\ 0 & d_2 & -c_2 & 0 \\ 0 & -b_2 & a_2 & 0 \\ c_2 & 0 & 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1a_2 \\ -c_1c_2 \\ c_1a_2 \\ a_1c_2 \end{pmatrix}$$

with a sign ambiguity. To go back, we write a primitive norm zero vector as (a, b, c, d). Using the above vector representation, we find that the corresponding cusp is $(a/c = -d/b, -c/b = a/d)$, where we choose fractions that don't have the form $0/0$. The following four cusps will be most important to us:

$$\pm(1, 0, 0, 0) \leftrightarrow (i\infty, i\infty)$$
$$\pm(0, 1, 0, 0) \leftrightarrow (0, 0)$$
$$\pm(0, 0, 1, 0) \leftrightarrow (0, i\infty)$$
$$\pm(0, 0, 0, 1) \leftrightarrow (i\infty, 0)$$

3.4.1. *Lorentzian case.* Borcherds's strategy for evaluating theta-lifts for orthogonal groups of indefinite signature is to split off hyperbolic subspaces of the real span of the lattice until one gets a definite space, for which the Grassmannian is a point, then build up lifts for increasingly larger lattices in terms of the lifts for the smaller ones.

Given a primitive norm zero vector $z \in M$, we form a singular lattice $z^\perp \subset M$, and a hyperbolic lattice $K = z^\perp/\mathbb{Z}z$. We choose a norm zero $z' \in M \otimes \mathbb{R}$ such that $(z, z') = 1$, and this lets us identify K with a sublattice of $M \otimes \mathbb{R}$ that lies in the orthogonal complement of z and z'. For example, when $z = (0, 0, 1, 0)$, we can choose $z' = (0, 0, 0, 1)$, and this identifies K with $II_{1,1}(N) \times \{0\} \subset M$. If instead $z = (1, 0, 0, 0)$, we can choose $z' = (0, 1, 0, 0) \in M^\vee$, identifying K with $\{0\} \times II_{1,1} \subset M$.

Once we have chosen z and K, we can make a second parametrization of $G(M)$ following [Borcherds-1998a] section 13. Given an oriented basis $(X_M, Y_M) \in G(M)$ satisfying $(Y_M, z) = 0$, $(X_M, z) > 0$, there exists a positive norm vector $Y \in K \otimes \mathbb{R}$ such that Y_M is a sum of Y and a multiple of z. Y is determined only by the choice of z and the choice of orientation of the basis. The set of positive norm vectors in $K \otimes \mathbb{R}$ has two connected components, and we choose a cone C to be the component corresponding to our choice of orientation. This identifies $G(M)$

with $K \otimes \mathbb{R} + iC \subset K \otimes \mathbb{C}$, and this identification is a special case of the general fact that the Grassmannian for $O_{2,n-2}$ is isometric to the Cartesian product of the Lorentz space $\mathbb{R}^{1,n-3}$ with its positive light cone. There is a map $K \otimes \mathbb{R} + iC \to P$ given by letting $Z = X + iY \in K \otimes \mathbb{R} + iC$ and setting $Z_M := (Z, -1, Q(Z)) \in K \otimes \mathbb{C} \oplus \mathbb{C}z' \oplus \mathbb{C}z \cong M \otimes \mathbb{C}$ to be the unique norm zero point having inner product 1 with z and projection Z.

Inside K, we choose a primitive norm zero vector w, and another norm zero vector $w' \in K^\vee$ satisfying $(w, w') = 1$. Following [Borcherds-1998a] section 5, we can construct functions F_M, F_K, and F_0 by combining individual components of F:

$$\begin{aligned} F_M &= \sum_{\delta \in M^\vee/M} F_\delta e_\delta = F \\ F_K &= \sum_{\lambda \in K^\vee/K} e_\lambda \sum_{\delta \in M^\vee/M, \delta|z^\perp = \lambda} F_\delta \\ F_0 &= \sum_{\lambda \in K^\vee/K, \lambda|(K \cap w^\perp)=0} \sum_{\delta \in M^\vee/M, \delta|(M \cap z^\perp)=\lambda} F_\delta \\ &= \sum_{\delta \in (M^\vee \cap (\mathbb{Q}w + \mathbb{Q}z))/M} F_\delta \end{aligned}$$

The notation $\delta|z^\perp = \lambda$ means that we lift λ to a set of linear functions $K \to \mathbb{Z}$, pull them back along the projection $z^\perp \to K$, lift δ to a set of linear functions $M \to \mathbb{Z}$, restrict them to $z^\perp$, and compare the elements of the two sets.

Lemma 3.20. *Let f be a principally normalized function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$, and let $F = F_M$ be the corresponding vector-valued function. If $z = (1,0,0,0)$, then F_K is a scalar function of the form $q^{-1} + O(q)$, and F_0 has the form $q^{-1} + O(q)$. If $z = (0,0,1,0)$ or $(0,0,0,1)$, and if $w = (1,0,0,0)$, then F_K has N^2 components, the N components in the span of w have the form $q^{-1} + O(q)$ (if it is the zero component) or $O(q)$ (otherwise), and F_0 has the form $q^{-1} + O(q)$.*

Proof. If $z = (1,0,0,0)$, $z^\perp = \{(a,0,c,d)\} \subset M$ and $K^\vee/K$ is trivial, so the sum is over $\lambda = 0$ and $\delta \in \{(a,0,c,d) + M\} \subset M^\vee/M$. Using the identification with the coordinate switch, we have $F_K = \sum_{a \in \mathbb{Z}/N\mathbb{Z}} F_{0,a} = \hat{F}_{0,0}$. Since f is principally normalized, this has the form $q^{-1} + O(q)$.

If $z = (0,0,1,0)$, then $z^\perp = \{(a,b,c,0)\} \subset M$, and $K^\vee/K \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. If $z = (0,0,0,1)$, then $z^\perp = \{a,b,0,d\} \subset M$, and $K^\vee/K$ is the same. In either case, each component of F_K is defined by a sum over δ that has exactly one summand. By the identification of $M^\vee/M$ with $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, the components in the span of w have the form $F_{0,a}$ for $a \in \mathbb{Z}/N\mathbb{Z}$. Since f is principally normalized, we have $\hat{F}_{0,k} = q^{-1} + O(q)$ for all k, and $F_{0,a} = \frac{1}{N}\sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(-ak/N)\hat{F}_{0,k} = \delta_{0,a}q^{-1} + O(q)$. Unwinding the sum, we find that $F_0 = \sum_{a \in \mathbb{Z}/N\mathbb{Z}} F_{0,a} = \hat{F}_{0,0} = q^{-1} + O(q)$. □

Lemma 3.21. *If $F_0 = q^{-1} + O(q)$, then the singular theta-lift for the zero lattice is $\Phi_0 = -8\pi$.*

Proof. By [Borcherds-1998a] Corollary 9.3, Φ_0 is $\frac{\pi}{3}$ times the constant term of $F_0 E_2$. We multiply $q^{-1} + O(q)$ by $1 - 24q + O(q^2)$ to get a constant term of -24. □

To describe the singular theta-lift Φ_K for the Lorentzian lattice, we need to introduce some notation. $G(K)$ is the space of positive-definite lines in $\mathbb{R}^{1,1}$, and can be identified with one connected component of the hyperbola of the norm one vectors $v \in K \otimes \mathbb{R}$. We write w_{v^+} to denote the orthogonal projection of the norm zero vector w to the line v^+ spanned by v, $c_\lambda(0)$ describes the constant terms in F_K, and $\mathbf{B}_2(x)$ is the periodic function that equals $x^2 - x + 1/6$ when $0 \le x \le 1$. Then by [Borcherds-1998a] Theorem 10.2, we have

$$\sqrt{2}|w_{v^+}|\Phi_K(v, F_K) = \Phi_0 + 4\pi w_{v^+}^2 \sum_{\lambda \in K^\vee/K, \lambda|(K \cap w^\perp)=0} c_\lambda(0)\mathbf{B}_2((\lambda, w'))$$

By [Borcherds-1998a] Theorem 6.3, Φ_K is a piecewise linear function on the positive cone of $G(K)$ in $K \otimes \mathbb{R}$.

Lemma 3.22. *If $F_0 = q^{-1} + O(q)$ and none of the components of F_K has a nonzero constant term, then $\Phi_K(v, F_K) = -8\sqrt{2}\pi|w'_{v^+}|$.*

Proof. Since the constant terms $c_\lambda(0)$ vanish, the sum in the above formula is zero. Hence, $\sqrt{2}|w_{v^+}|\Phi_K(v, F_K) = \Phi_0$, and this is -8π. We can write $v = mw' + nw$, so $v^2 = 2mn = 1$, $w_{v^+} = mv$, $|w_{v^+}| = m$, and $|w'_{v^+}| = n = 1/2m$. Then $\Phi(mw' + nw, F_K) = -8\sqrt{2}\pi n$. □

Following [Bruinier-2002], we define for each $\lambda \in K^\vee, n < 0$, a distinguished set of points $H(\lambda, n) = \{\lambda^\perp | c_\lambda(n) \neq 0, Q(\lambda) = n\}$. The complement of $\bigcup_{\lambda,n} H(\lambda, n)$ in $G(K)$ is a finite disjoint union of segments. Their positive cones in $K \otimes \mathbb{R}$ are called the Weyl chambers in K, and Φ_K is real analytic (in fact a piecewise polynomial) when restricted to a Weyl chamber. Given a Weyl chamber W, we write $(\lambda, W) > 0$ if the inner product of λ with any vector in the interior is positive. The set $\{\lambda \in K^\vee | c_\lambda(n) \neq 0 \text{ for some } n\}$ is contained in the disjoint union of $\{\lambda \in K^\vee | (\lambda, W) > 0\}$ with $\{\lambda \in K^\vee | (-\lambda, W) > 0\}$.

For each primitive norm zero vector $z \in M$ (equivalently, each cusp) and each Weyl chamber W, a Weyl vector $\rho(K, W, F_K)$ can be defined by $8\sqrt{2}\pi(\rho(K, W, F_K), v) = |v|\Phi_K(v/|v|, F_K)$. By [Borcherds-1998a] theorem 10.4, we have $\rho(K, W, F_K) = \rho_w w + \rho_{w'} w'$, where

$$\rho_{w'} = \text{constant term of } F_0(\tau)E_2(\tau)/24$$
$$\rho_w = \frac{1}{4} \sum_{\lambda \in (K^\vee \cap \mathbb{Q}w)/K} c_\lambda(0)\mathbf{B}_2((\lambda, w'))$$

Lemma 3.23. *If $F_0 = q^{-1} + O(q)$ and none of the components of F_K in the span of w has a nonzero constant term, then $\rho(K, W, F_K) = -w'$.*

Proof. Since $E_2(\tau) = 1 - 24q + \ldots$, we have $\rho_{w'} = -1$. Since all of the $c_\lambda(0)$ vanish in the sum, $\rho_w = 0$. □

3.4.2. *Signature 2,2.* We now consider the singular theta-lift Φ_M. This is a real-analytic function on $G(M) \cong \mathfrak{H} \times \mathfrak{H}$ with logarithmic singularities along certain divisors of the form $\lambda^\perp$, as described in theorem 6.2 of [Borcherds-1998a]. We are more interested in holomorphic functions Ψ satisfying $\Phi_M = -4\log|\Psi|$, because Ψ may be highly invariant and has zeroes and poles along the same divisors.

Theorem 3.24. *Let f be a principally normalized Hecke-monic function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$, and let F be the corresponding vector-valued function of type ρ_M. Suppose at least one of the following two conditions are satisfied:*

1. *$c_{0,k}(n) \in \mathbb{R}$ for all $k \in \mathbb{Z}/N\mathbb{Z}$ and all $n \in \mathbb{Q}$.*
2. *$c_{i,k}(n) \in \mathbb{R}$ for all $n < 0$, and all $i, k \in \mathbb{Z}/N\mathbb{Z}$.*

Then the function $f(1, g, \sigma) - f(1, g, \tau)$ satisfies the following properties:

1. *It is invariant under the the standard action of $\Gamma_0(N) \times \Gamma_0(N)$ on $\mathfrak{H} \times \mathfrak{H}$.*
2. *The only zeroes and poles lie on rational quadratic divisors $\lambda^\perp$ for $\lambda \in M$, $Q(\lambda) < 0$ and are zeroes of order*
$$\sum_{0 < x \in \mathbb{R}, x\lambda \in M^\vee} c_{x\lambda}(Q(\lambda)x^2).$$
3. *For each primitive norm zero vector $z \in M$, let m_z denote the smallest positive value of the inner product of z with a vector in M. For each Weyl chamber W of K, the restriction of $f(1, g, \sigma) - f(1, g, \tau)$ has an infinite product expansion converging when $Z = X + iY \in$*

$K \otimes \mathbb{R} + iC \subset K \otimes \mathbb{C}$ is in a neighborhood of the cusp of z and $Y \in W$. The product is some constant of absolute value

$$\prod_{\substack{k \in \mathbb{Z}/m_z\mathbb{Z} \\ k \neq 0}} (1 - e(k/m_z))^{c_{kz/m_z}(0)/2}$$

times

$$e((Z, \rho(K, W, F_K))) \prod_{\lambda \in K^\vee, (\lambda, W) > 0} \prod_{\delta \in M^\vee/M, \delta | z^\perp = \lambda} (1 - e((\lambda, Z) + (\delta, z')))^{c_\delta(Q(\lambda))}$$

Furthermore, if we let $e((w', Z)) = p$ and $e((w, Z)) = q$, then the expansions of $f(1, g, \sigma) - f(1, g, \tau)$ at $z = (i\infty, i\infty)$ and $(i\infty, 0)$ are given by the product formulas from Lemma 3.12 and Proposition 3.13:

$$f(1, g, \sigma) - f(1, g, \tau) = p^{-1} \prod_{m > 0, n \in \mathbb{Z}} \prod_{j \in \mathbb{Z}/N\mathbb{Z}} (1 - e(j/N) p^m q^n)^{c_{0,j}(mn)}$$

$$f(1, g, \sigma) - f(g, 1, \tau) = p^{-1} \prod_{m > 0, n \in \frac{1}{N}\mathbb{Z}} (1 - p^m q^n)^{c_{m,Nn}(mn)}.$$

Proof. Since f is principally normalized, the constant terms of $F_{0,k}$ vanish for all k, so in particular they are rational. If the first condition holds, Lemma 3.7 applies, and if the second condition holds, Lemma 3.8 applies, so in either case, all coefficients of F are real.

Since the q-expansion coefficients $c_\lambda(n)$ of F are real, and $c_0(0) = 0$, then the arguments in the proof of [Borcherds-1998a] Theorem 13.3 show that there is a multivalued function $\Psi_M(Z_M, F)$ that is holomorphic for Z_M in an open dense subset of $P := \{Z \in M \otimes \mathbb{C} | Q(Z) = 0, (Z, \overline{Z}) > 0\}$ with the following properties:

(1) $\Psi_M(Z_M, F)$ is invariant under the scaling action of $\mathbb{C}^\times$ on P, i.e., it descends to a multivalued function Ψ on an open dense subset of $G(M)$. Ψ is invariant up to phase under $\mathrm{Aut}(M, F) \supseteq \Gamma_0(N) \times \Gamma_0(N)$.

(2) The only zeroes and poles of Ψ lie on rational quadratic divisors $\lambda^\perp$ for $\lambda \in M$, $Q(\lambda) < 0$ and are zeroes of order

$$\sum_{0 < x \in \mathbb{R}, x\lambda \in M^\vee} c_{x\lambda}(Q(\lambda)x^2).$$

If for some λ this sum is not an integer, there is a branch on $\lambda^\perp$ with monodromy given by exponentiating $2\pi i$ times this sum. Ψ has single-valued holomorphic branches on any simply-connected neighborhood that has trivial intersection with the locus where this sum is not an element of $\mathbb{N}$.

(3) For each primitive norm zero vector $z \in M$, let m_z denote the smallest positive value of the inner product of z with a vector in M. For each Weyl chamber W of K, the restriction $\Psi_z(Z, F)$ has an infinite product expansion converging when $Z = X + iY \in K \otimes \mathbb{R} + iC \subset K \otimes \mathbb{C}$ is in a neighborhood of the cusp of z and $Y \in W$. The product is some constant of absolute value

$$\prod_{\substack{k \in \mathbb{Z}/m_z\mathbb{Z} \\ k \neq 0}} (1 - e(k/m_z))^{c_{kz/m_z}(0)/2}$$

times

$$e((Z, \rho(K, W, F_K))) \prod_{\lambda \in K^\vee, (\lambda, W) > 0} \prod_{\delta \in M^\vee/M, \delta | z^\perp = \lambda} (1 - e((\lambda, Z) + (\delta, z')))^{c_\delta(Q(\lambda))}$$

The branch locus of Ψ is nowhere dense, since it is contained in $\{Z \in \lambda^\perp | \lambda \in M^\vee, 0 > Q(\lambda) > k\}$ for any negative k that bounds the order of the poles of F.

The reader may notice a remarkable similarity between these properties of Ψ given by Borcherds's theorem and the properties we claim for $f(1,g,\sigma) - f(1,g,\tau)$ in the statement of our theorem. Indeed, our goal for the next two paragraphs is to show that Ψ and $f(1,g,\sigma) - f(1,g,\tau)$ are equal up to multiplication by an invertible constant.

We consider the product expansion at the cusp $z = (i\infty, i\infty) = (1,0,0,0)$. By Lemma 3.20 we have $F_K = q^{-1} + O(q)$ and $F_0 = q^{-1} + O(q)$. We fix $w = (0,0,1,0)$ and $w' = (0,0,0,1)$, and Lemma 3.23 implies $\rho(K,W,F_K) = -w'$, so $e((Z,\rho(K,W,F_K))) = p^{-1}$. The Weyl chambers are the two components of the complement of the hyperplane $(w-w')^\perp$, and we choose W to be the Weyl chamber whose boundary contains the null vector w. The set of $\lambda \in K^\vee$ satisfying $(\lambda, W) > 0$ are then those of the form $mw' + nw$, with $m > 0$. Here, $m_z = N$, and all $F_{0,j}$ have no constant terms, so $c_{jz/N}(0) = c_{0,j}(0) = 0$, and the constant multiple has norm 1. Choosing $z' = (0,1,0,0)$, we have:

$$\begin{aligned}
&p^{-1} \prod_{m>0, n\in\mathbb{Z}} \prod_{j\in\mathbb{Z}/N\mathbb{Z}} (1 - e(j/N)p^m q^n)^{c_{0,j}(mn)} \\
&= e((Z,-w')) \prod_{m>0, n\in\mathbb{Z}} \prod_{j\in\mathbb{Z}/N\mathbb{Z}} (1 - e(\frac{j}{N}z, z')e((mw',Z))e((nw,Z)))^{c_{jz}(mn)} \\
&= e((Z,\rho(F_K,W,K))) \prod_{\substack{\lambda\in K^\vee \\ (\lambda,W)>0}} \prod_{\delta\in M^\vee/M_{\delta|M\cap z^\perp=\lambda}} (1 - e((\lambda,Z) + (\delta,z')))^{c_\delta(Q(\lambda))}
\end{aligned}$$

The function Ψ has a unit magnitude constant ambiguity in its definition, so we may choose this constant to be one in a neighborhood of the cusp z. Since the above product is equal to the expansion of $f(1,g,\sigma) - f(1,g,\tau)$ as shown in Lemma 3.12, the two functions have identical analytic continuations, and Ψ is therefore single-valued. In particular, we find that the singular theta-lift is given by $\Phi_M = -4\log|f(1,g,\sigma) - f(1,g,\tau)|$.

We now consider the cusp $z = (i\infty, 0) = (0,0,0,1)$. If we choose $w = (1,0,0,0)$ and $w' = (0,1,0,0)$, Lemma 3.20 and Lemma 3.23 imply the Weyl vector is $-w'$, and $e((Z,\rho(K,W,F_K))) = p^{-1}$. In this case, the Weyl chambers may be more complicated, since there may be more singular terms in F_K, but again we choose the chamber whose boundary contains w. Here, $m_z = 1$, so the constant multiple has unit magnitude. Choosing $z' = (0,0,0,1)$ we have:

$$\begin{aligned}
&p^{-1} \prod_{m>0, n\in\frac{1}{N}\mathbb{Z}} (1 - p^m q^n)^{c_{m,Nn}(mn)} \\
&= e((Z,-w')) \prod_{m>0, n\in\frac{1}{N}\mathbb{Z}} (1 - e(mw',Z)e(nw,Z))^{c_{mw'+nw}(mn)} \\
&= e((Z,\rho(F_K,W,K))) \prod_{\substack{\lambda\in K^\vee \\ (\lambda,W)>0}} \prod_{\delta\in M^\vee/M_{\delta|M\cap z^\perp=\lambda}} (1 - e((\lambda,Z) + (\delta,z')))^{c_\delta(Q(\lambda))}
\end{aligned}$$

This matches the product expansion given in Proposition 3.13. □

To fit the results about Hecke-monic product expansions from Section 3.2 into the singular theta-lift framework, it now suffices to show that either the singularities of F have real coefficients, or that F has real coefficients along the span of w. We do not know how to do this in general, but it can be proved for the cases that interest us most.

Corollary 3.25. *Let g be a generator of $\mathbb{Z}/N\mathbb{Z}$, and let $\pi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{M}$ be a homomorphism to the monster simple group, such that the McKay-Thompson series $T_{\pi(g)}(\tau)$ is invariant under $\Gamma_0(N)$.*

Define functions $f_{(m)}$ by $f_{(m)}(\tau) = T_{\pi(g^m)}(\tau)$, and let F be the corresponding vector-valued function of type ρ_M. Then the first condition in Theorem 3.24 is satisfied. In particular, we have product expansions at all cusps with exponents given by coefficients of F, and the theta-lift Φ_M is equal to $-4\log|f(1,g,\sigma) - f(1,g,\tau)|$.

Proof. The McKay-Thompson series satisfy $T_{\pi(g^i)}(\tau) = \mathrm{Tr}(g^i q^{L_0-1}|V^\natural)$, where $L_0 - 1$ is a grading operator on the monster vertex algebra $V^\natural$, which we can view as a graded vector space with a homogeneous monster action. If V_n is the subspace of $V^\natural$ on which $L_0 - 1$ acts by multiplication by n, then the q^n coefficient of $T_{\pi(g^i)}(\tau)$ is $\mathrm{Tr}(g^i|V_n)$. The Fourier transform takes traces of powers of g to multiplicities of eigenvalues, so the coefficient $c_{0,k}(n)$ is the dimension of the subspace of V_n on which g acts by $e(k/N)$. These dimensions are non-negative integers, so the first condition is satisfied. The rest of the claims then follow from the theorem. □

Note: For the cases where $T_g(\tau)$ has squarefree level, this construction was suggested in Example 13.9 in [Borcherds-1998a], and the product formulas in question were proved in [Scheithauer-2008]. Scheithauer went a bit further in those cases, constructing multiple vector-valued forms, and writing down product expansions at all cusps.

Remark: There is a larger class of functions that we can fit into this framework, namely the completely replicable modular functions (see Section 3.3). Given such a function $f_{(1)}$ invariant under $\Gamma_0(N)$, we set $f_{(m)}$ to be its mth replicate, and by Corollary 5.6 in [Carnahan-2010], this defines a principally normalized Hecke-monic function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$. It is more difficult to verify the first condition given in Theorem 3.24, since the functions $\hat{F}_{0,j}$ do not appear as characters of a previously constructed representation of a group. Instead, we exhaustively computed the singular components of $\hat{F}$ using the description of all $n|h$-type invariance groups given in [Ferenbaugh-1993], and verified that $c_{i,k}(n) \in \{-1, 0, 1\}$ for all $n < 0$. Here 1 (resp. -1) occurs if and only if a Fricke involution acts by 1 (resp. -1) on $f_{(i)}$. Since these coefficients are real, the second condition in Theorem 3.24 holds.

3.5. **A monstrous example.** In this section, we describe an example of a product formula and an interpretation in vertex operator algebra language. The construction of the product formula is purely function-theoretic, and uses only the Hauptmodul property of some functions found in [Conway-Norton-1979]. However, our interpretation of these formulas in terms of algebraic structures rests on the construction in [Frenkel-Lepowsky-Meurman-1988] of the monster vertex operator algebra $V^\natural$ and the corresponding interpretation of McKay-Thompson series as characters.

As we mentioned in the introduction, if we forget some of the structure of $V^\natural$, we have a vector space with commuting actions of a semisimple operator L_0, which endows it with a grading, and the monster simple group $\mathbb{M}$, which acts homogeneously. We fix an element g of the monster simple group, and let $T_g(\tau) := \mathrm{Tr}(gq^{L_0-1}|V^\natural) \in \mathbb{C}((q))$ be the graded character. We will call this the McKay-Thompson series for g, although this is not how the series were originally defined (since the conceptual interpretation as the character of a known object rests on the construction of $V^\natural$ in [Frenkel-Lepowsky-Meurman-1988]). For each $m \geq 1$, we define $f_{(m)}$ to be $\mathrm{Tr}(g^m q^{L_0-1}|V^\natural)$. For any function f on the upper half-plane, we define its eigengroup to be the subgroup of elements of $SL_2(\mathbb{R})$ whose action on $\mathfrak{H}$ takes f to a constant multiple of itself. Conway and Norton [Conway-Norton-1979] introduced the notation $\Gamma_0(n|h)$ to denote the group of matrices

$$\left\{ \begin{pmatrix} a & b/h \\ cn & d \end{pmatrix} \mid ad - bcn/h = 1, a, b, c, d \in \mathbb{Z} \right\},$$

where $n = |g|$, and $h|(24, n)$. The group $\Gamma_0(n|h)$ contains $\Gamma_0(nh)$, is conjugate to $\Gamma_0(n/h)$, and can be extended to $\Gamma_0(n|h) + s_1, s_2, \ldots$ by adjoining some Atkin-Lehner involutions $W_{s_1}, W_{s_2}, \ldots$ for s_i exactly dividing nh.

Conway and Norton conjectured that (under the original interpretation of McKay-Thompson series, without the existence of $V^\natural$) the set of these functions $\{f_{(m)}\}_{m>0}$ satisfies the following property: $f_{(1)}$ has an eigengroup of the form $\Gamma_0(n|h) + s_1, s_2, \ldots$ for some h, and $f_{(m)}$ has eigengroup $\Gamma_0(n'|h') + s'_1, s'_2, \ldots$, where $n' = n/(n,m)$, $h' = h/(h,m)$, and $s'_i = s_i$ if and only if $s_i | \frac{n'}{h'}$ and $s'_i = 1$ otherwise. Frenkel, Lepowsky, and Meurman reinterpreted this assertion in terms of characters of the vertex operator algebra $V^\natural$ they constructed, proving many cases [Frenkel-Lepowsky-Meurman-1988], and Borcherds proved it in full, using the same interpretation [Borcherds-1992]. A consequence of this fact is that (setting $N = nh$), if $f_{(1)}$ is fixed by $\Gamma_0(N)$, then $f_{(m)}$ is fixed by $\Gamma_0(N/(N,m))$. Therefore, we can assemble these functions into a principally normalized Hecke-monic function on $\mathcal{M}_{Ell}^{\mathbb{Z}/N\mathbb{Z}}$, and take the Fourier transform to get a vector-valued function F. Note that N may be larger than $|g|$, since h may be greater than one. The eigengroups for all of the elements of the monster can be found in Table 2 of [Conway-Norton-1979].

In this setting, we can call $\hat{F}$ the "trace function" and F the "multiplicity function" since their components (conjecturally) describe the traces and multiplicities of eigenvalues for an action of $\mathbb{Z}/N\mathbb{Z}$ on the irreducible twisted modules of $V^\natural$. The details of twisted modules are outside the scope of this paper, but we remark that the notion was introduced in a restricted form in [Frenkel-Lepowsky-Meurman-1988], and subsequently generalized in the work of several authors. We will show in the next section that when we expand Ψ at $(i\infty, 0)$, the exponents in the product are the root multiplicities of a certain Lie algebra that we can define by generators and relations. Assuming some additional conjectures, there is a variant of the Goddard-Thorn no-ghost theorem that implies the root spaces are naturally isomorphic to the eigenspaces of the corresponding twisted modules. We hope to address this in future work.

We give a specific example to illuminate some of the objects in play. We will focus on the case of an element in conjugacy class 3C, since that case is interesting for several reasons:

- The centralizer of 3C in $\mathbb{M}$ is a direct product of $\mathbb{Z}/3\mathbb{Z}$ with Thompson's sporadic simple group Th, and since Th has no nontrivial central extension [ATLAS], the irreducible 3C-twisted module of $V^\natural$ has a natural action of Th.
- The McKay-Thompson series for 3C is a three-fold dilation of the character of the basic representation of affine E_8, and the character of the 3C-twisted module is (conjecturally) a 3-fold compression of the same. While Th embeds in $E_8(\mathbb{F}_3)$, it does not embed in $E_8(\mathbb{C})$ [Cohen-Griess-1987]. It is natural to ask whether the action of Th on the 3C-twisted module extends to an action of $E_8(\mathbb{F}_3)$, and if so, precisely where the obstruction to a 3-adic lift resides. This is analogous to an open problem in [Borcherds-1998b] concerning a certain mod 3 vertex algebra ${}^gV$ attached to class 3C.
- The dilation in the characters can be viewed as the 3-adic U_3 Hecke operator, arising from the Frobenius isogeny on mod 3 elliptic curves. The "Frobenius compression" arises in the proof of Corollary 4.8 in [Borcherds-Ryba-1995], where it is used to endow the mod 3 vertex algebra ${}^gV$ with a conformal vector. It would be interesting to have this apparent relationship between mod p phenomena and twisted modules fleshed out more.
- Most importantly for us, 3C is the conjugacy class of smallest order such that for an element g, the level of T_g is strictly greater than the order of g. The higher level creates a μ_3 ambiguity in the $SL_2(\mathbb{Z})$ compatibility formula in the generalized moonshine conjecture, since $T_g(-1/\tau)$ is not invariant under translation by 3. This behavior is called a *phase anomaly* in the mathematical physics literature. When we consider centralizers of elements of $\mathbb{M}$, this ambiguity manifests as the nontriviality of central extensions, i.e., in the analysis below, we are essentially pretending that g generates a cyclic group of order 9.

Let g be an element of order 3 in the conjugacy class 3C. Its McKay-Thompson series is

$$T_g(\tau) = q^{-1} + 248q^2 + 4124q^5 + 34752q^8 + \cdots = j(3\tau)^{1/3},$$

which has eigengroup

$$\Gamma_0(3|3) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{Z}) \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., this group acts by roots of unity. In fact, $y_3 = \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}$ acts by $e(-1/3)$, $x_3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ acts by $e(1/3)$, and the Fricke involution $\begin{pmatrix} 0 & -1 \\ 9 & 0 \end{pmatrix}$ fixes T_g. The fixing group, which is normal of index three in the eigengroup, contains $\Gamma_0(9)$ as an index 4 subgroup, and the Fricke involution together with x_3y_3 and $x_3^2y_3^2$ are representatives of the nontrivial cosets. Armed with the above information, we can construct a table whose entries are the components of $\hat{F}$. Rows and columns are indexed from 0 to 8 (mod 9).

J	$T_g(\tau)$	$T_g(\tau)$	J	$T_g(\tau)$	$T_g(\tau)$	J	$T_g(\tau)$	$T_g(\tau)$
$T_g(\frac{-1}{\tau})$	$T_g(\frac{-1}{\tau+1})$	$T_g(\frac{-1}{\tau+2})$	$T_g(\frac{-1}{\tau+3})$	$T_g(\frac{-1}{\tau+4})$	$T_g(\frac{-1}{\tau+5})$	$T_g(\frac{-1}{\tau+6})$	$T_g(\frac{-1}{\tau+7})$	$T_g(\frac{-1}{\tau+8})$
$T_g(\frac{-1}{\tau})$	$T_g(\frac{-1}{\tau+5})$	$T_g(\frac{-1}{\tau+1})$	$T_g(\frac{-1}{\tau+6})$	$T_g(\frac{-1}{\tau+2})$	$T_g(\frac{-1}{\tau+7})$	$T_g(\frac{-1}{\tau+3})$	$T_g(\frac{-1}{\tau+8})$	$T_g(\frac{-1}{\tau+4})$
J	$\zeta_3 T_g(\tau)$	$\zeta_3^2 T_g(\tau)$	J	$\zeta_3 T_g(\tau)$	$\zeta_3^2 T_g(\tau)$	J	$\zeta_3 T_g(\tau)$	$\zeta_3^2 T_g(\tau)$
$T_g(\frac{-1}{\tau})$	$T_g(\frac{-1}{\tau+7})$	$T_g(\frac{-1}{\tau+5})$	$T_g(\frac{-1}{\tau+3})$	$T_g(\frac{-1}{\tau+1})$	$T_g(\frac{-1}{\tau+8})$	$T_g(\frac{-1}{\tau+6})$	$T_g(\frac{-1}{\tau+4})$	$T_g(\frac{-1}{\tau+2})$
$T_g(\frac{-1}{\tau})$	$T_g(\frac{-1}{\tau+2})$	$T_g(\frac{-1}{\tau+4})$	$T_g(\frac{-1}{\tau+6})$	$T_g(\frac{-1}{\tau+8})$	$T_g(\frac{-1}{\tau+1})$	$T_g(\frac{-1}{\tau+3})$	$T_g(\frac{-1}{\tau+5})$	$T_g(\frac{-1}{\tau+7})$
J	$\zeta_3^2 T_g(\tau)$	$\zeta_3 T_g(\tau)$	J	$\zeta_3^2 T_g(\tau)$	$\zeta_3 T_g(\tau)$	J	$\zeta_3^2 T_g(\tau)$	$\zeta_3 T_g(\tau)$
$T_g(\frac{-1}{\tau})$	$T_g(\frac{-1}{\tau+4})$	$T_g(\frac{-1}{\tau+8})$	$T_g(\frac{-1}{\tau+3})$	$T_g(\frac{-1}{\tau+7})$	$T_g(\frac{-1}{\tau+2})$	$T_g(\frac{-1}{\tau+6})$	$T_g(\frac{-1}{\tau+1})$	$T_g(\frac{-1}{\tau+5})$
$T_g(\frac{-1}{\tau})$	$T_g(\frac{-1}{\tau+8})$	$T_g(\frac{-1}{\tau+7})$	$T_g(\frac{-1}{\tau+6})$	$T_g(\frac{-1}{\tau+5})$	$T_g(\frac{-1}{\tau+4})$	$T_g(\frac{-1}{\tau+3})$	$T_g(\frac{-1}{\tau+2})$	$T_g(\frac{-1}{\tau+1})$

The function J above denotes the $SL_2(\mathbb{Z})$-invariant function $T_1(\tau) = j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots$. Taking the Fourier transform, we can describe the vector-valued function F by another table (with the same indexing).

f^0	0	0	f^3	0	0	f^6	0	0
0	0	f^4	0	0	f^7	0	0	f^1
0	f^5	0	0	f^8	0	0	f^2	0
f^6	0	0	f^0	0	0	f^3	0	0
0	0	f^1	0	0	f^4	0	0	f^7
0	f^2	0	0	f^5	0	0	f^8	0
f^3	0	0	f^6	0	0	f^0	0	0
0	0	f^7	0	0	f^1	0	0	f^4
0	f^8	0	0	f^2	0	0	f^5	0

The functions are given by:

$$\begin{aligned}
f^0 &= (J + 2T_g)/3 = q^{-1} + 65628q + 7164752q^2 + \dots \\
f^1 &= q^{-1/9} + 34752q^{8/9} + 4530744q^{17/9} + \dots \\
f^2 &= 4124q^{5/9} + 1057504q^{14/9} + \dots \\
f^3 &= (J - T_g)/3 = 65628q + 7164504q^2 + \dots \\
f^4 &= 248q^{2/9} + 213126q^{11/9} + \dots \\
f^5 &= 248q^{2/9} + 213126q^{11/9} + \dots \\
f^6 &= (J - T_g)/3 = 65628q + 7164504q^2 + \dots \\
f^7 &= 4124q^{5/9} + 1057504q^{14/9} + \dots \\
f^8 &= q^{-1/9} + 34752q^{8/9} + 4530744q^{17/9} + \dots
\end{aligned}$$

The singular terms are f^0 with a pole of order 1, and $f^1 = f^8$, with poles of order 1/9. Since the residues are integers, F satisfies the second condition of Theorem 3.24. We can describe the zeros of Ψ by lifting the vector-valued function above to assign functions to each vector in $M^\vee$. Recall from section 3.1 that we identify $M = \mathbb{Z} \times 9\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $M^\vee = \frac{1}{9}\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Ψ has zeros of order one along rational quadratic divisors $\lambda^\perp$ for those $\lambda \in M$ of the following form: Any $\lambda \in M$ with $Q(\lambda) = -1$ is assigned f^0, and has a pole corresponding to divisors $\sigma = A\tau$ for $A \in \Gamma_0(9)$. Any $\lambda \in M$ with $Q(\lambda) = -9$, satisfying $\lambda/9 \in M^\vee$ with $\lambda/9$ assigned f^1 or f^8 has a pole corresponding to divisors $\sigma = A\tau$ for $A \in \begin{pmatrix} 0 & -1 \\ 9 & 0 \end{pmatrix} \Gamma_0(9)$. Any $\lambda \in M$ with $Q(\lambda) = -9$, satisfying $\lambda/3 \in M^\vee$ with $\lambda/3$ assigned f^0 has a pole corresponding to divisors $\sigma = A\tau$ for $A \in x_3y_3\Gamma_0(9)$ (if $\lambda/3 \equiv (1/3, 3, 0, 0) \bmod M$) or $A \in x_3^2y_3^2\Gamma_0(9)$ (if $\lambda/3 \equiv (2/3, 6, 0, 0) \bmod M$). Ψ also has a product expansion at the cusp $(0, 0, 0, 1) = (i\infty, 0)$ given by:

$$p^{-1} \prod_{m>0, n\in\frac{1}{9}\mathbb{Z}} (1 - p^m q^n)^{c(m,n)}$$

Here, $c(m, n)$ is given by $c_1(9mn) + c_2(mn)/3$ if $m/3$ and $3n$ are integers, and $c_1(9mn)$ otherwise, where $T_g(\tau) = \sum_{n\in Z} c_1(n)q^n$ and $\sum_{n\in Z} c_2(n)q^n = J(\tau) - T_g(\tau) = 196884q + 21493512q^2 + 864299970q^3 + 20245856256q^4 + \dots$.

This product is the expansion of $T_g(\sigma) - T_g(-1/\tau)$. Up to a change of grading, this product is equivalent to the Weyl-Kac-Borcherds denominator of one of the monstrous Lie algebras described in section 10 of [Borcherds-1992].

There is a good reason why most of the entries of F are zero, and it is because F descends to a vector-valued modular function for a lattice $L \times II_{1,1}$ that contains $M = II_{1,1}(9) \times II_{1,1}$ as an index three sublattice, where L is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ with the quadratic form $(a, b) \mapsto 3ab + b^2$. In particular, F is supported on $L^\vee/M$, which is an index three subgroup of $M^\vee/M$. Other elements whose McKay-Thompson series have level larger than their order (i.e., with $h > 1$ in the sense of [Conway-Norton-1979]) behave similarly.

For most elements, namely those with $h = 1$, the lattice quotients have the form $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. However, for 3C, $L^\vee/L \cong \mathbb{Z}/9\mathbb{Z}$, and we expect that the category of representations of the fixed point vertex algebra $(V^\natural)^g$ has a braided tensor structure, with fusion algebra naturally isomorphic to $\mathbb{Z}[L^\vee/L]$, such that each f^i equal to the graded dimension of the corresponding irreducible object. Since it is in general quite difficult to determine the representation theory of fixed point algebras, this interpretation is still conjectural.

We can view the vector-valued function F in a more holistic way, as a vector-valued character of an abelian intertwining algebra, in the sense of [Dong-Lepowsky-1993], and the theory of abelian intertwining algebras will still work if we replace $\langle g \rangle$ with any cyclic subgroup of $\mathbb{M}$, or with a choice of many non-cyclic abelian subgroups. There is an alternative viewpoint that extends to larger subgroups, given by vertex algebra objects (i.e., singular commutative rings) in a braided tensor category. We intend to show in forthcoming work that these objects are much easier to study than representations of fixed point algebras, once some conceptual hurdles are overcome. For the 3C case, there is more than one interesting way to view the braided category in question:

(1) The most straightforward view is that this is the category of $L^\vee/L$-graded vector spaces, where the braided monoidal structure is classified up to equivalence by the induced $\mathbb{Q}/\mathbb{Z}$-valued quadratic form. This view can be generalized to abelian subgroups.
(2) To connect this structure to the action of $\langle g \rangle$, one can view it as the Drinfeld center of the monoidal category of $\mathbb{Z}/3\mathbb{Z}$-graded vector spaces, with associator twisted by a nontrivial 3-cocycle with coefficients in $\mathbb{C}^\times$. Equivalently, the braided category is the category of modules of a twisted quantum double of $\mathbb{C}[\mathbb{Z}/3\mathbb{Z}]$ [Dijkgraaf-Pasquier-Roche-1990]. This viewpoint extends to treat nonabelian subgroups of the monster.

(3) The abelian category of $\mathbb{Z}/3\mathbb{Z}$-graded vector spaces can be viewed more naturally as vector bundles on the group $\mathbb{Z}/3\mathbb{Z}$, and the twisted associator on the convolution tensor structure describes a nontrivial 2-group structure, equivalent to the based loop space of a nontrivial $K(\mathbb{C}^\times, 2)$-torsor over $B\mathbb{Z}/3\mathbb{Z}$. Our monoidal category is given by sheaves on this 2-group with convolution, and its center is given by either conjugation-equivariant sheaves on the 2-group, or equivalently, sheaves on the free loop space of the torsor. If we expand our view to the abelian category $\mathcal{C}$ of all twisted modules of $V^\natural$, there is a natural strict action of a 2-group $\widetilde{\mathbb{M}}$ whose structure is described up to equivalence (following Sinh's conditions - see section 8 of [Baez-Lauda-2004]) by $\pi_0(\widetilde{\mathbb{M}}) \cong \mathbb{M}$, $\pi_1(\widetilde{\mathbb{M}}) \cong \mathrm{Aut}(Id_{\mathcal{C}})$, and a nontrivial element of $H^3(\pi_0(\widetilde{\mathbb{M}}), \pi_1(\widetilde{\mathbb{M}}))$, where the module structure is trivial.

The constant ambiguity in the generalized moonshine conjecture seems to arise from the nontriviality of the higher structure in $\widetilde{\mathbb{M}}$, and we can see it manifested in the 3C case. There is an action of the centralizer $C_{\widetilde{\mathbb{M}}}(g)$ on the category of g-twisted modules, which induces an action of a $\mathbb{C}^\times$ central extension of $C_{\mathbb{M}}(g)$ on any fixed irreducible g-twisted module, with a lift of g acting by the conformal operator $e(L_0)$. The class of the central extension is given by restriction of the loop transgression of a 3-cocycle on $\mathbb{M}$ (see section 3 of [Willerton-2008]). Further restriction of the 2-cocycle to the cyclic group generated by g is yields a trivializable cocycle, but any trivialization is incompatible with the assignment of $e(L_0)$ as a distinguished lift of g. When we pull back to the degree 3 central extension $\mathbb{Z}/9\mathbb{Z}$, the compatibility is achievable.

In general, when we deal with cyclic subgroups of $\mathbb{M}$, we can trivialize the restricted 2-group structure by passing to an abelian central extension, and in the above treatment of the 3C case, we did this by essentially pretending that g had order 9. Since Th has no nontrivial discrete central extensions, we can descend to an action of $\mathbb{Z}/9\mathbb{Z} \times Th$ on the irreducible twisted module. Geometrically, the nontriviality of the 3-cocycle implies that g-twisted modules are not canonically attached to ramified points on an $\mathbb{M}$-cover of an algebraic curve, because the twisted modules are not admissible in the sense of [Frenkel-Szczesny-2004]. Instead, one can either introduce a formalism of $\widetilde{\mathbb{M}}$-covers, or attach modules to nonzero one-jets at those points. The $e(L_0)$ incompatibility mentioned in the previous paragraph implies that the dependence on choice of 1-jet is nontrivial. We will treat the geometric formalism more thoroughly in [Carnahan-20??].

4. Lie algebras

In this section, we give a conditional proof of the generalized moonshine conjecture. It requires strong assumptions concerning group actions on certain Lie algebras, together with a technical hypothesis concerning genus 1 correlation functions.

In the first part, we construct Lie algebras whose denominator identities give the product formulas from the previous section. Then we introduce a notion of non-Fricke compatibility for Lie algebras, and show that the Lie algebras we constructed using McKay-Thompson series have positive subalgebras that are Fricke or non-Fricke compatible with cyclic group actions. We introduce some assumptions for groups acting on these Lie algebras, and show that under those assumptions, the resulting characters are weakly Hecke-monic. The weakly Hecke-monic functions are automatically genus zero in the Fricke case, but otherwise, we need an additional assumption.

4.1. **Denominator formulas.** Generalized Kac-Moody algebras were introduced by Borcherds in [Borcherds-1988]. The theory is very similar to that of Kac-Moody algebras, since the algebras are described by a Cartan matrix giving generators and relations, with the main distinction being the possibility of having imaginary simple roots, i.e., generators of non-positive norm. Like Kac-Moody algebras, they admit character formulas, but the formulas have correction terms for the imaginary simple roots. In the case of the trivial representation, we obtain the Weyl-Kac-Borcherds

denominator formula:

$$\sum_{w\in W} \epsilon(w)w(S) = e(\rho) \prod_{\alpha\in\Delta^+} (1-e(-\alpha))^{\mathrm{mult}(\alpha)}$$

Here, $S = e(\rho)\sum_s \epsilon(s)e(s)$, where s runs over all sums of pairwise orthogonal imaginary simple roots. From a homological algebra standpoint, this formula expresses an isomorphism of virtual vector spaces $H^*(E) \cong \wedge^*(E)$ arising from the Chevalley resolution of the trivial representation.

We are interested in the rank two setting, where the root lattice is Lorentzian of signature $(1,1)$. Since the real simple roots generate a Kac-Moody algebra, there are at most two of them, but the imaginary simple roots can have arbitrary multiplicity. For example, one can generate a large family of such algebras by choosing an arbitrary non-negative Laurent series of the form $f(q) = q^{-1} + \sum_{n>0} a_n q^n \in q^{-1}\mathbb{N}[[q]]$, setting a real simple root at $(1,-1)$, and imaginary simple roots of multiplicity a_n at $(1,n)$. This algebra has a Weyl group of order two, and the Weyl-Kac-Borcherds denominator formula equates $f(p)-f(q)$ with p^{-1} times a product of binomials of the form $1-p^m q^n$.

Lemma 4.1. *Suppose we are given a product formula of one of the following two types:*

(1) $f(p) - f(q^{1/N}) = p^{-1}\prod_{m>0,n\in\frac{1}{N}\mathbb{Z}}(1-p^m q^n)^{c(m,n)}$, *where* $f(q) = q^{-1} + \sum_{n=1}^{\infty} c(1,n/N)q^n \in q^{-1}\mathbb{N}[[q]]$.

(2) $f(p) - g(q^{1/N}) = p^{-1}\prod_{m>0,n\in\frac{1}{N}\mathbb{Z}}(1-p^m q^n)^{c(m,n)}$, *where* $f(q) = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{c(n,0)}$ *satisfies* $c(n,0)\in\mathbb{N}$ *for all* n, *and* $g(q) = \sum_{n\geq 0} c(1,n)q^n \in \mathbb{N}[[q]]$

Then there exists a unique generalized Kac-Moody algebra with that product formula as denominator formula.

Proof. This proof follows a strategy similar to the construction of the monster Lie algebra in [Jurisich-Lepowsky-Wilson-1995].

We begin by choosing the root lattice, whose underlying group is $\mathbb{Z}\times\frac{1}{N}\mathbb{Z}$, and whose rational bilinear form is induced from the rational quadratic form $Q((m,n)) = mn$. In particular, the inner product is given by $\langle (a,b/N),(c,d/N)\rangle = \frac{ad+bc}{N}$ for all integers a,b,c,d. We also introduce notation for describing the characters of vector spaces graded by the root lattice: A one dimensional vector space concentrated in degree $(m,n)\in\mathbb{Z}\times\frac{1}{N}\mathbb{Z}$ has character $p^m q^n$.

For the first type of product formula, we construct a Cartan matrix by allocating $c(1,n)$ rows and columns to degree $(1,n)$, and placing $-Nm-Nn$ in every entry whose row has degree $(1,n)$ and whose column has degree $(1,m)$, for each $m,n\in\frac{1}{N}\mathbb{Z}$. The resulting matrix has the following properties:

(1) The matrix is symmetric.
(2) There is exactly one non-negative diagonal entry. It is equal to 2, and corresponds to $c(1,-1/N)=1$.
(3) Off-diagonal entries are non-positive integers.

The three conditions for a generalized Cartan matrix listed in [Borcherds-1995a] are therefore satisfied by our matrix. We form the universal generalized Kac-Moody algebra described by this matrix, and let $\mathfrak{g}$ be the quotient by the center. Since the matrix has one positive diagonal entry, there is one real simple root, and the Weyl group has order 2. The Cartan subalgebra has dimension 2, and we may choose the Weyl vector to be $\rho = (-1,0)$, since that vector satisfies the defining property that $(\rho,\alpha_i) = -\frac{1}{2}(\alpha_i,\alpha_i)$ for all simple roots α_i. If f has nonzero terms in positive degree, then the Weyl vector is unique.

The left side of the denominator formula for $\mathfrak{g}$ has the form $\sum_{w\in W}\epsilon(w)w(S)$. Since all imaginary simple roots have negative norm, all sets of pairwise orthogonal imaginary simple roots have size

at most one, so

$$S = e(\rho)\sum_s \epsilon(s)e(s) = p^{-1}\sum_{n\in\frac{1}{N}\mathbb{Z}} c(1,n)pq^n.$$

The left side of the denominator formula is therefore $\sum_n c(1,n)q^n - \sum_n c(1,Nn)p^{Nn}$, which is the left side of the product formula. The right side of the denominator formula for $\mathfrak{g}$ has the form

$$e(\rho)\prod_{\alpha\in\Delta^+}(1-e(-\alpha))^{\mathrm{mult}(\alpha)}$$

The positive roots $\alpha\in\Delta^+$ have the property that $e(\alpha)=p^mq^n$ for some $m>0$ and $n\in\frac{1}{N}\mathbb{Z}$, so we can rewrite this expression as

$$p^{-1}\prod_{m>0,n\in\frac{1}{N}\mathbb{Z}}(1-p^mq^n)^{\mathrm{mult}(\alpha)}.$$

Since $f(p)-f(q^{1/N})$ has a unique product expansion in binomials of the form $(1-p^aq^b)$, the multiplicities of the roots α are given by $c(m,n)$.

For the second type, we construct a Cartan matrix by allocating $c(1,n)$ rows and columns to degree $(1,n)$, $c(m,0)$ rows and columns to degree $(m,0)$, and specifying the following entries:

(1) We place the integer $-Nm-Nn$ in every place with row degree $(1,n)$ and column degree $(1,m)$.
(2) We place the integer $-Nmn$ in every place with row degree $(1,n)$ and column degree $(m,0)$, or with row degree $(m,0)$ and column degree $(1,n)$.
(3) We place 0 in every place with row degree $(m,0)$ and column degree $(n,0)$.

This matrix is symmetric, with non-positive off-diagonal entries, and it has no positive diagonals, so it satisfies the criteria of [Borcherds-1995a]. As before, we form the universal generalized Kac-Moody algebra described by this matrix, and let $\mathfrak{g}$ be the quotient by its center. This algebra has no real simple roots, so its Weyl group is trivial. As in the first case, the Cartan subalgebra is two-dimensional and the Weyl vector ρ is given by $(-1,0)$.

Since the Weyl group of this algebra is trivial, the left side of the denominator formula for $\mathfrak{g}$ is just $S=e(\rho)\sum_s\epsilon(s)e(s)$. Unlike the first case, there are imaginary simple roots of norm zero, and they are orthogonal to each other, so the sum is somewhat more complicated. We note that the imaginary simple roots of degree $(1,n)$ for $n>0$ are not orthogonal to any others, so they appear in the sum once. The rest of the imaginary simple roots have degree $(m,0)$ for positive integers m, and these roots are mutually orthogonal. In other words, S is given by adding $\sum_{n=1}^\infty -c(1,n/N)pq^{n/N}$ to an alternating sum of over all finite sets of simple roots of degree $(m,0)$. The alternating sum can be viewed as the character of the exterior algebra whose basis is given by those simple roots, and this has the form of an infinite product. We get:

$$\begin{aligned}\sum_s\epsilon(s)e(s) &= \sum_{n=1}^\infty -c(1,n/N)pq^{n/N} + \prod_{m=1}^\infty(1-p^m)^{c(m,0)}\\ &= p(f(p)-g(q^{1/N}))\end{aligned}$$

Therefore, $S=f(p)-g(q^{1/N})$.

The right side of the denominator formula is similar to the first case:

$$p^{-1}\prod_{m>0,n\in\frac{1}{N}\mathbb{Z}}(1-p^mq^n)^{\mathrm{mult}(\alpha)}.$$

However, in this case, there are no real simple roots, so n can be restricted to range over positive numbers. As before, uniqueness of the product formula implies the multiplicities are given by the exponents $c(m,n)$. □

Remark: If the exponents in a product are non-integral or negative, we can view the product as a denominator formula for a more general Lie algebra object in virtual vector spaces. The monstrous Lie superalgebras in [Borcherds-1992] correspond to the setting of supervector spaces and negative integer exponents.

There is a distinguished subclass of these generalized Kac-Moody algebras, made of those whose denominator formulas have modular forms on the left, and products with coefficients of modular forms on the right. In contrast to the general case, it is expected that there are only finitely many examples of these.

Theorem 4.2. *For each $g \in \mathbb{M}$, the product expansion of $T_g(\sigma) - T_g(-1/\tau)$ as given in Proposition 3.13 is the denominator formula of a unique generalized Kac-Moody Lie algebra.*

Proof. We split this question into two cases:

(1) g is Fricke, i.e., if $T_g(\tau)$ is invariant under some $\tau \mapsto -1/N\tau$.
(2) g is non-Fricke, i.e., if $T_g(\tau)$ is regular at zero.

For the Fricke case, $T_g(\sigma) - T_g(-1/\tau) = T_g(\sigma) - T_g(\tau/N)$, so if we assume the coefficients of $T_g(\tau)$ (equivalently, those of $T_g(-1/\tau)$) are nonnegative integers, then the expansion has the form of the first case of Lemma 4.1. The coefficients are already known to be integers, so it suffices to show that they are nonnegative. By [Dong-Li-Mason-2000], there exists an irreducible g-twisted module of $V^\natural$, unique up to isomorphism, such that its character is a constant times the q-expansion $T_g(-1/\tau)$. Since the character has nonnegative integer coefficients and the leading term of $T_g(-1/\tau)$ is 1, this constant must be a positive rational number, and the coefficients of $T_g(\tau)$ are nonnegative.

For the non-Fricke case, the power series expansion of $T_g(\sigma) - T_g(-1/\tau)$ has no terms with any negative powers of q. By the second case of Lemma 4.1, it suffices to check that $T_g(-1/\tau)$ has nonnegative integer coefficients, and that $T_g(\tau)$ is an eta-product $\eta(a_1\tau)^{b_1}\eta(a_2\tau)^{b_2}\dots\eta(a_n\tau)^{b_n}$ whose exponents satisfy the following positivity condition:

- For any residue class k modulo $\mathrm{lcm}(\{a_1, \dots, a_n\})$,

$$\sum_{\substack{i \in \{1,\dots,n\} \\ a_i | k}} b_i \geq 0.$$

  Here, the notation $a_i|k$ means the order of k in $\mathbb{Z}/\mathrm{lcm}(\{a_1, \dots, a_n\})\mathbb{Z}$ divides the order of a_i.

The information necessary for verifying these conditions already exists in encoded form in Tables 2 and 3 in [Conway-Norton-1979]. Table 2 gives a "symbol" for each conjugacy class, and the non-Fricke elements are those whose symbol has the form $n-$, $n|h$ for $h \neq n$, $n + s_1, s_2, \dots$ where no s_i is equal to n, and $n|h + s_1, s_2, \dots$, where no s_i is equal to n/h. Table 3 gives eta-product expansions of some of the non-Fricke elements, and the rest are described in the "Harmonies and Symmetrizations" column. Checking this information in each case, we find that the positivity condition is satisfied, and that $T_g(-1/N\tau) = k_1 + k_2/T_g(\tau)$ for some positive integers k_1, k_2. The coefficients of $T_g(-1/\tau)$ are therefore integers, and either by using the results of [Dong-Li-Mason-2000] as in the Fricke case, or by analyzing the eta product expansion of $\frac{T_g(-1/\tau)-k_1}{k_2}$, we find that the coefficients of $T_g(-1/\tau)$ are non-negative. □

4.2. **Compatibility.** Let G be a group, and let g be an element of order N in the center of G. Suppose we have a collection $\mathcal{V} = \{V_k^{i,j/N} | i, j \in \mathbb{Z}/N\mathbb{Z}, k \in \frac{1}{N}\mathbb{Z}\}$ of G-modules, such that the action of g on $V_k^{i,j/N}$ is given by constant multiplication by the root of unity $e(j/N)$, and $\dim V_k^{i,j/N}$ grows subexponentially with k, i.e., for any $\epsilon > 0$, there is some $C > 0$ such that for all i, j, k, $\dim V_k^{i,j/N} < Ce^{\epsilon k}$.

From [Carnahan-2010], we say that a complex Lie algebra E is Fricke compatible with the triple $(G, g, \mathcal{V})$ if:

(1) E is graded by $\mathbb{Z}_{>0} \times \frac{1}{N}\mathbb{Z}$, with finite dimensional homogeneous components $E_{i,j}$. We write $E = \bigoplus_{i>0, j\in\frac{1}{N}\mathbb{Z}} E_{i,j} p^i q^j$, where p is a $(1,0)$ degree shift, and q is a $(0,1)$ degree shift. We can view this as a character decomposition under an action of a two dimensional torus H.
(2) E admits a homogeneous action of G by Lie algebra automorphisms, such that we have G-module isomorphisms $E_{i,j} \cong V^{i,j}_{1+ij}$
(3) The homology of E is given by:
   - $H_0(E) = \mathbb{C}$
   - $H_1(E) = \bigoplus_{n\in\frac{1}{N}\mathbb{Z}} V^{1,n}_{1+n} pq^n$
   - $H_2(E) = p\bigoplus_{m=1}^{\infty} V^{1,-1/N}_{1-1/N} \otimes V^{m,1/N}_{1+m/N} p^m$
   - $H_i(E) = 0$ for $i > 2$.
(4) $E_{1,-1/N} \cong V^{1,-1/N}_{1-1/N}$ is one dimensional.

We define here the complementary notion that is suited to non-Fricke elements of the monster.

Definition 4.3. A complex Lie algebra E is non-Fricke compatible with the triple $(G, g, \mathcal{V})$ if:

(1) E is graded by $\mathbb{Z}_{>0} \times \frac{1}{N}\mathbb{N}$, with finite dimensional homogeneous components $E_{i,j}$. We write $E = \bigoplus_{i\in\mathbb{N}, j\in\frac{1}{N}\mathbb{N}} E_{i,j} p^i q^j$, where p is a $(1,0)$ degree shift, and q is a $(0,1)$ degree shift. We can view this as a character decomposition under an action of a two dimensional torus H.
(2) E admits a homogeneous action of G by Lie algebra automorphisms, such that we have G-module isomorphisms $E_{i,j} \cong V^{i,j}_{1+ij}$
(3) The homology of E is given by:
   - $H_0(E) = \mathbb{C}$.
   - $H_1(E) = \bigoplus_{n\in\frac{1}{N}\mathbb{N}} V^{1,n}_{1+n} pq^n \oplus \bigoplus_{m=1}^{\infty} V^{m,0}_1 p^m$.
   - $H_i(E) = \bigwedge^i(\bigoplus_{m=1}^{\infty} V^{m,0}_1 p^m)$ for $i \geq 2$.

Remark: The last condition matches the higher homology of E with the higher homology of the abelian subalgebra $\bigoplus_{m>0} V^{m,0}_1 p^m$.

Proposition 4.4. *Let $G = \langle g\rangle$ be a cyclic group of order N, and let $\phi : G \to \mathbb{M}$ be a homomorphism. Let $T_g(\tau)$ denote the McKay-Thompson series for the automorphism of $V^\natural$ described by $\phi(g)$, and assume it is invariant under $\Gamma_0(N)$. Let $\mathcal{V}$ be the collection of G-modules defined by setting the dimension of $V^{i,j/N}_k$ to be the coefficient $c_{i,j}(k)$ of the vector-valued form associated to T_g, with g acting by multiplication by $e(j/N)$. Let W_g be the generalized Kac-Moody Lie algebra whose denominator formula is the product formula for the expansion of $T_g(\sigma) - T_g(-1/\tau)$, given in Proposition 3.13. Define an action of G on W_g by setting g to act on vectors of degree (m,n) by $e(n)$, and let E_g be the Lie subalgebra of W_g formed by the positive root spaces. If g is Fricke, then E_g is Fricke compatible with $(G, g, \mathcal{V})$, and if g is non-Fricke, then E_g is non-Fricke compatible with $(G, g, \mathcal{V})$.*

Proof. We split the problem into two cases:
First: Suppose g is Fricke. From Theorem 4.2, there is one real simple root in degree $(1, -1/N)$, and rest of the positive roots are imaginary with multiplicity $c_{n,m}(-mn)$ in degree (m,n) for all $m \in \mathbb{N}$, $n \in \frac{1}{N}\mathbb{N}$. E_g therefore satisfies the grading restriction condition, and $E_{1,-1/N}$ is one dimensional.

G acts by Lie algebra automorphisms, because the action of g is determined by grading in a multiplicative way, i.e, the action factors through the adjoint action of the torus H.

The action of G on root spaces of E_g is identified with the action on the spaces in $\mathcal{V}$, since g acts by $e(n)$ on $V^{m,n}_{1+mn}$ for all $m \in \mathbb{Z}, n \in \frac{1}{N}\mathbb{Z}$.

Since W_g is generalized Kac-Moody, the homology of E_g is given by a formula analogous to the denominator formula. The homology groups can be calculated by the rule at the end of section 4 in [Borcherds-1992], i.e., noting that Proposition 7.9 in [Garland-Lepowsky-1976] can be adapted to the case of generalized Kac-Moody algebras admitting a Weyl vector. This adaptation was executed in detail in the proof of Theorem 3.6 in [Jurisich-2004]. We find that $H_i(E_g)$ is the subspace of $\bigwedge^i(E_g)$ spanned by homogeneous vectors of degree r satisfying $(r+\rho)^2 = \rho^2$, or equivalently, $(r, r+2\rho) = 0$. ρ is the vector $(-1, 0)$, so the only degrees r for which the condition holds have the form $(m, 0)$ or $(1, n)$ for $m \geq 0$, $n \in \frac{1}{N}\mathbb{Z}$. Since E_g is supported in degrees $(a, b/N)$ for a a positive integer and $b \in \mathbb{Z}$, the only nonzero vectors of $\bigwedge(E_g)$ in degree $(1, n)$ are the simple roots in $E_g = \bigwedge^1(E_g)$. Since $E_{1,-1/N}$ is one dimensional, the only nonzero vectors of $\bigwedge(E_g)$ in degree $(m+1, 0)$ lie in $V^{m,1/N}_{1+m/N} \wedge V^{1,-1/N}_{1-1/N} \subset \bigwedge^2(E_g)$. Therefore the homology has the necessary form for Fricke-compatibility.

Second: Suppose g is non-Fricke. From Theorem 4.2, there are no real simple roots, and the positive roots are all imaginary with multiplicity $c_{n,m}(-mn)$ in degree (m, n) for all $m \in \mathbb{N}$, $n \in \frac{1}{N}\mathbb{N}$. E_g therefore satisfies the grading restriction condition.

By the same argument as in the first case, G acts by Lie algebra automorphisms, and there are G-module isomorphisms $E_{i,j} \cong V^{i,j}_{1+ij}$.

As in the Fricke case, $H_i(E_g)$ is the subspace of $\bigwedge^i(E_g)$ spanned by homogeneous vectors of degree r satisfying $(r, r+2\rho) = 0$, and the only degrees r for which this condition holds have the form $(m, 0)$ and $(1, n)$ for $m > 0$, $n \in \frac{1}{N}\mathbb{N}$. Since E_g is supported in degrees $(a, b/N)$ for a a positive integer and $b \in \mathbb{N}$, the only nonzero vectors of $\bigwedge(E_g)$ in degree $(1, n)$ are the degree $(1, n)$ vectors in $E_g = \bigwedge^1(E_g)$, and the only nonzero vectors in degree $(m, 0)$ are elements in the exterior algebra of $\bigoplus_{m=1}^{\infty} V_1^{m,0} p^m$. This has the form necessary to satisfy the non-Fricke condition. □

Corollary 4.5. *Under the hypotheses of Proposition 4.4,* $T_g(-1/\tau) = \sum_{n \in \frac{1}{N}\mathbb{Z}} \dim V^{1,n}_{1+n} q^n$.

Proof. By Proposition 4.4, the part of the homology of E_g concentrated in degree $(1, n)$ is $V^{1,n}_{1+n} \subset \bigwedge^1(E_g)$, in H_1. By Theorem 4.2, the denominator identity for W_g is given by $T_g(\sigma) - T_g(-1/\tau) = p^{-1} \prod_{m>0, n \in \frac{1}{N}\mathbb{Z}} (1 - p^m q^n)^{c_{m,Nn}(mn)}$. The pure-in-$q$ part of the left side is $T_g(-1/\tau)$, while the pure-in-q part of the right side is given by $\sum_{n \in \frac{1}{N}\mathbb{Z}} c_{1,Nn}(n) q^n$, which is equal to $\sum_{n \in \frac{1}{N}\mathbb{Z}} \dim E_{1,n} q^n$. We find that:

$$\begin{aligned} T_g(-1/\tau) &= \sum_{n \in \frac{1}{N}\mathbb{Z}} c_{1,Nn}(n) q^n \\ &= \sum_{n \in \frac{1}{N}\mathbb{Z}} \dim E_{1,n} q^n \\ &= \sum_{n \in \frac{1}{N}\mathbb{Z}} \dim V^{1,n}_{1+n} q^n \end{aligned}$$

□

4.3. **Twisted denominator formulas.** In this section, G is a finite group with an element g of order N in its center, and $\mathcal{V}$ is a collection of G-modules as described in the previous section.

Following [Carnahan-2010], we can define "formal orbifold partition functions" for any $h \in G$:

$$Z(g^k, g^l h^m, \tau) := \sum_{n \in \frac{1}{N}\mathbb{Z}} \sum_{\substack{r \in \frac{1}{N}\mathbb{Z}/\mathbb{Z} \\ n \in kr + \mathbb{Z}}} \mathrm{Tr}(g^l h^m | V^{k,r}_{1+n}) e(n\tau)$$

These are a priori formal power series in $q^{1/N}$, but they converge on $\mathfrak{H}$, by the subexponential growth condition in the definition of $\mathcal{V}$. We refer to the collection of these functions as Z.

Lemma 4.6. *Suppose $G = \langle g \rangle$ is a cyclic group of order N equipped with a homomorphism $\pi : G \to \mathbb{M}$, such that $T_g(\tau)$ has level N. Let $\mathcal{V}$ be the compatible family given in Proposition 4.4. Then for all $i \in \mathbb{Z}$, the orbifold partition function $Z(g, g^i, \tau)$ is equal to $T_g(\frac{-1}{\tau+i})$.*

Proof. By Corollary 4.5,

$$\begin{aligned} Z(g,1,\tau) &= \sum_{n \in \frac{1}{N}\mathbb{Z}} \mathrm{Tr}(1|V^{1,n}_{1+n}) e(n\tau) \\ &= T_g(-1/\tau) \end{aligned}$$

If we substitute $\tau + i$ for τ, and note that g^i acts on $V^{1,n}_{1+n}$ by multiplication by $e(in)$, we get:

$$\begin{aligned} T_g(\frac{-1}{\tau+i}) &= \sum_{n \in \frac{1}{N}\mathbb{Z}} \mathrm{Tr}(1|V^{1,n}_{1+n}) e(n\tau + in) \\ &= \sum_{n \in \frac{1}{N}\mathbb{Z}} \mathrm{Tr}(g^i|V^{1,n}_{1+n}) e(n\tau) \\ &= Z(g, g^i, \tau) \end{aligned}$$

□

Recall from [Carnahan-2010] Proposition 6.1, that for any Lie algebra E Fricke-compatible with the data $(G, g, \mathcal{V})$, and any $h \in G$, we had:

$$\begin{aligned} & p^{-1} + \sum_{m>0} \mathrm{Tr}(h|V^{1,-1/N}_{1-1/N}) \mathrm{Tr}(h|V^{m,1/N}_{1+m/N}) p^m - \sum_{n \in \frac{1}{N}\mathbb{Z}} \mathrm{Tr}(h|V^{1,n}_{n+1}) q^n \\ &= p^{-1} \exp\left(- \sum_{i>0} \sum_{m>0, n \in \frac{1}{N}\mathbb{Z}} \mathrm{Tr}(h^i|V^{m,n}_{1+mn}) p^{im} q^{in}/i \right) \end{aligned}$$

Proposition 4.7. *Suppose G and g are as described in the previous section, and suppose E is non-Fricke compatible with $(G, g, \mathcal{V})$. Then for any $h \in G$,*

$$\begin{aligned} & p^{-1} \exp\left(- \sum_{i>0} \sum_{m>0} \mathrm{Tr}(h^i|V^{m,0}_1) p^{im}/i \right) - \sum_{n \in \frac{1}{N}\mathbb{N}} \mathrm{Tr}(h|V^{1,n}_{1+n}) q^n = \\ &= p^{-1} \exp\left(- \sum_{i>0} \sum_{m>0, n \in \frac{1}{N}\mathbb{N}} \mathrm{Tr}(h^i|V^{m,n}_{1+mn}) p^{im} q^{in}/i \right). \end{aligned}$$

Proof. It suffices to show that the left and right sides of the equation describe $\mathrm{Tr}(h|H_\bullet(E))$ and $\mathrm{Tr}(h|\bigwedge^\bullet(E))$, respectively. The proof is essentially identical to [Carnahan-2010], Proposition 6.1. The exponential portions are a formal consequence of the definition of Adams operations. □

Recall from [Carnahan-2010] that we define equivariant Hecke operators by

$$T_n Z(g,h,\tau) = \frac{1}{n} \sum_{ad=n, 0 \le b < d} Z(g^d, g^{-b}h^a, \frac{a\tau + b}{d}).$$

If we apply the substitution $g \mapsto g^i, h \mapsto g^j$, we get the formula in the beginning of Section 3.2. A function Z on $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, G)/G \underset{\pm\mathbb{Z}}{\times} \mathfrak{H}$ is weakly Hecke-monic for (g, h) if $T_n Z(g, h, \tau)$ is a monic polynomial of degree n in $Z(g, h, \tau)$ for all $n > 0$. When g is Fricke, the combination of Proposition 4.4 in this paper and [Carnahan-2010], Proposition 6.2 directly imply $Z(g, h, \tau)$ is weakly Hecke-monic.

Proposition 4.8. *Let $\phi : \langle g\rangle \to \mathbb{M}$ be a homomorphism, such that $T_g(\tau)$ is invariant under $\Gamma_0(|g|)$ and $\phi(g)$ is a non-Fricke element of $\mathbb{M}$. Let G be a group with g in its center, and let $\mathcal{V}$ and E be defined as $\langle g\rangle$-modules as in Proposition 4.4. Suppose that the actions of $\langle g\rangle$ on $\mathcal{V}$ and E extend to actions of G, such that they are non-Fricke compatible. Then for any $h \in G$, Z is weakly Hecke-monic for (g, h).*

Proof. This is very similar to Proposition 6.2 in [Carnahan-2010]. We multiply both sides of the twisted denominator formula and take logarithms:

$$\begin{aligned}
&\log\left(\exp\left(-\sum_{i>0}\sum_{m>0}\mathrm{Tr}(h^i|V_1^{m,0})p^{im}/i\right) - p\sum_{n\in\frac{1}{N}\mathbb{N}}\mathrm{Tr}(h|V_{1+n}^{1,n})q^n\right)\\
&= -\sum_{i>0}\sum_{m>0,n\in\frac{1}{N}\mathbb{N}}\mathrm{Tr}(h^i|V_{1+mn}^{m,n})p^{im}q^{in}/i\\
&= -\sum_{m>0,n\in\frac{1}{N}\mathbb{N}}\sum_{0<a|(m,Nn)}\frac{1}{a}\mathrm{Tr}(h^a|V_{1+mn/a^2}^{m/a,n/a})p^m q^n\\
&= -\sum_{m>0}\sum_{ad=m}\frac{1}{a}\sum_{n\in\frac{1}{N}\mathbb{N}}\mathrm{Tr}(h^a|V_{1+dn}^{d,n})p^m q^{an}\\
&= -\sum_{m>0}\sum_{ad=m}\frac{1}{a}\sum_{0\le b<d}\frac{1}{d}\sum_{n\in\frac{1}{N}\mathbb{N}}\sum_{\substack{r\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}\\ n\in dr+\mathbb{Z}}}e(-br)\mathrm{Tr}(h^a|V_{1+n}^{d,r})e(br)q^{an/d}p^m\\
&= -\sum_{m>0}\frac{1}{m}\sum_{\substack{ad=m\\ 0\le b<d}}\sum_{n\in\frac{1}{N}\mathbb{N}}\sum_{\substack{r\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}\\ n\in dr+\mathbb{Z}}}\mathrm{Tr}(g^{-b}h^a|V_{1+n}^{d,r})e(n\frac{a\tau+b}{d})p^m\\
&= -\sum_{m>0}\frac{1}{m}\sum_{\substack{ad=m\\ 0\le b<d}}Z(g^d, g^{-b}h^a, \frac{a\tau+b}{d})p^m\\
&= -\sum_{m>0}T_m Z(g,h,\tau)p^m
\end{aligned}$$

Isolating the terms that are degree k in p on the first line yields a polynomial of degree k in Z, with leading coefficient $-1/k$. □

For the Fricke-compatible case, we used the analogous result to give a strong genus zero statement, in Proposition 6.3 of [Carnahan-2010]. This was possible because the function $Z(g, h, \tau)$ had a pole at infinity. Here, the question is more challenging, because the functions arising from non-Fricke compatible algebras are regular at infinity.

4.4. **Twisted modules - hypotheses.** We will use a little theory from vertex operator algebras and their twisted modules, but not in much depth. As we remarked earlier, twisted modules were introduced in [Frenkel-Lepowsky-Meurman-1988], and were essential in the construction of the monster vertex algebra $V^\natural$. While twisted modules have much subtle structure, we will only work with them explicitly in their capacity as vector spaces with a semisimple action of an operator L_0, whose eigenvalues produce a grading. We will describe actions of groups that respect some essential additional structure, namely an internal action of the Virasoro algebra (in which L_0 is a distinguished element acting semisimply), and such actions will be called conformal. Recall from before that the main player here is the monster vertex algebra $V^\natural$ endowed with a conformal

action of the monster simple group $\mathbb{M}$ with graded dimension $J(\tau)$ (as constructed and proved in [Frenkel-Lepowsky-Meurman-1988]), and whose graded characters $T_g(\tau) := Tr(gq^{L_0-1}|V)$ are the Hauptmoduln listed by Conway and Norton [Conway-Norton-1979] (as proved by Borcherds [Borcherds-1992]).

Let $G = \langle g \rangle \cong \mathbb{Z}/N\mathbb{Z}$ be a cyclic group with a conformal action on $V^\natural$, such that $f_{(1)}(\tau) := T_g(\tau) = Tr(gq^{L_0}|V^\natural)$ is invariant under $\Gamma_0(N)$. For each $m|N$, we let $f_{(m)}(\tau) := T_{g^m}(\tau)$, which is invariant under $\Gamma_0(N/m)$.

From [Dong-Li-Mason-2000], we have the following results:

(1) (Proposition 12.4 and Theorem 10.3) For each integer i, there is a unique isomorphism class of irreducible g^i-twisted module of $V^\natural$. We write it $V^\natural(g^i)$.
(2) (Theorems 8.1 and 5.4) Each twisted module $V^\natural(g^i)$ admits a conformal action of G compatible with the action of G on $V^\natural$ (i.e., producing a G-equivariant module structure), such that g^i acts as $e(L_0)$.
(3) $\mathrm{Tr}(g^j q^{L_0-1}|V^\natural(g^i)) = C(i,j)\hat{F}_{i,j}(\tau)$ for nonzero constants $C(i,j)$.

Remark: We wish to elaborate on the second result in the above list. For each $g \in \mathbb{M}$, the monster acts weakly on the category of g'-twisted modules, where g' ranges over conjugates of g, by fixing the underlying vector space and conjugating the action of $V^\natural$. This action is described in [Dong-Li-Mason-2000], and works in the following way: the action of $V^\natural$ on a twisted module $V^\natural(g)$ is a map $m : V^\natural \otimes V^\natural(g) \to V^\natural(g)((z^{1/|g|}))$ satisfying some additional relations like associativity and a monodromy condition. For any element $h \in \mathbb{M}$, we form an hgh^{-1} twisted module structure on $V^\natural(g)$ by $m^h(v \otimes a) := m(h \cdot v \otimes a)$. By the uniqueness theorem (first result above), the stabilizer of the isomorphism class of an irreducible g-twisted module $V^\natural(g)$ is the centralizer of g. By Schur's Lemma, there is therefore a canonical projective action of $C_{\mathbb{M}}(g)$, the centralizer of g in the monster, on $V^\natural(g)$, and this lifts to an action of a central extension $\mathbb{C}^\times.C_{\mathbb{M}}(g)$, where $\mathbb{C}^\times$ acts by scalars. In particular, each element of $C_{\mathbb{M}}(g)$ admits a nonunique lift to an automorphism of each $V^\natural(g^i)$, but the product of two lifts of elements differs from the lift of the product of those elements by a nonzero scalar factor. Since there are finitely many elements of $C_{\mathbb{M}}(g)$, we may choose these scalars to be finite-order roots of unity, and the action of $\mathbb{C}^\times.C_{\mathbb{M}}(g)$ descends to an action of a central extension $\widetilde{C_{\mathbb{M}}(g)}$ of finite order. If we restrict our attention to the cyclic group generated by g, we can lift to an action of a finite cyclic group that surjects to $\langle g \rangle$. The second result of [Dong-Li-Mason-2000] is in fact that any choice of lift of g^i on $V^\natural(g^i)$ is a scalar multiple of $e(L_0)$, but by virtue of the existence of the function $\hat{F}$, we may choose this lift to be given by an action of G, with the scalar multiple equal to one. The action of G on $V^\natural(g^i)$ for $(i,N) > 1$ is not uniquely defined by the $e(L_0)$ condition, but can be specified canonically by adding fusion information (to be treated later in this series).

We introduce some hypotheses on the structure of twisted modules.
Hypothesis A_g: $Tr(q^{L_0-1}|V^\natural(g^i)) = T_{g^i}(-1/\tau)$ for all i.
Hypothesis B_g: $C(i,j) = 1$ for all i,j.

Hypothesis B_g is a refined form of the generalized moonshine conjecture restricted to the cyclic group $\langle g \rangle$. Naturally, Hypothesis A_g is the $j = 0$ case of Hypothesis B_g.

Lemma 4.9. *If N is prime, then Hypothesis A_g implies Hypothesis B_g.*

Proof. From result number 2 listed above, the action of G on $V(g^i)$ is given by g^i acting by multiplication by $e(L_0)$. If $i \equiv 0 \pmod N$, this follows from the moonshine conjecture: $V(g^i) = V^\natural$, and $C(i,j) = \frac{\mathrm{Tr}(g^j q^{L_0-1}|V^\natural)}{\hat{F}_{0,j}(\tau)} = 1$. Otherwise, Hypothesis A_g implies $Tr(q^{L_0-1}|V^\natural(g^i)) = \hat{F}_{i,0}$. Since N is prime, there is a unique k such that $ik \equiv j \bmod N$. We write $V^\natural(g^i)_n$ for the subspace of

$V^\natural(g^i)$ on which L_0 acts by multiplication by n.

$$\begin{aligned}\hat{F}_{i,j}(\tau) &= \hat{F}_{i,0}(\tau+k)\\ &= \sum_{n\in\frac{1}{N}\mathbb{Z}} \dim V^\natural(g^i)_{n+1} e(n(\tau+k))\\ &= \sum_{n\in\frac{1}{N}\mathbb{Z}} e((n+1)k)\mathrm{Tr}(1|V^\natural(g^i)_{n+1})e(n\tau)\\ &= \sum_{n\in\frac{1}{N}\mathbb{Z}} \mathrm{Tr}(g^{ik}|V^\natural(g^i)_{n+1}q^n\\ &= \mathrm{Tr}(g^j q^{L_0-1}|V^\natural(g^i))\end{aligned}$$

□

When N is composite, we need a more detailed analysis of the action of g on $V^\natural(g^i)$ for $(i,N)>1$ to remove root-of-unity ambiguities.

Lemma 4.10. *Let $\mathcal{V}$ be the set of G-modules given by setting $V_k^{i,j/N}$ to be the subspace of $V^\natural(g^i)$ on which g acts by $e(j/N)$ and L_0 acts by k. If Hypothesis B_g holds, then E_g (as defined in Proposition 4.4) is Fricke compatible with $(G,g,\mathcal{V})$ if g is Fricke, and non-Fricke compatible with $(G,g,\mathcal{V})$ if g is non-Fricke.*

Proof. By hypothesis B_g, for any nonnegative integers i,j, $\hat{F}_{i,j}$ is the trace of $g^j q^{L_0-1}$ on $V^\natural(g^i)$. Taking the Fourier transform, we find that $c_{i,j}(ij/N)$ is equal to the dimension of both the $(i,j/N)$-root space of the Lie algebra W_g, and $V_{1+ij/N}^{i,j/N}$. The compatibility then follows from Proposition 4.4. □

Hypothesis C_g: Let $\mathcal{V}$ be the set of $\widetilde{C_{\mathbb{M}}(g)}$-modules given by setting $V_k^{i,j/N}$ to be the subspace of $V^\natural(g^i)$ on which g acts by $e(j/N)$ and L_0 acts by k. There is an action of $\widetilde{C_{\mathbb{M}}(g)}$ on W_g by homogeneous Lie algebra automorphisms, such that the Lie subalgebra E_g of positive roots is Fricke-compatible with $(\widetilde{C_{\mathbb{M}}(g)},g,\mathcal{V})$ if g is Fricke, and non-Fricke compatible with $(\widetilde{C_{\mathbb{M}}(g)},g,\mathcal{V})$ if g is non-Fricke.

Remark: Because $G\subset\widetilde{C_{\mathbb{M}}(g)}$, Hypothesis C_g clearly implies Hypothesis B_g (and hence A_g). I believe I have a proof that for all $g\in\mathbb{M}$, Hypotheses A_g is true, and Hypothesis B_g implies Hypothesis C_g. If everything works out, the proof should appear later in this series.

Remark: Borcherds has pointed out that there is an alternative way to endow these Lie algebras with group actions, namely by choosing an explicit twisted denominator identity for each conjugacy class in $\widetilde{C_{\mathbb{M}}(g)}$, in a way that is compatible with the series arising as characters of $\widetilde{C_{\mathbb{M}}(g)}$ on the simple roots. One needs only make this work for the first several simple roots, and then the compatibility follows from the Hecke-monic recursion. Assuming one knows the character tables of $\widetilde{C_{\mathbb{M}}(g)}$ for all $g\in\mathbb{M}$ and all genus zero functions that are expected to appear, this can be done by computational brute force. This strategy would yield the generalized moonshine conjecture without needing to appeal to twisted modules. Unfortunately, no one seems to know either of the necessary pieces of information at the time of writing.

4.5. **Conditional generalized moonshine.** In this section, we choose a homomorphism $\pi:\mathbb{Z}/N\mathbb{Z}\to\mathbb{M}$ sending a generator g to an element of $\mathbb{M}$, such that π lifts to a homomorphism to a central extension $\widetilde{C_{\mathbb{M}}(\pi(g))}$ of the centralizer of the image, that acts on all irreducible g^i-twisted modules of $V^\natural$. In particular, this implies $T_g(\tau)$ is invariant under $\Gamma_0(N)$.

Lemma 4.11. *Let $G = \widetilde{C_{\mathbb{M}}(g)}$, and assume hypothesis C_g is true. Then E_g is Fricke-compatible with $(G, g, \mathcal{V})$ if g is Fricke, and non-Fricke compatible with $(G, g, \mathcal{V})$ if g is non-Fricke. Furthermore, for any $h \in G$, Z is weakly Hecke-monic for (g, h).*

Proof. The compatibility follows from Proposition 4.4. The weakly Hecke-monic property follows from Proposition 4.8. □

By Proposition 6.3 in [Carnahan-2010], this implies $Z(g, h, \tau)$ is a congruence genus zero function if g is Fricke. This is not sufficient to prove the generalized moonshine conjecture with our current state of knowledge, since we do not have such control for non-Fricke g, and we do not have the $SL_2(\mathbb{Z})$-compatibility result. Dong, Li, and Mason give a weak form of compatibility in the following sense: Let V be a C_2 cofinite vertex operator algebra, let (g, h) be a commuting pair of automorphisms with $|g| = T, |h| = T_1$, and let $M(T, T_1)$ be the commutative ring of holomorphic modular forms (non-negative weight, regular at cusps) for the group $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) | b \in T\mathbb{Z}, c \in T_1\mathbb{Z}\}$. There is then a space $C_1(g, h)$ of functions $S : (V \otimes M(T, T_1)) \times \mathfrak{H}$ that contains traces of twisted modules and admits "slash" operations, such that for any $S \in C_1(g, h)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $S|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} \in C_1(g^a h^c, g^b h^d)$. If V is holomorphic and g-rational, then Theorem 10.1 of [Dong-Li-Mason-2000] implies $C_1(g, h)$ is spanned by these traces on twisted modules, and this yields a compatibility up to a constant, as predicted in the generalized moonshine conjecture. Unfortunately, g-rationality seems to be quite difficult to prove in general.

We make the following conjecture for any C_2-cofinite vertex operator algebra V with no negative weights, with g and h commuting finite order conformal automorphisms.

Hypothesis D: The space $C_1(g, h)$ (defined in section 5 of [Dong-Li-Mason-2000]) is spanned by a finite set of pseudo-traces (defined in section 4.1 of [Miyamoto-2004]) of lifts of h acting on g-twisted Verma modules. Furthermore, the subspace of functions with no logarithmic terms (i.e., powers of τ) is spanned by traces of lifts of h acting on irreducible g-twisted modules.

This hypothesis seems to be reasonable, in the sense that it is the "least common generalization" of Theorem 10.1 in [Dong-Li-Mason-2000] and theorem 5.5 in [Miyamoto-2004]. We hope to treat this in later work - most of the proof in [Miyamoto-2004] transfers to the twisted setting, but there are significant technical hurdles. In this paper, we are interested in the case $V = V^\natural$, where Theorem 10.3 in [Dong-Li-Mason-2000] implies the space of traces is one-dimensional.

Theorem 4.12. *Suppose hypothesis C_g holds for all $g \in \mathbb{M}$ and hypothesis D holds for $V^\natural$ and all commuting pairs (g, h). Then the generalized moonshine conjecture is true (where $SL_2(\mathbb{Z})$-compatibility is only given up to a nonzero constant).*

Proof. By Theorem 8.1 of [Dong-Li-Mason-2000], $Z(g, h, \tau)$ is a specialization of an element of $C_1(g, h)$, so for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $Z(g, h, \frac{a\tau+b}{c\tau+d})$ is a specialization of an element in $C_1(g^a h^c, g^b h^d)$. From the finiteness assertion in Hypothesis D, there is an upper bound $M \geq 0$ on the degree of τ in expansions of $Z(g, h, \tau)$ at any cusp.

By Theorem 6.5 of [Dong-Li-Mason-2000], the expansion of $Z(g, h, \frac{a\tau+b}{c\tau+d})$ has the form

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{\infty} S_{i,j,k} q^{k+r_j} (2\pi i \tau)^i.$$

We wish to show that $S_{i,j,k} = 0$ for $i > 0$. Suppose there are j, k and $i > 0$ such that $S_{i,j,k} \neq 0$, let i_0 be the largest value of i such that this holds. Let j_0 and k_0 be such that S_{i_0,j_0,k_0} is nonzero, and $k_0 + r_{j_0}$ is minimal with respect to this property. By Lemma 4.11,

$$pT_p Z(g, h, \frac{a\tau + b}{c\tau + d}) = Z(g, h^p, \frac{pa\tau + pb}{c\tau + d}) + \sum_{i=0}^{p-1} Z(g^p, g^{-i}h, \frac{a\tau + b}{pc\tau + pd} + i)$$

is a monic polynomial of degree p in $Z(g,h,\frac{a\tau+b}{c\tau+d})$, and therefore has a nontrivial power of τ of order pi_0. Each of the summands on the right side of the equation is a specialization of an element of some $C_1(g',h')$, possibly after an affine coordinate change. Affine coordinate changes fix the order of τ in the expansion, so the sum on the right side therefore has τ of order bounded above by M. Since $pi_0 > M$ for sufficiently large p, this is a contradiction.

The expansion of $Z(g,h,\frac{a\tau+b}{c\tau+d})$ therefore has no logarithmic terms, so by Hypothesis D, it is a nonzero constant multiple $Z(g^a h^c, g^b h^d, \tau)$. This proves the $SL_2(\mathbb{Z})$-compatibility up to a constant. If there exist coprime integers a, c such that $g^a h^c$ is Fricke, then Lemma 4.11 combined with Proposition 6.3 of [Carnahan-2010] implies $Z(g^a h^c, g^b h^d, \tau)$ is a genus zero function of finite level for any integers b, d satisfying $ad - bc = 1$. In this case, the $SL_2(\mathbb{Z})$-compatibility implies $Z(g,h,\tau)$ is also a genus zero function of finite level. If the integers a and c don't exist, then $Z(g,h,\tau)$ is regular at all cusps, and is therefore constant.

It remains to show that $Z(g,h,\tau) = J(\tau)$ if and only if $g = h = 1$. The leftward implication is clear - J is the character of $V^\natural$. The rightward implication is a consequence of the fact that the irreducible g-twisted modules of $V^\natural$ have strictly positive degree for $g \neq 1$, combined with the fact that nontrivial McKay-Thompson series have level greater than one. □

Remark: We don't need the full force of Hypothesis D, since the only part of the pseudo-trace statement that we use is the assertion that the powers of τ in the expansion of elements of $C_1(g,h)$ are bounded above.

Remark: This proof produces a slightly stronger statement of generalized moonshine: the constant ambiguity in the $SL_2(\mathbb{Z})$-compatibility can be made a root of unity for the non-constant functions, because weakly Hecke-monic functions have poles with unit norm residue. Unfortunately, we do not have enough information to restrict the ambiguity to a 24th root of unity.

5. Open problems

(1) Given a completely replicable modular function f, there is a canonical minimal cyclic group G for which f is the restriction of a principally normalized Hecke-monic function on $\mathcal{M}^G_{Ell}$. We could call this function the "Hecke-monic envelope" of f, since is it unique with respect to these conditions. Calculations of Queen and Norton have produced functions that show up as the characters $Z(g,h,\tau)$ in generalized moonshine, and many are $PSL_2^+(\mathbb{Q})$-transformations of functions that are replicable but not completely replicable. These are attached to Hecke-monic functions on $\mathcal{M}^G_{Ell}$ for G an abelian group that is not necessarily cyclic and not necessarily uniquely determined. Is there a characterization of the abelian groups that can be attached to either the replicable modular functions, or the functions that show up as $Z(g,h,\tau)$?

(2) The McKay-Thompson series satisfy the property that their expansions at zero have non-negative coefficients, and this is essential for constructing Lie algebras, since simple roots have non-negative multiplicity. Non-negativity also holds for many non-monstrous completely replicable functions, such as 9a and 49a (where functions are named following the conventions in [Alexander-Cummins-McKay-Simons-1992]), but others have negative coefficients and therefore cannot be used to generate Lie algebras. One obvious problem is: classify products with non-negative integer exponents. While the existence of Lie algebras does not distinguish monstrous from non-monstrous functions, there is a slightly stronger condition conjecturally satisfied by McKay-Thompson series (implied by hypothesis B_g), namely that the vector-valued form F have non-negative integer coefficients. We have not checked this condition for non-monstrous functions, but it is a weak form of demanding existence of twisted modules, i.e., assuming a uniqueness conjecture in [Frenkel-Lepowsky-Meurman-1988], McKay-Thompson series are distinguished by the property that they are characters of a conformal automorphism of a holomorphic vertex operator

algebra with central charge 24, whose identity character is given by J. Some non-monstrous cases of non-negativity may come from automorphisms of other vertex operator algebras, such as those arising from fixed-point-free automorphisms of Niemeier lattices.

(3) This is not a particularly new question, but it would be nice to have a functorial characterization of congruence genus zero quotients of $\mathfrak{H}$ via moduli of structured elliptic curves. For example, elements in the conjugacy class pA in the monster produce McKay-Thompson series that are invariant under $\Gamma_0^+(p)$, and the resulting modular curve $X_0^+(p) = \mathfrak{H}/\Gamma_0^+(p)$ parametrizes unordered pairs of elliptic curves, with dual degree p isogenies between them. Ogg pointed out in 1975 that the genus zero property of $X_0^+(p)$ characterizes those primes dividing $|\mathbb{M}|$. Substantial progress has been made in [Duncan-Frenkel-2009] using moduli of solid tori to characterize McKay-Thompson series, and giving a characterization of genus zero quotients in terms of Rademacher sums.

(4) When the exponents in a product formula are not in $\mathbb{N}$, we can no longer describe the identity as the denominator formula of a Lie algebras in vector spaces. However, if we have a suitable homomorphism from the Grothendieck semiring of a braided tensor category $\mathcal{C}$ to the complex numbers, we can sometimes employ the denominator identity to describe a Lie algebra object in $\mathcal{C}$ by checking that a pulled-back identity still holds. For example, the monstrous superalgebras constructed in [Borcherds-1992] are exactly of this form, when the category is that of super vector spaces. Are there interesting algebraic objects attached to more general products (such as those containing cyclotomic integers, giving rise to cyclic anyons), and do they arise in physical contexts?

(5) The correspondence between vector-valued modular functions for lattices and infinite products for $O(2,2)$ is far from a bijection, and its precise nature is far from clear. For example, when a McKay-Thompson series $T_g(\tau)$ has a regular cusp, then there are infinite product expansions of $T_g(\sigma) - T_g(\tau)$ involving such cusps that do not arise from lifting the vector-valued functions we introduced. In fact, one can use the exponents in a product to reverse-engineer Hecke-monic functions that are not principally normalized. Are there structures in conformal field theory that produce these residual products? We don't seem to have a enough a priori information in the Lie superalgebras to reconstruct all of the operator product expansions.

(6) Borcherds products like the ones in this paper appear in the string theory literature in many places, but with several different physical interpretations that are not obviously equivalent. One often sees connections between these algebras and BPS states, but there doesn't seem to be a BPS theory explicitly attached to $V^\natural$. Products in signature $(2,2)$ are also mentioned in [Dijkgraaf-1999], with an explanation that string theory partition functions associated to a torus depend not only on conformal class but on worldsheet volume and a two-form field. This expands the T-duality group from $SL_2(\mathbb{Z})$ to $SO_{2,2}(\mathbb{Z})$, which is commensurable with the symmetry groups of the automorphic functions we constructed. This is somewhat related to the previous question, in the sense that one might expect the space of physical states of the $V^\natural$ conformal field theory to be related to a residual Lie superalgebra by a duality. There is yet another interpretation of these Lie algebras as second quantizations in string field theory, arising from multiple bosonic string states (see, e.g., [Dijkgraaf-Moore-Verlinde-Verlinde-1997] for generalities and [Duncan-Frenkel-2009] for discussion of our specific case). The product formulas arising in permutation orbifold theory (e.g., in [Tuite-2008]) are very similar to those in this paper, but with minus signs on the coefficients, i.e., the products are reciprocals of what we see here. The reciprocal characters correspond to replacing $\bigwedge E$ in the Chevalley resolution with the symmetric algebra, suggesting a role for Koszul duality, and both the exterior and symmetric algebras on E (with a suitable shift) have canonical cocommutative DG coalgebra structures, but it's not clear if they tell us anything useful. Are these pictures related to each other?

(7) We can think of the Lie algebras in this paper as categorifications of their (twisted) denominator formulas. Lurie has asked what it means to categorify again, possibly with a view toward 2-equivariant elliptic cohomology and Hopkins-Kuhn-Ravenel characters. It's not clear what sort of algebraic gadget we should get, but it ought to involve as an intermediate step the whole collection of Lie algebras at once as a manifestation of the equivariant conformal field theory. Kremnizer has suggested some quiver-theoretic constructions, but I have been unable to find natural actions of finite groups on the resulting structures.

References

[Alexander-Cummins-McKay-Simons-1992] D. Alexander, C.Cummins, J. McKay, C. Simons, Completely Replicable Functions. *Groups, Combinatorics, and Geometry, Durham Symposium, 1990* London Math. Soc. Lecture Note Ser. **165**, Cambridge Univ. Press, Cambridge, 1992, 87–98.

[Baez-Lauda-2004] J. C. Baez, A. D. Lauda, Higher-Dimensional Algebra V: 2-Groups, *Theory and Applications of Categories* **12** (2004), 423–491.

[Borcherds-1988] R. Borcherds, Generalised Kac-Moody algebras, *J. Algebra* **115** no. 2 (1988), 501–512.

[Borcherds-1992] R. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992) 405–444.

[Borcherds-1995a] R. Borcherds, A characterization of generalized Kac-Moody algebras, *J. Algebra* **174** (1995) 1073–1079.

[Borcherds-1995b] R. Borcherds, Automorphic forms on Os+2,2(R) and infinite products, *Invent. Math.* **120** (1995) 161–213.

[Borcherds-Ryba-1995] R. Borcherds, A. J. Ryba, Modular Moonshine II *Duke Math. J.* **83** (1996), no. 2, 435-459

[Borcherds-1998a] R. Borcherds, Automorphic forms with singularities on Grassmannians, *Invent. Math.* **132** (1998), no. 3, 491–562.

[Borcherds-1998b] R. Borcherds, Modular Moonshine III *Duke Math. J.* **93** (1998), no. 1, 129–154.

[Bruinier-2002] J. Bruinier, *Borcherds Products on $O(2,l)$ and Chern Classes of Heegner Divisors*, Lecture Notes in Mathematics **1780** Springer-Verlag, Boston (2002).

[Carnahan-2010] S. Carnahan, Generalized Moonshine I: Genus zero functions, *Algebra and Number Theory* **4**:6 (2010) 649–679.

[Carnahan-20??] S. Carnahan, Generalized Moonshine III: Equivariant conformal blocks, *In preparation.*

[Cohen-Griess-1987] A. Cohen, R. Griess, On finite simple subgroups of the complex Lie group of type E_8, *Proc. Symp. Pure Math.* **47** (1987) 367–405.

[Conrad-20??] B. Conrad, *Modular forms, Cohomology, and the Ramanujan Conjecture* preprint

[Conway-Norton-1979] J. Conway, S. Norton, Monstrous Moonshine, *Bull. Lond. Math. Soc.* **11** (1979) 308–339.

[ATLAS] J. Conway, R. Wilson, R. Curtis, S. Norton, R. Parker, *Atlas of finite groups* Clarendon Press, Oxford (1985).

[Cummins-Gannon-1997] C. Cummins, T. Gannon, Modular equations and the genus zero property of modular functions, *Invent. Math.* **129** (1997), 413–443.

[Deligne-1969] P. Deligne, *Formes modulaires et representations ℓ-adiques* Seminaire Bourbaki *355* (1969) 139–172.

[Dijkgraaf-1999] R. Dijkgraaf, Fields, Strings, Matrices, and Symmetric Products, *ArXiv preprint:* hep-th/9912104.

[Dijkgraaf-Moore-Verlinde-Verlinde-1997] Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, Elliptic Genera of Symmetric Products and Second Quantized Strings, *Commun. Math. Phys.* **185** (1997) 197–209.

[Dijkgraaf-Pasquier-Roche-1990] R.Dijkgraaf, V. Pasquier, P. Roche, Quasi Hopf algebras, group cohomology and orbifold models, *Nucl. Phys. B* (Proc. Suppl.)**18**B (1990) 60–72.

[Dixon-Ginsparg-Harvey-1988] L. Dixon, P. Ginsparg, J. Harvey, Beauty and the Beast: Superconformal Symmetry in a Monster Module *Commun. Math. Phys.* **119** (1988) 221–241.

[Dong-Lepowsky-1993] C. Dong, J. Lepowsky, *Generalized vertex algebras and relative vertex operators*, Progress in Mathematics **112** Birkhuser Boston, Inc., Boston, MA, (1993).

[Dong-Li-Mason-1996] C. Dong, H. Li, G. Mason, Some twisted modules for the moonshine vertex operator algebra *Contemporary Math.* **193** (1996) 25–43.

[Dong-Li-Mason-2000] C. Dong, H. Li, G. Mason, Modular invariance of trace functions in orbifold theory, *Comm. Math. Phys.* **214** (2000) no. 1, 1–56.

[Duncan-Frenkel-2009] J. Duncan, I. Frenkel, Rademacher Sums, Moonshine, and Gravity, *ArXiv preprint:* 0907.4529.

[Ferenbaugh-1993] C. Ferenbaugh, The genus zero problem for $n|h$-type groups, *Duke Math. J.* **72** (1993) no. 1, 31–63.

[Frenkel-Szczesny-2004] E. Frenkel, M. Szczesny, Twisted modules over vertex algebras on algebraic curves, *Adv. Math.* **187** (2004), no. 1, 195–227.

[Frenkel-Lepowsky-Meurman-1988] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics **134** Academic Press, Inc., Boston, MA (1988).

[Garland-Lepowsky-1976] H. Garland, J. Lepowsky, Lie algebra homology and the Macdonald-Kac formulas, *Invent. math.* **34** (1976), 37–76.

[Grisenko-Nikulin-1997] V. Gritsenko, V. V. Nikulin, Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras, *Amer. J. Math.* **119** (1997) no. 1, 181–224.

[Gritsenko-Nikulin-1998] V. Gritsenko, V. V. Nikulin, Automorphic forms and Lorentzian Kac-Moody algebras I, II *Internat. J. Math.* **9** (1998) no. 2, 153–199, 201–275.

[Gunning-1990] R. Gunning, *Introduction to Holomorphic Functions in Several Complex Variables, Volume I*, Wadsworth, Inc., Belmont, CA (1990).

[Harvey-Moore-1995] J. Harvey, G. Moore, Algebras, BPS states, and strings, *ArXiv preprint:* hep-th/9510182

[Hoehn-2003] G. Hoehn, Generalized Moonshine for the Baby Monster, *preprint* (2003).

[Hopkins-Kuhn-Ravenel-2000] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel, Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, **13**:3 (2000) 553–594.

[Jurisich-Lepowsky-Wilson-1995] E. Jurisich, J. Lepowsky, R. L. Wilson, Realizations of the Monster Lie algebra *Selecta Math.* **1**:1 (1995) 129–161.

[Jurisich-2004] E. Jurisich, A resolution for standard modules of Borcherds Lie algebras *J. Pure Appl. Alg.* **192** (2004) 149–158.

[Kac-1990] V. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press (1990).

[Kozlov-1994] D. Kozlov, *On completely replicable functions* Master's Thesis (1994), Lund University, Sweden.

[McGraw-2003] W. J. McGraw, The rationality of vector valued modular forms associated with the Weil representation, *Math. Ann.* **326** (2003) No. 1, 105–122.

[Miyamoto-2004] M. Miyamoto, Modular invariance of vertex operator algebras satisfying C_2 cofiniteness, *Duke Math. J.* **122** (2004) 51–91.

[Norton-1984] S. Norton, More on Moonshine, *Computational Group Theory* ed. M. D. Atkinson (1984) 185–193, Academic Press.

[Norton-1987] S. Norton, Generalized moonshine, *Proc. Sympos. Pure Math.* **47** Part 1, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986) 209–210, Amer. Math. Soc., Providence, RI (1987).

[Norton-2001] S. Norton, From Moonshine to the Monster, *Proceedings on Moonshine and related topics (Montréal, QC, 1999)*, 163–171, CRM Proc. Lecture Notes **30**, Amer. Math. Soc., Providence, RI (2001).

[Queen-1981] L. Queen, Modular Functions arising from some finite groups, *Mathematics of Computation* **37** (1981) No. 156, 547–580.

[Scheithauer-2000] N. Scheithauer, The Fake Monster Superalgebra, *Adv. Math.* **151** (2000) 226–269.

[Scheithauer-2004] N. R. Scheithauer, Generalized Kac-Moody algebras, automorphic forms and Conway's group I, *Adv. Math.* **183** (2004) no. 2, 240–270.

[Scheithauer-2008] N. R. Scheithauer, Generalized Kac-Moody algebras, automorphic forms and Conway's group II, *J. Reine Angew. Math.* **625** (2008), 125–154.

[Scheithauer-2009] N. R. Scheithauer, The Weil representation of $SL_2(\mathbb{Z})$ and some applications. *Int. Math. Res. Not. IMRN* (2009) no. 8, 1488–1545.

[Tuite-2008] M. Tuite, Monstrous and Generalized Moonshine and Permutation Orbifolds, *ArXiv preprint:* 0811.4525

[Willerton-2008] S. Willerton The twisted Drinfeld double of a finite group via gerbes and finite groupoids, *Algebraic & Geometric Topology* **8** (2008) 1419–1457.

AEI-2012-058

Mathieu Moonshine and Orbifold K3s*

Matthias R. Gaberdiel
Institut für Theoretische Physik
ETH Zurich, 8093 Zurich, Switzerland

and

Roberto Volpato
Max-Planck-Institut für Gravitationsphysik
Am Mühlenberg 1, 14476 Golm, Germany

arXiv:1206.5143v1 [hep-th] 22 Jun 2012

Abstract: The current status of 'Mathieu Moonshine', the idea that the Mathieu group $\mathbb{M}_{24}$ organises the elliptic genus of K3, is reviewed. While there is a consistent decomposition of all Fourier coefficients of the elliptic genus in terms of Mathieu $\mathbb{M}_{24}$ representations, a conceptual understanding of this phenomenon in terms of K3 sigma-models is still missing. In particular, it follows from the recent classification of the automorphism groups of arbitrary K3 sigma-models that (i) there is no single K3 sigma-model that has $\mathbb{M}_{24}$ as an automorphism group; and (ii) there exist 'exceptional' K3 sigma-models whose automorphism group is not even a subgroup of $\mathbb{M}_{24}$. Here we show that all cyclic torus orbifolds are exceptional in this sense, and that almost all of the exceptional cases are realised as cyclic torus orbifolds. We also provide an explicit construction of a $\mathbb{Z}_5$ torus orbifold that realises one exceptional class of K3 sigma-models.

1 Introduction

In 2010, Eguchi, Ooguri and Tachikawa observed that the elliptic genus of K3 shows signs of an underlying Mathieu $\mathbb{M}_{24}$ group action [1]. In particular, they noted (see section 2 below for more details) that the Fourier coefficients of the elliptic genus can be written as sums of dimensions of irreducible $\mathbb{M}_{24}$ representations.† This intriguing observation is very reminiscent of the famous realisation of McKay and Thompson who noted that the Fourier expansion coefficients of the J-function can be written in terms of dimensions of representations of the Monster group [2, 3]. This led to a development that is now usually referred to as 'Monstrous Moonshine', see [4] for a nice review. One important

*Partially based on talk given by M.R.G. at the conference 'Conformal Field Theory, Automorphic Forms and Related Topics', Heidelberg, 19-23 September 2011.

†Actually, they did not just look at the Fourier coefficients themselves, but at the decomposition of the elliptic genus with respect to the elliptic genera of irreducible $\mathcal{N}=4$ superconformal representations. They then noted that these expansion coefficients (and hence in particular the usual Fourier coefficients) are sums of dimensions of irreducible $\mathbb{M}_{24}$ representations.

upshot of that analysis was that the J-function can be thought of as the partition function of a self-dual conformal field theory, the 'Monster conformal field theory' [5, 6], whose automorphism group is precisely the Monster group. The existence of this conformal field theory explains many aspects of Monstrous Moonshine although not all — in particular, the genus zero property is rather mysterious from this point of view.

In the Mathieu case, the situation is somewhat different compared to the early days of Monstrous Moonshine. It is by construction clear that the underlying conformal field theory *is* a K3 sigma-model (describing string propagation on the target space K3). However, this does not characterise the corresponding conformal field theory uniquely as there are many inequivalent such sigma-models — in fact, there is an 80-dimensional moduli space of such theories, all of which lead to the same elliptic genus. The natural analogue of the 'Monster conformal field theory' would therefore be a special K3 sigma-model whose automorphism group coincides with $\mathbb{M}_{24}$. Unfortunately, as we shall review here (see section 3), such a sigma-model does not exist: we have classified the automorphism groups of all K3 sigma-models, and none of them contains $\mathbb{M}_{24}$ [7]. In fact, not even all automorphism groups are contained in $\mathbb{M}_{24}$: the exceptional cases are the possibilities (ii), (iii) and (iv) of the theorem in section 3 (see [7]), as well as case (i) for nontrivial G'. Case (iii) was already shown in [7] to be realised by a specific Gepner model that is believed to be equivalent to a torus orbifold by $\mathbb{Z}_3$. Here we show that also cases (ii) and (iv) are realised by actual K3s — the argument in [7] for this relied on some assumption about the regularity of K3 sigma-models — and in both cases the relevant K3s are again torus orbifolds. More specifically, case (ii) is realised by an asymmetric $\mathbb{Z}_5$ orbifold of $\mathbb{T}^4$ (see section 5),[‡] while for case (iii) the relevant orbifold is by $\mathbb{Z}_3$ (see section 6).

Cyclic torus orbifolds are rather special K3s since they always possess a quantum symmetry whose orbifold leads back to $\mathbb{T}^4$. Using this property of cyclic torus orbifolds, we show (see section 4) that the group of automorphisms of K3s that are cyclic torus orbifolds is always exceptional; in particular, the quantum symmetry itself is never an element of $\mathbb{M}_{24}$. Although some 'exceptional' automorphism groups (contained in case (i) of the classification theorem) can also arise in K3 models that are not cyclic torus orbifolds, our observation may go a certain way towards explaining why only $\mathbb{M}_{24}$ seems to appear in the elliptic genus of K3.

We should mention that Mathieu Moonshine can also be formulated in terms of a mock modular form that can be naturally associated to the elliptic genus of K3 [1, 9, 10, 11]; this point of view has recently led to an interesting class of generalisations [12]. There are also indications that, just as for Monstrous Moonshine, Mathieu Moonshine can possibly be understood in terms of an underlying Borcherds-Kac-Moody algebra [13, 14, 15, 16].

2 Mathieu Moonshine

Let us first review the basic idea of 'Mathieu Moonshine'. We consider a conformal field theory sigma-model with target space K3. This theory has $\mathcal{N} = (4, 4)$ superconformal symmetry on the world-sheet. As a consequence, the space of states can be decomposed into representations of the $\mathcal{N} = 4$ superconformal algebra, both for the left- and the

[‡]Since the orbifold action is asymmetric, this evades various no-go-theorems (see e.g. [8]) that state that the possible orbifold groups are either $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, or $\mathbb{Z}_6$.

right-movers. (The left- and right-moving actions commute, and thus we can find a simultaneous decomposition.) The full space of states takes then the form

$$\mathcal{H} = \bigoplus_{i,j} N_{ij}\, \mathcal{H}_i \otimes \bar{\mathcal{H}}_j \ , \tag{2.1}$$

where i and j label the different $\mathcal{N}=4$ superconformal representations, and $N_{ij} \in \mathbb{N}_0$ denote the multiplicities with which these representations appear. The $\mathcal{N}=4$ algebra contains, apart from the Virasoro algebra L_n at $c=6$, four supercharge generators, as well as an affine $\hat{\mathfrak{su}}(2)_1$ subalgebra at level one; we denote the Cartan generator of the zero mode subalgebra $\mathfrak{su}(2)$ by J_0.

The full partition function of the conformal field theory is quite complicated, and is only explicitly known at special points in the moduli space. However, there exists some sort of partial index that is much better behaved. This is the so-called *elliptic genus* that is defined by

$$\phi_{\mathrm{K3}}(\tau,z) = \mathrm{Tr}_{\mathrm{RR}}\Big(q^{L_0-\frac{c}{24}}\, y^{J_0}\, (-1)^F\, \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\, (-1)^{\bar{F}} \Big) \equiv \phi_{0,1}(\tau,z) \ . \tag{2.2}$$

Here the trace is only taken over the Ramond-Ramond part of the spectrum (2.1), and the right-moving $\mathcal{N}=4$ modes are denoted by a bar. Furthermore, $q=\exp(2\pi i\tau)$ and $y=\exp(2\pi i z)$, F and $\bar{F}$ are the left- and right-moving fermion number operators, and the two central charges equal $c=\bar{c}=6$. Note that the elliptic genus does not actually depend on $\bar{\tau}$, although $\bar{q}=\exp(-2\pi i\bar{\tau})$ does; the reason for this is that, with respect to the right-moving algebra, the elliptic genus is like a Witten index, and only the right-moving ground states contribute. To see this one notices that states that are not annihilated by a supercharge zero mode appear always as a boson-fermion pair; the contribution of such a pair to the elliptic genus however vanishes because the two states contribute with the opposite sign (as a consequence of the $(-1)^{\bar{F}}$ factor). Thus only the right-moving ground states, i.e. the states that are annihilated by all right-moving supercharge zero modes, contribute to the elliptic genus, and the commutation relations of the $\mathcal{N}=4$ algebra then imply that they satisfy $(\bar{L}_0-\frac{\bar{c}}{24})\phi_{\mathrm{ground}}=0$; thus it follows that the elliptic genus is independent of $\bar{\tau}$. Note that this argument does not apply to the left-moving contributions because of the y^{J_0} factor. (The supercharges are 'charged' with respect to the J_0 Cartan generator, and hence the two terms of a boson-fermion pair come with different powers of y. However, if we also set $y=1$, the elliptic genus does indeed become a constant, independent of τ and $\bar{\tau}$.)

It follows from general string considerations that the elliptic genus defines a *weak Jacobi form of weight zero and index one* [17]. Recall that a weak Jacobi form of weight w and index m is a function [18]

$$\phi_{w,m} : \mathbb{H}_+ \times \mathbb{C} \to \mathbb{C} \ , \qquad (\tau,z) \mapsto \phi_{w,m}(\tau,z) \tag{2.3}$$

that satisfies

$$\phi_{w,m}\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^w\, e^{2\pi i m \frac{cz^2}{c\tau+d}}\, \phi_{w,m}(\tau,z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \ , \tag{2.4}$$

$$\phi(\tau, z+\ell\tau+\ell') = e^{-2\pi i m(\ell^2\tau+2\ell z)}\phi(\tau,z) \qquad \ell,\ell' \in \mathbb{Z} \ , \tag{2.5}$$

and has a Fourier expansion

$$\phi(\tau, z) = \sum_{n\geq 0,\, \ell\in\mathbb{Z}} c(n,\ell) q^n y^\ell \tag{2.6}$$

with $c(n,\ell) = (-1)^w c(n,-\ell)$. Weak Jacobi forms have been classified, and there is only one weak Jacobi form with $w = 0$ and $m = 1$. Up to normalisation $\phi_{\rm K3}$ must therefore agree with this unique weak Jacobi form $\phi_{0,1}$, which can explicitly be written in terms of Jacobi theta functions as

$$\phi_{0,1}(\tau, z) = 8 \sum_{i=2,3,4} \frac{\vartheta_i(\tau,z)^2}{\vartheta_i(\tau,0)^2} \ . \tag{2.7}$$

Note that the Fourier coefficients of $\phi_{\rm K3}$ are integers; as a consequence they cannot change continuously as one moves around in the moduli space of K3 sigma-models, and thus $\phi_{\rm K3}$ must be actually independent of the specific K3 sigma-model that is being considered, i.e. independent of the point in the moduli space. Here we have used that the moduli space is connected. More concretely, it can be described as the double quotient

$$\mathcal{M}_{\rm K3} = {\rm O}(\Gamma^{4,20}) \setminus {\rm O}(4,20) \ / \ {\rm O}(4) \times {\rm O}(20) \ . \tag{2.8}$$

We can think of the Grassmannian on the right

$${\rm O}(4,20) \ / \ {\rm O}(4) \times {\rm O}(20) \tag{2.9}$$

as describing the choice of a positive-definite 4-dimensional subspace $\Pi \subset \mathbb{R}^{4,20}$, while the group on the left, ${\rm O}(\Gamma^{4,20})$, leads to discrete identifications among them. Here ${\rm O}(\Gamma^{4,20})$ is the group of isometries of a given fixed unimodular lattice $\Gamma^{4,20} \subset \mathbb{R}^{4,20}$. (In physics terms, the lattice $\Gamma^{4,20}$ can be thought of as the D-brane charge lattice of string theory on K3.)

Let us denote by $\mathcal{H}^{(0)} \subset \mathcal{H}_{\rm RR}$ the subspace of (2.1) that consists of those RR states for which the right-moving states are ground states. (Thus $\mathcal{H}^{(0)}$ consists of the states that contribute to the elliptic genus.) $\mathcal{H}^{(0)}$ carries an action of the left-moving $\mathcal{N} = 4$ superconformal algebra, and at any point in moduli space, its decomposition is of the form

$$\mathcal{H}^{(0)} = 20 \cdot \mathcal{H}_{h=\frac{1}{4},j=0} \ \oplus \ 2 \cdot \mathcal{H}_{h=\frac{1}{4},j=\frac{1}{2}} \ \oplus \ \bigoplus_{n=1}^{\infty} D_n \, \mathcal{H}_{h=\frac{1}{4}+n,j=\frac{1}{2}} \ , \tag{2.10}$$

where $\mathcal{H}_{h,j}$ denotes the irreducible $\mathcal{N} = 4$ representation whose Virasoro primary states have conformal dimension h and transform in the spin j representation of $\mathfrak{su}(2)$. The multiplicities D_n are *not* constant over the moduli space, but the above argument shows that

$$A_n = {\rm Tr}_{D_n} (-1)^{\bar{F}} \tag{2.11}$$

are (where D_n is now understood not just as a multiplicity, but as a representation of the right-moving $(-1)^{\bar{F}}$ operator that determines the sign with which these states contribute to the elliptic genus). In this language, the elliptic genus then takes the form

$$\phi_{\rm K3}(\tau,z) = 20 \cdot \chi_{h=\frac{1}{4},j=0}(\tau,z) - 2 \cdot \chi_{h=\frac{1}{4},j=\frac{1}{2}}(\tau,z) + \sum_{n=1}^{\infty} A_n \cdot \chi_{h=\frac{1}{4}+n,j=\frac{1}{2}}(\tau,z) \ , \tag{2.12}$$

where $\chi_{h,j}(\tau,z)$ is the 'elliptic' genus of the corresponding $\mathcal{N}=4$ representation,

$$\chi_{h,j}(\tau,z) = \mathrm{Tr}_{\mathcal{H}_{h,j}}\left(q^{L_0-\frac{c}{24}}\, y^{J_0}(-1)^F\right) , \tag{2.13}$$

and we have used that $(-1)^{\bar{F}}$ takes the eigenvalues $+1$ and -1 on the 20- and 2-dimensional multiplicity spaces of the first two terms in (2.10), respectively.

The key observation of Eguchi, Ooguri & Tachikawa (EOT) [1] was that the A_n are sums of dimensions of $\mathbb{M}_{24}$ representation, in striking analogy to the original Monstrous Moonshine conjecture of [3]; the first few terms are

$$A_1 = 90 = \mathbf{45} + \overline{\mathbf{45}} \tag{2.14}$$

$$A_2 = 462 = \mathbf{231} + \overline{\mathbf{231}} \tag{2.15}$$

$$A_3 = 1540 = \mathbf{770} + \overline{\mathbf{770}} , \tag{2.16}$$

where $\mathbf{N}$ denotes a representation of $\mathbb{M}_{24}$ of dimension N. Actually, they guessed correctly the first six coefficients; from A_7 onwards the guesses become much more ambiguous (since the dimensions of the $\mathbb{M}_{24}$ representations are not that large) and they actually misidentified the seventh coefficient in their original analysis. (We will come back to the question of why and how one can be certain about the 'correct' decomposition shortly, see section 2.2.) The alert reader will also notice that the first two coefficients in (2.10), namely 20 and -2, are not directly $\mathbb{M}_{24}$ representations; the correct prescription is to introduce virtual representations and to write

$$20 = \mathbf{23} - 3\cdot\mathbf{1} , \qquad -2 = -2\cdot\mathbf{1} . \tag{2.17}$$

Recall that $\mathbb{M}_{24}$ is a sporadic finite simple group of order

$$|\mathbb{M}_{24}| = 2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23 = 244\,823\,040 . \tag{2.18}$$

It has 26 conjugacy classes (which are denoted by 1A, 2A, 3A, ..., 23A, 23B, where the number refers to the order of the corresponding group element) — see eqs. (2.19) and (2.20) below for the full list — and therefore also 26 irreducible representations whose dimensions range from $N=1$ to $N=10395$. The Mathieu group $\mathbb{M}_{24}$ can be defined as the subgroup of the permutation group S_{24} that leaves the extended Golay code invariant; equivalently, it is the quotient of the automorphism group of the $\mathfrak{su}(2)^{24}$ Niemeier lattice, divided by the Weyl group. Thought of as a subgroup of $\mathbb{M}_{24}\subset S_{24}$, it contains the subgroup $\mathbb{M}_{23}$ that is characterised by the condition that it leaves a given (fixed) element of $\{1,\ldots,24\}$ invariant.

2.1 Classical symmetries

The appearance of a Mathieu group in the elliptic genus of K3 is not totally surprising in view of the Mukai theorem [19, 20]. It states that any finite group of symplectic automorphisms of a K3 surface can be embedded into the Mathieu group $\mathbb{M}_{23}$. The symplectic automorphisms of a K3 surface define symmetries that act on the multiplicity spaces of the $\mathcal{N}=4$ representations, and therefore explain part of the above findings. However, it is also clear from Mukai's argument that they do not even account for the

full $\mathbb{M}_{23}$ group. Indeed, every symplectomorphism of K3 has at least five orbits on the set $\{1,\ldots,24\}$, and thus not all elements of $\mathbb{M}_{23}$ can be realised as a symplectomorphism. More specifically, of the 26 conjugacy classes of $\mathbb{M}_{24}$, 16 have a representative in $\mathbb{M}_{23}$, namely

$$\text{repr. in } \mathbb{M}_{23}: \quad \begin{array}{ll} \text{1A, 2A, 3A, 4B, 5A, 6A, 7A, 7B, 8A} & \text{(geometric)} \\ \text{11A, 14A, 14B, 15A, 15B, 23A, 23B} & \text{(non-geometric)} \end{array} , \tag{2.19}$$

where 'geometric' means that a representative can be (and in fact is) realised by a geometric symplectomorphism (i.e. that the representative has at least five orbits when acting on the set $\{1,\ldots,24\}$), while 'non-geometric' means that this is not the case. The remaining conjugacy classes do *not* have a representative in $\mathbb{M}_{23}$, and are therefore not accounted for geometrically via the Mukai theorem

$$\text{no repr. in } \mathbb{M}_{23}: \quad \text{2B, 3B, 4A, 4C, 6B, 10A, 12A, 12B, 21A, 21B} . \tag{2.20}$$

The classical symmetries can therefore only explain the symmetries in the first line of (2.19). Thus an additional argument is needed in order to understand the origin of the other symmetries; we shall come back to this in section 3.

2.2 Evidence for Moonshine

As was already alluded to above, in order to determine the 'correct' decomposition of the A_n multiplicity spaces in terms of $\mathbb{M}_{24}$ representations, we need to study more than just the usual elliptic genus. By analogy with Monstrous Moonshine, the natural objects to consider are the analogues of the McKay Thompson series [21]. These are obtained from the elliptic genus upon replacing

$$A_n = \dim R_n \;\to\; \mathrm{Tr}_{R_n}(g) \;, \tag{2.21}$$

where $g \in \mathbb{M}_{24}$, and R_n is the $\mathbb{M}_{24}$ representation whose dimension equals the coefficient A_n; the resulting functions are then (compare (2.12))

$$\phi_g(\tau,z) = \mathrm{Tr}_{\mathbf{23}-3\cdot\mathbf{1}}(g)\,\chi_{h=\frac{1}{4},j=0}(\tau,z) - 2\,\mathrm{Tr}_{\mathbf{1}}(g)\,\chi_{h=\frac{1}{4},j=\frac{1}{2}}(\tau,z) + \sum_{n=1}^{\infty}\mathrm{Tr}_{R_n}(g)\,\chi_{h=\frac{1}{4}+n,j=\frac{1}{2}}(\tau,z) \,. \tag{2.22}$$

The motivation for this definition comes from the observation that if the underlying vector space $\mathcal{H}^{(0)}$, see eq. (2.10), of states contributing to the elliptic genus were to carry an action of $\mathbb{M}_{24}$, $\phi_g(\tau,z)$ would equal the 'twining elliptic genus', i.e. the elliptic genus twined by the action of g

$$\phi_g(\tau,z) = \mathrm{Tr}_{\mathcal{H}^{(0)}}\Big(g\, q^{L_0-\frac{c}{24}}\, y^{J_0}\,(-1)^F\,\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\,(-1)^{\bar{F}}\Big) \,. \tag{2.23}$$

Obviously, a priori, it is not clear what the relevant R_n in (2.21) are. However, we have some partial information about them:

(i) For any explicit realisation of a symmetry of a K3 sigma-model, we can calculate (2.23) directly. (In particular, for some symmetries, the relevant twining genera had already been calculated in [22].)

(ii) The observation of EOT determines the first six coefficients explicitly.

(iii) The twining genera must have special modular properties.

Let us elaborate on (iii). Assuming that the functions $\phi_g(\tau, z)$ have indeed an interpretation as in (2.23), they correspond in the usual orbifold notation of string theory to the contribution

$$\phi_g(\tau, z) \;\longleftrightarrow\; e \underset{g}{\boxed{}} \tag{2.24}$$

where e is the identity element of the group. Under a modular transformation it is believed that these twining and twisted genera transform (up to a possible phase) as

$$h \underset{g}{\boxed{}} \quad \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \quad h^d g^c \underset{g^a h^b}{\boxed{}} \tag{2.25}$$

The twining genera (2.24) are therefore invariant (possibly up to a phase) under the modular transformations with

$$\gcd(a, o(g)) = 1 \qquad \text{and} \qquad c = 0 \mod o(g), \tag{2.26}$$

where $o(g)$ is the order of the group element g and we used that for $\gcd(a, o(g)) = 1$, the group element g^a is in the same conjugacy class as g or g^{-1}. (Because of reality, the twining genus of g and g^{-1} should be the same.) Since $ad - bc = 1$, the second condition implies the first, and we thus conclude that $\phi_g(\tau, z)$ should be (up to a possible multiplier system) a weak Jacobi form of weight zero and index one under the subgroup of $\mathrm{SL}(2, \mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \;:\; c = 0 \bmod (N) \right\} , \tag{2.27}$$

where $N = o(g)$. This is a relatively strong condition, and knowing the first few terms (for a fixed multiplier system) determines the function uniquely. In order to use this constraint, however, it is important to know the multiplier system. An ansatz (that seems to work, see below) was made in [23]

$$\phi_g\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = e^{\frac{2\pi i c d}{N h}}\, e^{\frac{2\pi i\, c z^2}{c\tau + d}}\, \phi_g(\tau, z) \;, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;, \tag{2.28}$$

where N is again the order of g and $h | \gcd(N, 12)$. The multiplier system is trivial ($h = 1$) if and only if g contains a representative in $\mathbb{M}_{23} \subset \mathbb{M}_{24}$. For the other conjugacy classes, the values are tabulated in table 1. It was noted in [10] that h equals the length of the shortest cycle (when interpreted as a permutation in S_{24}, see table 1 of [10]).

Using this ansatz, explicit expressions for all twining genera were determined in [23]; independently, the same twining genera were also found (using guesses based on the cycle shapes of the corresponding S_{24} representations) in [24]. (Earlier partial results had been obtained in [9] and [25].)

Class	2B	3B	4A	4C	6B	10A	12A	12B	21AB
h	2	3	2	4	6	2	2	12	3

Table 1: Value of h for the conjugacy classes in (2.28).

These explicit expressions for the twining genera then allow for a very non-trivial check of the EOT proposal. As is clear from their definition in (2.22), they determine the coefficients

$$\mathrm{Tr}_{R_n}(g) \qquad \text{for all } g \in \mathbb{M}_{24} \text{ and all } n \geq 1. \tag{2.29}$$

This information is therefore sufficient to *determine* the representations R_n, i.e. to calculate their decomposition into irreducible $\mathbb{M}_{24}$ representations, for all n. We have worked out the decompositions explicitly for the first 500 coefficients, and we have found that each R_n can be written as a direct sum of $\mathbb{M}_{24}$ representations with non-negative integer multiplicities [23]. (Subsequently [24] tested this property for the first 600 coefficients, and apparently Tachikawa has also checked it for the first 1000 coefficients.) Terry Gannon has informed us that this information is sufficient to prove that the same will then happen for all n [26]. In some sense this then proves the EOT conjecture.

3 Symmetries of K3 models

While the above considerations establish in some sense the EOT conjecture, they do not offer any insight into why the elliptic genus of K3 should exhibit an $\mathbb{M}_{24}$ symmetry. This is somewhat similar to the original situation in Monstrous Moonshine, after Conway and Norton had found the various Hauptmodules by somewhat similar techniques. Obviously, in the case of Monstrous Moonshine, many of these observations were subsequently explained by the construction of the Monster CFT (that possesses the Monster group as its automorphism group) [5, 6]. So we should similarly ask for a microscopic explanation of these findings.

In some sense it is clear what the analogue of the Monster CFT in the current context should be: we know that the function in question *is* the elliptic genus of K3. However, there is one problem with this. As we mentioned before, there is not just one K3 sigma-model, but rather a whole moduli space (see eq. (2.8)) of such CFTs. So the simplest explanation of the EOT observation would be if there is (at least) one special K3 sigma-model that has $\mathbb{M}_{24}$ as its automorphism group. Actually, the relevant symmetry group should commute with the action of the $\mathcal{N} = (4,4)$ superconformal symmetry (since it should act on the multiplicity spaces in $\mathcal{H}^{(0)}$, see eq. (2.10)). Furthermore, since the two $\mathcal{N} = 4$ representations with $h = \frac{1}{4}$ and $j = \frac{1}{2}$ are singlets — recall that the coefficient -2 transforms as $-2 = -2 \cdot \mathbf{1}$, see (2.17) — the automorphism must act trivially on the 4 RR ground states that transform in the $(\mathbf{2},\mathbf{2})$ representation of the $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ subalgebra of $\mathcal{N} = (4,4)$. Note that these four states generate the simultaneous half-unit spectral flows in the left- and the right-moving sector; the requirement that the symmetry leaves them invariant therefore means that spacetime supersymmetry is preserved.

Recall from (2.8) that the different K3 sigma-models are parametrised by the choice of a positive-definite 4-dimensional subspace $\Pi \subset \mathbb{R}^{4,20}$, modulo some discrete identifications. Let us denote by G_Π the group of symmetries of the sigma-model described by Π that

commute with the action of $\mathcal{N}=(4,4)$ and preserve the RR ground states in the $(\mathbf{2},\mathbf{2})$ (see above). It was argued in [7] that G_Π is precisely the subgroup of $\mathrm{O}(\Gamma^{4,20})$ consisting of those elements that leave Π pointwised fixed. The possible symmetry groups G_Π can then be classified following essentially the paradigm of the Mukai-Kondo argument for the symplectomorphisms of K3 surfaces [19, 20]. The outcome of the analysis can be summarised by the following theorem [7]:

Theorem: *Let G be the group of symmetries of a non-linear sigma-model on $K3$ preserving the $\mathcal{N}=(4,4)$ superconformal algebra as well as the spectral flow operators. One of the following possibilities holds:*

(i) $G=G'.G''$, *where G' is a subgroup of $\mathbb{Z}_2^{11}$, and G'' is a subgroup of $\mathbb{M}_{24}$ with at least four orbits when acting as a permutation on $\{1,\ldots,24\}$*

(ii) $G=5^{1+2}:\mathbb{Z}_4$

(iii) $G=\mathbb{Z}_3^4:A_6$

(iv) $G=3^{1+4}:\mathbb{Z}_2.G''$, *where G'' is either trivial, $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$.*

Here $G=N.Q$ means that N is a normal subgroup of G, and $G/N\cong Q$; when G is the semidirect product of N and Q, we denote it by $N:Q$. Furthermore, p^{1+2n} is an extra-special group of order p^{1+2n}, which is an extension of $\mathbb{Z}_p^{2n}$ by a central element of order p.

We will give a sketch of the proof below (see section 3.1), but for the moment let us comment on the implications of this result. First of all, our initial expectation from above is not realised: none of these groups $G\equiv G_\Pi$ contains $\mathbb{M}_{24}$. In particular, the twining genera for the conjugacy classes 12B, 21A, 21B, 23A, 23B of $\mathbb{M}_{24}$ cannot be realised by any symmetry of a K3 sigma-model. Thus we cannot give a direct explanation of the EOT observation along these lines.

Given that the elliptic genus is constant over the moduli space, one may then hope that we can explain the origin of $\mathbb{M}_{24}$ by 'combining' symmetries from different points in the moduli space. As we have mentioned before, this is also similar to what happens for the geometric symplectomorphisms of K3: it follows from the Mukai theorem that the Mathieu group $\mathbb{M}_{23}$ is the smallest group that contains all symplectomorphisms, but there is no K3 surface where all of $\mathbb{M}_{23}$ is realised, and indeed, certain generators of $\mathbb{M}_{23}$ can never be symmetries, see (2.19). However, also this explanation of the EOT observation is somewhat problematic: as is clear from the above theorem, not all symmetry groups of K3 sigma-models are in fact subgroups of $\mathbb{M}_{24}$. In particular, none of the cases (ii), (iii) and (iv) (as well as case (i) with G' non-trivial) have this property, as can be easily seen by comparing the prime factor decompositions of their orders to (2.18). The smallest group that contains all groups of the theorem is the Conway group Co_1, but as far as we are aware, there is no evidence of any 'Conway Moonshine' in the elliptic genus of K3.

One might speculate that, generically, the group G must be a subgroup of $\mathbb{M}_{24}$, and that the models whose symmetry group is not contained in $\mathbb{M}_{24}$ are, in some sense, special or 'exceptional' points in the moduli space. In order to make this idea precise, it is useful to analyse the exceptional models in detail. In [7], some examples have been provided of case (i) with non-trivial G' (a torus orbifold $\mathbb{T}^4/\mathbb{Z}_2$ or the Gepner model 2^4, believed to be

equivalent to a $\mathbb{T}^4/\mathbb{Z}_4$ orbifold), and of case (iii) (the Gepner model 1^6, which is believed to be equivalent to a $\mathbb{T}^4/\mathbb{Z}_3$ orbifold, see also [27]). For the cases (ii) and (iv), only an existence proof was given. In section 5, we will improve the situation by constructing in detail an example of case (ii), realised as an asymmetric $\mathbb{Z}_5$-orbifold of a torus $\mathbb{T}^4$. Furthermore, in section 6 we will briefly discuss the $\mathbb{Z}_3$-orbifold of a torus and the explicit realisation of its symmetry group, corresponding to cases (ii) and (iv) for any G''.

Notice that all the examples of exceptional models known so far are provided by torus orbifolds. In fact, we will show below (see section 4) that all cyclic torus orbifolds have exceptional symmetry groups. Conversely, we will prove that the cases (ii)–(iv) of the theorem are always realised by (cyclic) torus orbifolds. On the other hand, as we shall also explain, some of the exceptional models in case (i) are not cyclic torus orbifolds.

3.1 Sketch of the proof of the Theorem

In this subsection, we will describe the main steps in the proof of the above theorem; the details can be found in [7].

It was argued in [7] that the supersymmetry preserving automorphisms of the non-linear sigma-model characterised by Π generate the group $G \equiv G_\Pi$ that consists of those elements of $O(\Gamma^{4,20})$ that leave Π pointwise fixed. Let us denote by L^G the sublattice of G-invariant vectors of $L \equiv \Gamma^{4,20}$, and define L_G to be its orthogonal complement that carries a genuine action of G. Since $L^G \otimes \mathbb{R}$ contains the subspace Π, it follows that L^G has signature $(4, d)$ for some $d \geq 0$, so that L_G is a negative-definite lattice of rank $20 - d$. In [7], it is proved that, for any consistent model, L_G can be embedded (up to a change of sign in its quadratic form) into the Leech lattice Λ, the unique 24-dimensional positive-definite even unimodular lattice containing no vectors of squared norm 2. Furthermore, the action of G on L_G can be extended to an action on the whole of Λ, such that the sublattice $\Lambda^G \subset \Lambda$ of vectors fixed by G is the orthogonal complement of L_G in Λ. This construction implies that G must be a subgroup of $\mathrm{Co}_0 \equiv \mathrm{Aut}(\Lambda)$ that fixes a sublattice Λ^G of rank $4 + d$. Conversely, it can be shown that any such subgroup of $\mathrm{Aut}(\Lambda)$ is the symmetry group of some K3 sigma-model.

This leaves us with characterising the possible subgroups of the finite group $\mathrm{Co}_0 \equiv \mathrm{Aut}(\Lambda)$ that stabilise a suitable sublattice; problems of this kind have been studied in the mathematical literature before. In particular, the stabilisers of sublattices of rank at least 4 are, generically, the subgroups of $\mathbb{Z}_2^{11} : \mathbb{M}_{24}$ described in case (i) of the theorem above. The three cases (ii), (iii), (iv) arise when the invariant sublattice Λ^G is contained in some $\mathcal{S}$-lattice $S \subset \Lambda$. An $\mathcal{S}$-lattice S is a primitive sublattice of Λ such that each vector of S is congruent modulo $2S$ to a vector of norm 0, 4 or 6. Up to isomorphisms, there are only three kind of $\mathcal{S}$-lattices of rank at least four; their properties are summarised in the following table:

Name	type	rk S	Stab(S)	Aut(S)
$(A_2 \oplus A_2)'(3)$	$2^9 3^6$	4	$\mathbb{Z}_3^4 : A_6$	$\mathbb{Z}_2 \times (S_3 \times S_3).\mathbb{Z}_2$
$A_4^*(5)$	$2^5 3^{10}$	4	$5^{1+2} : \mathbb{Z}_4$	$\mathbb{Z}_2 \times S_5$
$E_6^*(3)$	$2^{27} 3^{36}$	6	$3^{1+4} : \mathbb{Z}_2$	$\mathbb{Z}_2 \times W(E_6)$.

Here, S is characterised by the type $2^p 3^q$, which indicates that S contains p pairs of opposite vectors of norm 4 (type 2) and q pairs of opposite vectors of norm 6 (type 3).

The group $\mathrm{Stab}(S)$ is the pointwise stabiliser of S in Co_0 and $\mathrm{Aut}(S)$ is the quotient of the setwise stabiliser of S modulo its pointwise stabiliser $\mathrm{Stab}(S)$. The group $\mathrm{Aut}(S)$ always contains a central $\mathbb{Z}_2$ subgroup, generated by the transformation that inverts the sign of all vectors of the Leech lattice. The lattice of type $2^{27}3^{36}$ is isomorphic to the weight lattice (the dual of the root lattice) of E_6 with quadratic form rescaled by 3 (i.e. the roots in $E_6^*(3)$ have squared norm 6), and $\mathrm{Aut}(S)/\mathbb{Z}_2$ is isomorphic to the Weyl group $W(E_6)$ of E_6. Similarly, the lattice of type $2^5 3^{10}$ is the weight lattice of A_4 rescaled by 5, and $\mathrm{Aut}(S)/\mathbb{Z}_2$ is isomorphic to the Weyl group $W(A_4) \cong S_5$ of A_4. Finally, the type $2^9 3^6$ is the 3-rescaling of a lattice $(A_2 \oplus A_2)'$ obtained by adjoining to the root lattice $A_2 \oplus A_2$ an element $(e_1^*, e_2^*) \in A_2^* \oplus A_2^*$, with e_1^* and e_2^* of norm $2/3$. The latter $\mathcal{S}$-lattice can also be described as the sublattice of vectors of $E_6^*(3)$ that are orthogonal to an $A_2(3)$ sublattice of $E_6^*(3)$. The group $\mathrm{Aut}(S)/\mathbb{Z}_2$ is the product $(S_3 \times S_3).\mathbb{Z}_2$ of the Weyl groups $W(A_2) = S_3$, and the $\mathbb{Z}_2$ symmetry that exchanges the two A_2 and maps e_1^* to e_2^*.

The cases (ii)–(iv) of the above theorem correspond to Λ^G being isomorphic to $A_4^*(5)$ (case ii), to $(A_2 \oplus A_2)'(3)$ (case iii) or to a sublattice of $E_6^*(3)$ different from $(A_2 \oplus A_2)'(3)$ (case iv). In the cases (ii) and (iii), G is isomorphic to $\mathrm{Stab}(S)$. In case (iv), $\mathrm{Stab}(S)$ is, generically, a normal subgroup of G, and $G'' \cong G/\mathrm{Stab}(S)$ is a subgroup of $\mathrm{Aut}(S) \cong \mathbb{Z}_2 \times W(E_6)$ that fixes a sublattice $\Lambda^G \subseteq E_6^*(3)$, with $\Lambda^G \neq (A_2 \oplus A_2)'(3)$, of rank at least 4. The only non-trivial subgroups of $\mathbb{Z}_2 \times W(E_6)$ with these properties are $G'' = \mathbb{Z}_2$, which corresponds to Λ^G being orthogonal to a single vector of norm 6 in $E_6^*(3)$ (a rescaled root), and $G'' = \mathbb{Z}_2^2$, which corresponds to Λ^G being orthogonal to two orthogonal vectors of norm 6.[§] If Λ^G is orthogonal to two vectors $v_1, v_2 \in E_6^*(3)$ of norm 6, with $v_1 \cdot v_2 = -3$, then $\Lambda^G \cong (A_2 \oplus A_2)'(3)$ and case (iii) applies.

4 Symmetry groups of torus orbifolds

In this section we will prove that all K3 sigma-models that are realised as (possibly left-right asymmetric) orbifolds of $\mathbb{T}^4$ by a cyclic group have an 'exceptional' group of symmetries, i.e. their symmetries are not a subgroup of $\mathbb{M}_{24}$. Furthermore, these torus orbifolds account for most of the exceptional models (in particular, for all models in the cases (ii)–(iv) of the theorem). On the other hand, as we shall also explain, there are exceptional models in case (i) that are not cyclic torus orbifolds.

Our reasoning is somewhat reminiscent of the construction of [28, 29] in the context of Monstrous Moonshine. Any $\mathbb{Z}_n$-orbifold of a conformal field theory has an automorphism g of order n, called the *quantum symmetry*, which acts trivially on the untwisted sector and by multiplication by the phase $\exp(\frac{2\pi i k}{n})$ on the k-th twisted sector. Furthermore, the orbifold of the orbifold theory by the group generated by the quantum symmetry g, is equivalent to the original conformal field theory [30]. This observation is the key for characterising K3 models that can be realised as torus orbifolds:

A K3 model $\mathcal{C}$ is a $\mathbb{Z}_n$-orbifold of a torus model if and only if it has a symmetry g of order n such that $\mathcal{C}/\langle g \rangle$ is a consistent orbifold equivalent to a torus model.

[§]The possibility $G'' = \mathbb{Z}_4$ that has been considered in [7] has to be excluded, since there are no elements of order 4 in $W(E_6)$ that preserve a four-dimensional sublattice of $E_6^*(3)$.

In order to see this, suppose that $\mathcal{C}_{K3}$ is a K3 sigma-model that can be realised as a torus orbifold $\mathcal{C}_{K3} = \tilde{\mathcal{C}}_{\mathbb{T}^4}/\langle\tilde{g}\rangle$, where $\tilde{g}$ is a symmetry of order n of the torus model $\tilde{\mathcal{C}}_{\mathbb{T}^4}$. Then $\mathcal{C}_{K3}$ possesses a 'quantum symmetry' g of order n, such that the orbifold of $\mathcal{C}_{K3}$ by g describes again the original torus model, $\tilde{\mathcal{C}}_{\mathbb{T}^4} = \mathcal{C}_{K3}/\langle g\rangle$.

Conversely, suppose $\mathcal{C}_{K3}$ has a symmetry g of order n, such that the orbifold of $\mathcal{C}_{K3}$ by g is consistent, i.e. satisfies the level matching condition — this is the case if and only if the twining genus ϕ_g has a trivial multiplier system — and leads to a torus model $\mathcal{C}_{K3}/\langle g\rangle = \tilde{\mathcal{C}}_{\mathbb{T}^4}$. Then $\mathcal{C}_{K3}$ itself is a torus orbifold since we can take the orbifold of $\tilde{\mathcal{C}}_{\mathbb{T}^4}$ by the quantum symmetry associated to g, and this will, by construction, lead back to $\mathcal{C}_{K3}$.

Thus we conclude that $\mathcal{C} \equiv \mathcal{C}_{K3}$ can be realised as a torus orbifold if and only if $\mathcal{C}$ contains a symmetry g such that (i) ϕ_g has a trivial multiplier system; and (ii) the orbifold of $\mathcal{C}$ by g leads to a torus model $\tilde{\mathcal{C}}_{\mathbb{T}^4}$. It is believed that the orbifold of $\mathcal{C}$ by any $\mathcal{N} = (4,4)$-preserving symmetry group, if consistent, will describe a sigma-model with target space either a torus $\mathbb{T}^4$ or a K3 manifold. The two cases can be distinguished by calculating the elliptic genus; in particular, if the target space is a torus, the elliptic genus vanishes. Actually, since the space of weak Jacobi forms of weight zero and index one is 1-dimensional, this condition is equivalent to the requirement that the elliptic genus $\tilde{\phi}(\tau, z)$ of $\tilde{\mathcal{C}} = \mathcal{C}/\langle g\rangle$ vanishes at $z = 0$.

Next we recall that the elliptic genus of the orbifold by a group element g of order $n = o(g)$ is given by the usual orbifold formula

$$\tilde{\phi}(\tau, z) = \frac{1}{n}\sum_{i,j=1}^{n} \phi_{g^i,g^j}(\tau, z) \ , \tag{4.1}$$

where $\phi_{g^i,g^j}(\tau, z)$ is the twining genus for g^j in the g^i-twisted sector; this can be obtained by a modular transformation from the untwisted twining genus $\phi_{g^d}(\tau, z)$ with $d = \gcd(i, j, n)$. As we have explained above, it is enough to evaluate the elliptic genus for $z = 0$. Then

$$\phi_{g^d}(\tau, z = 0) = \mathrm{Tr}_{\mathbf{24}}(g^d) \ , \tag{4.2}$$

where $\mathrm{Tr}_{\mathbf{24}}(g^d)$ is the trace of g^d over the 24-dimensional space of RR ground states, and since (4.2) is constant (and hence modular invariant) we conclude that

$$\tilde{\phi}(\tau, 0) = \frac{1}{n}\sum_{i,j=1}^{n} \mathrm{Tr}_{\mathbf{24}}(g^{\gcd(i,j,n)}) \ . \tag{4.3}$$

According to the theorem in section 3, all symmetry groups of K3 sigma-models are subgroups of Co_0 and, in fact, $\mathrm{Tr}_{\mathbf{24}}(g^d)$ coincides with the trace of $g^d \in \mathrm{Co}_0$ in the standard 24-dimensional representation of Co_0. Thus, the elliptic genus of the orbifold model $\tilde{\mathcal{C}} = \mathcal{C}/\langle g\rangle$ only depends on the conjugacy class of g in Co_0. The group Co_0 contains 167 conjugacy classes, but only 42 of them contain symmetries that are realised by some K3 sigma-model, i.e. elements that fix at least a four-dimensional subspace in the standard 24-dimensional representation of Co_0. If $\mathrm{Tr}_{\mathbf{24}}(g) \neq 0$ (this happens for 31 of the above 42 conjugacy classes), the twining genus $\phi_g(\tau, z)$ has necessarily a trivial multiplier system, and the orbifold $\mathcal{C}/\langle g\rangle$ is consistent. These classes are listed in the following table, together with the dimension of the space that is fixed by g, the trace over

the 24-dimensional representation, and the elliptic genus $\tilde{\phi}(\tau, z=0)$ of the orbifold model $\tilde{\mathcal{C}}$ (we underline the classes that restrict to $\mathbb{M}_{24}$ conjugacy classes):

Co_0-class	1A	2B	2C	3B	3C	4B	4E	4F	5B	5C	6G	6H	6I	6K	6L	6M	7B
dim fix	24	16	8	12	6	8	10	6	8	4	6	6	6	8	4	4	6
$\mathrm{Tr}_{\mathbf{24}}(g)$	24	8	−8	6	−3	8	4	−4	4	−1	−4	4	5	2	−2	−1	3
$\tilde{\phi}(\tau, 0)$	24	24	0	24	0	24	24	0	24	0	0	24	24	24	0	0	24

Co_0-class	8D	8G	8H	9C	10F	10G	10H	11A	12I	12L	12N	12O	14C	15D
dim fix	4	6	4	4	4	4	4	4	4	4	4	4	4	4
$\mathrm{Tr}_{\mathbf{24}}(g)$	4	2	−2	3	−2	2	3	2	2	1	−2	2	1	1
$\tilde{\phi}(\tau, 0)$	24	24	0	24	0	24	24	24	24	24	0	24	24	24

Note that the elliptic genus of the orbifold theory $\tilde{\mathcal{C}}$ is always 0 or 24, corresponding to a torus or a K3 sigma-model, respectively. Out of curiosity, we have also computed the putative elliptic genus $\tilde{\phi}(\tau, 0)$ for the 11 classes of symmetries g with $\mathrm{Tr}_{24}(g) = 0$ for which we do not expect the orbifold to make sense — the corresponding twining genus ϕ_g will typically have a non-trivial multiplier system, and hence the orbifold will not satisfy level-matching. Indeed, for almost none of these cases is $\tilde{\phi}(\tau, 0)$ equal to 0 or 24, thus signaling an inconsistency of the orbifold model:

Co_0-class	2D	3D	4D	4G	4H	6O	6P	8C	8I	10J	12P
dim fix	12	8	4	8	6	6	4	4	4	4	4
$\mathrm{Tr}_{\mathbf{24}}(g)$	0	0	0	0	0	0	0	0	0	0	0
$\tilde{\phi}(\tau, 0)$	12	8	0	12	6	12	4	12	6	12	12

The only exception is the class 4D, which might define a consistent orbifold (a torus model). It follows that a K3 model $\mathcal{C}$ is the $\mathbb{Z}_n$-orbifold of a torus model if and only if it contains a symmetry g in one of the classes

$$\begin{aligned} &2\mathrm{C},\ 3\mathrm{C},\ 4\mathrm{F},\ 5\mathrm{C},\ 6\mathrm{G},\ 6\mathrm{L},\ 6\mathrm{M},\ 8\mathrm{H},\ 10\mathrm{F},\ 12\mathrm{N}, \\ &4\mathrm{B},\ 4\mathrm{D},\ 6\mathrm{H},\ 6\mathrm{I},\ 8\mathrm{C},\ 8\mathrm{D},\ 9\mathrm{C},\ 10\mathrm{G},\ 10\mathrm{H},\ 12\mathrm{I},\ 12\mathrm{L},\ 12\mathrm{O} \end{aligned} \tag{4.4}$$

of Co_0.[¶] Here we have also included (in the second line) those classes of elements $g \in \mathrm{Co}_0$ for which $\mathcal{C}/\langle g^i \rangle$ is a torus model, for some power $i > 1$. Our main observation is now that none of the Co_0 classes in (4.4) restricts to a class in $\mathbb{M}_{24}$, i.e.

All K3 models that are realised as $\mathbb{Z}_n$-orbifolds of torus models are exceptional. In particular, the quantum symmetry is not an element of $\mathbb{M}_{24}$.

One might ask whether the converse is also true, i.e. whether all exceptional models are cyclic torus orbifolds. This is not quite the case: for example, the classification theorem

[¶] We should emphasise that for us the term 'orbifold' always refers to a conformal field theory (rather than a geometrical) construction. Although a non-linear sigma-model on a geometric orbifold $M/\mathbb{Z}_N$ always admits an interpretation as a CFT orbifold, the converse is not always true. In particular, there exist asymmetric orbifold constructions that do not have a direct geometric interpretation, see for example section 5.

of section 3 predicts the existence of models with a symmetry group $G \cong GL_2(3)$ (the group of 2×2 invertible matrices on the field $\mathbb{F}_3$ with 3 elements). The group G contains no elements in the classes (4.4), so the model is not a cyclic torus orbifold; on the other hand, G contains elements in the class 8I of Co_0, which does not restrict to $\mathbb{M}_{24}$. A second counterexample is a family of models with a symmetry g in the class 6O of Co_0. A generic point of this family is not a cyclic torus model (although some special points are), since the full symmetry group is generated by g and contains no elements in (4.4). Both these counterexamples belong to case (i) of the general classification theorem. In fact, we can prove that

The symmetry group G of a K3 sigma-model $\mathcal{C}$ contains a subgroup $3^{1+4} : \mathbb{Z}_2$ (cases (iii) and (iv) of the theorem) if and only if $\mathcal{C}$ is a $\mathbb{Z}_3$-orbifold of a torus model. Furthermore, $G = 5^{1+2} : \mathbb{Z}_4$ (case (ii)) if and only if $\mathcal{C}$ is a $\mathbb{Z}_5$-orbifold of a torus model.

The proof goes as follows. All subgroups of Co_0 of the form $3^{1+4} : \mathbb{Z}_2$ (respectively, $5^{1+2} : \mathbb{Z}_4$) contain an element in the class 3C (resp., 5C), and therefore the corresponding models are $\mathbb{Z}_3$ (resp., $\mathbb{Z}_5$) torus orbifolds. Conversely, consider a $\mathbb{Z}_3$-orbifold of a torus model. Its symmetry group G contains the quantum symmetry g in class 3C of Co_0. (It must contain a symmetry generator of order three whose orbifold leads to a torus, and 3C is then the only possibility.) The sublattice $\Lambda^{\langle g \rangle} \subset \Lambda$ fixed by g is the $\mathcal{S}$-lattice $2^{27}3^{36}$ [31]. From the classification theorem, we know that G is the stabiliser of a sublattice $\Lambda^G \subset \Lambda$ of rank at least 4. Since $\Lambda^G \subseteq \Lambda^{\langle g \rangle}$, G contains as a subgroup the stabiliser of $\Lambda^{\langle g \rangle}$, namely $3^{1+4} : \mathbb{Z}_2$.

Analogously, a $\mathbb{Z}_5$ torus orbifold always has a symmetry in class 5C, whose fixed sublattice $\Lambda^{\langle g \rangle}$ is the $\mathcal{S}$-lattice $2^5 3^{10}$ [31]. Since $\Lambda^{\langle g \rangle}$ has rank 4 and is primitive, $\Lambda^G = \Lambda^{\langle g \rangle}$ and the symmetry group G must be the stabiliser $5^{1+2} : \mathbb{Z}_4$ of $\Lambda^{\langle g \rangle}$.

It was shown in [7] that the Gepner model $(1)^6$ corresponds to the case (ii) of the classification theorem. It thus follows from the above reasoning that it must indeed be equivalent to a $\mathbb{Z}_3$-orbifold of $\mathbb{T}^4$, see also [27]. (We shall also study these orbifolds more systematically in section 6.) In the next section, we will provide an explicit construction of a $\mathbb{Z}_5$-orbifold of a torus model and show that its symmetry group is $5^{1+2} : \mathbb{Z}_4$, as predicted by the above analysis.

5 A K3 model with symmetry group $5^{1+2} : \mathbb{Z}_4$

In this section we will construct a supersymmetric sigma-model on $\mathbb{T}^4$ with a symmetry g of order 5 commuting with an $\mathcal{N} = (4,4)$ superconformal algebra and acting asymmetrically on the left- and on the right-moving sector. The orbifold of this model by g will turn out to be a well-defined SCFT with $\mathcal{N} = (4,4)$ (in particular, the level matching condition is satisfied) that can be interpreted as a non-linear sigma-model on K3. We will argue that the group of symmetries of this model is $G = 5^{1+2}.\mathbb{Z}_4$, one of the exceptional groups considered in the general classification theorem.

5.1 The torus model

Let us consider a supersymmetric sigma-model on the torus $\mathbb{T}^4$. Geometrically, we can characterise the theory in terms of a metric and a Kalb-Ramond field, but it is actually more convenient to describe it simply as a conformal field theory that is generated by the following fields: four left-moving $u(1)$ currents $\partial X^a(z)$, $a = 1, \ldots, 4$, four free fermions $\psi^a(z)$, $a = 1, \ldots, 4$, their right-moving analogues $\bar{\partial} X^a(\bar{z}), \tilde{\psi}^a(\bar{z})$, as well as some winding-momentum fields $V_\lambda(z, \bar{z})$ that are associated to vectors λ in an even unimodular lattice $\Gamma^{4,4}$ of signature $(4, 4)$. The mode expansions of the left-moving fields are

$$\partial X^a(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} \ , \qquad \psi^a = \sum_{n\in\mathbb{Z}+\nu} \psi_n z^{-n-\frac{1}{2}} \ , \tag{5.1}$$

where $\nu = 0, 1/2$ in the R- and NS-sector, respectively. Furthermore, we have the usual commutation relations

$$[\alpha^a_m, \alpha^b_n] = m\, \delta^{ab}\, \delta_{m,-n} \qquad\qquad \{\psi^a_m, \psi^b_n\} = \delta^{ab}\, \delta_{m,-n} \ . \tag{5.2}$$

Analogous statements also hold for the right-moving modes $\tilde{\alpha}_n$ and $\tilde{\psi}_n$. The vectors $\lambda \equiv (\lambda_L, \lambda_R) \in \Gamma^{4,4}$ describe the eigenvalues of the corresponding state with respect to the left- and right-moving zero modes α^a_0 and $\tilde{\alpha}^a_0$, respectively. In these conventions the inner product on $\Gamma^{4,4}$ is given as

$$(\lambda, \lambda') = \lambda_L \cdot \lambda'_L - \lambda_R \cdot \lambda'_R \ . \tag{5.3}$$

5.1.1 Continuous and discrete symmetries

Any torus model contains an $\hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{u}}(1)^4$ current algebra, both on the left and on the right. Here, the $\hat{\mathfrak{u}}(1)^4$ currents are the ∂X^a, $a = 1, \ldots, 4$, while $\hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{su}}(2)_1 = \hat{\mathfrak{so}}(4)_1$ is generated by the fermionic bilinears

$$a^3 := \bar{\psi}^{(1)}\psi^{(1)} + \bar{\psi}^{(2)}\psi^{(2)} \qquad a^+ := \bar{\psi}^{(1)}\bar{\psi}^{(2)} \qquad a^- := -\psi^{(1)}\psi^{(2)} \ , \tag{5.4}$$

$$\hat{a}^3 := \bar{\psi}^{(1)}\psi^{(1)} - \bar{\psi}^{(2)}\psi^{(2)} \qquad \hat{a}^+ := \bar{\psi}^{(1)}\psi^{(2)} \qquad \hat{a}^- := -\psi^{(1)}\bar{\psi}^{(2)} \ , \tag{5.5}$$

where

$$\psi^{(1)} = \tfrac{1}{\sqrt{2}}\,(\psi^1 + i\psi^2) \qquad\qquad \psi^{(2)} = \tfrac{1}{\sqrt{2}}\,(\psi^3 + i\psi^4) \tag{5.6}$$

$$\bar{\psi}^{(1)} = \tfrac{1}{\sqrt{2}}\,(\psi^1 - i\psi^2) \qquad\qquad \bar{\psi}^{(2)} = \tfrac{1}{\sqrt{2}}\,(\psi^3 - i\psi^4) \ . \tag{5.7}$$

At special points in the moduli space, where the $\Gamma^{4,4}$ lattice contains vectors of the form $(\lambda_L, 0)$ with $\lambda_L^2 = 2$, the bosonic $\mathfrak{u}(1)^4$ algebra is enhanced to some non-abelian algebra $\mathfrak{g}$ of rank 4. There are generically 16 (left-moving) supercharges; they form four $(\mathbf{2}, \mathbf{2})$ representations of the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ zero mode algebra from (5.4) and (5.5). Altogether, the chiral algebra at generic points is a *large* $\mathcal{N} = 4$ superconformal algebra.

We want to construct a model with a symmetry g of order 5, acting non-trivially on the fermionic fields, and commuting with the *small* $\mathcal{N} = 4$ subalgebras both on the left and on the right. A small $\mathcal{N} = 4$ algebra contains an $\hat{\mathfrak{su}}(2)_1$ current algebra and four supercharges in two doublets of $\mathfrak{su}(2)$. The symmetry g acts by an $\mathrm{O}(4, \mathbb{R})$ rotation on

the left-moving fermions ψ^a, preserving the anti-commutation relations (5.2). Without loss of generality, we may assume that $\psi^{(1)}$ and $\bar{\psi}^{(1)}$ are eigenvectors of g with eigenvalues ζ and ζ^{-1}, where ζ is a primitive fifth root of unity

$$\zeta^5 = 1 \ , \tag{5.8}$$

and that the $\hat{\mathfrak{su}}(2)_1$ algebra preserved by g is (5.4). This implies that g acts on all the fermionic fields by

$$\psi^{(1)} \mapsto \zeta\, \psi^{(1)} \ , \quad \bar{\psi}^{(1)} \mapsto \zeta^{-1} \bar{\psi}^{(1)} \ , \quad \psi^{(2)} \mapsto \zeta^{-1} \psi^{(2)} \ , \quad \bar{\psi}^{(2)} \mapsto \zeta\, \bar{\psi}^{(2)} \ . \tag{5.9}$$

Note that the action of g on the fermionic fields can be described by $e^{\frac{2\pi i k}{5}\hat{a}^3_0}$ for some $k = 1, \ldots, 4$, where $\hat{a}^3$ is the current in the algebra (5.5). The four g-invariant supercharges can then be taken to be

$$\sqrt{2}\sum_{i=1}^{2} J^{(i)}\bar{\psi}^{(i)} \ , \quad \sqrt{2}\sum_{i=1}^{2} \bar{J}^{(i)}\psi^{(i)} \ , \quad \sqrt{2}(\bar{J}^{(1)}\bar{\psi}^{(2)} - \bar{J}^{(2)}\bar{\psi}^{(1)}) \ , \quad \sqrt{2}(J^{(1)}\psi^{(2)} - J^{(2)}\psi^{(1)}) \ , \tag{5.10}$$

where $J^{(1)}, \bar{J}^{(1)}, J^{(2)}, \bar{J}^{(2)}$ are suitable (complex) linear combinations of the left-moving currents ∂X^a, $a = 1, \ldots, 4$. In order to preserve the four supercharges, g must act with the same eigenvalues on the bosonic currents

$$J^{(1)} \mapsto \zeta\, J^{(1)} \ , \quad \bar{J}^{(1)} \mapsto \zeta^{-1} \bar{J}^{(1)} \ , \quad J^{(2)} \mapsto \zeta^{-1} J^{(2)} \ , \quad \bar{J}^{(2)} \mapsto \zeta\, \bar{J}^{(2)} \ . \tag{5.11}$$

A similar reasoning applies to the right-moving algebra with respect to an eigenvalue $\tilde{\zeta}$, with $\tilde{\zeta}^5 = 1$. For the symmetries with a geometric interpretation, the action on the left- and right-moving bosonic currents is induced by an $\mathrm{O}(4,\mathbb{R})$-transformation on the scalar fields X^a, $a, = 1, \ldots, 4$, representing the coordinates on the torus; then ζ and $\tilde{\zeta}$ are necessarily equal. In our treatment, we want to allow for the more general case where $\zeta \neq \tilde{\zeta}$.

The action of g on J^a and $\tilde{J}^a$ induces an $\mathrm{O}(4,4,\mathbb{R})$-transformation on the lattice $\Gamma^{4,4}$. The transformation g is a symmetry of the model if and only if it induces an automorphism on $\Gamma^{4,4}$. In particular, it must act by an invertible integral matrix on any lattice basis. The requirement that the trace of this matrix (and of any power of it) must be integral, leads to the condition that

$$2(\zeta^i + \zeta^{-i} + \tilde{\zeta}^i + \tilde{\zeta}^{-i}) \in \mathbb{Z} \ , \tag{5.12}$$

for all $i \in \mathbb{Z}$. For g of order 5, this condition is satisfied by

$$\zeta = e^{\frac{2\pi i}{5}} \qquad \text{and} \qquad \tilde{\zeta} = e^{\frac{4\pi i}{5}} \ , \tag{5.13}$$

and this solution is essentially unique (up to taking powers of it or exchanging the definition of ζ and ζ^{-1}). Eq. (5.13) shows that a supersymmetry preserving symmetry of order 5 is necessarily left-right asymmetric, and hence does not have a geometric interpretation.

It is now clear how to construct a torus model with the symmetries (5.9) and (5.11). First of all, we need an automorphism g of $\Gamma^{4,4}$ of order five. Such an automorphism

must have eigenvalues $\zeta, \zeta^2, \zeta^3, \zeta^4$, each corresponding to two independent eigenvectors $v^{(1)}_{\zeta^i}, v^{(2)}_{\zeta^i}$, $i = 1, \ldots, 4$, in $\Gamma^{4,4} \otimes \mathbb{C}$. Given the discussion above, see in particular (5.13), we now require that the vectors

$$v^{(1)}_{\zeta^1} \ , \quad v^{(2)}_{\zeta^1} \ , \quad v^{(1)}_{\zeta^4} \ , \quad v^{(2)}_{\zeta^4} \tag{5.14}$$

span a positive-definite subspace of $\Gamma^{4,4} \otimes \mathbb{C}$ (i.e. correspond to the left-movers), while the vectors

$$v^{(1)}_{\zeta^2} \ , \quad v^{(2)}_{\zeta^2} \ , \quad v^{(1)}_{\zeta^3} \ , \quad v^{(2)}_{\zeta^3} \tag{5.15}$$

span a negative-definite subspace of $\Gamma^{4,4} \otimes \mathbb{C}$ (i.e. correspond to the right-movers).

An automorphism g with the properties above can be explicitly constructed as follows. Let us consider the real vector space with basis vectors $x_1, \ldots, x_4$, and $y_1, \ldots, y_4$, and define a linear map g of order 5 by

$$g(x_i) = x_{i+1} \ , \qquad g(y_i) = y_{i+1} \ , \qquad\qquad i = 1, \ldots, 3 \ , \tag{5.16}$$

and

$$g(x_4) = -(x_1 + x_2 + x_3 + x_4) \ , \qquad g(y_4) = -(y_1 + y_2 + y_3 + y_4) \ . \tag{5.17}$$

A g-invariant bilinear form on the space is uniquely determined by the conditions

$$(x_i, x_i) = 0 = (y_i, y_i) \ , \qquad i = 1, \ldots, 4 \tag{5.18}$$

and

$$(x_1, x_2) = 1 \ , \quad (x_1, x_3) = (x_1, x_4) = -1 \ , \qquad (y_1, y_2) = 1 \ , \quad (y_1, y_3) = (y_1, y_4) = -1 \ , \tag{5.19}$$

as well as

$$(x_1, y_1) = 1 \ , \quad \text{and} \quad (x_i, y_1) = 0 \ , \quad (i = 2, 3, 4) \ . \tag{5.20}$$

The lattice spanned by these basis vectors is an indefinite even unimodular lattice of rank 8 and thus necessarily isomorphic to $\Gamma^{4,4}$. The g-eigenvectors can be easily constructed in terms of the basis vectors and one can verify that the eigenspaces have the correct signature.

This torus model has an additional $\mathbb{Z}_4$ symmetry group that preserves the superconformal algebra and normalises the group generated by g. The generator h of this group acts by

$$h(x_i) := g^{1-i}(x_1 + x_4 + 2y_1 + y_2 + y_3 + y_4) \ , \tag{5.21}$$

$$h(y_i) := g^{1-i}(-2x_1 - x_2 - x_3 - x_4 - y_1 - y_3 - y_4) \ , \qquad i = 1, \ldots, 4 \ , \tag{5.22}$$

on the lattice vectors. The g-eigenvectors $v^{(a)}_{\zeta^i}$, $a = 1, 2$, $i = 1, \ldots, 4$ can be defined as

$$v^{(1)}_{\zeta^i} := \sum_{j=0}^{4} \zeta^{-ij} g^j (x_1 + h(x_1)) \ , \qquad v^{(2)}_{\zeta^i} := \sum_{j=0}^{4} \zeta^{-ij} g^j (x_1 - h(x_1)) \ , \tag{5.23}$$

so that

$$h(v^{(1)}_{\zeta^i}) = -v^{(2)}_{\zeta^{-i}} \ , \qquad h(v^{(2)}_{\zeta^i}) = v^{(1)}_{\zeta^{-i}} \ . \tag{5.24}$$

Correspondingly, the action of h on the left-moving fields is

$$\psi^{(1)} \mapsto -\psi^{(2)} \ , \quad \psi^{(2)} \mapsto \psi^{(1)} \ , \quad \bar\psi^{(1)} \mapsto -\bar\psi^{(2)} \ , \quad \bar\psi^{(2)} \mapsto \bar\psi^{(1)} \ , \tag{5.25}$$

$$J^{(1)} \mapsto -J^{(2)} \ , \quad J^{(2)} \mapsto J^{(1)} \ , \quad \bar J^{(1)} \mapsto -\bar J^{(2)} \ , \quad \bar J^{(2)} \mapsto \bar J^{(1)} \ , \tag{5.26}$$

and the action on the right-moving fields is analogous. It is immediate to verify that the generators of the superconformal algebra are invariant under this transformation.

5.2 The orbifold theory

Next we want to consider the orbifold of this torus theory by the group $\mathbb{Z}_5$ that is generated by g.

5.2.1 The elliptic genus

The elliptic genus of the orbifold theory can be computed by summing over the $\mathrm{SL}(2,\mathbb{Z})$ images of the untwisted sector contribution, which in turn is given by

$$\phi^U(\tau,z) = \frac{1}{5}\sum_{k=0}^{4} \phi_{1,g^k}(\tau,z) \ , \tag{5.27}$$

where

$$\phi_{1,g^k}(\tau,z) = \mathrm{Tr}_{\mathrm{RR}}(g^k\, q^{L_0-\frac{c}{24}}\bar q^{\tilde L_0-\frac{\tilde c}{24}} y^{2J_0}(-1)^{F+\tilde F}) \ . \tag{5.28}$$

The $k=0$ contribution, i.e. the elliptic genus of the original torus theory, is zero. Each g^k-contribution, for $k=1,\ldots,4$, is the product of a factor coming from the ground states, one from the oscillators and one from the momenta

$$\phi_{1,g^k}(\tau,z) = \phi^{\mathrm{gd}}_{1,g^k}(\tau,z)\,\phi^{\mathrm{osc}}_{1,g^k}(\tau,z)\,\phi^{\mathrm{mom}}_{1,g^k}(\tau,z) \ . \tag{5.29}$$

These contributions are, respectively,

$$\phi^{\mathrm{gd}}_{1,g^k}(\tau,z) = y^{-1}(1-\zeta^k y)(1-\zeta^{-k}y)(1-\zeta^{2k})(1-\zeta^{-2k}) = 2y^{-1}+2y+1 \ , \tag{5.30}$$

$$\phi^{\mathrm{osc}}_{1,g^k}(\tau,z) = \prod_{n=1}^{\infty} \frac{(1-\zeta^k yq^n)(1-\zeta^{-k}yq^n)(1-\zeta^k y^{-1}q^n)(1-\zeta^{-k}y^{-1}q^n)}{(1-\zeta^k q^n)^2(1-\zeta^{-k}q^n)^2} \ , \tag{5.31}$$

and

$$\phi^{\mathrm{mom}}_{1,g^k}(\tau,z) = 1 \ , \tag{5.32}$$

where the last result follows because the only g-invariant state of the form (k_L,k_R) is the vacuum $(0,0)$. Thus we have

$$\phi_{1,g^k}(\tau,z) = 5\,\frac{\vartheta_1(\tau,z+\frac{k}{5})\,\vartheta_1(\tau,z-\frac{k}{5})}{\vartheta_1(\tau,\frac{k}{5})\,\vartheta_1(\tau,-\frac{k}{5})} \ , \tag{5.33}$$

where

$$\vartheta_1(\tau,z) = -iq^{1/8}y^{-\frac12}(y-1)\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \ , \tag{5.34}$$

is the first Jacobi theta function. Modular transformations of $\phi_{1,g^k}(\tau,z)$ reproduce the twining genera in the twisted sector

$$\phi_{g^l,g^k}(\tau,z) = \mathrm{Tr}_{\mathcal{H}^{(l)}}\left(g^k\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}} y^{2J_0}(-1)^{F+\tilde{F}}\right) , \tag{5.35}$$

and using the modular properties of the theta function we obtain

$$\phi_{g^l,g^k}(\tau,z) = 5\,\frac{\vartheta_1(\tau,z+\frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau,z-\frac{k}{5}-\frac{l\tau}{5})}{\vartheta_1(\tau,\frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau,-\frac{k}{5}-\frac{l\tau}{5})} , \tag{5.36}$$

for $k,l\in\mathbb{Z}/5\mathbb{Z}$, $(k,l)\neq(0,0)$. The elliptic genus of the full orbifold theory is then

$$\phi_{\mathrm{orb}}(\tau,z) = \frac{1}{5}\sum_{k,l\in\mathbb{Z}/5\mathbb{Z}}\phi_{g^l,g^k}(\tau,z) = \sum_{\substack{k,l\in\mathbb{Z}/5\mathbb{Z}\\(k,l)\neq(0,0)}}\frac{\vartheta_1(\tau,z+\frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau,z-\frac{k}{5}-\frac{l\tau}{5})}{\vartheta_1(\tau,\frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau,-\frac{k}{5}-\frac{l\tau}{5})} . \tag{5.37}$$

Since $\phi_{g^k,g^l}(\tau,0)=5$ for all $(k,l)\neq(0,0)$, we have

$$\phi_{\mathrm{orb}}(\tau,0) = \frac{1}{5}\sum_{\substack{k,l\in\mathbb{Z}/5\mathbb{Z}\\(k,l)\neq(0,0)}} 5 = 24 , \tag{5.38}$$

which shows that the orbifold theory is a non-linear sigma-model on K3. In particular, the untwisted sector has 4 RR ground states, while each of the four twisted sectors contains 5 RR ground states. For the following it will be important to understand the structure of the various twisted sectors in detail.

5.2.2 The twisted sectors

In the g^k-twisted sector, let us consider a basis of g-eigenvectors for the currents and fermionic fields. For a given eigenvalue ζ^i, $i\in\mathbb{Z}/5\mathbb{Z}$, of g^k, the corresponding currents $J^{i,a}$ and fermionic fields $\psi^{i,b}$ (where a,b labels distinct eigenvectors with the same eigenvalue) have a mode expansion

$$J^{i,a}(z) = \sum_{n\in\frac{i}{5}+\mathbb{Z}}\alpha_n^{i,a}z^{-n-1} , \qquad \psi^{i,a}(z) = \sum_{r\in\frac{i}{5}+\nu+\mathbb{Z}}\psi_r^{i,a}z^{-r-1/2} , \tag{5.39}$$

where $\nu=1/2$ in the NS- and $\nu=0$ in the R-sector. The ground states of the g^k-twisted sector are characterised by the conditions

$$\alpha_n^{i,a}|m,k\rangle = \tilde{\alpha}_n^{i,a}|m,k\rangle = 0 , \qquad \forall\ n>0,\ i,\ a , \tag{5.40}$$

$$\psi_r^{i,b}|m,k\rangle = \tilde{\psi}_r^{i,b}|m,k\rangle = 0 , \qquad \forall\ r>0,\ i,\ b , \tag{5.41}$$

where $|m,k\rangle$ denotes the mth ground state in the g^k-twisted sector. Note that since none of the currents are g-invariant, there are no current zero modes in the g^k-twisted sector, and similarly for the fermions. For a given k, the states $|m,k\rangle$ have then all the same conformal dimension, which can be calculated using the commutation relation $[L_{-1},L_1]=2L_0$ or

read off from the leading term of the modular transform of the twisted character (5.35). In the g^k-twisted NS-NS-sector the ground states have conformal dimension

$$\text{NS-NS } g^k\text{-twisted:} \qquad h = \frac{k}{5} \qquad \text{and} \qquad \tilde{h} = \frac{2k}{5} \ , \tag{5.42}$$

while in the RR-sector we have instead

$$\text{R-R } g^k\text{-twisted:} \qquad h = \tilde{h} = \frac{1}{4} \ . \tag{5.43}$$

In particular, level matching is satisfied, and thus the asymmetric orbifold is consistent [32]. The full g^k-twisted sector is then obtained by acting with the negative modes of the currents and the fermionic fields on the ground states $|m, k\rangle$.

Let us have a closer look at the ground states of the g^k twisted sector; for concreteness we shall restrict ourselves to the case $k = 1$, but the modifications for general k are minor (see below). The vertex operators $V_\lambda(z, \bar{z})$ associated to $\lambda \in \Gamma^{4,4}$, act on the ground states $|m, 1\rangle$ by

$$\lim_{z\to 0} V_\lambda(z, \bar{z})|m, 1\rangle = e_\lambda |m, 1\rangle \ , \tag{5.44}$$

where e_λ are operators commuting with all current and fermionic oscillators and satisfying

$$e_\lambda \, e_\mu = \epsilon(\lambda, \mu) \, e_{\lambda+\mu} \ , \tag{5.45}$$

for some fifth root of unity $\epsilon(\lambda, \mu)$. The vertex operators V_λ and V_μ must be local relative to one another, and this is the case provided that (see the appendix)

$$\frac{\epsilon(\lambda, \mu)}{\epsilon(\mu, \lambda)} = C(\lambda, \mu) \ , \tag{5.46}$$

where

$$C(\lambda, \mu) = \prod_{i=1}^{4} (\zeta^i)^{(g^i(\lambda), \mu)} = \zeta^{(P_g(\lambda), \mu)} \qquad \text{with} \qquad P_g(\lambda) = \sum_{i=1}^{4} i g^i(\lambda) \ . \tag{5.47}$$

The factor $C(\lambda, \mu)$ has the properties

$$C(\lambda, \mu_1 + \mu_2) = C(\lambda, \mu_1) \, C(\lambda, \mu_2) \ , \qquad C(\lambda_1 + \lambda_2, \mu) = C(\lambda_1, \mu) \, C(\lambda_2, \mu) \ , \tag{5.48}$$

$$C(\lambda, \mu) = C(\mu, \lambda)^{-1} \ , \tag{5.49}$$

$$C(\lambda, \mu) = C(g(\lambda), g(\mu)) \ . \tag{5.50}$$

Because of (5.48), $C(\lambda, 0) = C(0, \lambda) = 1$ for all $\lambda \in \Gamma^{4,4}$, and we can set

$$e_0 = 1 \ , \tag{5.51}$$

so that $\epsilon(0, \lambda) = 1 = \epsilon(\lambda, 0)$. More generally, for the vectors λ in the sublattice

$$R := \{\lambda \in \Gamma^{4,4} \mid C(\lambda, \mu) = 1 \ , \ \text{for all } \mu \in \Gamma^{4,4}\} \subset \Gamma^{4,4} \ , \tag{5.52}$$

we have $C(\lambda + \mu_1, \mu_2) = C(\mu_1, \mu_2)$, for all $\mu_1, \mu_2 \in \Gamma^{4,4}$, so that we can set

$$e_{\mu+\lambda} = e_\mu \ , \qquad \forall \lambda \in R \ , \quad \mu \in \Gamma^{4,4} \ . \tag{5.53}$$

Thus, we only need to describe the operators corresponding to representatives of the group $\Gamma^{4,4}/R$. The vectors $\lambda \in R$ are characterised by

$$(P_g(\lambda), \mu) \equiv 0 \mod 5 \ , \qquad \text{for all } \mu \in \Gamma^{4,4} \ , \tag{5.54}$$

and since $\Gamma^{4,4}$ is self-dual this condition is equivalent to

$$P_g(\lambda) \in 5\,\Gamma^{4,4} \ . \tag{5.55}$$

Since g has no invariant subspace, we have the identity

$$1 + g + g^2 + g^3 + g^4 = 0 \tag{5.56}$$

that implies (see (5.47))

$$P_g \circ (1-g) = (1-g) \circ P_g = -5 \cdot \mathbf{1} \ . \tag{5.57}$$

Thus, $\lambda \in R$ if and only if

$$P_g(\lambda) = P_g \circ (1-g)(\tilde{\lambda}) \ , \tag{5.58}$$

for some $\tilde{\lambda} \in \Gamma^{4,4}$, and since P_g has trivial kernel (see (5.57)), we finally obtain

$$R = (1-g)\,\Gamma^{4,4} \ . \tag{5.59}$$

Since also $(1-g)$ has trivial kernel, R has rank 8 and $\Gamma^{4,4}/R$ is a finite group. Furthermore,

$$|\Gamma^{4,4}/R| = \det(1-g) = 25 \ , \tag{5.60}$$

and, since $5\,\Gamma^{4,4} \subset R$, the group $\Gamma^{4,4}/R$ has exponent 5. The only possibility is

$$\Gamma^{4,4}/R \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \ . \tag{5.61}$$

Let $x, y \in \Gamma^{4,4}$ be representatives for the generators of $\Gamma^{4,4}/R$. By (5.49), we know that $C(x,x) = C(y,y) = 1$, so that $C(x,y) \neq 1$ (otherwise C would be trivial over the whole $\Gamma^{4,4}$), and we can choose x, y such that

$$C(x,y) = \zeta \ . \tag{5.62}$$

Thus, the ground states form a representation of the algebra of operators generated by e_x, e_y, satisfying

$$e_x^5 = 1 = e_y^5 \ , \qquad e_x e_y = \zeta e_y e_x \ . \tag{5.63}$$

The group generated by e_x and e_y is the extra-special group 5^{1+2}, and all its non-abelian irreducible representations[‖] are five dimensional.

In particular, for the representation on the g-twisted ground states, we can choose a basis of e_x-eigenvectors

$$|m;1\rangle \quad \text{with} \quad e_x|m;1\rangle = \zeta^m |m;1\rangle \ , \qquad m \in \mathbb{Z}/5\mathbb{Z} \ , \tag{5.64}$$

[‖] We call a representation non-abelian if the central element does not act trivially.

and define the action of the operators e_y by

$$e_y|m;1\rangle = |m+1;1\rangle \ . \tag{5.65}$$

For any vector $\lambda \in \Gamma^{4,4}$, there are unique $a, b \in \mathbb{Z}/5\mathbb{Z}$ such that $\lambda = ax + by + (1-g)(\mu)$ for some $\mu \in \Gamma^{4,4}$ and we define**

$$e_\lambda := e_x^a e_y^b \ . \tag{5.66}$$

Since $g(\lambda) = ax + by + (1-g)(g(\mu) - ax - by)$, by (5.53) we have

$$e_{g(\lambda)} = e_\lambda \ , \tag{5.67}$$

so that, with respect to the natural action $g(e_\lambda) := e_{g(\lambda)}$, the algebra is g-invariant. This is compatible with the fact that all ground states have the same left and right conformal weights h and $\tilde{h}$, so that the action of $g = e^{2\pi i(h-\tilde{h})}$ is proportional to the identity.

The construction of the g^k-twisted sector, for $k = 2, 3, 4$, is completely analogous to the g^1-twisted case, the only difference being that the root ζ in the definition of $C(\lambda, \mu)$ should be replaced by ζ^k. Thus, one can define operators $e_x^{(k)}, e_y^{(k)}$ on the g^k-twisted sector, for each $k = 1, \ldots, 4$, satisfying

$$(e_x^{(k)})^5 = 1 = (e_y^{(k)})^5 \ , \qquad e_x^{(k)} e_y^{(k)} = \zeta^k e_y^{(k)} e_x^{(k)} \ . \tag{5.68}$$

The action of these operators on the analogous basis $|m;k\rangle$ with $m \in \mathbb{Z}/5\mathbb{Z}$ is then

$$e_x^{(k)}|m;k\rangle = \zeta^m |m;k\rangle \ , \quad e_y^{(k)}|m;k\rangle = |m+k;k\rangle \ . \tag{5.69}$$

5.2.3 Spectrum and symmetries

The spectrum of the actual orbifold theory is finally obtained from the above twisted sectors by projecting onto the g-invariant states; technically, this is equivalent to restricting to the states for which the difference of the left- and right- conformal dimensions is integer, $h - \tilde{h} \in \mathbb{Z}$. In particular, the RR ground states (5.69) in each (twisted) sector have $h = \tilde{h} = 1/4$, so that they all survive the projection. Thus, the orbifold theory has 4 RR ground states in the untwisted sector (the spectral flow generators), forming a $(\mathbf{2}, \mathbf{2})$ representation of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, and 5 RR ground states in each twisted sector, which are singlets of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. In total there are therefore 24 RR ground states, as expected for a non-linear sigma-model on K3. (Obviously, we are here just reproducing what we already saw in (5.38).)

Next we want to define symmetry operators acting on the orbifold theory. First we can construct operators e_λ associated to $\lambda \in \Gamma^{4,4}$, that will form the extra special group 5^{1+2}. They are defined to act by $e_\lambda^{(k)}$ on the g^k-twisted sector. The action of the untwisted sector preserves the subspaces $\mathcal{H}_m^U$, $m \in \mathbb{Z}/5\mathbb{Z}$, of states with momentum of the form $\lambda = nx + my + (1-g)(\mu)$, for some $n \in \mathbb{Z}$ and $\mu \in \Gamma^{4,4}$. Let us denote by $T_{m;k}$ a generic vertex operator associated with a g^k-twisted state, $k = 1, \ldots, 4$, with e_x-eigenvalue ζ^m,

**The ordering of e_x and e_y in this definition is arbitrary; however, any other choice corresponds to a redefinition $\tilde{e}_\lambda = c(\lambda) e_\lambda$, for some fifth root of unity $c(\lambda)$, that does not affect the commutation relations $\tilde{e}_\lambda \tilde{e}_\mu = C(\lambda, \mu) \tilde{e}_\mu \tilde{e}_\lambda$.

$m \in \mathbb{Z}/5\mathbb{Z}$, and by $T_{m;0}$ a vertex operator associated with a state in $\mathcal{H}^U_m$. Consistency of the OPE implies the fusion rules

$$T_{m;k} \times T_{m';k'} \to T_{m+m';k+k'} \ . \tag{5.70}$$

These rules are preserved by the maps

$$T_{m;k} \mapsto e_\lambda \, T_{m;k} \, e_\lambda^{-1} \ , \qquad \lambda \in \Gamma^{4,4} \ , \tag{5.71}$$

which therefore define symmetries of the orbifold theory. As we have explained above, these symmetries form the extra-special group 5^{1+2}.

Finally, the symmetries (5.21), (5.22), (5.25) and (5.26) of the original torus theory induce a $\mathbb{Z}_4$-group of symmetries of the orbifold. Since $h^{-1}gh = g^{-1}$, the space of g-invariant states of the original torus theory is stabilised by h, so that h restricts to a well-defined transformation on the untwisted sector of the orbifold. Furthermore, h maps the g^k- to the g^{5-k}-twisted sector. Eqs. (5.21) and (5.22) can be written as

$$h(x_1) = 2x_1 + (1-g)(-x_1 - x_2 - x_3 + y_1 + y_2 + y_3 + y_4) \ , \tag{5.72}$$
$$h(y_1) = 2y_1 + (1-g)(-x_1 - x_2 - x_3 - x_4 - 2y_1 - y_2 - y_3 - y_4) \ . \tag{5.73}$$

It follows that the action of h on the operators $e_\lambda^{(k)}$, $k = 1, \ldots, 4$ must be

$$h e_x^{(k)} h^{-1} = e_{2x}^{(5-k)} \ , \qquad h e_y^{(k)} h^{-1} = e_{2y}^{(5-k)} \ , \tag{5.74}$$

and it is easy to verify that this transformation is compatible with (5.68). Correspondingly, the action on the twisted sector ground states is

$$h|m;k\rangle = |3m;5-k\rangle \ , \tag{5.75}$$

and it is consistent with (5.70).

Thus the full symmetry group is the semi-direct product

$$G = 5^{1+2} : \mathbb{Z}_4 \ , \tag{5.76}$$

where the generator $h \in \mathbb{Z}_4$ maps the central element $\zeta \in 5^{1+2}$ to ζ^{-1}.

All of these symmetries act trivially on the superconformal algebra and on the spectral flow generators, and therefore define symmetries in the sense of the general classification theorem. Indeed, G agrees precisely with the group in case (ii) of the theorem. Thus our orbifold theory realises this possibility.

6 Models with symmetry group containing $3^{1+4} : \mathbb{Z}_2$

Most of the torus orbifold construction described in the previous section generalises to symmetries g of order different than 5. In particular, one can show explicitly that orbifolds of $\mathbb{T}^4$ models by a symmetry g of order 3 contain a group of symmetries $3^{1+4} : \mathbb{Z}_2$, so that they belong to one of the cases (iii) and (iv) of the theorem, as expected from the discussion in section 4.

We take the action of the symmetry g on the left-moving currents and fermionic fields to be of the form (5.9) and (5.11), where ζ is a now a third root of unity; analogous transformations hold for the right-moving fields with respect to a third root of unity $\tilde{\zeta}$. In this case, eq.(5.12) can be satisifed by

$$\zeta = \tilde{\zeta} = e^{\frac{2\pi i}{3}} \ , \tag{6.1}$$

so that the action is left-right symmetric and g admits an interpretation as a geometric $\mathrm{O}(4,\mathbb{R})$-rotation of order 3 of the torus $\mathbb{T}^4$. For example, the torus $\mathbb{R}^4/(A_2 \oplus A_2)$, where A_2 is the root lattice of the $su(3)$ Lie algebra, with vanishing Kalb-Ramond field, admits such an automorphism.

The orbifold by g contains 6 RR ground states in the untwisted sector. In the kth twisted sector, $k = 1, 2$, the ground states form a representation of an algebra of operators $e^{(k)}_\lambda$, $\lambda \in \Gamma^{4,4}$, satisfying the commutation relations

$$e^{(k)}_\lambda e^{(k)}_\mu = C(\lambda,\mu)^k e^{(k)}_\mu e^{(k)}_\lambda \ , \tag{6.2}$$

where

$$C(\lambda,\mu) = \zeta^{(P_g(\lambda),\mu)} \ , \qquad P_g = g + 2g^2 \ . \tag{6.3}$$

As discussed in section 5.2.2, we can set

$$e^{(k)}_{\lambda+\mu} = e^{(k)}_\mu \ , \qquad \forall \lambda \in R, \mu \in \Gamma^{4,4} \ , \tag{6.4}$$

where

$$R = (1-g)\Gamma^{4,4} \ . \tag{6.5}$$

(Note that $\Gamma^{4,4}$ contains no g-invariant vectors). The main difference with the analysis of section 5.2.2 is that, in this case,

$$\Gamma^{4,4}/R \cong \mathbb{Z}_3^4 \ . \tag{6.6}$$

In particular, we can find vectors $x_1, x_2, y_1, y_2 \in \Gamma^{4,4}$ such that

$$C(x_i, y_j) = \zeta^{\delta_{ij}} \ , \qquad C(x_i, x_j) = C(y_i, y_j) = 1 \ . \tag{6.7}$$

The corresponding operators obey the relations

$$e^{(k)}_{x_i} e^{(k)}_{y_j} = \zeta^{k\delta_{ij}} e^{(k)}_{y_j} e^{(k)}_{x_i} \ , \quad e^{(k)}_{x_i} e^{(k)}_{x_j} = e^{(k)}_{x_j} e^{(k)}_{x_i} \ , \quad e^{(k)}_{y_i} e^{(k)}_{y_j} = e^{(k)}_{y_j} e^{(k)}_{y_i} \ , \tag{6.8}$$

as well as

$$(e^{(k)}_{x_i})^3 = 1 = (e^{(k)}_{y_i})^3 \ . \tag{6.9}$$

These operators generate the extra-special group 3^{1+4} of exponent 3, and the kth-twisted ground states form a representation of this group. We can choose a basis $|m_1, m_2; k\rangle$, with $m_1, m_2 \in \mathbb{Z}/3\mathbb{Z}$, of simultaneous eigenvectors of $e^{(k)}_{x_1}$ and $e^{(k)}_{x_2}$, so that

$$e^{(k)}_{x_i}|m_1, m_2; k\rangle = \zeta^{m_i}|m_1, m_2; k\rangle \ , \qquad e^{(k)}_{y_i}|m_1, m_2; k\rangle = |m_1 + k\delta_{1i}, m_2 + k\delta_{2i}; k\rangle \ . \tag{6.10}$$

The resulting orbifold model has 9 RR ground states in each twisted sector, for a total of $6 + 9 + 9 = 24$ RR ground states, as expected for a K3 model. As in section 5.2.3, the

group 3^{1+4} generated by $e_\lambda^{(k)}$ extends to a group of symmetries of the whole orbifold model. Furthermore, the $\mathbb{Z}_2$-symmetry that flips the signs of the coordinates in the original torus theory induces a symmetry h of the orbifold theory, which acts on the twisted sectors by

$$h\,|m_1, m_2; k\rangle = |-m_1, -m_2; k\rangle \ . \tag{6.11}$$

We conclude that the group G of symmetries of any torus orbifold $\mathbb{T}^4/\mathbb{Z}_3$ contains a subgroup $3^{1+4} : \mathbb{Z}_2$, and is therefore included in the cases (iii) or (iv) of the classification theorem. This obviously ties in nicely with the general discussion of section 4.

7 Conclusions

In this paper we have reviewed the current status of the EOT conjecture concerning a possible $\mathbb{M}_{24}$ symmetry appearing in the elliptic genus of K3. We have explained that, in some sense, the EOT conjecture has already been proven since twining genera, satisfying the appropriate modular and integrality properties, have been constructed for all conjugacy classes of $\mathbb{M}_{24}$. However, the analogue of the Monster conformal field theory that would 'explain' the underlying symmetry has not yet been found. In fact, no single K3 sigma-model will be able to achieve this since none of them possesses an automorphism group that contains $\mathbb{M}_{24}$.

Actually, the situation is yet further complicated by the fact that there are K3 sigma-models whose automorphism group is *not even a subgroup of* $\mathbb{M}_{24}$; on the other hand, the elliptic genus of K3 does not show any signs of exhibiting 'Moonshine' with respect to any larger group. As we have explained in this paper, most of the exceptional automorphism groups (i.e. automorphism groups that are not subgroups of $\mathbb{M}_{24}$) appear for K3s that are torus orbifolds. In fact, all cyclic torus orbifolds are necessarily exceptional in this sense, and (cyclic) torus orbifolds account for all incarnations of the cases (ii) – (iv) of the classification theorem of [7] (see section 4). We have checked these predictions by explicitly constructing an asymmetric $\mathbb{Z}_5$ orbifold that realises case (ii) of the theorem (see section 5), and a family of $\mathbb{Z}_3$ orbifolds realising cases (iii) and (iv) of the theorem (see section 6). Incidentally, these constructions also demonstrate that the exceptional cases (ii)-(iv) actually appear in the K3 moduli space — in the analysis of [7] this conclusion relied on some assumption about the regularity of K3 sigma-models.

The main open problem that remains to be understood is why precisely $\mathbb{M}_{24}$ is 'visible' in the elliptic genus of K3, rather than any smaller (or indeed bigger) group. Recently, we have constructed (some of) the twisted twining elliptic genera of K3 [33], i.e. the analogues of Simon Norton's generalised Moonshine functions [34]. We hope that they will help to shed further light on this question.

Acknowledgements

We thank Terry Gannon, Gerald Hoehn, and Michael Tuite for useful conversations. The research of MRG is supported in parts by the Swiss National Science Foundation.

A Commutation relations in the twisted sector

The vertex operators $V_\lambda(z,\bar z)$ in the g-twisted sector can be defined in terms of formal exponentials of current oscillators

$$E^\pm_\lambda(z,\bar z) := \exp\Big(\sum_{\substack{r\in\frac15\mathbb{Z}\\ \pm r>0}} (\lambda_L\cdot\alpha)^{(r)}_r \frac{z^{-r}}{r}\Big)\exp\Big(\sum_{\substack{r\in\frac15\mathbb{Z}\\ \pm r>0}} (\lambda_R\cdot\tilde\alpha)^{(r)}_r \frac{\bar z^{-r}}{r}\Big)\ , \tag{A.1}$$

where $(\lambda_L\cdot\alpha)^{(r)}_r$ and $(\lambda_R\cdot\tilde\alpha)^{(r)}_r$ are the r-modes of the currents

$$(\lambda_L\cdot\partial X)^{(r)} := \frac15\sum_{i=0}^4 \zeta^{5ir}\lambda_L\cdot g^i(\partial X) = \frac15\sum_{i=0}^4 \zeta^{5ir} g^{-i}(\lambda_L)\cdot\partial X\ , \tag{A.2}$$

$$(\lambda_R\cdot\bar\partial X)^{(r)} := \frac15\sum_{i=0}^4 \bar\zeta^{5ir}\lambda_R\cdot g^i(\bar\partial X) = \frac15\sum_{i=0}^4 \bar\zeta^{5ir} g^{-i}(\lambda_R)\cdot\bar\partial X\ . \tag{A.3}$$

Then we can define

$$V_\lambda(z,\bar z) := E^-_\lambda(z,\bar z)\, E^+_\lambda(z,\bar z)\, e_\lambda\ , \tag{A.4}$$

where the operators e_λ commute with all current oscillators and satisfy

$$e_\lambda\, e_\mu = \epsilon(\lambda,\mu)\, e_{\lambda+\mu}\ , \tag{A.5}$$

for some fifth root of unity $\epsilon(\lambda,\mu)$. The commutator factor

$$C(\lambda,\mu) := \frac{\epsilon(\lambda,\mu)}{\epsilon(\mu,\lambda)}\ , \tag{A.6}$$

can be determined by imposing the locality condition

$$V_\lambda(z_1,\bar z_1)\, V_\mu(z_2,\bar z_2) = V_\mu(z_2,\bar z_2)\, V_\lambda(z_1,\bar z_1)\ . \tag{A.7}$$

To do so, we note that the commutation relations between the operators $E^\pm_\lambda$ can be computed, as in [35], using the Campbell-Baker-Hausdorff formula

$$E^+_\lambda(z_1,\bar z_1)E^-_\mu(z_2,\bar z_2) = E^-_\mu(z_2,\bar z_2)E^+_\lambda(z_1,\bar z_1) \prod_{i=0}^4 [(1-\zeta^{-i}(\tfrac{z_1}{z_2})^{\frac15})^{g^i(\lambda)_L\cdot\mu_L}(1-\bar\zeta^{-i}(\tfrac{\bar z_1}{\bar z_2})^{\frac15})^{g^i(\lambda)_R\cdot\mu_R}]\ . \tag{A.8}$$

Using (A.8) and $e_\lambda e_\mu = C(\lambda,\mu)e_\mu e_\lambda$, the locality condition is then equivalent to

$$C(\lambda,\mu)\prod_{i=0}^4 \frac{(1-\zeta^{-i}(\frac{z_1}{z_2})^{\frac15})^{g^i(\lambda)_L\cdot\mu_L}(1-\bar\zeta^{-i}\frac{\bar z_1^{1/5}}{\bar z_2^{1/5}})^{g^i(\lambda)_R\cdot\mu_R}}{(1-\zeta^{i}(\frac{z_2}{z_1})^{\frac15})^{g^i(\lambda)_L\cdot\mu_L}(1-\bar\zeta^{i}(\frac{\bar z_2}{\bar z_1})^{\frac15})^{g^i(\lambda)_R\cdot\mu_R}} = 1\ , \tag{A.9}$$

that is

$$C(\lambda,\mu)\Big(-\frac{z_1^{1/5}}{z_2^{1/5}}\Big)^{\sum_i g^i(\lambda)_L\cdot\mu_L}\Big(-\frac{\bar z_1^{1/5}}{\bar z_2^{1/5}}\Big)^{\sum_i g^i(\lambda)_R\cdot\mu_R}\prod_{i=0}^4[(\zeta^{-i})^{g^i(\lambda)_L\cdot\mu_L}(\bar\zeta^{-i})^{g^i(\lambda)_R\cdot\mu_R}] = 1\ . \tag{A.10}$$

Since $\Gamma^{4,4}$ has no g-invariant vector, we have the identities

$$\sum_{i=0}^{4} g^i(\lambda)_L = 0 = \sum_{i=0}^{4} g^i(\lambda)_R \ , \tag{A.11}$$

and hence finally obtain

$$C(\lambda,\mu) = \prod_{i=0}^{4} (\zeta^i)^{g^i(\lambda_L)\cdot\mu_L - g^i(\lambda_R)\cdot\mu_R} \ . \tag{A.12}$$

References

[1] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 surface and the Mathieu group M_{24}*, Exper. Math. **20** (2011) 91 [arXiv:1004.0956 [hep-th]].

[2] J.G. Thompson, *Some numerology between the Fischer-Griess Monster and the elliptic modular function*, Bull. Lond. Math. Soc. **11** (1979) 352.

[3] J.H. Conway and S. Norton, *Monstrous Moonshine*, Bull. Lond. Math. Soc. **11** (1979) 308.

[4] T. Gannon, *Moonshine beyond the Monster: The Bridge connecting Algebra, Modular Forms and Physics*, Cambridge University Press (2006).

[5] R. Borcherds, *Vertex algebras, Kac-Moody algebras and the Monster*, Proc. Nat. Acad. Sci. U.S.A. **83** (1986) 3068.

[6] I. Frenkel, J. Lepowski and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press (1986).

[7] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Symmetries of K3 sigma models*, arXiv:1106.4315 [hep-th], to appear in Commun. Number Theory Phys.

[8] M.A. Walton, *Heterotic string on the simplest Calabi-Yau manifold and its orbifold limits*, Phys. Rev. D **37** (1988) 377.

[9] M.C.N. Cheng, *K3 Surfaces, N=4 dyons, and the Mathieu group M_{24}*, Commun. Number Theory Phys. **4** (2010) 623 [arXiv:1005.5415 [hep-th]].

[10] M.C.N. Cheng and J.F.R. Duncan, *On Rademacher sums, the largest Mathieu group, and the holographic modularity of moonshine*, arXiv:1110.3859 [math.RT].

[11] M.C.N. Cheng and J.F.R. Duncan, *The largest Mathieu group and (mock) automorphic forms*, arXiv:1201.4140 [math.RT].

[12] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, *Umbral moonshine*, arXiv:1204.2779 [math.RT].

[13] S. Govindarajan, *BKM Lie superalgebras from counting twisted CHL dyons*, JHEP **1105** (2011) 089 [arXiv:1006.3472 [hep-th]].

[14] S. Govindarajan, *Brewing moonshine for Mathieu*, arXiv:1012.5732 [math.NT].

[15] S. Govindarajan, *Unravelling Mathieu moonshine*, arXiv:1106.5715 [hep-th].

[16] S. Hohenegger and S. Stieberger, *BPS saturated string amplitudes: K3 elliptic genus and Igusa cusp form*, Nucl. Phys. B **856** (2012) 413 [arXiv:1108.0323 [hep-th]].

[17] T. Kawai, Y. Yamada, and S.-K. Yang, *Elliptic genera and $\mathcal{N} = 2$ superconformal field theory*, Nucl. Phys. B **414** (1994) 191 [arXiv:hep-th/9306096].

[18] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Birkhäuser (1985).

[19] S. Mukai, *Finite groups of automorphisms of $K3$ surfaces and the Mathieu group*, Invent. Math. **94** (1988) 183.

[20] S. Kondo, *Niemeier lattices, Mathieu groups and finite groups of symplectic automorphisms of K3 surfaces*, Duke Math. Journal **92** (1998) 593, appendix by S. Mukai.

[21] J.G. Thompson, *Finite groups and modular functions*, Bull. London Math. Soc. **11** (1979) 347.

[22] J.R. David, D.P. Jatkar and A. Sen, *Product representation of dyon partition function in CHL models*, JHEP **0606** (2006) 064 [arXiv:hep-th/0602254].

[23] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu moonshine in the elliptic genus of K3*, JHEP **1010** (2010) 062 [arXiv:1008.3778 [hep-th]].

[24] T. Eguchi and K. Hikami, *Note on twisted elliptic genus of K3 surface*, Phys. Lett. B **694** (2011) 446 [arXiv:1008.4924 [hep-th]].

[25] M.R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu twining characters for K3*, JHEP **1009** (2010) 058 [arXiv:1006.0221 [hep-th]].

[26] T.J. Gannon, *Much ado about Mathieu*, to appear.

[27] M. Fluder, *Symmetries of non-linear sigma models on K3*, Master thesis, ETH Zürich, August 2011.

[28] M.P. Tuite, *Monstrous moonshine and the uniqueness of the moonshine module*, arXiv:hep-th/9211069.

[29] M.P. Tuite, *On the relationship between monstrous moonshine and the uniqueness of the moonshine module*, Commun. Math. Phys. **166** (1995) 495 [arXiv:hep-th/9305057].

[30] P.H. Ginsparg, *Applied conformal field theory,* arXiv:hep-th/9108028.

[31] R. T. Curtis, *On subgroups of $\cdot O$. II. Local structure*, J. Algebra **63** (1980) 413.

[32] K.S. Narain, M.H. Sarmadi and C. Vafa, *Asymmetric orbifolds*, Nucl. Phys. B **288** (1987) 551.

[33] M.R. Gaberdiel, D. Persson, H. Ronellenfitsch and R. Volpato, *in preparation*.

[34] S.P. Norton, *Generalised moonshine*, Proc. Symp. Pure Math. **47** (1987) 209.

[35] J. Lepowsky, *Calculus of twisted vertex operators*, Proc. Nat. Acad. Sci. U.S.A. **82** (1985) 8295.

arXiv:1211.7074v1 [hep-th] 29 Nov 2012

AEI-2012-197

GENERALISED MATHIEU MOONSHINE

MATTHIAS R. GABERDIEL, DANIEL PERSSON, HENRIK RONELLENFITSCH, AND ROBERTO VOLPATO

Abstract. The Mathieu twisted twining genera, i.e. the analogues of Norton's generalised Moonshine functions, are constructed for the elliptic genus of K3. It is shown that they satisfy the expected consistency conditions, and that their behaviour under modular transformations is controlled by a 3-cocycle in $H^3(M_{24}, U(1))$, just as for the case of holomorphic orbifolds. This suggests that a holomorphic VOA may be underlying Mathieu Moonshine.

Contents

1. Introduction 2
1.1. Monstrous Moonshine 2
1.2. Mathieu Moonshine 3
1.3. Generalised Monstrous Moonshine 4
1.4. Generalised Mathieu Moonshine 4
1.5. Outline 5
2. Twisted Twining Genera 6
2.1. Properties of the twisted twining genera 6
2.2. Cohomological obstructions 8
2.3. Classification of independent twisted twining genera 10
2.4. Modular properties of the twisted twining genera 12
2.5. Explicit twining genera 14
3. Holomorphic Orbifolds and Group Cohomology 14
3.1. Review of holomorphic orbifolds 15
3.2. Application to Mathieu Moonshine 18
3.3. Determining the cohomology class 19
3.4. Computation of the twisted twining characters: an example 22
3.5. Projective representations 24
4. K3 Orbifolds 25

4.1. The relabelling phenomenon 26
4.2. An example 28
4.3. Computation of a twisted twining genus 29
5. Conclusions 32
5.1. Summary 32
5.2. Open problems and future work 33
Acknowledgements 35
Appendix A. Details on the unobstructed twisted twining genera 35
A.1. The character $\phi_{2B,4A}$ (group 13) $\phi_{4B,4A}$ (groups 27 and 28) 35
A.2. The cases $\phi_{3A,3B}$ (group 31) and $\phi_{3A,3A}$ (group 32) 37
Appendix B. Some group cohomology 39
Appendix C. Projective representations of finite groups 41
C.1. Central extension and orbifolds 42
Appendix D. Character Tables for $C_{M_{24}}(g)$ and decompositions 43
Appendix E. Decompositions of twisted sectors 44
References 44

1. Introduction

1.1. **Monstrous Moonshine.** In mathematics and physics the word 'Moonshine' has come to represent various surprising and deep connections between a priori unrelated fields, such as number theory, representation theory of finite groups, algebra and quantum field theory. The first and most well-known example of such a connection is, of course, Conway and Norton's *Monstrous Moonshine conjecture* [1]. Their starting point was the observation of McKay that the first few Fourier coefficients of the modular invariant J-function $J(\tau)$ are dimensions of representations of the Monster group $\mathbb{M}$, the largest finite simple sporadic group. Conway and Norton then conjectured that to each element $g \in \mathbb{M}$ of the Monster, one can associate a function $T_g(\tau)$ (the so-called McKay-Thompson series) on the upper half-plane $\mathbb{H}_+$ that is invariant under some subgroup $\Gamma_g \subset SL(2,\mathbb{R})$. They also conjectured that the invariance group Γ_g must be genus zero, meaning that, as a Riemann surface, the quotient $\Gamma_g \backslash \mathbb{H}_+$ is topologically a sphere. Furthermore, they conjectured that $T_g(\tau)$ is in fact the Hauptmodul for Γ_g.

Subsequently, Frenkel, Lepowsky and Meurman (FLM) [2] constructed a graded $\mathbb{M}$-module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$, such that the dimension of the graded subspaces $a(n) = \dim V_n^\natural$ reproduce

precisely the Fourier coefficients in the q-expansion of the J-function, $J(\tau) = \sum_{n=-1}^{\infty} a(n)q^n$, $q = e^{2\pi i\tau}$. The coefficients $a_g(n)$ in the Fourier expansion of the McKay-Thompson series are then identified with the characters $\mathrm{Tr}_{V_n^\natural}(g)$. Physically, the FLM-module is the $\mathbb{Z}_2$-orbifold of the open bosonic string compactified on the Leech torus $(\mathbb{R}^{24}/\Lambda_{\mathrm{Leech}})$, where Λ_{Leech} is the Leech lattice. The FLM-construction provided important clues into the moonshine conjectures, but fell short of proving them. The final proof was found by Borcherds [3], making heavy use of the FLM-module $V^\natural$, but also introducing yet another set of ingredients to the story, namely generalised Kac-Moody algebras and automorphic denominator identities. Thus, by the time the original conjectures were proven, Monstrous Moonshine encompassed not only the realms of finite groups and modular forms, but also string theory and infinite-dimensional algebras, for a nice review see [4].

1.2. **Mathieu Moonshine.** Recently, a very interesting new moonshine phenomenon was conjectured by Eguchi, Ooguri and Tachikawa (EOT) [5]: they observed that the first few Fourier coefficients of the elliptic genus of K3 are dimensions of representations of the largest Mathieu group M_{24}. This suggests the existence of a new moonshine-type connection between Mathieu groups, Jacobi forms and K3 surfaces. The analogues of the McKay-Thompson series, the *twining elliptic genera* $\phi_g(\tau, z)$, $g \in M_{24}$, were constructed in a series of papers [6, 7, 8, 9], and it was shown that they are all weak Jacobi forms of weight 0 and index 1 (with multiplier system) for subgroups $\Gamma_0(N_g) \subset SL(2,\mathbb{Z})$, where N_g is the order of the element g. (The elliptic genus ϕ_{K3} itself corresponds to taking g to be the identity element.) The compatibility with M_{24}-representations was checked up to the first 600 coefficients [8, 9]; according to Gannon [10] this is sufficient to prove that all Fourier coefficients of the elliptic genus of K3 are dimensions of M_{24} representations. It is also shown in [10] that the multiplicities of the real M_{24} representations are always even, see also [11].

Although in a certain sense the above results establish the Mathieu Moonshine conjecture, there are many aspects of it that are much less understood compared to Monstrous Moonshine. For instance, the genus zero property of Monstrous Moonshine does not seem to hold for the M_{24}-twining genera since some of the invariance groups $\Gamma_0(N_g)$ are not genus zero.* More importantly, the analogue of the FLM module $V^\natural$ is not yet known for Mathieu Moonshine, i.e. we do not know of any CFT with automorphism group M_{24} whose elliptic

*It was proposed in [12] that the correct generalisation of the genus zero property is a certain Rademacher-summability condition, which does seem to be satisfied in all cases.

genus reproduces the elliptic genus of K3. In particular, none of the K3-sigma models have this property since M_{24} is not contained in the automorphism group of any of them [13].

1.3. **Generalised Monstrous Moonshine.** A few years after the original Monstrous Moonshine conjectures, Norton proposed a generalisation that he dubbed *generalised Moonshine* [14]. He suggested that for each commuting pair of Monster group elements $g, h \in \mathbb{M}$ there exists a function $f(g, h; \tau)$ that is also invariant under a genus zero subgroup of $SL(2, \mathbb{R})$. These functions generalise the McKay-Thompson series to which they reduce for $g = e$. Furthermore, they transform into one another under a simultaneous action of $SL(2, \mathbb{Z})$ on $(\tau; g, h) \in \mathbb{H}_+ \times \mathbb{M} \times \mathbb{M}$. Finally, the Fourier coefficients in the q-expansion of $f(g, h; \tau)$ are conjectured to be dimensions of (projective) representations of the centraliser of g in $\mathbb{M}$, $C_{\mathbb{M}}(g) = \{k \in \mathbb{M} : gk = kg\}$. Although this conjecture has been proven in special cases [15, 16], the general case is still open.[†]

The generalised Moonshine conjecture was given a physical interpretation by Dixon, Ginsparg and Harvey [21] that was later elaborated upon by Tuite [16]. They showed that the Norton series $f(g, h; \tau)$ arises naturally as the character in the twisted sector $V_g^\natural$ of an orbifold of the Monster CFT $V^\natural$ by the element $g \in \mathbb{M}$, 'twined' by the group element h; in standard CFT language, they can therefore be interpreted as

$$f(g, h; \tau) = {}_g\underset{h}{\square} = \mathrm{Tr}_{V_g^\natural}\left(h\, q^{L_0 - 1}\right) \ . \tag{1}$$

Many of the properties conjectured by Norton then follow from standard holomorphic orbifold considerations.

1.4. **Generalised Mathieu Moonshine.** In this paper, we show that Norton's generalisation also applies to Mathieu Moonshine. More specifically, for every pair of commuting group elements $g, h \in M_{24}$, we construct 'twisted twining genera' $\phi_{g,h} : \mathbb{H}_+ \times \mathbb{C} \to \mathbb{C}$ that either vanish or are weak Jacobi forms of weight 0 and index 1 for some $\Gamma_{g,h} \subset SL(2, \mathbb{Z})$. For $g = e$, they reduce to the twining genera of [6, 7, 8, 9], and they transform under the modular group $SL(2, \mathbb{Z})$ into one another. The multiplier phases that appear in these transformations behave as though these twisted twining genera were twisted twining characters of a holomorphic orbifold; in particular, they are controlled by a 3-cocycle $\alpha \in H^3(M_{24}, U(1))$ via a formula that was first written down by Dijkgraaf and Witten in [22]. Furthermore, the Fourier coefficients of the twisted elliptic genera $f(g, e; \tau)$ equal dimensions of projective representations of the centraliser $C_{M_{24}}(g)$, and the central extension that characterises the

[†]Carnahan has announced a series of papers [17, 18, 19, 20] which he claims will lead to a complete proof.

projective representation is again determined by the cohomology class $\alpha \in H^3(M_{24}, U(1))$.

The idea for using the cohomology group $H^3(M_{24}, U(1))$ and the formalism of Dijkgraaf and Witten in order to understand the multiplier phases of the twisted twining genera was first suggested to us by Terry Gannon in 2011 [23], see also [10]. Mason has also speculated [24] that $H^3(\mathbb{M}, U(1))$ should play a similar role in the context of generalised Monstrous Moonshine, although this has, to our knowledge, not yet been worked out (partially, because $H^3(\mathbb{M}, U(1))$ is unknown).

We should also mention that Mason has previously proposed a version of generalised Moonshine for M_{24} [25], where the role of the Norton series $f(g, h; \tau)$ is played by products of eta functions $\eta(\tau)$. The precise relation between our twisted twining genera and Mason's eta products will be explained elsewhere [26].

1.5. **Outline.** The paper is organised as follows. In section 2 we introduce the twisted twining genera $\phi_{g,h}$, and discuss their general properties, in particular the expected behaviour under modular transformations. We then explain how many independent twisted twining genera there are, and list all of them explicitly, see table 3. In section 3 we review the structure of holomorphic orbifolds and explain in particular, the role of the 3-cocycle in characterising the various transformation properties of the twisted twining characters. We then postulate that the transformation properties of the twisted twining genera are similarly constrained, and determine the underlying 3-cocycle in $H^3(M_{24}, U(1))$. This then allows us to find all twisted twining genera explicitly. We also check that the resulting functions are compatible with the requirement that they arise from the appropriate projective representation of the centraliser. In section 4 we subject our results to two independent consistency checks. First, for group elements whose orbifold leads again to a K3 sigma-model, we calculate the twining genera of the orbifold from the twisted twining genera of the original theory, and show that we reproduce answers from [8, 9]. However, as it turns out, the relevant group element is sometimes different from what one would have expected, and we explain this 'relabelling' phenomenon in quite some detail in section 4.1, see also section 4.2 for an explicit example. Secondly, we explain the vanishing of some of the twisted twining genera from a geometrical point of view, see section 4.3. Finally we end with some conclusions and open problems in section 5. We have relegated some of the more technical material to various appendices to which we refer throughout the main body of the paper.

Moreover, all character tables and decompositions of the twisted twining genera can be found in the TEX-source file in the arXiv repository of this paper.

2. Twisted Twining Genera

Let us begin by introducing the twisted twining genera that are the main object of this paper. Suppose we are given a K3 sigma-model $\mathcal{H}$, i.e. a CFT describing string propagation on K3, whose automorphism group contains two commuting elements $g, h \in M_{24}$. Then we can consider the orbifold of the sigma-model by g, and in particular, define the g-twisted sector $\mathcal{H}_g$. Since g and h commute, h gives rise to an action on $\mathcal{H}_g$, and we can define the *twisted twining genus* $\phi_{g,h}(\tau, z)$ by

$$\phi_{g,h}(\tau, z) = \operatorname{Tr}_{\mathcal{H}_g}\Big(h(-1)^{J_0+\bar{J}_0} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} y^{J_0}\Big) \ , \tag{2}$$

where $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$. We expect that $\phi_{g,h} : \mathbb{H}_+ \times \mathbb{C} \to \mathbb{C}$ is holomorphic in both τ and z. Furthermore, since the elliptic genus is independent of the choice of the underlying K3 sigma-model, we expect that the same is true for these twisted twining genera, i.e. we expect that (2) does not depend on the choice of $\mathcal{H}$ (as long as g and h are automorphisms of $\mathcal{H}$). By construction, for $g = e$ the identity element in M_{24}, the twisted twining genus $\phi_{e,h}$ agrees with the twining genus ϕ_h considered in [6, 7, 8, 9]; in particular, for $g = h = e$ $\phi_{e,e}$ is just the elliptic genus of K3.

Unfortunately, while for some commuting pairs (g, h) an actual K3 sigma-model for which both g and h are automorphisms can be found, this is not true in general [13]; in fact, this problem already arises for the usual twining genera, i.e. for the pairs (e, h). In the spirit of the EOT conjecture, we shall nevertheless assume that twisted twining genera $\phi_{g,h}$ can be defined — albeit not directly by a formula of the form (2) — for *all* commuting pairs $g, h \in M_{24}$. The fact that our construction will be successful is an a posteriori justification for this assumption.

2.1. Properties of the twisted twining genera.

The definition of the twisted twining genera $\phi_{g,h}$ in terms of (2) suggests that they should satisfy the following properties:

(A) Elliptic and modular properties:

$$\phi_{g,h}(\tau, z + \ell\tau + \ell') = e^{-2\pi i(\ell^2\tau + 2\ell z)}\, \phi_{g,h}(\tau, z) \qquad \ell, \ell' \in \mathbb{Z} \tag{3}$$

$$\phi_{g,h}\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = \chi_{g,h}\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big)\, e^{2\pi i \frac{cz^2}{c\tau+d}}\, \phi_{h^c g^a, h^d g^b}(\tau, z) \ , \quad \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in SL(2, \mathbb{Z}) \tag{4}$$

for a certain multiplier $\chi_{g,h} : SL(2,\mathbb{Z}) \to U(1)$. In particular, each $\phi_{g,h}$ is a weak Jacobi form of weight 0 and index 1 with multiplier $\chi_{g,h}$ under a subgroup $\Gamma_{g,h}$ of $SL(2,\mathbb{Z})$ (see [27] for the definitions).

(B) Invariance under conjugation of the pair g, h in M_{24},

$$\phi_{g,h}(\tau,z) = \xi_{g,h}(k)\, \phi_{k^{-1}gk,k^{-1}hk}(\tau,z) \ , \qquad k \in M_{24} \ , \tag{5}$$

where $\xi_{g,h}(k)$ is a phase.

(C) If $g \in M_{24}$ has order N, the twisted twining genera $\phi_{g,h}$ have an expansion of the form

$$\phi_{g,h}(\tau,z) = \sum_{\substack{r\in\lambda_g+\mathbb{Z}/N \\ r\geq 0}} \mathrm{Tr}\,_{\mathcal{H}_{g,r}}\big(\rho_{g,r}(h)\big)\, \mathrm{ch}_{h=\frac{1}{4}+r,\ell}(\tau,z) \ , \tag{6}$$

where $\lambda_g \in \mathbb{Q}$, and $\mathrm{ch}_{h,\ell}(\tau,z)$ are elliptic genera of Ramond representations of the $\mathcal{N} = 4$ superconformal algebra at central charge $c = 6$. (Here $\ell = \frac{1}{2}$, except possibly for $h = \frac{1}{4}$, where $\ell = 0$ is also possible — if both $\ell = 0, \frac{1}{2}$ appear for $r = 0$, it is understood that there are two such terms in the above sum. We use the same conventions for the elliptic genera as e.g. in [28].) Furthermore, each vector space $\mathcal{H}_{g,r}$ is finite dimensional, and it carries a projective representation $\rho_{g,r}$ of the centraliser $C_{M_{24}}(g)$ of g in M_{24}, such that

$$\rho_{g,r}(g) = e^{2\pi i r} \ , \qquad \rho_{g,r}(h_1)\, \rho_{g,r}(h_2) = c_g(h_1,h_2)\, \rho_{g,r}(h_1 h_2) \ , \tag{7}$$

for all $h_1, h_2 \in C_{M_{24}}(g)$. Here $c_g : C_{M_{24}}(g) \times C_{M_{24}}(g) \to U(1)$ is independent of r, and satisfies the cocycle condition

$$c_g(h_1,h_2)\, c_g(h_1h_2,h_3) = c_g(h_1,h_2h_3)\, c_g(h_2,h_3) \tag{8}$$

for all $h_1, h_2, h_3 \in C_{M_{24}}(g)$.

(D) For $g = e$, where e is the identity element of M_{24}, the functions $\phi_{e,h}$ correspond to the twining genera considered in [6]–[9]. In particular, $\phi_{e,e}$ is the K3 elliptic genus.

Since the elliptic genus behaves essentially like the character of a holomorphic CFT — in particular, it is modular invariant and holomorphic by itself — it is natural to believe that the same will be true for the twisted twining genera, i.e. that they will be analogous to twisted twining characters of a holomorphic CFT. As we will review in more detail in section 3, the modular properties of the twisted twining characters of a holomorphic CFT are controlled by a 3-cocycle $\alpha : G \times G \times G \to U(1)$ representing a cohomology class in the

third cohomology group $H^3(G,U(1))$. (Some background material about group cohomology can be found in appendix B.) We will therefore postulate that

(E) The multipliers $\chi_{g,h}$, the phases $\xi_{g,h}$, and the 2-cocycles c_g associated with the projective representations $\rho_{g,r}$ are completely determined (by the same formulas as for holomorphic orbifolds) in terms of a 3-cocycle α representing a class in $H^3(M_{24},U(1))$.

The third cohomology group of M_{24} was only recently computed with the result [29][‡]

$$H^3(M_{24},U(1)) \cong \mathbb{Z}_{12} \ . \tag{9}$$

The fact that this group is known explicitly plays a crucial role in our analysis. The specific cohomology class $[\alpha] \in H^3(M_{24},U(1))$ that is relevant in our context is uniquely determined by the condition that it reproduces the multiplier system for the twining genera $\phi_{e,h}$ as described in [8], namely

$$\chi_{e,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = e^{\frac{2\pi i c d}{o(h)\ell(h)}} \ , \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(o(h)) \ . \tag{10}$$

Here, $o(h)$ is the order of h and $\ell(h)$ is the length of the smallest cycle, when $h \in M_{24}$ is regarded as a permutation of 24 symbols [12]. Indeed, since $\ell(12\mathrm{B}) = 12$, it follows that α must correspond to the primitive generator of $H^3(M_{24},U(1))$. The main result of our paper can now be stated as follows:

There exists a unique set of functions $\phi_{g,h}$ (unique up to redefinitions by (g,h)-dependent but otherwise constant phases) and a unique cohomology class $[\alpha] \in H^3(M_{24},U(1))$ such that all conditions (A)–(E) are satisfied.

In the following we want to explain how this set of functions can be determined, and which explicit form it takes. As it turns out, a surprisingly large number of the twisted twining genera vanish, as we shall now explain.

2.2. **Cohomological obstructions.** For some pairs of commuting elements $g,h \in M_{24}$, the transformation properties above can only be satisfied if $\phi_{g,h}$ vanishes identically. In this case, we will say that the corresponding twisted twining genus is *obstructed*. As will be

[‡]Note that for a finite group G one has the isomorphisms

$$H_{n-1}(G,\mathbb{Z}) \cong H^n(G,\mathbb{Z}) \ , \qquad H^n(G,\mathbb{Z}) \cong H^{n-1}(G,U(1)) \ ,$$

which in particular imply that $H_3(M_{24},\mathbb{Z}) \cong H^3(M_{24},U(1))$.

shown in more detail in section 3, these obstructions are also controlled by the cohomology class of the cocycle $\alpha \in H^3(M_{24}, U(1))$.[§]

In order to understand the origin of these obstructions, let us first derive some general consequences of our assumptions (A)–(E). First we note that the $SL(2,\mathbb{Z})$ action on the twisted twining characters can be extended to $GL(2,\mathbb{Z})$ by setting

$$\phi^*_{g,h}(\tau,z) = \chi_{g,h}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\, \phi_{g,h^{-1}}(\tau,z) \ , \tag{11}$$

where $\phi^*_{g,h}(\tau,z)$ is obtained by taking the complex conjugate of all Fourier coefficients of $\phi_{g,h}(\tau,z)$

$$\phi^*_{g,h}(\tau,z) = \overline{\phi_{g,h}(-\bar{\tau},-\bar{z})} \ . \tag{12}$$

The identity (11) follows from eq. (6), together with the observation that, for any projective representation R of a finite group G, $\mathrm{Tr}_R(h^{-1})$ equals $\overline{\mathrm{Tr}_R(h)}$ up to a phase, which we have denoted by $\chi_{g,h}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. As we will see, this phase also depends on the 3-cocycle α. We also note that since the $\mathcal{N}=4$ characters are invariant under $z \to -z$, we have the identity

$$\phi_{g,h}(\tau,-z) = \phi_{g,h}(\tau,z) \ . \tag{13}$$

With these preparations we can now describe two possible kinds of obstructions.

Obstruction 1: Let us consider pairwise commuting g,h,k. Then, by (5), we have

$$\phi_{g,h}(\tau,z) = \xi_{g,h}(k)\, \phi_{g,h}(\tau,z) \ , \tag{14}$$

and if

$$\xi_{g,h}(k) \neq 1 \ , \tag{15}$$

it follows that $\phi_{g,h}(\tau,z) = 0$.

Obstruction 2: Suppose there are $g,h,k \in M_{24}$, with g and h commuting and

$$k^{-1}g^{-1}k = g \ , \qquad k^{-1}h^{-1}k = h \ , \tag{16}$$

i.e. the commuting pair (g,h) is conjugate within M_{24} to the pair (g^{-1},h^{-1}). Then, by eqs. (44) and (42) (see section 3 for details), we obtain the relations

$$\phi_{g,h}(\tau,z) = \chi_{g,h}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\, \phi_{g^{-1},h^{-1}}(\tau,-z) \tag{17}$$

$$= \chi_{g,h}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\, \xi_{g^{-1},h^{-1}}(k)\, \phi_{g,h}(\tau,-z) \ . \tag{18}$$

Therefore, if

$$\chi_{g,h}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\, \xi_{g^{-1},h^{-1}}(k) \neq 1 \ , \tag{19}$$

[§]The fact that such obstructions might exist was first suggested to us by T. Gannon.

eq. (13) implies $\phi_{g,h}(\tau,z)=0$.

As we shall see, these two obstructions are responsible for the fact that most twisted twining genera vanish.

2.3. **Classification of independent twisted twining genera.** Our next aim is to enumerate all possible independent twisted twining genera. Let us denote the set of commuting group elements by

$$\mathcal{P}=\{(g,h)\in M_{24}\times M_{24}\mid gh=hg\}\ . \tag{20}$$

This set carries an action of $GL(2,\mathbb{Z})\times M_{24}$, given by

$$(g,h)\mapsto(k^{-1}(g^ah^c)k,k^{-1}(g^bh^d)k)\ ,\quad \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in GL(2,\mathbb{Z})\ ,\quad k\in M_{24}\ , \tag{21}$$

and the twisted twining genera associated to different (g,h)'s in the same orbit are related to one another by modular transformations, see eqs. (4), (5), and/or by complex conjugation, see eq. (11). We therefore want to describe the set of orbits of $\mathcal{P}$ under the action (21).

First we note that the $GL(2,\mathbb{Z})$ orbit of a pair (g,h) consists of all possible pairs of generators of the abelian group $\langle g,h\rangle\subset M_{24}$. Thus the orbits of $GL(2,\mathbb{Z})\times M_{24}$ are in one-to-one correspondence with the conjugacy classes of abelian subgroups of M_{24} generated by two elements, i.e.

$$\bar{\mathcal{P}}=\mathcal{P}/(GL(2,\mathbb{Z})\times M_{24})=\{M_{24}-\text{conjugacy classes of groups }\langle g,h\rangle\subset M_{24},gh=hg\}\ . \tag{22}$$

This description allows a complete classification of the orbits $[g,h]\in\bar{\mathcal{P}}$: there are 55 such orbits, 21 of which correspond to cyclic subgroups, i.e. to subgroups of the form $[e,h]$. The associated twisted twining genera are therefore just the twining genera $\phi_{e,h}$, for which explicit expressions were already derived in [8, 9]. Thus we only need to construct the remaining 34 genuinely twisted twining genera. We have tabulated the corresponding conjugacy classes of groups $\langle g,h\rangle$ in table 1. There we have described their structure as an abelian group, i.e. as $\mathbb{Z}_m\times\mathbb{Z}_n$, the M_{24} classes of all its elements (excluding the identity), the order of the centraliser $C(g,h)$, the index $|N(g,h)|/|C(g,h)|$ of the centraliser in the normaliser of $\langle g,h\rangle$ in M_{24}, and the lengths of the orbits of $\langle g,h\rangle\subset M_{24}$ when acting as a group of permutations of 24 objects. Finally, the last column gives the conjugacy classes of the non-cyclic maximal subgroups. For example, group 27 has three distinct non-cyclic maximal subgroups, all of the form $\mathbb{Z}_2\times\mathbb{Z}_4$; two of them are conjugated to group 12 and one is conjugated to group 11.

#	Structure	Elements	$\lvert C(g,h)\rvert$	$\frac{\lvert N(g,h)\rvert}{\lvert C(g,h)\rvert}$	Orbits on **24**	Max subgr.
1.	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$(2A)(2B)^2(4B)^4(8A)^8$	16	8	$2^2 \cdot 4^1 \cdot 16^1$	30
2.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2B)^3$	96	6	4^6	
3.	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$(2B)^3(5A)^4(10A)^{12}$	20	12	$4^1 \cdot 20^1$	33
4.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2A)^3$	1536	6	$1^8 \cdot 4^4$	
5.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2A)^3$	1536	6	2^{12}	
6.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2A)(2B)^2$	512	2	$2^4 \cdot 4^4$	
7.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2A)^3$	128	6	$1^4 \cdot 2^6 \cdot 4^2$	
8.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2A)(2B)^2$	128	2	$2^4 \cdot 4^4$	
9.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)^3(4B)^4$	64	8	$1^4 \cdot 2^2 \cdot 8^2$	4
10.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)^3(4A)^4$	64	8	$2^4 \cdot 8^2$	4
11.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)^3(4B)^4$	64	8	$2^4 \cdot 4^4$	5
12.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)^3(4A)^4$	64	8	4^6	5
13.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)(2B)^2(4A)^4$	64	8	$4^2 \cdot 8^2$	6
14.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2A)^2(2B)$	256	2	$2^8 \cdot 4^2$	
15.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)(2B)^2(4C)^4$	32	4	$4^2 \cdot 8^2$	6
16.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)^2(2B)(4A)^2(4B)^2$	32	2	$2^2 \cdot 4^3 \cdot 8^1$	14
17.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)(2B)^2(4A)^4$	32	8	$4^2 \cdot 8^2$	6
18.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$(2A)^3(3A)^2(6A)^6$	12	12	$1^2 \cdot 3^2 \cdot 4^1 \cdot 12^1$	4
19.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$(2A)^3(3A)^2(6A)^6$	12	12	$2^3 \cdot 6^3$	5
20.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$(2B)^3(3B)^2(6B)^6$	12	12	12^2	33
21.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$(2B)^3(3B)^2(6B)^6$	12	12	12^2	2
22.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)^3(4B)^4$	16	8	$1^2 \cdot 2^3 \cdot 4^2 \cdot 8^1$	7
23.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)(2B)^2(4C)^4$	16	4	$4^2 \cdot 8^2$	8
24.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2A)^3(4B)^{12}$	16	96	$1^4 \cdot 4^1 \cdot 16^1$	$9, 9, 9$
25.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2A)^3(4B)^{12}$	16	96	4^6	$11, 11, 11$
26.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)(2B)^2(4B)^4$	16	8	$2^4 \cdot 8^2$	8
27.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2A)^3(4A)^8(4B)^4$	16	32	$4^2 \cdot 8^2$	$11, 12, 12$
28.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2A)^3(4A)^8(4B)^4$	16	32	$2^2 \cdot 4^1 \cdot 16^1$	$9, 10, 10$
29.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(2A)(2B)^2(4A)^4(4C)^8$	16	16	$8^1 \cdot 16^1$	$13, 15, 15$
30.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$(2A)(2B)^2(4B)^4$	64	8	$2^4 \cdot 8^2$	6
31.	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(3A)^2(3B)^6$	9	12	$3^2 \cdot 9^2$	
32.	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(3A)^8$	9	48	$1^3 \cdot 3^4 \cdot 9^1$	
33.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2B)^3$	3840	6	4^6	
34.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2B)^3$	64	6	4^6	

TABLE 1. The 34 conjugacy classes of abelian subgroups of rank 2 in M_{24}.

For M_{24} it turns out that the orbits of $\mathcal{P}$ under $SL(2,\mathbb{Z})\times M_{24}$ are exactly the same as those under $GL(2,\mathbb{Z})\times M_{24}$, i.e.[¶]

$$\bar{\mathcal{P}} = \mathcal{P}/(GL(2,\mathbb{Z})\times M_{24}) = \mathcal{P}/(SL(2,\mathbb{Z})\times M_{24})\ . \tag{23}$$

Thus the twisted twining genera in each orbit in $\bar{\mathcal{P}}$ are just related by phases, see eqs. (4) and (5), and we do not need to invoke eq. (11).

2.4. **Modular properties of the twisted twining genera.** For each commuting pair $(g,h)\in\mathcal{P}$, let us denote by $\tilde{\Gamma}_{g,h}\subset SL(2,\mathbb{Z})\times M_{24}$ the group of elements (γ,k) that leave the pair (g,h) fixed or map it to its inverse (g^{-1},h^{-1})

$$\tilde{\Gamma}_{g,h} = \left\{ \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), k \right) \in SL(2,\mathbb{Z})\times M_{24} \mid \left(k^{-1}(g^a h^c)k,\ k^{-1}(g^b h^d)k\right) = (g,h) \text{ or } (g^{-1},h^{-1}) \right\}\ . \tag{24}$$

It follows from (4) and (5), together with (13) as well as (44) below, that the corresponding twisted twining genus $\phi_{g,h}$ will be invariant (up to a phase) under $\Gamma_{g,h}\equiv\pi(\tilde{\Gamma}_{g,h})$, where π denotes the projection of $SL(2,\mathbb{Z})\times M_{24}$ onto its first factor, i.e. onto $SL(2,\mathbb{Z})$.

Each twisted twining genus $\phi_{g,h}$ belongs to a (finite-dimensional) vector representation of $SL(2,\mathbb{Z})$, that is spanned by the functions $\{\phi_{(g,h)\gamma}(\tau,z)\}$, where $\gamma\in SL(2,\mathbb{Z})$. Since every $\gamma\in\Gamma_{g,h}$ acts trivially (i.e. up to a phase) on $\phi_{g,h}$, the $SL(2,\mathbb{Z})$ representation is spanned by $\{\phi_{(g,h)\gamma}(\tau,z)\}_{\gamma\in\Gamma_{g,h}\backslash SL(2,\mathbb{Z})}$, where γ runs over a set of representatives for the cosets in $\Gamma_{g,h}\backslash SL(2,\mathbb{Z})$. Each of these functions $\phi_{(g,h)\gamma}$ is a weak Jacobi form of weight 0 and index 1 for the congruence subgroup $\gamma^{-1}\Gamma_{g,h}\gamma\in SL(2,\mathbb{Z})$. We have tabulated for each orbit $[g,h]\in\bar{\mathcal{P}}$ the functions $\phi_{(g,h)\gamma}$ (starting with $\phi_{g,h}$) and the invariance group $\Gamma_{g,h}$ in table 2. (Note, however, that some of these functions vanish identically; in particular, this will be the case if there is an obstruction.)

Because of (11), most of the functions satisfy a 'reality' condition of the form

$$\phi^*_{g,h} = \zeta\,\phi_{g,h}\ , \tag{25}$$

for some constant phase ζ, which implies that the Fourier coefficients of $\zeta^{1/2}\phi_{g,h}$ are all real. Note that even if $\phi^*_{g,h}$ and $\phi_{g,h}$ are not proportional to one another, they are necessarily related by a modular transformation because of (23); if this is the case, they are listed as distinct functions in the same orbit in $\bar{\mathcal{P}}$. For example, in group 1, there are 4 distinct 'real'

[¶]For a generic group G different from M_{24}, some $SL(2,\mathbb{Z})\times G$ orbits might be strictly smaller than the $GL(2,\mathbb{Z})\times G$ orbits.

#	Structure	Functions	$\Gamma_{g,h}$
1.	$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\phi_{2B,8A}, \phi_{8A,2B}, \phi_{4B,8A}, \phi_{8A,4B}, \phi_{8A,8A}, \phi^*_{8A,8A}$	$\Gamma_0(4)$
2.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2B,2B}$	$\Gamma(1)$
3.	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	$2 \times \phi_{2B,10A},\ 4 \times \phi_{10A,10A},\ 4 \times \phi^*_{10A,10A},\ 2 \times \phi_{10A,2B}$	$\Gamma_{2B,10A}$
4.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2A,2A}$	$\Gamma(1)$
5.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2A,2A}$	$\Gamma(1)$
6.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2A,2B},\ \phi_{2B,2A},\ \phi_{2B,2B}$	$\Gamma_0(2)$
7.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2A,2A}$	$\Gamma(1)$
8.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2A,2B},\ \phi_{2B,2A},\ \phi_{2B,2B}$	$\Gamma_0(2)$
9.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4B},\ \phi_{4B,2A},\ \phi_{4B,4B}$	$\Gamma_0(2)$
10.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4A},\ \phi_{4A,2A},\ \phi_{4A,4A}$	$\Gamma_0(2)$
11.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4B},\ \phi_{4B,2A},\ \phi_{4B,4B}$	$\Gamma_0(2)$
12.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4A},\ \phi_{4A,2A},\ \phi_{4A,4A}$	$\Gamma_0(2)$
13.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2B,4A},\ \phi_{4A,2B},\ \phi_{4A,4A}$	$\Gamma_0(2)$
14.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2B,2A},\ \phi_{2A,2B},\ \phi_{2A,2A}$	$\Gamma_0(2)$
15.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4C}, \phi_{4C,2A}, \phi_{4C,2B}, \phi_{2B,4C}, \phi_{4C,4C}, \phi^*_{4C,4C}$	$\Gamma_0(4)$
16.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4A}, \phi_{2A,4B}, \phi_{2B,4A}, \phi_{2B,4B}, \phi_{4A,2A}, \phi_{4A,2B},$ $\phi_{4B,2A}, \phi_{4B,2B}, \phi_{4B,4A}, \phi^*_{4B,4A}, \phi_{4A,4B}, \phi^*_{4A,4B}$	$\Gamma_{2A,4A}$
17.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2B,4A},\ \phi_{4A,2B},\ \phi_{4A,4A}$	$\Gamma_0(2)$
18.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2A,6A}, \phi_{6A,2A}, \phi_{6A,6A}, \phi^*_{6A,6A}$	$\Gamma_0(3)$
19.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2A,6A}, \phi_{6A,2A}, \phi_{6A,6A}, \phi^*_{6A,6A}$	$\Gamma_0(3)$
20.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2B,6B}, \phi_{6B,2B}, \phi_{6B,6B}, \phi^*_{6B,6B}$	$\Gamma_0(3)$
21.	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\phi_{2B,6B}, \phi_{6B,2B}, \phi_{6B,6B}, \phi^*_{6B,6B}$	$\Gamma_0(3)$
22.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4B},\ \phi_{4B,2A},\ \phi_{4B,4B}$	$\Gamma_0(2)$
23.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2A,4C}, \phi_{4C,2A}, \phi_{4C,2B}, \phi_{2B,4C}, \phi_{4C,4C}, \phi^*_{4C,4C}$	$\Gamma_0(4)$
24.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4B,4B}$	$\Gamma(1)$
25.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4B,4B}$	$\Gamma(1)$
26.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2B,4B},\ \phi_{4B,2B},\ \phi_{4B,4B}$	$\Gamma_0(2)$
27.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4B,4A},\ \phi_{4A,4B},\ \phi_{4A,4A}$	$\Gamma_0(2)$
28.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\phi_{4B,4A},\ \phi_{4A,4B},\ \phi_{4A,4A}$	$\Gamma_0(2)$
29.	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$2 \times \phi_{4A,4C}, 2 \times \phi_{4C,4A}, \phi_{4C,4C}, \phi^*_{4C,4C}$	$\Gamma_{4A,4C}$
30.	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\phi_{2B,4B},\ \phi_{4B,2B},\ \phi_{4B,4B}$	$\Gamma_0(2)$
31.	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\phi_{3A,3B}, \phi_{3B,3A}, \phi_{3B,3B}, \phi^*_{3B,3B}$	$\Gamma_0(3)$
32.	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\phi_{3A,3A}$	$\Gamma(1)$
33.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2B,2B}$	$\Gamma(1)$
34.	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\phi_{2B,2B}$	$\Gamma(1)$

TABLE 2. The independent functions and the group $\Gamma_{g,h}$ for each $[g,h] \in \bar{P}$.

functions $\phi_{2B,8A}$, $\phi_{8A,2B}$, $\phi_{4B,8A}$, $\phi_{8A,4B}$, while $\phi_{8A,8A}$ is not proportional to its complex conjugate $\phi^*_{8A,8A}$. These six functions correspond to the six cosets of $\Gamma_0(4)\backslash SL(2,\mathbb{Z})$.

Most of the modular groups $\Gamma_{g,h}$ are of the form $\Gamma(1) = SL(2,\mathbb{Z})$ or

$$\Gamma_0(N) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid c \equiv 0 \mod N \} , \tag{26}$$

or conjugates of $\Gamma_0(N)$ in $SL(2,\mathbb{Z})$. The exceptions are the group in case 3, where

$$\Gamma_{2B,10A} = \bigcup_{i\in\mathbb{Z}/3\mathbb{Z}, j\in\mathbb{Z}/4\mathbb{Z}} \left(\begin{smallmatrix} 1 & 1 \\ -5 & -4 \end{smallmatrix}\right)^i \left(\begin{smallmatrix} -3 & -1 \\ 10 & 3 \end{smallmatrix}\right)^j \Gamma_{2,10} , \tag{27}$$

is a subgroup of index 12 in $SL(2,\mathbb{Z})$ and

$$\Gamma_{2,10} = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid a \equiv 1,\ b \equiv 0 \mod 2,\ c \equiv 0,\ d \equiv 1 \mod 10 \} , \tag{28}$$

is the group of elements $\gamma \in SL(2,\mathbb{Z})$ such that $(g,h)\cdot\gamma = (g,h)$; the group in case 16, with

$$\Gamma_{2A,4A} = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid b \equiv 0 \mod 2, c \equiv 0 \mod 4 \} , \tag{29}$$

which is a conjugate of $\Gamma_0(8)$ in $SL(2,\mathbb{R})$; and the group in case 29, with

$$\Gamma_{4A,4C} = \langle \left(\begin{smallmatrix} -1 & 1 \\ -2 & 1 \end{smallmatrix}\right) , \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) , \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) , \left(\begin{smallmatrix} 3 & -2 \\ -4 & 3 \end{smallmatrix}\right) \rangle . \tag{30}$$

2.5. **Explicit twining genera.** We have now reduced our problem to finding 34 functions $\phi_{g,h}$ that are weak Jacobi forms with respect to $\Gamma_{g,h}$ up to some multiplier phases. As will be explained in the following section 3, our assumption (E) allows us to determine the precise form of these multipliers explicitly. Then most of the 34 functions vanish because of the obstructions, see table 3. For those that do not, the modular properties are strong enough to determine them explicitly, see also table 3. These explicit results are one of the main results of this paper, and their derivation is sketched in section 3, see in particular section 3.4, as well as appendix A. As will be explained below in section 3.5, these functions are also compatible with the expected projective representation of the centraliser $C_{M_{24}}(g)$ on the g-twisted sector, see property (C) in section 2.

3. Holomorphic Orbifolds and Group Cohomology

In this section we review the modular properties of holomorphic orbifolds since, according to our assumption (E), these are also relevant for the twisted twining genera of K3. We then explain how, in our context, the known multiplier phases of the twining genera determine the underlying 3-cocycle $\alpha \in H^3(M_{24}, U(1))$ uniquely, and how to obtain from this our explicit results given in table 3. Finally, we shall check that these formulae indeed give rise to the appropriate projective representations of the centraliser $C_{M_{24}}(g)$.

#	$\phi_{g,h}$	Obstr.	#	$\phi_{g,h}$	Obstr.
1.	$\phi_{2B,8A} = 2\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no	18.	0	2
2.	0	2	19.	0	2
3.	0	2	20.	0	2
4.	0	1	21.	0	2
5.	0	1	22.	0	1
6.	0	1	23.	0	1
7.	0	1	24.	0	2
8.	0	1	25.	0	2
9.	0	1	26.	0	1
10.	0	1	27.	$\phi_{4B,4A} = 2\sqrt{2}\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no
11.	0	1	28.	$\phi_{4B,4A} = 2\sqrt{2}\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no
12.	0	1	29.	0	2
13.	$\phi_{2B,4A} = 4\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2$	no	30.	0	2
14.	0	1	31.	$\phi_{3A,3B} = 0$	no
15.	0	1	32.	$\phi_{3A,3A} = 0$	no
16.	0	1	33.	0	2
17.	0	1	34.	0	1

TABLE 3. For each of the 34 cases we give here the explicit result for the twisted twining genus, as well as where applicable, the obstruction that it responsible for its vanishing.

3.1. **Review of holomorphic orbifolds.** Suppose $\mathcal{C}$ is a self-dual VOA, and G is a group of $\mathcal{C}$-automorphisms. We are interested in the 'orbifold' of $\mathcal{C}$ by G. To this end we consider the G-invariant sub-VOA of $\mathcal{C}$,

$$\mathcal{C}^G = \{\psi \in \mathcal{C} : g\psi = \psi,\, \forall g \in G\} \ . \tag{31}$$

Each representation of $\mathcal{C}$ gives rise to a representation of $\mathcal{C}^G$. In addition, there are new representations of $\mathcal{C}^G$ that appear in the different twisted sectors $\mathcal{C}_A$; since $\mathcal{C}$ is holomorphic there is a unique twisted sector $\mathcal{C}_A$ for each conjugacy class A in G.

3.1.1. *Twisted twining characters.* Each twisted sector $\mathcal{C}_A$ defines a representation of the G-invariant sub-VOA $\mathcal{C}^G$, but it is typically not irreducible. In particular, for each g_A in

the conjugacy class $g_A \in [A]$, we can project $\mathcal{C}_{g_A}$ onto a $\mathcal{C}^G$-invariant subspace using any (projective) character χ of $C_G(g_A)$. Thus we can label the irreducible representations of $\mathcal{C}^G$ by pairs (A, χ). In physics, by the orbifold of $\mathcal{C}$ by G one usually means the theory where one chooses $\chi = 1$ in each twisted sector, i.e. the full partition function takes the form

$$Z_{\text{orb}}(\tau) = \sum_A \frac{1}{|C_G(A)|} \sum_{h \in C_G(g_A)} Z_{g_A,h}(\tau) \ , \tag{32}$$

where

$$Z_{g,h}(\tau) \equiv {}_g\underset{h}{\square} = \text{Tr}_{\mathcal{C}_g}\left(\rho_g(h)\, q^{L_0 - \frac{c}{24}}\right) \ , \tag{33}$$

and ρ_g denotes the action of $C_G(g)$ on $\mathcal{C}_g$. We should mention that this action is usually *not* canonically defined; for example, what is usually referred to as discrete torsion [30] can be thought of as being an ambiguity in the definition of ρ_g. Incidentally, the fact that the action is not canonically defined will be important to us later, see in particular section 4.1. For the moment though, we want to assume that such an action has been chosen.

3.1.2. *Projective representations.* As alluded to above, the twisted sectors $\mathcal{C}_g$ typically do not form genuine representations of $C_G(g)$ but only *projective representations*. A projective representation ρ of a finite group G is characterised by the relation (see also appendix C for some introductory comments)

$$\rho(g_1)\, \rho(g_2) = c(g_1, g_2)\, \rho(g_1 g_2) \ , \tag{34}$$

where $c : G \times G \to U(1)$ is a (normalised) 2-cocycle representing a cohomology class in $H^2(G, U(1))$. Genuine representations of finite groups are classified, up to equivalence, by their characters $\text{Tr}_\rho(g)$. This is, however, no longer true for projective representations since taking the trace over a projective representation does not lead to a class function, i.e. a function that is invariant under conjugation by elements in G. Instead, for projective representations one has

$$\text{Tr}_\rho(hgh^{-1}) = \frac{c(h, h^{-1}gh)}{c(g, h)}\, \text{Tr}_\rho(g) \ , \tag{35}$$

as follows from the defining relation (34) together with the cyclicity of the trace (see e.g. [31] for a nice discussion). For the case of the holomorphic orbifold, this then results in the conjugation relation

$$Z_{g,h}(\tau) = \frac{c_g(h, k)}{c_g(k, k^{-1}hk)}\, Z_{k^{-1}gk, k^{-1}hk}(\tau) \ , \tag{36}$$

where the 2-cocycle c_g represents a class in $H^2(C_G(g), U(1))$ specifying the projective representation of the centralizer $C_G(g)$ on $\mathcal{C}_g$, and we have assumed that k and g commute so

that both expressions are evaluated in the same twisted sector. (In fact, (36) is even true if k and g do not commute, see [22].) If in addition k commutes with h, (36) reduces to

$$Z_{g,h}(\tau) = \frac{c_g(h,k)}{c_g(k,h)}\, Z_{g,h}(\tau) \ , \tag{37}$$

which therefore implies that $Z_{g,h}$ vanishes unless the 2-cocycle is *regular*,

$$c_g(h,k) = c_g(k,h) \ . \tag{38}$$

This is the holomorphic orbifold version of the first type of obstruction discussed in section 2.2 above.

3.1.3. *The underlying cohomology.* The possible choices for the 2-cocycles c_g in the various twisted sectors are constrained; for example this follows from demanding consistency of the OPE involving fields from different twist sectors.[‖] In particular, it was argued in [22] that the various consistent choices are in one-to-one correspondence with elements in the third cohomology group $[\alpha] \in H^3(G, U(1))$, see also [32, 33, 34, 35] for subsequent work. The 3-cocycle α determines distinguished elements $c_g \in H^2(C_G(g), U(1))$ via the formula [22, 32]

$$c_g(h_1,h_2) = \frac{\alpha(g,h_1,h_2)\,\alpha(h_1,h_2,(h_1h_2)^{-1}g(h_1h_2))}{\alpha(h_1,h_1^{-1}gh_1,h_2)} \ . \tag{39}$$

Since α fixes all 2-cocylces c_g via (39), it is clear that the choice of α determines the behaviour of the twisted twining characters under conjugation (36). Furthermore, α also determines all the multiplier phases that appear under modular transformations, since it follows from [32] that[**]

$$\begin{aligned} Z_{g,h}(\tau+1) &= c_g(g,h)\, Z_{g,gh}(\tau) \ , \\ Z_{g,h}(-1/\tau) &= \overline{c_h(g,g^{-1})}\, Z_{h,g^{-1}}(\tau) \ . \end{aligned} \tag{40}$$

The choice of a particular representative α in the class $[\alpha] \in H^3(G, U(1))$ corresponds to a choice of normalisation for the action of h on the g-twisted sector. To see this, let $\tilde{\alpha}$ represent a 3-cocycle in the same cohomology class $[\alpha] \in H^3(G, U(1))$, so that $\tilde{\alpha} = \alpha \cdot \partial\beta$ (see eq. (151)) for some 2-cochain β. Then the associate 2-cocycles c_g and $\tilde{c}_g$ differ (for each $g \in G$) only by a 2-coboundary i.e. $\tilde{c}_g = c_g \cdot \partial\gamma_g$, with $\gamma_g(h) = \frac{\beta(h,g)}{\beta(g,h)}$, see appendix B. Thus,

[‖]This origin of the constraint was suggested in [33].

[**]Note that (40) is obtained from [32] upon working with the inverse 3-cohomology element. We also thank Terry Gannon for explaining these formulae to us. It is believed (and we have verified this in simple cases) that (40) follows from the transformation rules for the irreducible orbifold characters, see [34, eqs. (5.23) and (5.24)], but we have not proven this in complete generality.

the 3-cocycle $\tilde{\alpha}$ is associated with the modular properties of the functions $\tilde{Z}_{g,h} = \gamma_g(h) Z_{g,h}$, which, by (33), simply correspond to a different normalisation $\tilde{\rho}_g(h) = \gamma_g(h)\rho_g(h)$ for the action of h on the g-twisted sector.

3.2. **Application to Mathieu Moonshine.** Given what we said before, it is now natural to postulate that for the twisted twining genera of K3, the various multiplier phases also come from a 3-cocycle $\alpha \in H^3(M_{24}, U(1))$. This is to say, we postulate that the projective representation appearing in (C) is the one determined by c_g, given by (39). Furthermore, $\chi_{g,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in (4) is determined by (40), and $\xi_{g,h}(k)$ in (5) agrees with (37), where in both cases c_g is again the function determined by (39) for a fixed $\alpha \in H^3(M_{24}, U(1))$. More explicitly, we therefore propose that

$$\begin{aligned} \phi_{g,h}\left(-\tfrac{1}{\tau}, \tfrac{z}{\tau}\right) &= \overline{c_h(g, g^{-1})}\, e^{2\pi i \frac{z^2}{\tau}}\, \phi_{h,g^{-1}}(\tau, z) \ , \\ \phi_{g,h}(\tau+1, z) &= c_g(g,h)\, \phi_{g,gh}(\tau, z) \ , \end{aligned} \tag{41}$$

with $c_g(h_1, h_2)$ given by (39), and

$$\phi_{g,h}(\tau, z) = \frac{c_g(h,k)}{c_g(k, k^{-1}hk)}\, \phi_{k^{-1}gk, k^{-1}hk}(\tau, z) \ , \qquad k \in M_{24} \ . \tag{42}$$

Moreover, we postulate that

$$\phi_{g,h^{-1}}(\tau, z) = c_g(h, h^{-1})\, \phi^*_{g,h}(\tau, z) \ , \tag{43}$$

where $\phi^*_{g,h}(\tau, z)$ is defined in (12).

Note that the charge conjugation operator $C = S^2$ acts on the genera $\phi_{g,h}(\tau, z)$ by flipping the sign of the second argument z. In fact, writing $\tau = -\frac{1}{\tilde{\tau}}$ and $z = \frac{\tilde{z}}{\tilde{\tau}}$, we get from twice applying (41)

$$\begin{aligned} \phi_{g,h}(\tau, z) = \phi_{g,h}(-\tfrac{1}{\tilde{\tau}}, \tfrac{\tilde{z}}{\tilde{\tau}}) &= \frac{e^{2\pi i \frac{\tilde{z}^2}{\tilde{\tau}}}}{c_h(g, g^{-1})}\, \phi_{h,g^{-1}}(\tilde{\tau}, \tilde{z}) \\ &= \frac{e^{-2\pi i \frac{z^2}{\tau}}}{c_h(g, g^{-1})}\, \phi_{h,g^{-1}}(-\tfrac{1}{\tau}, -\tfrac{z}{\tau}) = \frac{1}{c_h(g, g^{-1}) c_{g^{-1}}(h, h^{-1})}\, \phi_{g^{-1},h^{-1}}(\tau, -z) \ . \end{aligned} \tag{44}$$

Together with (13), this then leads to

$$\phi_{g,h}(\tau, z) = \frac{1}{c_h(g, g^{-1})\, c_{g^{-1}}(h, h^{-1})}\, \phi_{g^{-1},h^{-1}}(\tau, z) \ , \tag{45}$$

thus reproducing (17). This identity played a key role in the derivation of obstruction 2 in section 2.2.

Combining modular transformations with conjugation one obtains a whole set of relations among the twisted twining genera, in particular

$$\chi_{g,h}(\gamma_1\gamma_2) = \chi_{g,h}(\gamma_1)\,\chi_{(g,h)\gamma_1}(\gamma_2)\ , \qquad \gamma_1,\gamma_2 \in SL(2,\mathbb{Z})\ , \tag{46}$$

$$\chi_{g,h}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \frac{1}{c_h(g,g^{-1})}\ , \tag{47}$$

$$\chi_{g,h}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = c_g(g,h)\ , \tag{48}$$

$$\chi_{g,h}\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \frac{1}{c_h(g,g^{-1})c_{g^{-1}}(h,h^{-1})}\ , \tag{49}$$

$$\xi_{g,h}(k) = \frac{c_g(h,k)}{c_g(k,k^{-1}hk)}\ , \tag{50}$$

$$\chi_{g,h}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \frac{1}{c_g(h,h^{-1})}\ . \tag{51}$$

While our proposal about the multiplier phases is certainly natural, we do not have a direct proof for it. However, the fact that with this assumption we shall find consistent answers is in our opinion very convincing evidence in favour of this ansatz.

3.3. **Determining the cohomology class.** In order to have explicit formulae for the multipliers, the next step consists of determining $[\alpha] \in H^3(M_{24}, U(1))$. As explained following (9), the cohomology class $[\alpha]$ is already uniquely determined by the multiplier phases of the original twining genera. However, already these define an overdetermined problem, and we therefore first need to check that there is a consistent choice for α that is compatible with these multiplier phases.

First we note that the normalisation condition for α, see eq. (152), implies that $c_e(g,h) = 1$ for all $g,h \in M_{24}$. Thus $\phi_{e,g}(\tau+1,z) = \phi_{e,g}(\tau,z)$ and $\phi_{e,g}(\tau,z) = \phi_{e,k^{-1}gk}(\tau,z)$ for all $k \in M_{24}$, as expected. Furthermore, the representation ρ_e in the decomposition (6) is a genuine representation of $C_{M_{24}}(e) = M_{24}$.

Next we observe that eq. (41) implies

$$\phi_{g^a,g^b}(\tau + o(g^a), z) = \kappa(g^a,g^b)\,\phi_{g^a,g^b}(\tau,z)\ , \tag{52}$$

where

$$\kappa(g^a,g^b) := \prod_{k=1}^{o(g^a)} c_{g^a}(g^a, g^{b+ak}) = \prod_{k=1}^{o(g^a)} \alpha(g^a, g^{b+ak}, g^a)\ . \tag{53}$$

Therefore, for any $g \in M_{24}$, $\phi_{e,g}$ obeys the expected modular properties if and only if

$$\kappa(g^a,g^b) = e^{-\frac{2\pi i}{\ell(g^a)}}\ , \tag{54}$$

for all $a, b \in \mathbb{Z}$, where $\ell(g^a)$ is the length of the smallest cycle of g^a when considered as a permutation of 24 objects. It is easy to verify that $\kappa(g^a, g^b)$ only depends on the cohomology class of the 3-cocycle α. Thus we have to prove that there is a unique class $[\alpha] \in H^3(M_{24}, U(1))$ such that (54) is satisfied for all $g \in M_{24}$, $a, b \in \mathbb{Z}$.

In order to show that such an α exists, we first need two basic facts about group cohomology. Given two finite groups G and H and a homomorphism $\iota : H \to G$, we can define the induced map ι^* from $U(1)$-cochains α for G to $U(1)$-cochains $\iota^*(\alpha)$ for H

$$\iota^*(\alpha)(g_1, \ldots, g_n) = \alpha(\iota(g_1), \ldots, \iota(g_n)) \ , \tag{55}$$

which induces a well-defined map on cohomologies ([40], Chapter III.8). A particular case is when $H = G$ and the map $H \to G$ is conjugation $\iota_k(g) = k^{-1}gk$ by an element $k \in G$. In this case, the induced map on the cohomology ι_k^* acts as the identity, i.e. $[\iota^*(\alpha)] = [\alpha]$ (see [40], Chapter III Proposition 8.3). A second interesting case is when H is a subgroup of G with ι the natural inclusion; in this case, $\mathrm{Res} \equiv \iota^*$ is called a restriction map. In particular, if H is a Sylow p-subgroup $\mathrm{Syl}_p\, G$ of G,[††] then the image of the restriction map

$$\mathrm{Res}_p : H^n(G, U(1)) \to H^n(\mathrm{Syl}_p(G), U(1)) \tag{56}$$

is isomorphic to the p-part of $H^n(G, U(1))$ ([40], Chapter III Theorem 10.3). For $G = M_{24}$, using that $H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12}$ [29], this implies

$$\mathrm{Res}_2(H^3(M_{24}, U(1))) \cong \mathbb{Z}_4 \ , \qquad \mathrm{Res}_3(H^3(M_{24}, U(1))) \cong \mathbb{Z}_3 \ , \tag{57}$$

while $\mathrm{Res}_p(H^3(M_{24}, U(1)))$ is trivial for all primes $p > 3$.

With these preparations we can now prove the existence of a class $[\alpha] \in H^3(M_{24}, U(1))$ such that (54) is satisfied. From the invariance of the cohomology classes under the conjugation map ι_k^*, it follows that the multiplier in (54) only depends on the conjugacy class of g. Next we note that in order to compute $\kappa(g^a, g^b)$, it is sufficient to consider the restriction of the cocycle α to the group $\langle g \rangle$ generated by g. For a cyclic group $\mathbb{Z}_N$, the third cohomology $H^3(\mathbb{Z}_N, U(1))$ is isomorphic to $\mathbb{Z}_N$, and a set of cocycle representatives for the cohomology classes are given by [41, 34]

$$\omega_q(g^a, g^b, g^c) = e^{2\pi i \frac{qa}{o(g)^2}([b]+[c]-[b+c])} \ , \qquad q = 0, \ldots, o(g) - 1 \ , \tag{58}$$

where $[\cdot] : \mathbb{Z} \to \{0, \ldots, o(g) - 1\}$ is the reduction modulo $o(g)$.

[††]Recall that a p-group (p prime) is a finite group with order a power of p; a Sylow p-subgroup of a group G is a p-subgroup of G that is maximal, i.e. that is not properly contained in any other p-subgroup. For each p, all Sylow p-subgroups of G are isomorphic and conjugate to one another inside G.

Let us assume, without loss of generality, that $1 \leq a < o(g) - 1$ and $0 \leq b < a$, and that a divides $o(g)$; in fact, if this was not true, we can rewrite both g^a and g^b as powers of another $\tilde{g}$ in $\langle g \rangle$ for which this is then true. The multiplier corresponding to a 3-cocycle ω_q is

$$\begin{aligned}\kappa(g^a, g^b) &= \prod_{k=0}^{o(g^a)-1} \omega_q(g^a, g^{b+ak}, g^a) = e^{2\pi i \frac{qa}{o(g)^2} \sum_{k=0}^{o(g^a)-1}([b+ak]+[a]-[b+a(k+1)])} \\ &= e^{2\pi i \frac{qa}{o(g)^2}([b]+o(g^a)[a]-[b+ao(g^a)])} = e^{2\pi i \frac{qa}{o(g)}} \ ,\end{aligned} \tag{59}$$

where we have used that $o(g^a) = o(g)/a$ when a divides $o(g)$. Thus, $\kappa(g^a, g^b) \equiv \kappa(g^a)$ is independent of b, and $\kappa(g^a) = \kappa(g)^a$ whenever a divides $o(g)$. The latter property is consistent with (54) thanks to the identity

$$\frac{1}{\ell(g^a)} \equiv \frac{a}{\ell(g)} \mod 1 \ , \tag{60}$$

that holds for all elements $g \in M_{24}$ and for all divisors a of $o(g)$.*

It therefore follows that it is sufficient to prove (54) for $a = 1$ and $b = 0$, and for one representative g for each conjugacy class. Furthermore, thanks to $\kappa(g^a) = \kappa(g)^a$, we can restrict to classes whose order is a prime power. In other words, we only need to check (54) for a Sylow p-subgroup $\mathrm{Syl}_p(M_{24})$ and for each prime p.

Using the software `GAP` [38] with the package `HAP` [39] implemented, we have computed a basis of cocycle representatives for $H^3(\mathrm{Syl}_2(M_{24}), U(1))$ and $H^3(\mathrm{Syl}_3(M_{24}), U(1))$. Note that any 3-cocycle in the image of Res_p has the property that, for any $g \in \mathrm{Syl}_p(M_{24})$, the corresponding multiplier $\kappa(g)$ depends only on the conjugacy class of g in M_{24}. We have verified that there is a unique $\mathbb{Z}_4$ subgroup of $H^3(\mathrm{Syl}_2(M_{24}), U(1))$ and a unique $\mathbb{Z}_3$ subgroup of $H^3(\mathrm{Syl}_3(M_{24}), U(1))$ satisfying this property. Therefore these subgroups must correspond to the restrictions $\mathrm{Res}_p(H^3(M_{24}, U(1)))$, for $p = 2$ and $p = 3$, respectively. Finally, for both $p = 2$ and $p = 3$, we have verified that there is a unique cohomology class $[\alpha_p] \in \mathrm{Res}_p(H^3(M_{24}, U(1)))$ that satisfies (54) for all g in $\mathrm{Syl}_p(M_{24})$. Thus, the unique class $[\alpha] \in H^3(M_{24}, U(1))$ that has the property that $\mathrm{Res}_p([\alpha]) = [\alpha_p]$ for all primes p, satisfies (54) for all $g \in M_{24}$. This concludes the proof.

The above techniques are also useful in order to compute the obstructions and the multipliers for a given twisted twining genus $\phi_{g,h}$. To this end we note that it is sufficient to consider the restriction of the cocycle α to the normaliser $N(g, h)$ of $\langle g, h \rangle$. If this normaliser is contained in some Sylow p-subgroup, one can simply use the cocycle α_p determined above.

*Note that this property is special for M_{24}, and is not true for a generic permutation.

Otherwise, one can use GAP to compute a basis for the cohomology of $N(g,h)$ and check which cohomology classes reproduce the correct multipliers for all genera of the form $\phi_{e,k}$, $k \in N(g,h)$. In all cases we considered, there was only one cohomology class with this property, which must therefore be the restriction of the class $[\alpha] \in H^3(M_{24}, U(1))$ to the cohomology of $N(g,h)$.

Once the restriction of $[\alpha]$ to $N(g,h)$ is determined, one can check the existence of obstructions for all 34 groups of table 1. The results are collected in table 3. For the unobstructed twisted twining genera $\phi_{g,h}$, one can compute the precise multiplier system. It turns out that this information is sufficient to determine $\phi_{g,h}$ up to normalisation. Finally, the normalisation can be fixed (up to a phase) by requiring that a decomposition (6) exists. An example is worked out in section 3.4, and the other cases are considered in appendix A.

The GAP-files containing the computations described above are available online [42].

3.4. **Computation of the twisted twining characters: an example.** As we have explained in section 2.5, the knowledge about the multiplier phases implies that most of the twisted twining genera vanish. However, some of them are non-zero, and in this section we illustrate how one of these non-zero twisted twining genera, namely the one associated to $g \in 2\mathrm{B}$ and $h \in 8\mathrm{A}$, can be determined explicitly; the other cases are dealt with in appendix A. Let us work with the conventions that the three permutations

$$(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23)(24) \ ,$$
$$(3,17,10,7,9)(4,13,14,19,5)(8,18,11,12,23)(15,20,22,21,16)(1)(2)(6)(24) \ ,$$
$$(1,24)(2,23)(3,12)(4,16)(5,18)(6,10)(7,20)(8,14)(9,21)(11,17)(13,22)(15,19)$$

generate $M_{24} \subset S_{24}$. As representative for $g \in 2\mathrm{B}$ we can then take

$$g = (1,10)(2,14)(3,8)(4,5)(6,22)(7,20)(9,18)(11,23)(12,24)(13,19)(15,16)(17,21) \ , \quad (61)$$

while for $h \in 8\mathrm{A}$ we consider

$$h = (2,14)(3,9,8,18)(4,6,21,19,15,24,20,11)(5,22,17,13,16,12,7,23)(1)(10) \ . \quad (62)$$

The elements g, h generate the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_8$ corresponding to case 1 of table 1. There are two interesting conjugation relations, namely

$$(k^{-1}gh^4k \, , \, k^{-1}h^{-1}k) = (g,h) \ , \qquad (r^{-1}gr \, , \, r^{-1}ghr) = (g,h) \ , \quad (63)$$

where

$$k = (1,10)(2,14)(3,18)(4,11)(5,13)(6,20)(7,12)(8,9)(15,19)(16,23)(17,22)(21,24)\ ,$$
$$r = (1,2)(3,9)(6,22)(8,18)(10,14)(11,23)(12,24)(13,19)(4)(5)(7)(15)(16)(17)(20)(21)\ . \tag{64}$$

Thus, we can use the general formulae (41) and their inverses to deduce that[†]

$$\phi_{g,h}(\tau+1) = c_g(g,h)\phi_{g,gh}(\tau) = c_g(g,h)\frac{c_g(gh,r)}{c_g(r,h)}\phi_{g,h}(\tau) = i\phi_{g,h}(\tau)\ , \tag{65}$$

where we have used the explicit form of the multiplier factors from section 3.3. We also have (recall that $g = g^{-1}$)

$$\begin{aligned}\phi_{g,h}(\tfrac{\tau}{-4\tau+1}) =& \frac{\phi_{h,g}(4-\frac{1}{\tau})}{c_h(g,g)} = \frac{\prod_{i=1}^3 c_h(h,gh^i)}{c_h(g,g)}\phi_{h,gh^4}(-\tfrac{1}{\tau}) = \frac{\prod_{i=1}^3 c_h(h,gh^i)}{c_h(g,g)c_{gh^4}(h,h^{-1})}\phi_{gh^4,h^{-1}}(\tau)\\ =& \frac{\prod_{i=1}^3 c_h(h,gh^i)}{c_h(g,g)c_{gh^4}(h,h^{-1})}\frac{c_{g^4h}(h^{-1},k)}{c_{g^4h}(k,h)}\phi_{g,h}(\tau) = -\phi_{g,h}(\tau)\ .\end{aligned} \tag{66}$$

This implies that $\phi_{g,h}(\tau,z)$ is a Jacobi form under the group $\Gamma_0(4)$ with the above multipliers. Since the multiplier system is non-trivial, the constant $\phi_{g,h}(\tau,0)$ must vanish and by $\phi_{g,h}(\tau,z) = \phi_{g,h}(\tau,-z)$ it follows that $\phi_{g,h}$ has a double zero at $z=0$. On the other hand, the Jacobi form

$$\psi(\tau,z) \equiv \frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2} \tag{67}$$

has weight 0 and index 1 under $\Gamma_0(4)$ (with trivial multiplier). Since it only has a double zero at $z=0$ (mod $\mathbb{Z}+\tau\mathbb{Z}$), it follows that

$$F(\tau) \equiv \frac{\phi_{g,h}(\tau,z)}{\psi(\tau,z)} \tag{68}$$

is a modular function of weight 0 under $\Gamma_0(4)$ (with multiplier system (65)–(66)) that is holomorphic in the interior of $\mathbb{H}_+$. Let us consider the asymptotic behaviour of $F(\tau)$ at the boundary $\Gamma_0(4)\backslash(\mathbb{Q}\cup\{\infty\})$ of the compactification $\Gamma_0(4)\backslash\bar{\mathbb{H}}_+$. This boundary consists of three points (cusps) of the form $\gamma_c\cdot\infty$, where $\gamma_\infty = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\gamma_0 = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $\gamma_{1/2} = \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$. The twisted twining genus $\phi_{g,h}$ has the Fourier expansion at these cusps

$$\phi_{g,h}(\gamma_\infty(\tau,z)) = \phi_{g,h}(\tau,z) = A_\infty\, q^{1/4}(2-y-y^{-1}) + O(q^{5/4}) \tag{69}$$

$$\phi_{g,h}(\gamma_0(\tau,z)) \sim \phi_{h,g}(\tau,z) = A_0\, q^{1/8}(2-y-y^{-1}) + O(q^{3/8}) \tag{70}$$

$$\phi_{g,h}(\gamma_{1/2}(\tau,z)) \sim \phi_{gh^2,h}(\tau,z) = A_{1/2}\, q^{1/4}(2-y-y^{-1}) + O(q^{3/4})\ , \tag{71}$$

[†]For brevity of presentation we omit the dependence of z, as well as the factors $e^{2\pi i\frac{cz^2}{c\tau+d}}$, in the following formulae.

where $\sim$ denotes equality up to a phase, and we have included the lowest non-negative powers of q^r that are compatible with the multiplier system. It follows from (6) that the constants A_∞, A_0, $A_{1/2}$ equal

$$A_\infty = \operatorname{Tr}_{\mathcal{H}_{g,r=\frac{1}{4}}}(\rho_{g,\frac{1}{4}}(h)) \ , \quad A_0 = \operatorname{Tr}_{\mathcal{H}_{h,r=\frac{1}{8}}}(\rho_{h,\frac{1}{8}}(g)) \ , \quad A_{1/2} = \operatorname{Tr}_{\mathcal{H}_{gh^2,r=\frac{1}{4}}}(\rho_{gh^2,\frac{1}{4}}(h)) \ , \tag{72}$$

and from the Fourier expansion of $\phi_{h,e}$ we know that $\dim \mathcal{H}_{h,r=\frac{1}{8}} = 1$, so that $|A_0| = 1$. On the other hand, $\psi(\tau, z)$ satisfies at the cusps

$$\psi(\gamma_\infty(\tau,z)) = \psi(\gamma_{1/2}(\tau,z)) = \frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2} = \frac{1}{4}(2-y-y^{-1}) + O(q) \tag{73}$$

$$\psi(\gamma_0(\tau,z)) = \frac{\vartheta_1(\tau,z)^2}{\vartheta_4(\tau,0)^2} = q^{1/4}(2-y-y^{-1}) + O(q^{3/4}) \ , \tag{74}$$

and we therefore obtain

$$F(\gamma_\infty \cdot \tau) = 4\, A_\infty\, q^{1/4} + O(q^{5/4}) \tag{75}$$

$$F(\gamma_0 \cdot \tau) \sim A_0\, q^{-1/8} + O(q^{1/8}) \tag{76}$$

$$F(\gamma_{1/2} \cdot \tau) \sim 4\, A_{1/2}\, q^{1/4} + O(q^{3/4}) \ . \tag{77}$$

Up to a phase, there is a unique modular form with the correct modular properties and the expected Fourier expansion at the cusps, namely

$$F(\tau) = 8\frac{\eta(2\tau)^6}{\eta(\tau)^6} \ . \tag{78}$$

Thus we conclude that up to an overall phase we have

$$\phi_{2\mathrm{B},8\mathrm{A}}(\tau,z) = 8\frac{\eta(2\tau)^6}{\eta(\tau)^6}\frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2} = 2\frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2 \ , \tag{79}$$

where we used the identity $\vartheta_2(\tau,0)^2 = 4\frac{\eta(2\tau)^4}{\eta(\tau)^2}$. The other non-trivial cases can be worked out similarly, see appendix A for the details.

3.5. **Projective representations.** As we have argued above, see property (C) in section 2, the twisted sector $\mathcal{H}_g$ should carry a projective representation of the centraliser $C_{M_{24}}(g)$, whose 2-cocycle is given by c_g as determined in (39). Given our explicit knowledge of all twisted twining genera as well as c_g, this can now be tested as in [8] (where the corresponding analysis was performed for the case of the twining genera): if each $\mathcal{H}_{g,r}$ is a projective representation of $C_{M_{24}}(g)$, we can decompose it as

$$\mathcal{H}_{g,r} = \bigoplus_j h_{g,r}^{(j)}\, R_j \ , \tag{80}$$

where $h^{(j)}_{g,r}$ is the multiplicity with which the projective representation R_j (whose projectivity is characterised by c_g) appears in $\mathcal{H}_{g,r}$. On the other hand, using the orthogonality of group characters (see appendix C), we can calculate $h^{(j)}_{g,r}$ from the knowledge of the twisted twining genera explicitly. The consistency condition is then that all $h^{(j)}_{g,r}$ are indeed (non-negative) integers. We have done this analysis for all twisted sectors and for the first 500 levels (i.e. the first 500 values for r), and the multiplicities are indeed always non-negative integers. The explicit results for the first 20 levels as well as the (projective) character tables of all non-isomorphic centralisers $C_{M_{24}}(g)$ are given in appendix D, but commented out in the LaTeX source code. The decompositions for all g and for the first 500 levels are available online [42].

4. K3 Orbifolds

There are at least two further consistency checks on our proposal that can be fairly easily analysed. Suppose we consider a K3 sigma-model $\mathcal{C}$ whose automorphism group contains the cyclic subgroup generated by g. Then we can consider the orbifold of $\mathcal{C}$ by $G = \langle g \rangle$. A priori, it is not guaranteed that this orbifold is consistent — indeed, since these symmetries effectively act asymmetrically, they generically suffer from the level-matching problem, and hence may be inconsistent [36]. However, as was already mentioned in [37], the level-matching condition is satisfied provided that the twining genus ϕ_g has a trivial multiplier system. This is in particular the case if $g \in M_{23}$. In that case, the resulting orbifold theory $\widehat{\mathcal{C}} = \mathcal{C}/G$ is again a K3 sigma model, as was also shown in [37].

Suppose then that the original $\mathcal{C}$ has, in addition to $g \in M_{23}$, another commuting group element h in its automorphism group. Then h also gives rise to a symmetry $\hat{h}$ of the orbifold theory $\widehat{\mathcal{C}}$, and we can calculate the twining genus of $\hat{h}$ by the usual orbifold formula

$$\phi_{e,\hat{h}} = \frac{1}{o(g)} \sum_{i,j=0}^{o(g)-1} \phi_{g^i,hg^j} \ . \tag{81}$$

This leads to a non-trivial consistency condition: given our explicit knowledge of all twisted twining genera, we can calculate the right-hand-side explicitly, and this must agree, for every h, with *one* of the twining genera of [8, 9]. We have checked that this is indeed true; an example is illustrated in section 4.2.

Naively, one may have guessed that $\hat{h}$ should simply agree with h, i.e. that the left-hand-side of (81) is the twining genus associated to the *same* group element $h \in M_{24}$. However, this is in general not the case, as will be explained in the following section 4.1, see also section 4.2

for an explicit example. Whenever this happens we will say that the group element h has been *relabelled* in the orbifold theory.

The other consistency check is even more obvious. As was mentioned at the beginning of section 2, there exist K3 sigma-models with a commuting pair of automorphisms (g, h). For them (2) can be calculated directly, and thus compared to our answers. We shall perform this computation for some simple cases in section 4.3. In particular, we shall concentrate on examples where our obstruction analysis predicts that the associated twisted twining genus must vanish. The fact that we can reproduce this result using elementary methods is a good consistency check on our analysis.

4.1. **The relabelling phenomenon.** Suppose that a K3 sigma-model $\mathcal{C}$ contains in its automorphism group an element g such that the orbifold of $\mathcal{C}$ by g, $\hat{\mathcal{C}} = \mathcal{C}/\langle g\rangle$ is consistent. In general, each g^r-twisted sector carries a projective representation of $H \equiv C_{M_{24}}(g)$. Equivalently, we can think of the untwisted and twisted sectors as carrying genuine representations of some central extension $\tilde{H}$ of H. In fact, the central extension can be chosen to be of the form (see also appendix C.1 for more details)

$$1 \to \langle \tilde{Q}\rangle \cong \mathbb{Z}_N \to \tilde{H} \xrightarrow{\rho} H \to 1 \ . \tag{82}$$

Here ρ is the representation in the untwisted sector, i.e. on the spectrum of $\mathcal{C}$, N is the order $o(g)$ of g and the generator $\tilde{Q}$ (that we call the *quantum symmetry*) of the central $\mathbb{Z}_N$ acts by $e^{2\pi i r/N}$ on the g^r-twisted sector. Note that there is always a lift $\tilde{g} \in \tilde{H}$ of g that acts by $e^{2\pi i k/N}$ on the states of conformal weight $h - 1/4 = k/N$.

In the orbifold theory $\hat{\mathcal{C}}$, all states are invariant under the action of $\tilde{g}$ — this is just the precise way of imposing the usual requirement that the orbifold theory consists of 'g-invariant' states only. Since $\tilde{g}$ is in the center of $\tilde{H}$, the spectrum of $\hat{\mathcal{C}}$ carries a well-defined representation $\hat{\rho}$ of $\tilde{H}$, such that $\hat{\rho}(\tilde{g})$ is the identity. In fact, it is easy to see that $\ker \hat{\rho} = \langle \tilde{g}\rangle$, because $\tilde{Q}$ acts non-trivially on the twisted sectors, and each non-trivial element in $H/\langle g\rangle$ lifts to an element of $\tilde{H}$ that acts non-trivially on the g-invariant untwisted sector, which in turn is part of the spectrum of $\hat{\mathcal{C}}$. Thus, we have an exact sequence

$$1 \to \langle \tilde{g}\rangle \cong \mathbb{Z}_N \to \tilde{H} \xrightarrow{\hat{\rho}} \hat{H} \to 1 \ . \tag{83}$$

The group $\hat{H} \equiv \tilde{H}/\langle \tilde{g}\rangle$ defines a fully fledged symmetry of the orbifold CFT $\hat{\mathcal{C}}$. For the case at hand, both the original theory and the orbifold are K3 sigma models, and thus all

of these symmetry groups should be subgroups of M_{24},[‡]

$$\hat{H} \subseteq M_{24} \ . \tag{84}$$

This assumption allows us to determine $\hat{H}$ (at least as an abstract group). In fact, the order of $\hat{H}$ is given by

$$|\hat{H}| = \frac{|\tilde{H}|}{N} = |H| = |C_{M_{24}}(g)| \ . \tag{85}$$

The quantum symmetry $Q \equiv \hat{\rho}(\tilde{Q})$ must be in the same M_{24}-conjugacy class as g, since it has the same eigenvalues on the 24-dimensional representation of RR ground states. Since all elements of $\hat{H}$ commute with Q, we can conclude by (85) that

$$\hat{H} = C_{M_{24}}(Q) \cong C_{M_{24}}(g) = H \ . \tag{86}$$

This construction allows us to give a precise interpretation of (81). For each symmetry $h \in H$ in the model $\mathcal{C}$, one chooses a lift $\tilde{h} \in \tilde{H}$, with $\rho(\tilde{h}) = h$ (this amounts to choosing the phases of the twisted twining genera $\phi_{g,h}$). Then the formula for the twining genus of the symmetry $\hat{\rho}(\tilde{h})$ in the orbifold model $\hat{\mathcal{C}}$ takes the form

$$\phi_{e,\hat{\rho}(\tilde{h})} = \frac{1}{o(g)} \sum_{i,j=0}^{o(g)-1} \phi_{g^i,\tilde{h}\tilde{g}^j} \ . \tag{87}$$

After these preparations we can now explain the relabelling phenomenon. While it follows from (86) that $\hat{H}$ and H are isomorphic, they are not canonically isomorphic; in fact, we only have a canonical isomorphism between

$$H/\langle g \rangle \cong \tilde{H}/\langle \tilde{g}, \tilde{Q} \rangle \cong \hat{H}/\langle Q \rangle \ , \tag{88}$$

but this does not necessarily lift to a canonical isomorphism $\Phi : \hat{H} \to H$, with $\Phi(Q) = g$. Put differently, in general it is not possible to find an isomorphism $\Phi : \hat{H} \to H$, with $\Phi(Q) = g$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \tilde{H} \arrow[dl, "\hat{\rho}"'] \arrow[dr, "\rho"] & \\
\hat{H} \arrow[rr, dashed, "\Phi"] \arrow[dr] & & H \arrow[dl] \\
 & \tilde{H}/\langle \tilde{g}, \tilde{Q} \rangle &
\end{tikzcd}$$

[‡]We are assuming here that the relevant K3 sigma model is 'non-exceptional' in the terminology of [37].

For example, this will be the case if for any lift $\tilde{h}$ with $\rho(\tilde{h}) = h$, the corresponding symmetry $\hat{\rho}(\tilde{h})$ is not in the same M_{24} conjugacy class as h. In that case, the twining genus obtained by formula (87) does not agree with $\phi_{e,h}$. Whenever this is the case, we shall say that the conjugacy class of h has been *relabelled* in the orbifold theory.

4.2. **An example.** As an explicit example of the relabelling phenomenon let us consider the case where g is in the class 2A of M_{24} (so that $N = o(g) = 2$). Then the group $\tilde{H}$ is a central extension of H by the $\mathbb{Z}_2$ group generated by $\tilde{Q}$. We denote by $\tilde{g}$ the lift of g in $\tilde{H}$, such that $\tilde{g}$ and $\tilde{Q}$ have the following eigenvalues

$\tilde{Q}$	$\tilde{g}$	sector	model
$+1$	$+1$	untwisted	$\mathcal{C}, \hat{\mathcal{C}}$
$+1$	-1	untwisted	$\mathcal{C}$
-1	$+1$	twisted, $q^{\mathbb{Z}}$	$\hat{\mathcal{C}}$
-1	-1	twisted, $q^{\mathbb{Z}+\frac{1}{2}}$	$-$

(89)

As expected from the discussion in the previous section, the group $\tilde{H}/\tilde{Q}$ is isomorphic to the centraliser in M_{24} of an element of class 2A.

Now consider an element $h \in H$ in class 8A, with $h^4 = g$. The trace of h over the 24-dimensional representation of ground states in the original model $\mathcal{C}$ is $+2$. Furthermore, since hg is in the same class, also the trace over hg equals $+2$, from which we conclude that on the ground states of $\mathcal{C}$ we have

$$\mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=1,\tilde{g}=1}}(h) = +2 \ , \qquad \mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=1,\tilde{g}=-1}}(h) = 0 \ , \tag{90}$$

where $\mathcal{H}^0_{\tilde{Q}=\pm1,\tilde{g}=\pm1}$ are the eigenspaces of $\tilde{Q}$ and $\tilde{g}$ at conformal weight $h_L - \frac{1}{4} = 0$. The lifts $\tilde{h}$ and $\tilde{Q}\tilde{h}$ in $\tilde{H}$ are such that $\tilde{h}^4 = (\tilde{Q}\tilde{h})^4 = \tilde{g}$. On dimensional grounds, the twisted sector at $h_L - 1/4 = 0$ must correspond to a certain 8-dimensional irreducible representation of $\tilde{H}$, where the lifts $\tilde{h}$ and $\tilde{Q}\tilde{h}$ have trace ± 2. We notice that, up to a sign, this trace is independent of the choice of the central extension, and is in agreement with the expectation from the twisted twining genus $\phi_{g,h} \equiv \phi_{h^4,h}$. On the ground states in the orbifold theory $\hat{\mathcal{C}}$, the trace of $\tilde{h}$ is now

$$\mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=1,\tilde{g}=1}}(\tilde{h}) + \mathrm{Tr}_{\mathcal{H}^0_{\tilde{Q}=-1,\tilde{g}=1}}(\tilde{h}) = 2 \pm 2 \ , \tag{91}$$

and thus there is no choice for $\tilde{h}$ such that the trace is equal to $+2$, as required for an element in class 8A. In fact, the images $\hat{h}$ and $Q\hat{h}$ in $\hat{H} \cong C_{M_{24}}(Q)$ have order 4 and belong

to the classes 4A and 4B. In particular, while the pair of commuting elements g, h in the original model generate the cyclic group

$$\langle g, h\rangle = \langle h\rangle \cong \mathbb{Z}_8 \ , \tag{92}$$

both the groups $\langle Q, \hat{h}\rangle$ and $\langle Q, Q\hat{h}\rangle$ correspond to group 16 in our general list. Thus, there cannot exist an isomorphism $\Phi : \hat{H} \to H$ such that $\Phi(\hat{\rho}(\tilde{h})) = h$ or $\Phi(\hat{\rho}(\tilde{Q}\tilde{h})) = h$.

Let us verify the consistency of eq. (87) for this case, see the comments at the beginning of section 4. The twining genera $\phi_{e,\hat{\rho}(\tilde{h})}$ and $\phi_{e,\hat{\rho}(\tilde{Q}\tilde{h})}$ are given by

$$\phi_{e,\hat{\rho}(\tilde{h})}(\tau, z) = \frac{1}{2}(\phi_{e,h}(\tau, z) + \phi_{e,h^5}(\tau, z) + \phi_{h^4,h}(\tau, z) + \phi_{h^4,h^5}(\tau, z)) \ , \tag{93}$$

$$\phi_{e,\hat{\rho}(\tilde{Q}\tilde{h})}(\tau, z) = \frac{1}{2}(\phi_{e,h}(\tau, z) + \phi_{e,h^5}(\tau, z) - \phi_{h^4,h}(\tau, z) - \phi_{h^4,h^5}(\tau, z)) \ . \tag{94}$$

Using the identity

$$\phi_{e,8\mathrm{A}}(\tau, z) = \frac{1}{2}(\phi_{e,4\mathrm{B}}(\tau, z) + \phi_{e,4\mathrm{A}}(\tau, z)) \ , \tag{95}$$

it is easy to verify that

$$\phi_{e,\hat{\rho}(\tilde{h})} = \phi_{e,4\mathrm{B}}, \qquad \phi_{e,\hat{\rho}(\tilde{Q}\tilde{h})} = \phi_{e,4\mathrm{A}} \ , \tag{96}$$

as expected. This example also shows that the consistency of eq. (87) gives, in general, highly non-trivial conditions on the twisted twining genera.

We have verified explicitly that a similar phenomenon also occurs for the orbifold by an element g in class 4B.

4.3. **Computation of a twisted twining genus.** In this section we shall calculate some twisted twining genera directly. In particular, we shall concentrate on some simple cases where our general analysis predicts that they must vanish (because of some obstruction). As we shall see, we can reach the same conclusion from first principles.

Let us consider a K3 σ-model $\mathcal{C}$ whose automorphism group contains two commuting elements g, h of order m and n, respectively, generating the non-cyclic group $G = \langle g, h\rangle$. If these symmetries have a geometric origin, i.e. if they are induced by automorphisms of the K3 target space that fix the holomorphic 2-form, then the group G must be a subgroup of M_{23} with at least five orbits over the 24-dimensional representation, as follows from the Mukai theorem [43]. These conditions are satisfied by the groups 4 and 7 ($\mathbb{Z}_2 \times \mathbb{Z}_2$), groups 9 and 22 ($\mathbb{Z}_2 \times \mathbb{Z}_4$), group 18 ($\mathbb{Z}_2 \times \mathbb{Z}_6$), group 24 ($\mathbb{Z}_4 \times \mathbb{Z}_4$), and group 32 ($\mathbb{Z}_3 \times \mathbb{Z}_3$) of our table 1. In all these cases, the twisted twining genus vanishes, $\phi_{g,h} = 0$, see table 3. In the following we want to verify this independently.

In the general $\mathbb{Z}_n \times \mathbb{Z}_m$ case, the twisted twining genus $\phi_{g,h}$ is expected to be a weak Jacobi form of weight 0 and index 1 under the congruence subgroup

$$\Gamma_{m,n} \equiv \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid a \equiv 1, b \equiv 0 \mod m,\ c \equiv 0, d \equiv 1 \mod n \} \ . \tag{97}$$

Note that this group is in general smaller than the group $\Gamma_{g,h}$ defined below (124), because we are not using the identifications under conjugation (5). Since both g and h are elements of M_{23}, the multiplier system for $\phi_{g,h}$ must be trivial (since $H^3(M_{23}, U(1)) \cong 0$).
Because of the triviality of the multiplier system, it follows that any modular image $\phi_{g',h'}$ of $\phi_{g,h}$, where $(g',h') = (g,h)\cdot\gamma$ for some $\gamma \in SL(2,\mathbb{Z})$, has a Fourier expansion of the form

$$\phi_{g',h'}(\tau,z) = \sum_{n=0}^{\infty} \sum_{\ell} c_{g',h'}(n,\ell)\, q^{\frac{n}{o(g')}}\, y^{\ell} \ , \tag{98}$$

where $o(g')$ is the order of g'. Let us assume

$$c_{g',h'}(0,\ell) = 0 \quad \forall \ell \in \mathbb{Z} \ , \tag{99}$$

for any modular image $(g',h') = (g,h)\cdot\gamma$; we will prove this below. Then it follows that

$$\phi_{g,h}(\tau,0) = \sum_{\ell} c_{g,h}(0,\ell) = 0 \ . \tag{100}$$

Because of (13), $\phi_{g,h}(\tau,z) = \phi_{g,h}(\tau,-z)$, and we can conclude that $\phi_{g,h}$ has a double zero at $z=0$. Next we recall that $\vartheta_1(\tau,z)^2/\eta(\tau)^2$ is a Jacobi form of weight 0 and index 1 under $SL(2,\mathbb{Z})$ (with a non-trivial multiplier). It has a double zero at $z=0$ (mod $\mathbb{Z}+\tau\mathbb{Z}$), and is non-vanishing elsewhere, so that

$$F_{g,h}(\tau) = \frac{\eta(\tau)^2 \phi_{g,h}(\tau,z)}{\vartheta_1(\tau,z)^2} \tag{101}$$

is a modular function of weight 0 under $\Gamma_{m,n}$ (with non-trivial multiplier), which is holomorphic in the interior of the upper-half plane $\mathbb{H}_+$. Let us consider the asymptotics of $F_{g,h}$ at the boundary $\Gamma_{m,n}\backslash(\mathbb{Q}\cup\infty)$ of the compactification $\Gamma_{m,n}\backslash\bar{\mathbb{H}}_+$. The boundary is the union of points (cusps) of the form $\gamma(\infty)$, where γ runs through the representatives of the double cosets $\Gamma_{m,n}\backslash SL(2,\mathbb{Z})/\Gamma^{\infty}$. Here, Γ^{∞} is the parabolic subgroup of $SL(2,\mathbb{Z})$ that fixes $\infty \in \bar{\mathbb{H}}_+$ and is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The Jacobi form $\vartheta_1(\tau,z)^2/\eta(\tau)^2$ has the same asymptotic behaviour at each cusp, namely

$$\frac{\vartheta_1(\gamma\cdot(\tau,z))^2}{\eta(\gamma\cdot\tau)^2} \overset{\tau\to\infty}{\sim} q^{1/6}(2-y-y^{-1}) + O(q^{7/6}) \ , \tag{102}$$

up to a constant coefficient. By (98) and (99), it then follows that

$$F_{g,h}(\gamma\cdot\tau) \sim \frac{\eta(\tau)^2 \phi_{g',h'}(\tau,z)}{\vartheta_1(\tau,z)^2} \overset{\tau\to\infty}{\sim} O(q^{\frac{1}{o(g')}-\frac{1}{6}}) \ , \tag{103}$$

where $(g', h') = (g, h)\gamma$ and $o(g')$ is the order of g'. For all the groups we are interested in, we have $o(g') \leq 6$, and the inequality is strict for at least one g'. It follows that $F_{g,h}$ has no poles and at least one zero at the boundary of $\Gamma_{m,n}\backslash\bar{\mathbb{H}}_+$. Since it is holomorphic in the interior, it must therefore vanish identically, and we conclude that

$$\phi_{g,h}(\tau, z) = 0 \ , \tag{104}$$

as predicted by our results.

It therefore only remains to prove (99). The coefficient $c_{g,h}(0, \ell)$ corresponds to the trace of h (or, more generally, of the lift $\tilde{h}$ to a central extension of the group $\langle g, h\rangle$) on the RR g-twisted states with conformal weight $h_L = h_R = 1/4$ and weight ℓ under the $\mathfrak{su}(2)_L$ subalgebra of the left-moving $\mathcal{N} = 4$ algebra. The g-twisted ground states are g-invariant (the conformal weight satisfies $h_L - 1/4 \in \mathbb{Z}$), so they belong to the spectrum of the orbifold theory $\mathcal{C}/\langle g\rangle$.

Any sigma-model whose target space is a K3 manifold X contains 24 RR ground states corresponding to the harmonic forms on X. Thus, the real vector space spanned by these states can be identified with the real cohomology $H^*(X, \mathbb{R})$ on X. The orbifold model $\hat{\mathcal{C}} = \mathcal{C}/\langle g\rangle$ corresponds to a non-linear sigma-model on the geometric orbifold $\hat{X} = X/\langle g\rangle$, which is a singular K3 surface. In particular, for each point of X that is fixed by some non-trivial subgroup $\langle g^r\rangle \subseteq \langle g\rangle$, the orbifold $\hat{X}$ has a singularity of type A_{n-1}, where n is the order of g^r. By blowing up all these singularities, one obtains again a smooth K3 manifold.

The RR ground states in the untwisted sector of $\hat{\mathcal{C}}$ are obtained by projecting onto the g-invariant subspace of $H^*(X, \mathbb{R})$, while the ground states in the twisted sectors are the Poincaré duals of the cycles corresponding to the exceptional divisors in the blow-up of $\hat{X}$. More precisely, the resolution of each A_{n-1} singularity of $\hat{\mathcal{C}}$ gives rise to $n-1$ rational curves $\mathbb{P}^1$, representing elements in the homology of the blow-up. The cohomology classes dual to these cycles span an $(n-1)$-dimensional subspace of RR ground states for $\hat{\mathcal{C}}$, one in each g^{ir} twisted sector, for $i = 1, \ldots, n-1$. Thus, we arrive at the usual statement that the ground states of the g^k-twisted sector are in one to one correspondence with the g^k-fixed points of the target space.

Any other symplectic automorphism h of X that commutes with g acts as a permutation on the g^r-fixed points. This action induces a permutation on the singularities of $\hat{X}$, and

therefore on the twisted RR ground states dual to the exceptional cycles.[§] Therefore, the trace of h over the g-twisted sector only gets a non-zero contribution from the singularities of $\hat{X}$ that are fixed by h, i.e. only from the points of X that are fixed by both g and h. It is known (see, for example Proposition 1.5 of [43]) that a finite group of symplectic automorphisms of a K3 manifold X that fixes at least one point must be a subgroup of $SL(2,\mathbb{C})$. On the other hand, all finite abelian subgroups of $SL(2,\mathbb{C})$ are cyclic. Thus we conclude that if $\langle g,h\rangle$ is abelian but not cyclic, the trace of h over the g-twisted ground fields is 0 and (99) follows.

Our results also agree with the analysis of [44], where it was shown that the twisted twining genus $\phi_{g,h}$ for a pair of commuting symplectic automorphisms g,h of a K3 manifold X is given by a sum of contributions corresponding to the points fixed by both g and h. If the abelian group $\langle g,h\rangle$ is not cyclic, then there are no fixed points and $\phi_{g,h}=0$.

5. Conclusions

5.1. **Summary.** Inspired by the generalised Monstrous Moonshine idea of Norton [14] we have in this paper established 'generalised Mathieu Moonshine' for the elliptic genus of K3. More specifically, we have found, for each commuting pair $(g,h)\in M_{24}$, explicit expressions for the twisted twining genera $\phi_{g,h}:\mathbb{H}_+\times\mathbb{C}\to\mathbb{C}$, and we have verified that these functions satisfy conditions (A)–(D) of section 2.1. In particular, the twisted twining genera $\phi_{g,h}(\tau,z)$ are weak Jacobi forms of weight 0 and index 1 for certain subgroups $\Gamma_{g,h}\subset SL(2,\mathbb{Z})$ with a multiplier system $\chi_{g,h}:\Gamma_{g,h}\to U(1)$.

One of the key insights of our work is that these multiplier phases are all determined in terms of a class $[\alpha]\in H^3(M_{24},U(1))$ via a formula (see eq. (39)) that was first derived by Dijkgraaf and Witten in the context of holomorphic orbifolds [22]. We have also shown that the twisted twining genera $\phi_{g,h}$ are compatible with a decomposition of the g-twisted sector into (projective) representations of the centraliser $C_{M_{24}}(g)$. The particular central extension of $C_{M_{24}}(g)$ associated with this projective representation is also determined by the cohomology class $[\alpha]\in H^3(M_{24},U(1))$.

As it turned out, many of the twisted twining genera vanish, and in most cases this follows from the structure of the class $[\alpha]\in H^3(M_{24},U(1))$. While these sorts of cohomological obstructions should also arise in the context of other holomorphic orbifolds (in particular

[§]More precisely, one should choose a lift $\tilde{h}$ of h to a central extension of the group of symmetries of $\mathcal{C}$; then, $\tilde{h}$ acts on the g-twisted sector by a permutation times an overall phase that depends on the particular lift. Since the phase is not important for our argument, we will neglect this subtlety.

in Monstrous Moonshine), our results provide, to our knowledege, the first example where they have been explicitly verified. We have also confirmed the vanishing of some of these twisted twining genera independently, using geometrical arguments.

5.2. **Open problems and future work.** The elliptic genus of K3 is closely related to the physics of 1/4 BPS-dyons in $\mathcal{N} = 4$ string theory. The generating function of elliptic genera of symmetric products of K3's is the famous Igusa cusp form Φ_{10}, whose inverse is the partition function of $\mathcal{N} = 4$ dyons [45, 46]. This fact lies at the heart of the recent progress in understanding wall-crossing of multi-centered BPS-states in $\mathcal{N} = 4$ string theory using mock modular forms [47]. It is therefore natural to wonder about the corresponding physical interpretation of the twisted twining genera $\phi_{g,h}$. In the special case of a trivial twist $g = e$, it has been shown that for some elements h the twining genera $\phi_{e,h}$ correspond to BPS-indices in CHL-orbifolds [6, 48, 49, 50]. We may therefore expect that the twisted twining genera $\phi_{g,h}$ should be related to the counting of 'twisted dyons' in CHL-models. Indeed for some pairs (g, h) we have verified this by showing that the twisted CHL-indices of Sen [44] are compatible with our $\phi_{g,h}$. It would be interesting to determine more generally whether $\phi_{g,h}$ have a physical interpretation in terms of CHL-models or some generalisation thereof.

In Borcherds' proof of the Monstrous Moonshine conjecture he introduced a new class of infinite-dimensional (super-)Lie algebras that he called generalised Kac-Moody algebras (GKM) [3]. The key idea was that the multiplicative lift of the modular J-function could be interpreted as the denominator formula for a specific GKM, now commonly called the Monster Lie algebra, whose root system carries an action of the Monster group $\mathbb{M}$. The generalised moonshine ideas of Norton [14] suggest that similar GKMs should exist for each class $[g] \in \mathbb{M}$ [17, 18, 51]. In the context of Mathieu Moonshine the role of generalised Kac-Moody algebras has so far not been understood. Although there is a GKM associated with the elliptic genus of K3 [52], this algebra does not seem to carry any natural action of M_{24} due to the fact that its denominator formula is $1/\sqrt{\Phi_{10}}$ rather than $1/\Phi_{10}$. For some conjugacy classes $[g] \in M_{24}$ the associated second-quantised twining genera Φ_g also give rise to denominator formulas of GKMs [53], which should presumably be identified with the wall-crossing algebras found in CHL-models [54, 55, 56, 57, 48]. Although these observations are very suggestive, the precise role of the GKMs for Mathieu Moonshine remains to be understood.

A crucial point in our analysis was that the multiplier phases are controlled by a class $[\alpha] \in H^3(M_{24}, U(1))$, from which also the various obstructions could be read off. It follows from the work of [22, 34] that similarly a cohomology class in $H^3(\mathbb{M}, U(1))$ should underly generalised Monstrous Moonshine, and it would be very interesting to see whether this point of view may lead to new insights. In particular, some of Norton's generalised Moonshine functions $f(g, h; \tau)$ are known to vanish, and it would be natural to expect that this is due to some cohomological obstructions similar to those we have found in this paper.[¶] Some related ideas were already put forward some time ago by Mason [24], but very little has been done explicitly, mainly because the group $H^3(\mathbb{M}, U(1))$ seems to be poorly understood.
In a similar vein, it would also be interesting to investigate the generalised versions of the recently discovered Umbral Moonshine observations [58], of which the M_{24}/K3-moonshine is a special case. In particular, one might expect that there is a relation between the Rademacher summability condition of [12] and the cohomological obstructions we have described here.

The fact that there is a class in $H^3(M_{24}, U(1))$ determining the properties of the twisted twining genera $\phi_{g,h}$ represents strong evidence for the idea that a holomorphic VOA $\mathcal{H}$ underlies Mathieu Moonshine. Indeed, the original motivation for suspecting the relevance of the cohomology group $H^3(M_{24}, U(1))$ was based on the formal analogy with holomorphic orbifolds [22, 34]. In a sense we are therefore in a similar situation now as for Monstrous Moonshine before the discovery of the Monster VOA $V^\natural$ [2]. However, the story appears to be more subtle since the natural expectation that $\mathcal{H}$ would arise as a superconformal σ-model on K3 does not hold [13]. On the other hand, this is perhaps not too surprising, given the fact that the modular J-function corresponds to the *full* partition function of a CFT, while the elliptic genus of K3 only receives contribution from a *subset* of the physical states of the $\mathcal{N} = (4, 4)$ superconformal theory on K3. Hence it is tempting to speculate that $\mathcal{H}$ should correspond to something like the algebra of BPS-states of string theory on K3 (in the sense of Harvey and Moore [59, 60]). In particular, one might therefore hope that $\mathcal{H}$ can be constructed using some kind of topological sigma model, e.g. the 'half-twisted' topological σ-model on K3 [61]. Mathematically, this sigma model (or rather its large radius limit) can be viewed as a bundle of VOAs known as the chiral de Rham complex, Ω_X^{ch} [62], and in [63] it was shown that the elliptic genus of K3 can be obtained by taking the graded trace over the corresponding cohomology $H^*(\Omega_X^{ch})$. It is therefore tempting to speculate

[¶]We thank Terry Gannon for suggestions along these lines, see also [10].

that one may be able to construct an action of M_{24} on the cohomology of Ω_X^{ch}, at least for some choice of K3 surface X.

We hope to return to these and related issues in future work.

Acknowledgements

We are indebted to Terry Gannon for the suggestion that the 3rd cohomology of M_{24} should control the modular properties of the twisted twining genera, as well as for many useful explanations and directions to the relevant literature. We are furthermore grateful to Mathieu Dutour and Graham Ellis for helpful correspondence on using the computer package `HAP` for calculating the cohomology of finite groups. We also thank Nils Carqueville, Miranda Cheng, Jeff Harvey, Stefan Hohenegger, Axel Kleinschmidt, Sameer Murthy, Daniel Roggenkamp, Ashoke Sen and Don Zagier for useful discussions. D.P. would like to thank the Albert Einstein Institute in Golm for hospitality while part of this work was carried out. The work of M.R.G. is partially supported by the Swiss National Science Foundation, and the visit of Henrik Ronellenfitsch at ETH was supported by the Pauli Center.

Appendix A. Details on the unobstructed twisted twining genera

In this appendix, following the example of section 3.4, we will determine the twisted twining genera $\phi_{g,h}$ for the groups 13, 27, 28, 31 and 32 in Table 1. The genus $\phi_{2\mathrm{B},8\mathrm{A}}$ for group 1 was obtained in section 3.4, while for all the other groups $\phi_{g,h}$ vanishes due to some obstruction.

A.1. **The character $\phi_{2\mathrm{B},4\mathrm{A}}$ (group 13) $\phi_{4\mathrm{B},4\mathrm{A}}$ (groups 27 and 28).** In this section we will consider the twisted twining genera $\phi_{g,h}$ for groups 13, 27 and 28; as we will see, it turns out that these three characters have exactly the same modular properties. For group 13 g is in class 2B and h in class 4A, and we can choose

$$\begin{aligned} g &= (1,10)(2,14)(3,8)(4,16)(5,15)(6,12)(7,21)(9,18)(11,13)(17,20)(19,23)(22,24)\ , \\ h &= (1,2)(3,18)(4,7,15,17)(5,20,16,21)(6,19,24,11)(8,9)(10,14)(12,23,22,13)\ . \end{aligned} \tag{105}$$

For the groups 27 and 28, g is in class 4B and h in class 4A and we can choose

$$g = (3,8)(4,21,15,20)(5,17,16,7)(6,19,24,11)(9,18)(12,23,22,13)\ , \tag{106}$$

for both groups (from now on, we drop the trivial cycles). For group 27 we can take

$$h = (1,2,10,14)(3,9,8,18)(4,5,15,16)(6,13)(7,21,17,20)(11,22)(12,19)(23,24)\ , \tag{107}$$

and for group 28

$$h = (1,10)(2,14)(3,9)(4,11,5,13)(6,17,12,21)(7,22,20,24)(8,18)(15,19,16,23) \ . \quad (108)$$

In all the three cases we have the relations

$$(k^{-1}gk, k^{-1}ghk) = (g,h) \ , \qquad (r^{-1}gh^{-2}r, r^{-1}hr) = (g,h) \ , \quad (109)$$

where, for group 13

$$k = (1,9,2,3)(6,13,22,19)(7,21)(8,10,18,14)(11,24,23,12)(17,20) \ , \quad (110)$$

$$r = (3,8)(4,11,5,13)(6,20,12,7)(9,18)(15,19,16,23)(17,24,21,22) \ . \quad (111)$$

for group 27

$$k = (3,9)(4,12,5,6)(7,11,20,13)(8,18)(15,22,16,24)(17,19,21,23) \ , \quad (112)$$

$$r = (1,3)(2,9)(4,20)(5,7)(8,10)(14,18)(15,21)(16,17) \ . \quad (113)$$

and for group 28

$$k = (1,2,10,14)(3,9,8,18)(5,16)(6,11,24,19)(7,17)(12,23,22,13) \ , \quad (114)$$

$$r = (1,2)(3,8)(6,12)(7,20)(9,18)(10,14)(17,21)(22,24) \ . \quad (115)$$

Using the explicit knowledge of the 3-cocycle α we can derive the modular properties of the twisted twining genus $\phi_{g,h}$, including the multiplier system. It turns out that in all these three cases $\phi_{g,h}$ is invariant under $\Gamma_0(2)$ with the same multiplier

$$\phi_{g,h}(\tau+1) = c_g(g,h)\frac{c_g(gh,k)}{c_g(k,h)}\phi_{g,h}(\tau) = i\phi_{g,h}(\tau) \ , \quad (116)$$

$$\begin{aligned}\phi_{g,h}(\frac{\tau}{2\tau+1}) &= \frac{1}{c_h(h,h^{-1}g)c_h(h,h^{-2}g)}\frac{c_h(h^{-2}g,r)}{c_h(r,g)}\phi_{g,h}(\tau) \\ &= -i\phi_{g,h}(\tau) \ . \end{aligned} \quad (117)$$

This implies that $\phi_{g,h}$ can be expressed as

$$\phi_{g,h}(\tau,z) = \frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2}F(\tau) \ , \quad (118)$$

for some $\Gamma_0(2)$-invariant function $F(\tau)$. The group $\Gamma_0(2)$ has two cusps at ∞ and 0. To proceed we study the asymptotic behavior at infinity

$$\phi_{g,h}(\tau,z) = A_\infty\, q^{1/4}(2-y-y^{-1}) + \cdots \quad (119)$$

$$\frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,0)^2} = \frac{1}{4}\,(2-y-y^{-1}) + \cdots \ , \quad (120)$$

which implies that $F(\tau) \sim 4A_\infty\, q^{1/4} + \cdots$ has a zero at $\tau \to \infty$. The behavior at $\tau = 0$ is

$$\phi_{g,h}(\gamma_0(\tau,z)) \sim \phi_{h,g^{-1}}(\tau,z) = A_0\, q^{1/8}\,(2 - y - y^{-1}) + \cdots \tag{121}$$

$$\frac{\vartheta_1(\tau,z)^2}{\vartheta_4(\tau,0)^2} = q^{1/4}\,(2 - y - y^{-1}) + \cdots\ , \tag{122}$$

implying that $F(\gamma_0 \cdot \tau) \sim A_0\, q^{-1/8} + \cdots$ has a pole at 0. A function with the correct modular properties and behaviour at the cusps is

$$F(\tau) = 4A_\infty\, \frac{\eta(2\tau)^6}{\eta(\tau)^6}\ , \tag{123}$$

so that

$$\phi_{g,h}(\tau,z) = 4A_\infty \frac{\eta(2\tau)^6}{\eta(\tau)^6}\frac{\vartheta_1(\tau,z)^2}{\vartheta_2(\tau,z)^2} = A_\infty \frac{\eta(2\tau)^2}{\eta(\tau)^4}\vartheta_1(\tau,z)^2\ . \tag{124}$$

The constants A_∞ for the three cases are determined (up to the phase) by ensuring a decomposition into projective characters of $C_{M_{24}}(g)$. In particular, $A_\infty = 4$ for group 13 and $A_\infty = 2\sqrt{2}$ for groups 27 and 28.

A.2. **The cases $\phi_{3\mathrm{A},3\mathrm{B}}$ (group 31) and $\phi_{3\mathrm{A},3\mathrm{A}}$ (group 32).** In this section we show that the unobstructed twisted twining genera $\phi_{3\mathrm{A},3\mathrm{A}}$ and $\phi_{3\mathrm{A},3\mathrm{B}}$ must vanish by modular arguments.

The group $\langle g,h\rangle = \langle 3\mathrm{A}, 3\mathrm{B}\rangle$ (type 31) is generated by

$$g = (2,23,11)(3,20,10)(5,14,9)(6,16,13)(7,19,22)(12,24,18) \tag{125}$$

in class 3A and

$$h = (1,15,17)(2,19,16)(3,18,5)(4,21,8)(6,11,7)(9,10,24)(12,14,20)(13,23,22) \tag{126}$$

in class 3B. The following conjugation relation holds

$$(k^{-1}gk, k^{-1}ghk) = (g,h)\ , \tag{127}$$

where

$$k = (2,7,16)(3,20,10)(4,21,8)(5,9,14)(6,11,22)(13,23,19)\ . \tag{128}$$

The modular properties of the associated twisted twining genus are

$$\phi_{g,h}(\tau+1) = c_g(g,h)\frac{c_g(gh,k)}{c_g(k,h)}\phi_{g,h}(\tau) = e^{\frac{4\pi i}{3}}\phi_{g,h}(\tau)\ , \tag{129}$$

and

$$\phi_{g,h}(\frac{\tau}{-3\tau+1}) = \prod_{i=0}^{2} c_h(h,h^i g^{-1})\phi_{g,h}(\tau) = e^{\frac{4\pi i}{3}}\phi_{g,h}(\tau)\ , \tag{130}$$

where we have used the explicit form for the multiplier phases as determined in terms of the 3-cocycle $\alpha \in H^3(M_{24}, U(1))$. We conclude that, up to the multiplier, $\phi_{g,h}$ is invariant under $\Gamma_0(3)$ and hence can be written as

$$\phi_{g,h}(\tau, z) = \frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^2} F_{g,h}(\tau) \tag{131}$$

for some $\Gamma_0(3)$-invariant function $F_{g,h}(\tau)$. The asymptotics for $\tau \to \infty$ is

$$\phi_{g,h}(\tau, z) \sim A_\infty \, q^{\frac{2}{3}} + \cdots \ , \tag{132}$$

for some (possibly vanishing) constant A_∞, while the asymptotic behavior at 0 is given by

$$\phi_{h,g}(\tau, z) \sim A_0 \, q^{\frac{2}{9}} + \cdots \ . \tag{133}$$

Since both at ∞ and 0 we have

$$\frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^2} \sim q^{1/6}(2 - y - y^{-1}) + \cdots \ , \tag{134}$$

we deduce that $F_{g,h}$ has a zero at both 0 and ∞ and no poles; hence it must vanish. We therefore conclude

$$\phi_{3\mathrm{A},3\mathrm{B}}(\tau, z) = 0 \ , \tag{135}$$

as claimed.

The $(3\mathrm{A}, 3\mathrm{A})$-case (group 32) is analogous. Here, the group is generated by

$$g = (2, 23, 11)(3, 20, 10)(5, 14, 9)(6, 16, 13)(7, 19, 22)(12, 24, 18) \tag{136}$$

and

$$h = (2, 22, 13)(4, 21, 8)(5, 14, 9)(6, 23, 7)(11, 19, 16)(12, 18, 24) \ , \tag{137}$$

and we have the relations

$$(k^{-1}gk, k^{-1}ghk) = (g, h) \ , \qquad (r^{-1}hr, r^{-1}g^{-1}r) = (g, h) \tag{138}$$

where

$$k = (1, 17, 15)(3, 9, 24)(5, 18, 20)(6, 16, 13)(7, 22, 19)(10, 14, 12) \ , \tag{139}$$

$$r = (3, 4, 10, 21)(5, 18)(6, 16, 19, 7)(8, 20)(9, 24, 14, 12)(11, 22, 23, 13) \ . \tag{140}$$

The modular properties of $\phi_{g,h}$ are

$$\phi_{g,h}(\tau + 1) = c_g(g, h) \, \frac{c_g(gh, k)}{c_g(k, h)} \, \phi_{g,h}(\tau) = e^{\frac{4\pi i}{3}} \, \phi_{g,h}(\tau) \ , \tag{141}$$

$$\phi_{g,h}(-1/\tau) = \frac{1}{c_h(g, g^{-1})} \frac{c_h(g^{-1}, r)}{c_h(r, h)} \phi_{g,h}(\tau) = \phi_{g,h}(\tau) \ . \tag{142}$$

Thus $\phi_{g,h}$ is invariant (up to a multiplier) under $SL(2,\mathbb{Z})$, and $\phi_{g,h} \sim A\, q^{2/3}$ at ∞, so that

$$F_{3\mathrm{A},3\mathrm{A}}(\tau) \equiv \phi_{g,h}(\tau,z)\frac{\eta(\tau)^2}{\vartheta_1(\tau,z)^2} \sim A\, q^{1/2} + \cdots \ . \tag{143}$$

Thus, $F_{3\mathrm{A},3\mathrm{A}}$ has a zero and no poles and therefore must vanish, thus leading to

$$\phi_{3\mathrm{A},3\mathrm{A}}(\tau,z) = 0 \ . \tag{144}$$

Appendix B. Some group cohomology

In this section, we will review the basic definitions and properties of group cohomology, see for example [40] for an introduction to the subject.

In general, group cohomology is defined in terms of a finite group G and a G-module A, i.e. an abelian group with an action of G satisfying certain compatibility properties. For our purposes, it is sufficient to consider $A = U(1)$ with the trivial action of G. An n-cochain ψ (with $n \geq 0$) for G with values in $U(1)$ is simply a function from $G \times \ldots \times G$ (n times) to $U(1)$; a 0-cochain is just an element of $U(1)$. On the space $C^n(G, U(1))$ of n-cochains one can define a *coboundary* operator $\partial_n : C^n(G, U(1)) \to C^{n+1}(G, U(1))$ by

$$\begin{aligned}\partial_n \psi\,(g_1,\ldots,g_{n+1}) &= \psi(g_2,\ldots,g_{n+1}) \\ &\quad \times \prod_{i=1}^{n} \psi(g_1,\ldots,g_{i-1},g_i g_{i+1},g_{i+2},\ldots,g_{n+1})^{(-1)^i}\, \psi(g_1,\ldots,g_n)^{(-1)^{n+1}} \ .\end{aligned} \tag{145}$$

The cochains in the kernel of ∂_n are called n-cocycles (or co-closed n-cochains), while the cochains in the image of ∂_{n-1} are called n-coboundaries. The coboundary operator satisfies $\partial_{n+1} \circ \partial_n = 1$ (the trivial element of $U(1)$)), so that we can define the cohomology, as usual, as the quotient of the space of cocycles modulo coboundaries

$$H^n(G, U(1)) = \frac{\ker \partial_n}{\mathrm{Im}\partial_{n-1}} \ , \tag{146}$$

where $n > 0$, and we use the convention that $H^0(G, U(1)) = U(1)$, and that every 0-cochain is coclosed, ie. $\partial_0 = 1$.

In particular, for a 1-cochain γ, the condition $\partial_1 \gamma = 1$ is simply $\gamma(g)\gamma(h) = \gamma(gh)$ for all $g, h \in G$, so that $H^1(G, U(1))$ is just the group of homomorphism from G to $U(1)$. A 2-cochain $\beta : G \times G \to U(1)$ is co-closed (and hence defines a cocycle) provided it satisfies

$$\beta(g_1, g_2 g_3)\, \beta(g_2, g_3) = \beta(g_1 g_2, g_3)\, \beta(g_1, g_2) \tag{147}$$

for $g_1, g_2, g_3 \in G$. The second cohomology $H^2(G, U(1))$ then consists of the co-closed 2-cochains, modulo the ambiguity

$$\beta(g_1, g_2) \to \beta(g_1, g_2) \frac{\gamma(g_1)\gamma(g_2)}{\gamma(g_1 g_2)} \ , \tag{148}$$

where γ is an arbitrary 1-cochain, i.e. an arbitrary function $\gamma : G \to U(1)$. The second cohomology of a group G classifies the inequivalent projective representations of G, see appendix C. A 3-cochain α,

$$\alpha \ : \ G \times G \times G \ \to \ U(1) \tag{149}$$

is closed provided it satisfies

$$\alpha(g_1, g_2, g_3)\, \alpha(g_1, g_2 g_3, g_4)\, \alpha(g_2, g_3, g_4) = \alpha(g_1 g_2, g_3, g_4)\, \alpha(g_1, g_2, g_3 g_4) \ . \tag{150}$$

In the cohomology group $H^3(G, U(1))$ the 3-cocycles are then identified modulo

$$\alpha(g_1, g_2, g_3) \to \alpha(g_1, g_2, g_3)\, \frac{\beta(g_1, g_2 g_3)\beta(g_2, g_3)}{\beta(g_1 g_2, g_3)\beta(g_1, g_2)} \ . \tag{151}$$

Note that the multiplying factor is trivial if β is closed, i.e. if it satisfies the 2-cocycle condition (147). We shall always work with normalised cocycles, i.e. we shall assume that

$$\alpha(e, g_1, g_2) = \alpha(g_1, e, g_2) = \alpha(g_1, g_2, e) = 1 \tag{152}$$

for all $g_1, g_2 \in G$, where e denotes the identity element of G.

Given a 3-cocycle α, we can define, for any $h \in G$, a 2-cochain $c_h : G \times G \to U(1)$ via

$$c_h(g_1, g_2) = \frac{\alpha(h, g_1, g_2)\, \alpha(g_1, g_2, (g_1 g_2)^{-1} h (g_1 g_2))}{\alpha(g_1, g_1^{-1} h g_1, g_2)} \ . \tag{153}$$

It is shown in [22] that c_h defines a 2-cocycle of the centraliser $C_G(h) \subset G$ (i.e. the subgroup of all elements g_1, g_2 which commute with h). When $g_1, g_2 \in C_G(h)$, we have the simplified expression

$$c_h(g_1, g_2) = \frac{\alpha(h, g_1, g_2)\, \alpha(g_1, g_2, h)}{\alpha(g_1, h, g_2)} \ , \qquad g_1, g_2 \in C_G(h) \ . \tag{154}$$

Under the 'gauge transformation' (151), c_h transforms as

$$c_h(g_1, g_2) \to c_h(g_1, g_2) \frac{\gamma_h(g_1)\gamma_h(g_2)}{\gamma_h(g_1 g_2)} \ , \qquad g_1, g_2 \in C_G(h) \ , \tag{155}$$

where we defined the 1-cochain γ_h by

$$\gamma_h(g) \equiv \frac{\beta(g, h)}{\beta(h, g)} \ . \tag{156}$$

This is indeed of the form (148), and hence, for all $h \in G$, c_h defines a map

$$c_h \ : \ H^3(G, U(1)) \ \to H^2(C_G(h), U(1)) \ . \tag{157}$$

Appendix C. Projective representations of finite groups

This section follows [34, 64]. A projective representation is a map $\rho : G \to \mathrm{End}(V)$ from a group G to the endomorphism group of some complex vector space V satisfying

$$\rho(g)\,\rho(h) = \beta(g,h)\,\rho(gh)\ , \quad \beta(g,h) \in U(1)\ . \tag{158}$$

It follows from the associativity of the group multiplication that the phase $\beta(g,h)$ satisfies the 2-cocycle condition

$$\beta(g,h)\,\beta(gh,k) = \beta(h,k)\,\beta(g,hk)\ . \tag{159}$$

There is the freedom of rescaling the maps $\rho(g)$ as

$$\rho(g) \mapsto \gamma(g)\rho(g)\ , \quad \gamma(g) \in U(1)\ . \tag{160}$$

For the 2-cocylce β this leads to the rescaling by a coboundary

$$\beta(g,h) \to \beta(g,h)\,\gamma(gh)\,\gamma(g)^{-1}\,\gamma(h)^{-1}\ . \tag{161}$$

In particular, one can always choose a rescaling such that $\rho(e)$ is the identity matrix in $\mathrm{End}(V)$; with this choice, the cocycle β is normalised. Two projective representations whose 2-cocycles differ only by a coboundary are called projectively equivalent, since their representation matrices are related by a rescaling. Therefore, the inequivalent projective representations of G are captured by the second group cohomology $H^2(G, U(1))$.

As for genuine linear representations, one may define the character of a projective representation as the trace

$$\chi(g) = \mathrm{Tr}_V(\rho(g))\ . \tag{162}$$

Due to the modified composition law, projective characters are in general not class functions, but differ by a phase on different representatives of one conjugacy class. However, it is always possible to choose a cocycle representative such that the characters are indeed class functions.

Assuming that $gh = hg$, we have, using the projective composition law,

$$\chi(g) = \chi(h^{-1}gh) = \frac{\beta(gh, h^{-1})}{\beta(h^{-1}, gh)}\,\chi(g)\ . \tag{163}$$

Using the cocycle condition, it is easy to see that this condition on the characters is equivalent to the obstuction of the first kind (14). Also, the obstruction does not depend on the specific cocycle representative $\beta(g,h)$ or the class representative g. Classes whose projective characters vanish are called β-irregular. It may be shown that to each cocycle $\beta(g,h)$, there

exist exactly as many linearly inequivalent irreducible projective representations as there are β-regular conjugacy classes.

Two irreducible characters belonging to representations with the same cocycle $\beta(g,h)$ satisfy an orthogonality relation of the form

$$\sum_{[g]} |K_g|^{-1}\, \chi(g)^*\, \tilde{\chi}(g) = \begin{cases} 1 & \chi = \tilde{\chi} \\ 0 & \text{otherwise,} \end{cases} \tag{164}$$

where the sum is taken over a set of (β-regular) class representatives of G, and $|K_g|$ is the size of the conjugacy class of g.

There is a natural connection between projective representations and central extensions of a group G. We say that $\tilde{G}$ is a central extention of G if there is an abelian subgroup $A \subset Z(\tilde{G})$ of the center $Z(\tilde{G})$ of $\tilde{G}$ such that the short sequence

$$0 \to A \to \tilde{G} \to G \to 0 \tag{165}$$

is exact, or equivalently that $G = \tilde{G}/A$. The simplest central extension is of course $\tilde{G} = A \times G$, but this is in general not the only possibility. It can be shown that the non-isomorphic central extensions of G by A are also classified by $H^2(G,A)$. Each set of projectively inequivalent projective representations of G corresponds to a set of genuine representations of some extension $\tilde{G}$.

In some sense there exists a maximal central extension S for each G, the Schur cover or 'Darstellungsgruppe'. It is characterised by the property that any projective representation of G corresponds to a genuine representation of S. The Schur cover is, in general, not unique, unless G is a perfect group, i.e. it coincides with the commutator subgroup. Otherwise, the Schur covers are related by isoclinism.

C.1. **Central extension and orbifolds.** Let $\mathcal{C}$ be a K3 sigma-model and suppose that its automorphism group contains an element g such that the orbifold of $\mathcal{C}$ by g, $\hat{\mathcal{C}} = \mathcal{C}/\langle g\rangle$ is consistent. In this subsection, we will show that all projective representations of $H \equiv C_{M_{24}}(g)$ on the g^r-twisted sectors are equivalent to genuine representations of a central extension $\tilde{H}$ of the form described in section 4.1, eq. (82).

By (52) and (59), the level-matching condition is satisfied if and only if the restriction of the class $[\alpha] \in H^3(M_{24}, U(1))$ to $H^3(\langle g\rangle, U(1))$ is trivial. Each g^r-twisted sector carries a, in general projective, representation ρ_{g^r} of $H \equiv C_{M_{24}}(g)$, with underlying 2-cocycle c_{g^r} given

by (39). The various cocycles c_{g^r} are related to one another by the identity

$$c_{g^r}(h,k) = \frac{f_{g,r}(h) f_{g,r}(k)}{f_{g,r}(hk)}\, c_g(h,k)^r \ , \qquad h,k \in C(g) \ , \tag{166}$$

where

$$f_{g,r}(h) = \prod_{i=1}^{r-1} c_h(g,g^i) \ . \tag{167}$$

Here we have used the definition (39) of c_g and applied repeatedly the condition that the 3-cocycle α is closed. Thanks to the level-matching condition, we can choose the representative cocycle α to be identically 1 when restricted to $\langle g \rangle$, so that $f_{g,r}(g^i) = 1$ for all i. Furthermore, by changing α by a coboundary $\partial\beta$ as in (151), the functions $f_{g,r}(h)$ transform as

$$f_{g,r}(h) \to f_{g,r}(h)\, \frac{\beta(g^r,h)\beta(h,g)^r}{\beta(h,g^r)\beta(g,h)^r} \ . \tag{168}$$

Thus, we can choose β in such a way that $f_{g,r}(h) = 1$ for all $h \in C(g)$, so that

$$c_{g^r}(h,k) = c_g(h,k)^r \ , \qquad h,k \in C(g) \ . \tag{169}$$

In particular, since $c_e(h,k) = 1$, $c_g(h,k)$ is always an N-th root of unity, where $N = o(g)$.

Appendix D. Character Tables for $C_{M_{24}}(g)$ and decompositions

This section describes the projective character tables of the centraliser $H \equiv C_{M_{24}}(g)$ for each conjugacy class of M_{24}, corresponding to the projective equivalence class determined by the 2-cocycle c_g. The procedure to compute these tables is as follows.

Let Schur(H) denote a Schur cover of H, i.e. a central extension of H fitting into the exact sequence

$$1 \to M(H) \to \mathrm{Schur}(H) \to H \to 1 \ , \tag{170}$$

where $M(H) = H_2(H,\mathbb{Z}) = (H^2(H,U(1)))^*$ is the Schur multiplier. The Schur cover Schur(H) and its character table can be easily computed (we used `GAP` [38] for this). Any irreducible projective representation ρ of H is equivalent to a genuine irreducible representation $\tilde{\rho}$ of Schur(H) and vice versa. Thus, one can associate a class of $H^2(H,U(1))$ to each irreducible representation of Schur(H). The projective character table we are interested in is obtained from the one of Schur(H) by keeping only the representations associated to the class $[c_g]$, and by choosing a lift in Schur(H) of each conjugacy class of H (different choices of the lifts lead to projectively equivalent representations).

It only remains to determine which irreducible representations of Schur(H) correspond to the class $[c_g]$. Consider an irreducible projective representation ρ of H associated with the 2-cocycle c_g. For any commuting pair $h, k \in H$, $hk = kh$, we have

$$\rho(h)\rho(k)\rho(h)^{-1}\rho(k)^{-1} = \frac{c_g(h,k)}{c_g(k,h)} \ . \tag{171}$$

(Note that $\frac{c_g(h,k)}{c_g(k,h)}$ only depends on the class $[c_g]$ and not on the representative cocycle c_g.) Thus, any representation σ of Schur(H) in the class $[c_g]$ must satisfy

$$\sigma(\tilde{h}\tilde{k}\tilde{h}^{-1}\tilde{k}^{-1}) = \frac{c_g(h,k)}{c_g(k,h)} \ , \tag{172}$$

where $\tilde{h}, \tilde{k}$ are some lifts of h, k in Schur(H). It is easy to see that the commutator $\tilde{h}\tilde{k}\tilde{h}^{-1}\tilde{k}^{-1}$ is an element of $M(H)$, and that it does not depend of the choice of the lifts. In fact, the commutators of this form generate $M(H)$, and if a representation σ satisfies (172) for a set of generators of $M(H) = (H^2(H, U(1)))^*$, then it must be associated with the class $[c_g]$. This condition can be easily tested for each irreducible representation of Schur(H), given our explicit knowledge of the cocycle c_g.

Since the actual tables take up a lot of space, we have commented them out in the LaTeX-source file that is stored on the arXiv repository.

Appendix E. Decompositions of twisted sectors

In this section we give the decomposition of the first 20 representation spaces $\mathcal{H}_g^n$ for all non-isomorphic twisted sectors. In each table, the integer in the i'th row and the column labelled r is the multiplicity $h_{i,r}$ of the irreducible representation χ_i (numbered as in the corresponding character table) in the representation $\rho_{g,r}$ of eq. (6).

Again, the tables are rather lengthy, and we have therefore commented them out in the LaTeX-source file that is available from the arXiv repository.

References

[1] J.H. Conway and S.P. Norton, "Monstrous Moonshine," Bull. London Math. Soc. **11**, 308 (1979).

[2] I. Frenkel, J. Lepowsky and A. Meurman, "Vertex operator algebras and the Monster," Pure and Applied Mathematics **134**, Academic Press, Inc., Boston, MA, (1988).

[3] R.E. Borcherds, "Monstrous Moonshine and monstrous Lie superalgebras," Invent. Math. **109**, 405 (1992).

[4] T.J. Gannon, "Moonshine beyond the Monster: The bridge connecting algebra, modular forms and physics," Cambridge University Presse (2006).

[5] T. Eguchi, H. Ooguri and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," Exper. Math. **20**, 91 (2011) [arXiv:1004.0956 [hep-th]].

[6] M.C.N. Cheng, "K3 Surfaces, N=4 dyons, and the Mathieu group M_{24}," Commun. Number Theory Phys. **4**, 623 (2010) [arXiv:1005.5415 [hep-th]].

[7] M.R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu twining characters for K3," JHEP **1009**, 058 (2010) [arXiv:1006.0221 [hep-th]].

[8] M.R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," JHEP **1010**, 062 (2010) [arXiv:1008.3778 [hep-th]].

[9] T. Eguchi and K. Hikami, "Note on twisted elliptic genus of K3 surface," Phys. Lett. B **694**, 446 (2011) [arXiv:1008.4924 [hep-th]].

[10] T. Gannon, "Much ado about Mathieu," arXiv:1211.5531 [math.RT].

[11] T. Creutzig, G. Höhn and T. Miezaki, "The McKay-Thompson series of Mathieu Moonshine modulo two," arXiv:1211.3703v1 [math.NT].

[12] M.C.N. Cheng and J.F.R. Duncan, "On Rademacher sums, the largest Mathieu group, and the holographic modularity of Moonshine," arXiv:1110.3859 [math.RT].

[13] M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Symmetries of K3 sigma models," Comm. Number Theory and Phys. **6** (1) (2012) [arXiv:1106.4315 [hep-th]].

[14] S. Norton, "Generalized moonshine," Proc. Sympos. Pure Math. **47**, 209 Amer. Math. Soc., Providence, RI (1987).

[15] G. Höhn, "Generalized moonshine for the baby Monster," Habilitation thesis, Mathematisches Institut, Universität Freiburg (2003), http://www.math.ksu.edu/∼gerald/papers/baby8.ps.

[16] M.P. Tuite, "Generalized moonshine and abelian orbifold constructions," arXiv:hep-th/9412036.

[17] S. Carnahan, "Generalized Moonshine I: Genus zero functions," arXiv:0812.3440 [math.RT].

[18] S. Carnahan, "Generalized Moonshine II: Borcherds products," arXiv:0908.4223 [math.RT].

[19] S. Carnahan, "Generalized Moonshine III: Equivariant intertwining operators," to appear.

[20] S. Carnahan, "Generalized Moonshine IV: Monstrous Lie algebras," arXiv:1208.6254 [math.RT].

[21] L.J. Dixon, P.H. Ginsparg and J.A. Harvey, "Beauty and the beast: superconformal symmetry in a Monster module," Commun. Math. Phys. **119**, 221 (1988).

[22] R. Dijkgraaf and E. Witten, "Topological gauge theories and group cohomology," Commun. Math. Phys. **129**, 393 (1990).

[23] T. Gannon, talk at 'Mathieu Moonshine', ETH Zürich, July 2011.

[24] G. Mason, "Orbifold conformal field theory and the cohomology of the Monster," unpublished.

[25] G. Mason, "On a system of elliptic modular forms attached to the largest Mathieu group," Nagoya Math. J. **118**, 177 (1990).

[26] D. Persson and R. Volpato, "Generalised Mathieu moonshine: Borcherds lifts and eta products," in progress.

[27] M. Eichler and D. Zagier, "The theory of Jacobi forms," Birkhäuser (1985).

[28] T. Eguchi and A. Taormina, "Character formulas for the N=4 superconformal algebra," Phys. Lett. B **200**, 315 (1988).

[29] M. Dutour Sikirić and G. Ellis, "Wythoff polytopes and low-dimensional homology of Mathieu groups," J. Algebra **322**, 4143 (2009).

[30] C. Vafa, "Modular invariance and discrete torsion on orbifolds," Nucl. Phys. B **273**, 592 (1986).

[31] S. Willerton, "The twisted Drinfeld double of a finite group via gerbes and finite groupoids," Algebr. Geom. Topol. **8**, 1419 (2008) [arXiv:math/0503266].

[32] P. Bantay, "Orbifolds and Hopf algebras," Phys. Lett. B **245**, 477 (1990).

[33] P. Roche, V. Pasquier and R. Dijkgraaf, "Quasi-Hopf algebras, group cohomology and orbifold models," Nucl. Phys. Proc. Suppl. **18B**, 60 (1990).

[34] A. Coste, T. Gannon and P. Ruelle, "Finite group modular data," Nucl. Phys. B **581**, 679 (2000) [arXiv:hep-th/0001158].

[35] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, "Defect lines, dualities, and generalised orbifolds," in *XVIth International Congress on Mathematical Physics*, 608–613, World Sci. Publ., Hackensack, NJ (2010) [arXiv:0909.5013 [math-ph]].

[36] K.S. Narain, M.H. Sarmadi and C. Vafa, "Asymmetric orbifolds," Nucl. Phys. B **288**, 551 (1987).

[37] M.R. Gaberdiel and R. Volpato, "Mathieu moonshine and orbifold K3s," arXiv:1206.5143 [hep-th].

[38] The GAP Group, "GAP – Groups, Algorithms, and Programming, version 4.5.6," (2012) http://www.gap-system.org.

[39] G. Ellis, "HAP (Homological Algebra Programming) – a GAP package, version 1.10.8," (2012) http://www.gap-system.org/Packages/hap.html.

[40] K.S. Brown, "Cohomology of groups," Graduate Texts in Mathematics **87**, Springer, New York (1994).

[41] M.D.F. de Wild Propitius, "Topological interactions in broken gauge theories," hep-th/9511195.

[42] See the ancillary files attached with the preprint version of this paper at http://arxiv.org.

[43] S. Mukai, "Finite groups of automorphisms of $K3$ surfaces and the Mathieu group," Invent. Math. **94**, 183 (1988).

[44] A. Sen, "Discrete information from CHL black holes," JHEP **1011**, 138 (2010) [arXiv:1002.3857 [hep-th]].

[45] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, "Counting dyons in N=4 string theory," Nucl. Phys. B **484**, 543 (1997) [arXiv:hep-th/9607026].

[46] D. Shih, A. Strominger and X. Yin, "Recounting dyons in N=4 string theory," JHEP **0610**, 087 (2006) [arXiv:hep-th/0505094].

[47] A. Dabholkar, S. Murthy and D. Zagier, "Quantum black holes, wall crossing, and mock modular forms," arXiv:1208.4074 [hep-th].

[48] S. Govindarajan, "BKM Lie superalgebras from counting twisted CHL dyons," JHEP **1105**, 089 (2011) [arXiv:1006.3472 [hep-th]].

[49] S. Govindarajan, "Brewing moonshine for Mathieu," arXiv:1012.5732 [math.NT].

[50] S. Govindarajan, "Unravelling Mathieu moonshine," Nucl. Phys. B **864**, 823 (2012) [arXiv:1106.5715 [hep-th]].

[51] J.F. Duncan and I.B. Frenkel, "Rademacher sums, moonshine and gravity," Commun. Num. Theor. Phys. **5**, 849 (2011) [arXiv:0907.4529 [math.RT]].

[52] V.A. Gritsenko and V.V. Nikulin, "Siegel automorphic form corrections of some Lorentzian Kac–Moody Lie algebras," [arXiv:alg-geom/9504006].

[53] V.A. Gritsenko and F. Clery, "The Siegel modular forms of genus 2 with the simplest divisor," [arXiv:0812.3962 [math.NT]].

[54] M.C.N. Cheng and E.P. Verlinde, "Wall crossing, discrete attractor flow, and Borcherds algebra," SIGMA **4**, 068 (2008) [arXiv:0806.2337 [hep-th]].

[55] M.C.N. Cheng and A. Dabholkar, "Borcherds-Kac-Moody symmetry of N=4 dyons," Commun. Num. Theor. Phys. **3**, 59 (2009) [arXiv:0809.4258 [hep-th]].

[56] S. Govindarajan and K. Gopala Krishna, "Generalized Kac-Moody algebras from CHL dyons," JHEP **0904**, 032 (2009) [arXiv:0807.4451 [hep-th]].

[57] S. Govindarajan and K. Gopala Krishna, "BKM Lie superalgebras from dyon spectra in Z_N CHL orbifolds for composite N," JHEP **1005**, 014 (2010) [arXiv:0907.1410 [hep-th]].

[58] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, "Umbral moonshine," arXiv:1204.2779 [math.RT].

[59] J.A. Harvey and G.W. Moore, "Algebras, BPS states, and strings," Nucl. Phys. B **463**, 315 (1996) [arXiv:hep-th/9510182].

[60] J.A. Harvey and G.W. Moore, "On the algebras of BPS states," Commun. Math. Phys. **197**, 489 (1998) [arXiv:hep-th/9609017].

[61] A. Kapustin, "Chiral de Rham complex and the half-twisted sigma-model," arXiv:hep-th/0504074.

[62] F. Malikov, V. Schechtman and A. Vaintrob, "Chiral de Rham complex," Commun. Math. Phys. **204**, 439 (1999) [arXiv:math/9803041].

[63] L.A. Borisov and L. Libgober, "Elliptic genera of toric varieties and applications to mirror symmetry," Invent. Math. **140**, 453 (2000) [arXiv:math/9904126].

[64] G. Karpilovsky, "Group representations: volume 2," North-Holland Mathematical Studies **177**, Elsevier (1993).

Institut für Theoretische Physik, ETH Zürich, CH-8093 Zürich, Switzerland

E-mail address: gaberdiel@itp.phys.ethz.ch

Fundamental Physics, Chalmers University of Technology, SE-412 96, Gothenburg, Sweden

E-mail address: daniel.persson@chalmers.se

URL: http://www.danper.se

Max-Planck-Institut für Dynamik und Selbstorganisation (MPIDS), 37077 Göttingen, Germany

E-mail address: henrik.ronellenfitsch@ds.mpg.de

Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, 14476 Golm, Germany

E-mail address: roberto.volpato@aei.mpg.de

SU/ITP-14/17

Mock Modular Mathieu Moonshine Modules

Miranda C. N. Cheng[1], Xi Dong[2], John F. R. Duncan[3], Sarah Harrison[2], Shamit Kachru[2], and Timm Wrase[2]

[1]Institute of Physics and Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, the Netherlands*

[2]Stanford Institute for Theoretical Physics, Department of Physics, and Theory Group, SLAC, Stanford University, Stanford, CA 94305, USA

[3]Department of Mathematics, Applied Mathematics and Statistics, Case Western Reserve University, Cleveland, OH 44106, USA

arXiv:1406.5502v2 [hep-th] 10 Aug 2015

Abstract

We construct super vertex operator algebras which lead to modules for moonshine relations connecting the four smaller sporadic simple Mathieu groups with distinguished mock modular forms. Starting with an orbifold of a free fermion theory, any subgroup of Co_0 that fixes a 3-dimensional subspace of its unique non-trivial 24-dimensional representation commutes with a certain $\mathcal{N} = 4$ superconformal algebra. Similarly, any subgroup of Co_0 that fixes a 2-dimensional subspace of the 24-dimensional representation commutes with a certain $\mathcal{N} = 2$ superconformal algebra. Through the decomposition of the corresponding twined partition functions into characters of the $\mathcal{N} = 4$ (resp. $\mathcal{N} = 2$) superconformal algebra, we arrive at mock modular forms which coincide with the graded characters of an infinite-dimensional $\mathbb{Z}$-graded module for the corresponding group. The Mathieu groups are singled out amongst various other possibilities by the moonshine property: requiring the corresponding weak Jacobi forms to have certain asymptotic behaviour near cusps. Our constructions constitute the first examples of explicitly realized modules underlying moonshine phenomena relating mock modular forms to sporadic simple groups. Modules for other groups, including the sporadic groups of McLaughlin and Higman–Sims, are also discussed.

*On leave from CNRS, Paris.

Contents

1 Introduction 2

2 The Free Field Theory 5

3 The Superconformal Algebras 8

4 Global Symmetries 12

5 Twining the Module 15

6 The $\mathcal{N} = 4$ Decompositions 19

7 The $\mathcal{N} = 2$ Decompositions 24

8 Mathieu Moonshine 28

9 Discussion 32

A Jacobi Theta Functions 34

B Character Tables 35

C Coefficient Tables 45

D Decomposition Tables 59

References 91

1 Introduction

The investigation of moonshine connecting modular objects, sporadic groups, and 2d conformal field theories has been revitalized in recent years by the discovery of several new classes of examples. While monstrous moonshine [1–7] remains the best understood and prototypical case, a new class of umbral moonshines tying mock modular forms to automorphism groups of Niemeier lattices has recently been uncovered [8,9]. (Cf. also [10].) The best studied example, and the first to be discovered, involves the group M_{24} and was discovered through the study of the elliptic genus of K3 [11]. The twining functions have been constructed in [12–14] and were proved to be the graded characters of an infinite-dimensional M_{24}-module in [15]. Steps towards a better and deeper understanding of this mock modular moonshine can be found in [16–27], and particularly in [28], where the importance of K3 surface geometry for all cases of umbral

moonshine is elucidated. Possible connections to space-time physics in string theory have been discussed in [29–33]. Evidence for a deep connection between monstrous and umbral moonshine has appeared in [34,35].

In none of these cases, however, has a connection to an underlying conformal field theory[1] (whose Hilbert space furnishes the underlying module) been established. The goal of this paper is to provide first examples of mock modular moonshine for sporadic simple groups G, where the underlying G-module can be explicitly constructed in the state space of a simple and soluble conformal field theory.

Our starting point is the Conway module sketched in [2], studied in detail in [37], and revisited recently in [38]. The original construction was in terms of a supersymmetric theory of bosons on the E_8 root lattice, but this has the drawback of obscuring the true symmetries of the model. In [37], a different formulation of the same theory, as a $\mathbb{Z}_2$ orbifold of the theory of 24 free chiral fermions, was introduced. A priori, the theory has a Spin(24) symmetry. However, one can also view this theory as an $\mathcal{N}=1$ superconformal field theory. The choice of $\mathcal{N}=1$ structure breaks the Spin(24) symmetry to a subgroup. In [37] it was shown that the subgroup preserving the natural choice of $\mathcal{N}=1$ structure is precisely the Conway group Co_0, a double cover of the sporadic group Co_1. In [38] it was shown that this action can be used to attach a normalized principal modulus (i.e. normalized Hauptmodul) for a genus zero group to every element of Co_0.

In this paper, we show that generalizations of the basic strategy of [37,38] can be used to construct a wide variety of new examples of mock modular moonshine. Instead of choosing an $\mathcal{N}=1$ superconformal structure, we choose larger extended chiral algebras $\mathcal{A}$. The subgroup of Spin(24) that commutes with a given choice can be determined by simple geometric considerations; in the cases of interest to us, it will be a subgroup that preserves point-wise a 2-plane or a 3-plane in the 24 dimensional representation of Co_0, or equivalently, a subgroup that acts trivially on two or three of the free fermions in some basis. In the rest of the paper, we will use $\mathbf{24}$ to denote the unique non-trivial 24-dimensional representation of Co_0, and use the term n-plane to refer to a n-dimensional subspace in $\mathbf{24}$.

It is natural to ask about the role in moonshine, or geometry, of n-planes in $\mathbf{24}$ for other values of n. One of the inspirations for our analysis here is the recent result of Gaberdiel–Hohenegger–Volpato [17] which indicates the importance of 4-planes in $\mathbf{24}$ for non-linear K3 sigma models. The relationship between their results and the Co_0-module considered here is studied in [35], where connections to umbral moonshine for various higher n are also established. We refer the reader to §9, or the recent articles [39,40], for a discussion of the interesting case that $n=1$.

[1] See however the recent work [36] which constructs the super vertex operator algebra underlying the $X=E_8^3$ case of umbral moonshine.

In this work we will focus on the cases where $\mathcal{A}$ is an $\mathcal{N} = 4$ or $\mathcal{N} = 2$ superconformal algebra, though other possibilities exist. In the first case, we demonstrate that any subgroup of Co_0 that preserves a 3-plane in the 24-dimensional representation can stabilise an $\mathcal{N} = 4$ structure. The groups that arise are discussed in e.g. Chapter 10 of the book by Conway and Sloane [41]. They include in particular the Mathieu groups M_{22} and M_{11}. In the second case, where the group need only fix a 2-plane in order to preserve an $\mathcal{N} = 2$ superconformal algebra, there are again many possibilities (again, see [41]), including the larger Mathieu groups M_{23} and M_{12}. Note that the larger the superconformal algebra we wish to preserve, the smaller the global symmetry group is. Corresponding to certain specific choices of the $\mathcal{N} = 0, 1, 2, 4$ algebras, we have the global symmetry groups $\mathrm{Spin}(24) \supset Co_0 \supset M_{23} \supset M_{22}$.

We should stress again that there are other Co_0 subgroups that preserve $\mathcal{N} = 4$ resp. $\mathcal{N} = 2$ superconformal algebras arising from 3-planes resp. 2-planes in **24**. Some examples are: the group $U_4(3)$ for the former case, and the McLaughlin (McL) and Higman–Sims (HS) sporadic groups, and also $U_6(2)$ for the latter case. However, only for the Mathieu groups do the twined partition functions of the module display uniformly a special property, which we regard as an essential feature of the moonshine phenomena. Namely, all the mock modular forms obtained via twining by elements of the Mathieu groups are encoded in Jacobi forms that are constant in the elliptic variable, in the limit as the modular variable tends to any cusp other than the infinite one. This property also holds for the Jacobi forms of the Mathieu moonshine mentioned above, and may be regarded as a counter-part to the genus zero property of monstrous moonshine, as we explain in more detail in §8.

The importance of this property is its predictive power: it allows us to write down trace functions for the actions of Mathieu group elements with little more information than a certain fixed multiplier system, and the levels of the functions we expect to find. A priori these are just guesses, but the constructions we present here verify their validity.

This may be compared to the predictive power of the genus zero property of monstrous moonshine: if $\Gamma < SL_2(\mathbb{R})$ determines a genus zero quotient of the upper-half plane, and if the stabilizer of $i\infty$ in Γ is generated by $\pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, then there is a unique Γ-invariant holomorphic function satisfying $T_\Gamma(\tau) = q^{-1} + O(q)$ as $\tau \to i\infty$, for $q = e^{2\pi i\tau}$. The miracle of monstrous moonshine, and the content of the moonshine conjectures of Conway–Norton [1], is that for suitable choices of Γ, the function T_Γ is the trace of an element of the monster on some graded infinite-dimensional module (namely, the moonshine module of [2, 3]). The optimal growth property formulated in [9] plays the analogous predictive role in umbral moonshine, and is similar to the special property we formulate for the Mathieu moonshine considered here.

We mention here that although the moonshine conjectures have been proven in the monstrous case by Borcherds [4], and verified in [15, 20, 42] (see also [10]) for umbral moonshine, conceptual explanations of the genus zero property of monstrous moonshine, and of the analogous properties

of umbral moonshine, and the Mathieu moonshine studied here, remain to be determined. An approach to establishing the genus zero property of monstrous moonshine via quantum gravity is discussed in [43].

The organization of the paper is as follows. We begin in §2 with a review of the module discussed in [38]. In §3, we describe methods for endowing this module with $\mathcal{N}=4$ and $\mathcal{N}=2$ structure. In §4, we discuss what this does to the manifest symmetry group of the model, reducing the symmetry from Co_0 to a variety of other possible groups which preserve a 3-plane (respectively 2-plane) in the **24** of Co_0. In §4 and §5, we discuss the action of these Co_0-subgroups on the modules and compute the corresponding twining functions. We identify M_{23}, M_{22}, McL, HS, $U_6(2)$ and $U_4(3)$ as some of the most interesting Co_0-subgroups preserving some extended superconformal algebra. In §6 and §7, we discuss in some detail the decomposition of the graded partition function of our chiral conformal field theory into characters of irreducible representations of the $\mathcal{N}=4$ and $\mathcal{N}=2$ superconformal algebras. In §8, we discuss the special property we require from a moonshine twining function, and show how this property singles out the Mathieu groups in our setup. We close with a discussion in §9. The appendices contain a number of tables: character tables for the various groups we discuss, tables of coefficients of the vector-valued mock modular forms that arise as our twining functions, and tables describing the decompositions of our modules into irreducible representations of the various groups.

2 The Free Field Theory

The chiral 2d conformal field theory that will play a starring role in this paper has two different constructions. The first is described in [3] and starts with 8 free bosons X^i compactified on the 8-dimensional torus given by the E_8 root lattice, together with their Fermi superpartners ψ^i. One then orbifolds by the $\mathbb{Z}_2$ symmetry

$$(X^i, \psi^i) \to (-X^i, -\psi^i) \ . \tag{2.1}$$

Note that, in more mathematical terms, a chiral 2d conformal field theory can be understood to mean a super vertex operator algebra (usually assumed to be simple, and of CFT-type in the sense of [44]), together with a (simple) canonically-twisted module. These two spaces are referred to as the Neveu–Schwarz (NS) and Ramond (R) sectors of the theory, respectively. The compactification of free bosons on the torus defined by a lattice L manifests, in the NS sector, as the usual lattice vertex algebra construction, and their Fermi superpartners are then realized by a Clifford module, or free fermion, super vertex algebra, where the underlying orthogonal space comes equipped with an isometric embedding of L. In the cases under consideration, there is a unique simple canonically-twisted module up to equivalence (cf. e.g. [37]), and hence a unique

choice of R sector.

The orbifold procedure is described in detail in the language of vertex algebra in [37]. In what follows, a *field of dimension* d is a vertex operator or intertwining operator attached to a vector v in the NS or R sector, respectively, with $L(0)v = dv$. A *current* is a field of dimension 1. A field is called *primary* if its corresponding vector is a highest weight for the Virasoro (Lie) algebra. A *ground state* is an $L(0)$-eigenvector of minimal eigenvalue.

The E_8 construction just described has manifest $\mathcal{N} = 1$ supersymmetry, in the sense that the Neveu–Schwarz and Ramond algebras act naturally on the NS and R sectors, respectively. After orbifolding we obtain a $c = 12$ theory with no primary fields of dimension $\frac{1}{2}$. The partition functions (i.e., graded dimensions) of this free field theory can easily be computed. For example, the NS sector partition function is given by

$$\begin{aligned} Z_{\mathrm{NS},E8}(\tau) &= \frac{1}{2}\left(\frac{E_4(\tau)\theta_3(\tau,0)^4}{\eta(\tau)^{12}} + 16\frac{\theta_4(\tau,0)^4}{\theta_2(\tau,0)^4} + 16\frac{\theta_2(\tau,0)^4}{\theta_4(\tau,0)^4}\right) && (2.2)\\ &= q^{-1/2} + 0 + 276q^{1/2} + 2048q + 11202q^{3/2} + \cdots , && (2.3) \end{aligned}$$

where E_4 is the weight 4 Eisenstein series, being the theta series of the E_8 lattice, $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function, and θ_i are the Jacobi theta functions recorded in Appendix A. We have also set $q = \mathrm{e}(\tau)$ and we use the shorthand notation $\mathrm{e}(x) = e^{2\pi i x}$ throughout this paper.

One recognizes representations of the Co_1 sporadic group appearing in the q-series (2.3): apart from 276, which is the minimal dimension of a faithful irreducible representation (cf. [45]), one can also observe

$$\begin{aligned} 2048 &= 1 + 276 + 1771 , && (2.4)\\ 11202 &= 1 + 276 + 299 + 1771 + 8855 , && (2.5)\\ &\cdots \end{aligned}$$

In fact, this theory has a $\mathrm{Co}_0 \cong 2.\mathrm{Co}_1$ symmetry, which we call non-manifest since the action of Co_0 is not obvious from the given description. Note that we sometimes use n or $\mathbb{Z}_n$ to denote $\mathbb{Z}/n\mathbb{Z}$ depending on the context.

A better realization, for our purposes, was discussed in detail in [37] (cf. also [38]). The E_8 orbifold theory is equivalent to a theory of 24 free chiral fermions $\lambda_1, \lambda_2, \ldots, \lambda_{24}$, also orbifolded by the $\mathbb{Z}_2$ symmetry $\lambda_\alpha \to -\lambda_\alpha$. This gives an alternative description of the Conway module above. The partition function from this "free fermion" point of view is more naturally written as

$$Z_{\mathrm{NS,fermion}}(\tau) = \frac{1}{2}\sum_{i=2}^{4}\frac{\theta_i^{12}(\tau,0)}{\eta^{12}(\tau)} \ . \qquad (2.6)$$

This is equal to (2.2) according to non-trivial identities satisfied by theta functions. Note that $\theta_1(\tau, 0) = 0$.

The free fermion theory has a manifest Spin(24) symmetry, but not a manifest $\mathcal{N} = 1$ supersymmetry. However, one can construct an $\mathcal{N} = 1$ supercurrent as follows. There is a unique (up to scale) NS ground state, but there are $2^{12} = 4096$ linearly independent Ramond sector ground states, which may be obtained by acting on a given fixed R sector ground state with the fermion zero modes $\lambda_i(0)$. It will be convenient to label the resulting 4096 Ramond sector ground states by vectors $\mathbf{s} \in \tilde{\mathbb{F}}_2^{12}$, where $\tilde{\mathbb{F}}_2 = \{-1/2, 1/2\}$.

We therefore have 4096 spin fields of dimension $\frac{3}{2}$ which implement the flow from the NS to the R sector. Denoting these fields as $\mathcal{W}_{\mathbf{s}}$, one can try to find a linear combination

$$W = \sum_{\mathbf{s} \in \tilde{\mathbb{F}}_2^{12}} c_{\mathbf{s}} \mathcal{W}_{\mathbf{s}} \tag{2.7}$$

which will serve as an $\mathcal{N} = 1$ supercurrent (i.e., field whose modes generate actions of the Neveu–Schwarz and Ramond super Lie algebras). As demonstrated in [37], and as we will review in the next section, there exists a set of values $c_{\mathbf{s}}$ such that the operator product expansion of W and the stress tensor T close properly, defining actions of the Neveu–Schwarz and Ramond algebras.

Any choice of W breaks the Spin(24) symmetry, since the Ramond sector ground states split into two 2048-dimensional irreducible representation of Spin(24). It is proven in [37] that the subgroup of Spin(24) that stabilizes a suitably chosen $\mathcal{N} = 1$ supercurrent is exactly the Conway group Co_0. In brief, the method of [37] is to identify a certain elementary abelian subgroup of order 2^{12} in Spin(24) (which should be regarded as a copy of the extended Golay code in Spin(24)). The action of this subgroup on the Ramond sector ground states singles out a particular choice of W, with the property that it is not annihilated by the zero mode of any dimension 1 field in the theory. It follows from this (cf. Proposition 4.8 of [37]) that the subgroup of Spin(24) that stabilizes W is a reductive algebraic group of dimension 0, and hence finite. On the other hand, one can show (cf. Proposition 4.7 of [37]) that this group contains Co_0, by virtue of the choice of subgroup 2^{12}. We obtain that the full stabilizer of W in Spin(24) is Co_0 by verifying (cf. Proposition 4.9 of [37]) that Co_0 is a maximal subgroup, subject to being finite.

In the rest of this paper, we extend this idea as follows. Instead of choosing an $\mathcal{N} = 1$ supercurrent and viewing the theory as an $\mathcal{N} = 1$ super conformal field theory, we choose various other super extensions of the Virasoro algebra. We will argue that $\mathcal{N} = 4$ and $\mathcal{N} = 2$ superconformal presentations of the theory are in one to one correspondence with choices of subgroups of Co_0 which fix a 3-plane (respectively, 2-plane) in the 24 dimensional representation. This leads us naturally to theories with various interesting symmetry groups, whose twining functions are easily computed in terms of the partition function (or elliptic genus) of the free

fermion conformal field theory. These functions in turn are expressed nicely in terms of mock modular forms, and thus we establish mock modular moonshine relations for subgroups of Co_0 via this family of modules.

3 The Superconformal Algebras

We first discuss the largest superconformal algebra (SCA) we will consider, which gives rise to smaller global symmetry groups. We will construct an $\mathcal{N}=4$ SCA in the free fermion orbifold theory. Our strategy is to first construct the $SU(2)$ fields, and act with them on an $\mathcal{N}=1$ supercurrent to generate the full $\mathcal{N}=4$ SCA. We consequently obtain actions of the $\mathcal{N}=2$ SCA by virtue of its embeddings in the $\mathcal{N}=4$ SCA. In this process, we break the Co_0 symmetry group down to a proper subgroup as we will discuss in §4. We refer the reader to [46–48] for background on the $\mathcal{N}=4$ and $\mathcal{N}=2$ superconformal algebras.

We start with 24 real free fermions $\lambda_1, \lambda_2, \ldots, \lambda_{24}$. Picking out the first three fermions, we obtain the currents J_i:

$$J_i = -i\epsilon_{ijk}\lambda_j\lambda_k\,, \qquad i,j,k \in \{1,2,3\}\,. \tag{3.1}$$

They form an affine $SU(2)$ algebra with level 2 as may be seen from their operator product expansion (OPE),

$$J_i(z)J_j(0) \sim \frac{1}{z^2}\delta_{ij} + \frac{i}{z}\epsilon_{ijk}J_k(0)\,. \tag{3.2}$$

The next step is to pick an $\mathcal{N}=1$ supercurrent and act with J_i on it. As we reviewed in §2, an $\mathcal{N}=1$ supercurrent exists in this model and may be written as a linear combination of spin fields. Moreover, it may be chosen so that its stabilizer in Spin(24) is precisely Co_0. We will present a very general version of the construction now, and then extend it to find the $\mathcal{N}=4$ SCA.

To write the $\mathcal{N}=1$ supercurrent explicitly, we first group the 24 real fermions into 12 complex ones and bosonize them:

$$\psi_a \equiv 2^{-1/2}(\lambda_{2a-1}+i\lambda_{2a}) \cong e^{iH_a}\,, \quad \bar{\psi}_a \equiv 2^{-1/2}(\lambda_{2a-1}-i\lambda_{2a}) \cong e^{-iH_a}\,, \quad a=1,2,\cdots,12\,. \tag{3.3}$$

In terms of the bosonic fields $\mathbf{H}=(H_1,\ldots,H_{12})$, an $\mathcal{N}=1$ supercurrent W may be written as

$$W = \sum_{\mathbf{s}\in\tilde{\mathbb{F}}_2^{12}} w_{\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p})\,, \tag{3.4}$$

where each component of $\mathbf{s}=(s_1,s_2,\cdots,s_{12})$ takes the values $\pm 1/2$, and the coefficients $w_{\mathbf{s}}$ belong to $\mathbb{C}$. We have introduced cocycle operators $c_{\mathbf{s}}(\mathbf{p})$ to ensure that the fields with integer spins (i.e., corresponding to even parity vectors) commute with all other operators, and the fields

with half integral spins (corresponding to odd parity vectors) anticommute amongst themselves. The cocycle operators depend on the zero-mode operators $\mathbf{p}$ which are characterized by the commutation relation

$$\left[\mathbf{p}, e^{i\mathbf{k}\cdot\mathbf{H}}\right] = \mathbf{k}e^{i\mathbf{k}\cdot\mathbf{H}}\,, \tag{3.5}$$

where $\mathbf{k} = (k_1, \ldots, k_{12})$ is an arbitrary 12-tuple of complex numbers.

The associativity and closure of the OPE of the "dressed" vertex operators

$$V_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{H}} c_{\mathbf{k}}(\mathbf{p}) \tag{3.6}$$

requires that

$$c_{\mathbf{k}}(\mathbf{p}+\mathbf{k}')c_{\mathbf{k}'}(\mathbf{p}) = \epsilon(\mathbf{k},\mathbf{k}')c_{\mathbf{k}+\mathbf{k}'}(\mathbf{p}), \tag{3.7}$$

where the $\epsilon(\mathbf{k},\mathbf{k}')$ satisfy the 2-cocycle condition

$$\epsilon(\mathbf{k},\mathbf{k}')\,\epsilon(\mathbf{k}+\mathbf{k}',\mathbf{k}'') = \epsilon(\mathbf{k}',\mathbf{k}'')\,\epsilon(\mathbf{k},\mathbf{k}'+\mathbf{k}'')\,. \tag{3.8}$$

Moreover, in order for $V_{\mathbf{k}}$ to have the desired (anti)commutation relation, the condition

$$\epsilon(\mathbf{k},\mathbf{k}') = (-1)^{\mathbf{k}\cdot\mathbf{k}'+\mathbf{k}^2\mathbf{k}'^2}\epsilon(\mathbf{k}',\mathbf{k}) \tag{3.9}$$

should be imposed. An explicit description of the cocycle for a general vertex operator $e^{i\mathbf{k}\cdot\mathbf{H}}$ may be chosen as

$$c_{\mathbf{k}}(\mathbf{p}) = e^{i\pi\mathbf{k}\cdot\mathbf{M}\cdot\mathbf{p}} \tag{3.10}$$

according to [49]. In our case, $\mathbf{M}$ is a 12×12 matrix that has the block form

$$\mathbf{M} = \begin{pmatrix} M_4 & 0 & 0 \\ 1_4 & M_4 & 0 \\ 1_4 & 1_4 & M_4 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \tag{3.11}$$

where 1_4 is the 4×4 matrix with all entries set to 1.

Generically, the OPE of W with itself is

$$W(z)W(0) \sim \bar{w}w\left[\frac{1}{z^3} + \frac{T(0)}{4z}\right] + \frac{\bar{w}\Gamma^{\alpha\beta\gamma\delta}w}{96z}\lambda_\alpha\lambda_\beta\lambda_\gamma\lambda_\delta(0)\,, \tag{3.12}$$

where we have defined

$$\bar{w}_{\mathbf{s}} = w_{-\mathbf{s}} c_{-\mathbf{s}}(\mathbf{s}), \tag{3.13}$$

$$\Gamma^{\alpha\beta\gamma\delta}_{\mathbf{s}\mathbf{s}'} = \left(\Gamma^\alpha\Gamma^\beta\Gamma^\gamma\Gamma^\delta\right)_{\mathbf{s}\mathbf{s}'} c_{-\mathbf{s}}(\mathbf{s}'-\mathbf{s}) \left(-2s_{\lceil\frac{\alpha}{2}\rceil+1}\right)\cdots\left(-2s_{\lceil\frac{\beta}{2}\rceil}\right)\left(-2s_{\lceil\frac{\gamma}{2}\rceil+1}\right)\cdots\left(-2s_{\lceil\frac{\delta}{2}\rceil}\right), \tag{3.14}$$

for $\alpha < \beta < \gamma < \delta$. The other components of $\Gamma^{\alpha\beta\gamma\delta}$ are defined by the requirement that it is totally antisymmetrized. For W to be an $\mathcal{N} = 1$ supercurrent, the last terms in (3.12) must vanish,

$$\bar{w}\Gamma^{\alpha\beta\gamma\delta} w = 0\,, \qquad \forall \alpha,\beta,\gamma,\delta \in \{1,2,\cdots,12\}\,, \tag{3.15}$$

and the first two terms must have the correct normalization,

$$\bar{w}w = \sum_{\mathbf{s}\in\tilde{\mathbb{F}}_2^{12}} w_{-\mathbf{s}} w_{\mathbf{s}} c_{\mathbf{s}}(-\mathbf{s}) = 8\,. \tag{3.16}$$

From now on we presume to be chosen a solution $(w_{\mathbf{s}})$ such that W is an $N = 1$ supercurrent stabilized by Co_0, as described in §2.

We may now act with the $SU(2)$ currents J_i on our $\mathcal{N} = 1$ supercurrent W. In order to do this, write the $SU(2)$ currents in (3.1) in bosonized form,

$$J_1 = -\frac{1}{2}\left(e^{iH_1} - e^{-iH_1}\right)\left(e^{iH_2}e^{i\pi p_1} + e^{-iH_2}e^{-i\pi p_1}\right)\,, \tag{3.17}$$

$$J_2 = \frac{i}{2}\left(e^{iH_1} + e^{-iH_1}\right)\left(e^{iH_2}e^{i\pi p_1} + e^{-iH_2}e^{-i\pi p_1}\right)\,, \tag{3.18}$$

$$J_3 = i\partial H_1\,. \tag{3.19}$$

Here we have included the cocycles $e^{\pm i\pi p_1}$. We now extract the singular terms of the OPEs,

$$J_i(z)W(0) \sim -\frac{i}{2z}W_i(0)\,, \tag{3.20}$$

where W_i are slightly modified combinations of spin fields,

$$W_1 = -\sum_{\mathbf{s}} 2s_2 w_{R\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p})\,, \tag{3.21}$$

$$W_2 = i\sum_{\mathbf{s}} 4s_1 s_2 w_{R\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p})\,, \tag{3.22}$$

$$W_3 = i\sum_{\mathbf{s}} 2s_1 w_{\mathbf{s}} e^{i\mathbf{s}\cdot\mathbf{H}} c_{\mathbf{s}}(\mathbf{p})\,, \tag{3.23}$$

and where $R\mathbf{s} \equiv (-s_1, -s_2, s_3, \cdots, s_{12})$.

We claim that all three W_i defined above are valid $\mathcal{N} = 1$ supercurrents. This is because we may obtain, for instance, W_3 from W by rotating the 1-2 plane by π, and the conditions (3.15)

and (3.16), for being an $\mathcal{N}=1$ supercurrent, are invariant under $SO(24)$ rotations. We may obtain W_2 and W_3 similarly. This shows that each of the W_i is an $\mathcal{N}=1$ supercurrent.

Furthermore, we can check using the identity (3.15) that the OPEs of the W_i are given by

$$W_i(z)W_j(0) \sim \delta_{ij}\left[\frac{8}{z^3}+\frac{2}{z}T(0)\right]+2i\epsilon_{ijk}\left[\frac{2}{z^2}J_k(0)+\frac{1}{z}\partial J_k(0)\right], \tag{3.24}$$

$$W(z)W_i(0) \sim -2i\left(\frac{2}{z^2}+\frac{\partial}{z}\right)J_i(0), \tag{3.25}$$

$$J_i(z)W_j(0) \sim \frac{i}{2z}(\delta_{ij}W+\epsilon_{ijk}W_k). \tag{3.26}$$

This shows that W, W_i, and J_i, together with the stress tensor T, defined as

$$T=-\frac{1}{2}\sum_\alpha \lambda_\alpha\partial\lambda_\alpha=-\frac{1}{2}\sum_a \partial H_a\partial H_a, \tag{3.27}$$

form an $\mathcal{N}=4$ SCA with central charge $c=12$. We may recombine the four $\mathcal{N}=1$ supercurrents W, W_i into the more conventional $\mathcal{N}=4$ supercurrents

$$W_1^\pm \equiv 2^{-1/2}(W\pm iW_3), \qquad W_2^\pm \equiv \pm 2^{-1/2}i(W_1\pm iW_2), \tag{3.28}$$

which transform according to the representation $\mathbf{2}+\bar{\mathbf{2}}$ of $SU(2)$. In terms of these supercurrents we obtain the standard (small) $\mathcal{N}=4$ SCA with central charge $c=12$, characterized by the following set of OPEs:

$$T(z)T(0) \sim \frac{6}{z^4}+\frac{2}{z^2}T(0)+\frac{1}{z}\partial T(0), \tag{3.29}$$

$$T(z)W_a^\pm(0) \sim \frac{3}{2z^2}W_a^\pm(0)+\frac{1}{z}\partial W_a^\pm(0), \tag{3.30}$$

$$T(z)J_i(0) \sim \frac{1}{z^2}J_i(0)+\frac{1}{z}\partial J_i(0), \tag{3.31}$$

$$W_a^+(z)W_b^-(0) \sim \delta_{ab}\left[\frac{8}{z^3}+\frac{2}{z}T(0)\right]-2\sigma^i_{ab}\left[\frac{2}{z^2}J_i(0)+\frac{1}{z}\partial J_i(0)\right], \tag{3.32}$$

$$W_a^+(z)W_b^+(0) \sim W_a^-(z)W_b^-(0) \sim 0, \tag{3.33}$$

$$J_i(z)W_a^+(0) \sim -\frac{1}{2z}\sigma^i_{ab}W_b^+(0), \tag{3.34}$$

$$J_i(z)W_a^-(0) \sim \frac{1}{2z}\sigma^{i*}_{ab}W_b^-(0), \tag{3.35}$$

$$J_i(z)J_j(0) \sim \frac{1}{z^2}\delta_{ij}+\frac{i}{z}\epsilon_{ijk}J_k(0). \tag{3.36}$$

Here σ^i are the Pauli matrices.

Now we can generalize our formula for the partition function (2.6) to include a grading by the $U(1)$ charge under the Cartan generator of the $SU(2)$. The $U(1)$ charge operator J_0 is, by

definition, twice the zero-mode of the J_3 current. From this and the definition $J_3 = -i\lambda_1\lambda_2 = \psi_1\bar{\psi}_1$, we see that under J_0 the complex fermion ψ_1 has charge 2 while the other 11 complex fermions are neutral. Therefore, the $U(1)$-graded NS sector partition function becomes

$$Z_{\rm NS}(\tau, z) = \frac{1}{2}\sum_{i=2}^{4} \frac{\theta_i(\tau, 2z)\ \theta_i(\tau, 0)^{11}}{\eta(\tau)^{12}}\ . \tag{3.37}$$

In the above discussion, we have chosen the first three fermions out of a total of 24 to generate a set of $SU(2)$ currents. Together with an $\mathcal{N} = 1$ supercurrent they generate a full $\mathcal{N} = 4$ SCA. It is clear that we are free to choose any three fermions for this purpose. In fact, we could choose an arbitrary three-dimensional subspace of the 24-dimensional vector space spanned by the fermions, and obtain an $\mathcal{N} = 4$ SCA. For a given $\mathcal{N} = 1$ supercurrent, not all choices of 3-plane are equivalent, as we will see in §4.

Observe that we could instead have chosen to single out only two real fermions, and construct a $U(1)$ current algebra instead of an $SU(2)$ current algebra. Completely analogous manipulations then show that each such choice provides an $\mathcal{N} = 2$ superconformal algebra. As a result we can equip the Co_0 theory with $\mathcal{N} = 2$ structure in such a way that the global symmetry group is broken to subgroups G of Co_0 which stabilize 2-planes in **24**.

To summarize the results of this section, we have shown how to construct an $\mathcal{N} = 1$ supercurrent for the chiral conformal field theory described, in the previous section, as an orbifold of 24 free fermions. We have also shown how choices of 2- and 3-planes in the space spanned by the generating fermions give rise to actions of the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal algebras (respectively) on the theory. As reviewed in §2, a suitable choice of $\mathcal{N} = 1$ structure reduces the global symmetry of the theory to Co_0. In the next section we will discuss the finite simple groups that appear when we impose the richer, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal structures.

4 Global Symmetries

Enhancing the $\mathcal{N} = 1$ structure of the theory to $\mathcal{N} = 4$ breaks the Co_0 symmetry. We now show that for a specific choice of 3-plane in **24**, resulting in a specific copy of the $\mathcal{N} = 4$ SCA, the stabilising subgroup of Co_0 is the sporadic group M_{22}. Similarly, for a specific choice of 2-plane, resulting in a specific copy of the $\mathcal{N} = 2$ SCA, the stabilising subgroup of Co_0 is the sporadic group M_{23}. This amounts to a proof that the model described in §2 results in an infinite-dimensional M_{22} (resp. M_{23})-module underlying the mock modular forms described in §6 (resp. §7) arising from its interpretation as an $\mathcal{N} = 4$ (resp. $\mathcal{N} = 2$) module. More generally, we establish the modules for 3- (2-)plane-fixing subgroups of the largest Mathieu group M_{24} by fixing a specific copy of $\mathcal{N} = 4$ ($\mathcal{N} = 2$) SCA.

Recall that the theory regarded as an $\mathcal{N} = 0$ theory (i.e., with no extension of the Virasoro

action) has a Spin(24) symmetry resulting from the $SO(24)$ rotations on the 24-dimensional space, and a suitable choice of $\mathcal{N}=1$ supercurrent breaks the Spin(24) group down to its subgroup Co_0. The group Co_0 is the automorphism group of the Leech lattice Λ_{Leech}, and various interesting subgroups of Co_0 can be identified as stabilizers of suitably chosen lattice vectors in Λ_{Leech}. To study the automorphism group of the module when fixing more structure—more supersymmetries in this case—it will therefore be useful to describe the enhanced supersymmetries in terms of Leech lattice vectors.

In Chapter 10 of [41] it is shown that if we choose an appropriate tetrahedron in the Leech lattice, whose edges have lengths squared $16\times(2,2,2,2,3,3)$ in the normalisation described below, the subgroup of Co_0 that leaves all vertices of the tetrahedron invariant is M_{22}. To be more precise, let $\mathbf{e}_\gamma$, for $\gamma\in\{1,2,\ldots,24\}$, be an orthonormal basis of $\mathbb{R}^{24}$, and choose a copy $\mathcal{G}$ of the extended binary Golay code in $\mathcal{P}(\{1,\ldots,24\})$. Then we may realize Λ_{Leech} as the lattice generated by the vectors $2\sum_{\gamma\in C}\mathbf{e}_\gamma$ for $C\in\mathcal{G}$ together with $-4\mathbf{e}_1+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$. (One can show that all 24 vectors of the form $-4\mathbf{e}_\alpha+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ are in Λ_{Leech}.) Define the tetrahedron $T_{\{\alpha,\beta\}}$ to be that whose four vertices are $O=0$, $X_\alpha=4\mathbf{e}_\alpha+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$, $X_\beta=4\mathbf{e}_\beta+\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ and $P_{\alpha\beta}=4\mathbf{e}_\alpha+4\mathbf{e}_\beta$, for any $\alpha,\beta\in\{1,2,\ldots,24\}$ with $\alpha\neq\beta$. For every such $T_{\{\alpha,\beta\}}$, the subgroup fixing every vertex is a copy of M_{22}, a sporadic simple group of order $2^7\cdot3^2\cdot5\cdot7\cdot11=443,520$ and the subgroup of M_{24} fixing $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$.

From the above discussion, it is clear that given $\{\alpha,\beta\}$, a copy of M_{22} stabilises the real span of $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$ and $\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$. Given a suitable choice of the $\mathcal{N}=1$ superconformal algebra, this copy of $\mathbb{R}^3$ in $\mathbf{24}$ then determines, up to rotations, the three fermions, denoted $\lambda_{1,2,3}$, from which the $SU(2)$ current algebra was built in §3. By definition then, a copy of M_{22} leaves the $\mathcal{N}=4$ superconformal algebra invariant.

A natural question is: what is the symmetry group G that fixes a given choice of $\mathcal{N}=2$ superconformal structure? Given the above description of the M_{22} action, we can choose the $\mathbb{R}^2\subset\mathbb{R}^3$ generated by $\mathbf{e}_\alpha$ and $\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ and use the two free fermions lying in the $\mathbb{R}^2$ to construct the $\mathcal{N}=2$ sub-algebra of the $\mathcal{N}=4$ SCA. Specifically, the $U(1)$ action is rotation of the $\mathbb{R}^2$. From the above discussion, it is not hard to see that there is a copy of M_{23} fixing $\mathbf{e}_\alpha$ and $\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$ and hence stabilising the $\mathcal{N}=2$ structure. Recall that M_{23} is a sporadic simple group of order $2^7\cdot3^2\cdot5\cdot7\cdot11\cdot23=10,200,960$. In terms of the Leech lattice, it corresponds to the fact that the stabiliser of the triangle in Λ_{Leech} whose edges have lengths squared $16\times(6,3,2)$, with vertices chosen to be O, X_α and $2\sum_{\gamma=1}^{24}\mathbf{e}_\gamma$, is a copy of M_{23} inside the copy of Co_0 stabilising Λ_{Leech}.

This furnishes a proof that the theory described in §2 leads to modules for M_{22} and M_{23} which explicitly realize the mock modular forms to be defined in §6 and §7.

We should mention that by stabilizing different choices of geometric structure, other than the tetrahedron and triangle just discussed, leading to M_{22} and M_{23}, respectively, we can determine

other global symmetry groups G. Indeed, our method constructs a G-module with $\mathcal{N} = 4$ ($\mathcal{N} = 2$) superconformal symmetry for any subgroup $G < Co_0$ which fixes a 3-plane (2-plane) in **24**. Since, as we will see in §6 (§7), such modules furnish assignments of mock modular forms to the elements of their global symmetry groups, it is an interesting question to classify the global symmetry groups $G < Co_0$ that can arise. We conclude this section with a discussion of some of these possibilities. Certainly a full classification is beyond the scope of this work, so we restrict our attention (mostly) to sporadic simple examples.

Indeed, the Conway group Co_0 is a rich source of sporadic simple groups, for no less than 12 of the 26 sporadic simple groups are involved in Co_0 (cf. [45]), in the sense that they may be obtained by taking quotients of subgroups of Co_0. Of these 12, all but 3 are actually realised as subgroups, and 6 of these 9 sporadic simple groups appear as subgroups of Co_0 fixing (at least) a 2-plane in **24**. These six 2-plane fixing groups are the smaller Mathieu groups, M_{23}, M_{22}, M_{12} and M_{11}, the Higman–Sims group HS, and the McLaughlin group McL. Some 2-planes they fix are described explicitly in Chapter 10 of [41].

$\mathcal{N} = 4$ modules

From the character tables (cf. [45]) of the six sporadic 2-plane fixing subgroups of Co_0 it is clear that M_{22} and M_{11} are the only examples that fix a 3-plane. Even though M_{11} is not a subgroup of M_{22}, it turns out that the mock modular forms attached to M_{11} by our $\mathcal{N} = 4$ construction (and the analysis of §6) are a proper subset of those attached to M_{22}, since the conjugacy classes of Co_0 appearing in a 3-plane-fixing subgroup M_{11} are a proper subset of those appearing in a subgroup M_{22}. For this reason we focus on M_{22} when discussing mock modular forms attached to sporadic simple groups via the $\mathcal{N} = 4$ construction in this work.

If we expand our attention to simple, not necessarily sporadic subgroups of Co_0, then there is one example which is larger than M_{22} (which has order 443,520). Namely, the group $U_4(3)$, with order 3,265,920, can arise as the stabilizer of a suitably chosen 3-plane in the **24** of Co_0 [41]. The $U_4(3)$ characters are presented in Table 20, the coefficients in the associated (twined) vector valued mock modular forms in Tables 3 and 4, and the decomposition of the module into irreducible representations of the group in Tables 26 and 27.

As we shall see in §8, the Jacobi forms attached to M_{22} (and therefore also those attached to M_{11}) by the $\mathcal{N} = 4$ construction are distinguished in that they satisfy a natural analogue of the genus zero condition of monstrous moonshine. By contrast, this property does not hold for all the Jacobi forms arising from $U_4(3)$. This is the main reason for our focus on M_{22} in the context of $\mathcal{N} = 4$ supersymmetry.

$\mathcal{N} = 2$ modules

We have focused on the example of M_{23}, with order 10,200,960, in this section. Since M_{22} and M_{11} are subgroups of M_{23} we do not consider them further in the context of $\mathcal{N} = 2$ structures.

Of the remaining sporadic simple 2-plane-fixing subgroups of Co_0, the largest is the McLaughlin group McL, which is actually considerably larger than M_{23}, having order 898,128,000. Its characters are presented in Table 21, the coefficients of the (twined) mock modular forms in Tables 7 and 8, and the decomposition of the module into irreducible representations of the group in Tables 33-38.

The next largest example, also larger than M_{23}, is the Higman-Sims group HS, with order 44,352,000. Its characters are presented in Table 22, the coefficients of the (twined) mock modular forms in Tables 9 and 10, and the decomposition of the module into irreducible representations of the group in Tables 39-44.

If we expand our attention to simple groups fixing a 2-plane in **24** then there is one example larger than McL. Namely, the group $U_6(2)$, of order 9,196,830,720, fixes any triangle in **24** whose three sides are vectors of minimal length in the Leech lattice. The characters of $U_6(2)$ are given in Tables 17-19, the coefficients of the (twined) mock modular forms in Tables 11-14, and the decomposition of the module into irreducible representations of the group in Tables 45-54.

In direct analogy with the case of $\mathcal{N}=4$ structure, it will develop in §8 that the Jacobi forms attached to M_{12} and M_{23} satisfy a natural analogue of the genus zero condition of monstrous moonshine, and, contrastingly, this property fails in general for the modular forms arising from the other, non-Mathieu, 2-plane-fixing simple groups mentioned above. For these reasons, and since M_{12} is relatively small, we focus on M_{23} in our discussion of $\mathcal{N}=2$ supersymmetry.

5 Twining the Module

In the last sections, we have described how to equip the orbifolded free fermion theory with $\mathcal{N}=4$ and $\mathcal{N}=2$ superconformal structures. In this section we will use the Ramond sector of our theory to attach two variable formal power series—the g-twined graded R sector partition function, cf. (5.10)—to each element $g \in Co_0$ that preserves at least a 2-plane in **24**.

Let us denote the Ramond sector by V, and let us choose a $U(1)$ charge operator J_0. This will be twice the Cartan generator of the $SU(2)$ in the $\mathcal{N}=4$ case, or the single $U(1)$ generator in the case of $\mathcal{N}=2$ SCA. Then it is natural to define the Ramond-sector $U(1)$-graded partition function, or *elliptic genus*,

$$Z(\tau,z) = \mathrm{Tr}_V(-1)^F q^{L_0-c/24} y^{J_0} \tag{5.1}$$

$$= \frac{1}{2}\frac{1}{\eta^{12}(\tau)}\sum_{i=2}^{4}(-1)^{i+1}\theta_i(\tau,2z)\theta_i^{11}(\tau,0) \tag{5.2}$$

$$= \frac{1}{2}\frac{E_4(\tau)\theta_1^4(\tau,z)}{\eta^{12}(\tau)} + 8\sum_{i=2}^{4}\left(\frac{\theta_i(\tau,z)}{\theta_i(\tau,0)}\right)^4, \tag{5.3}$$

where we have introduced a chemical potential for the J_0 charges and set $y = \mathrm{e}(z)$ for $z \in \mathbb{C}$. Also, we define $(-1)^F$ as an operator on V by requiring that it act as Id on the untwisted free fermion contribution to V, and as $-\mathrm{Id}$ on the twisted fermion contribution.

As is expected for 2d conformal field theories with $\mathcal{N} \geq 2$ supersymmetry, the elliptic genus (5.1) transforms as a Jacobi form of weight 0, index $m = \frac{c}{6}$ and level 1. Explicitly, and since $c = 12$ in our case, this means that $Z|_2(\lambda, \mu) = Z$ for all $\lambda, \mu \in \mathbb{Z}$, and $Z|_{0,2}\gamma = Z$ for all $\gamma \in SL_2(\mathbb{Z})$, where the elliptic and modular slash operators are defined by

$$\begin{aligned}(\phi|_m(\lambda,\mu))(\tau,z) &= e(m(\lambda^2\tau + 2\lambda z))\phi(\tau, z+\lambda\tau+\mu),\\ (\phi|_{k,m}\gamma)(\tau,z) &= e(-m\tfrac{cz^2}{c\tau+d})(c\tau+d)^{-k}\phi\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}\right),\end{aligned} \tag{5.4}$$

respectively, for $\lambda, \mu \in \mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. (A Jacobi form of level N is only required to satisfy $\phi|_{k,m}\gamma = \phi$ for γ in the congruence subgroup $\Gamma_0(N)$ (cf. (6.24)).)

As we have seen in §2, the two different ways of writing this function, (5.2) and (5.3), are intuitively connected more closely with the free fermion and E_8 root lattice descriptions of the theory, respectively. Of course, the $U(1)$-graded NS sector partition function (3.37) is related to the above, graded Ramond sector partition function by a spectral flow transformation

$$Z_{\mathrm{NS}}(\tau, z) = q^{1/2}y^{-2}Z(\tau, z - \tfrac{\tau+1}{2}). \tag{5.5}$$

There is a natural way in which one can twine the above function under certain subgroups of Co_0. From the previous discussions, we see that the representation **24** plays a central role in the way various subgroups of Co_0 act on the model. Let's denote by $\ell_{g,k}$ and $\bar{\ell}_{g,k}$, for $k = 1, \ldots 12$, the 12 complex conjugate pairs of eigenvalues of $g \in Co_0$ when acting on **24**. This information is conveniently encoded in the so-called Frame shape of g, given by

$$\Pi_g = \prod_n {L_n}^{m_n}\,, \quad 1 \leq L_1 < L_2 < L_3 \ldots\,, \quad \text{and} \;\; m_n \in \mathbb{Z}, m_n \neq 0\,,$$

satisfying $\sum_n L_n m_n = 24$, through the fact that the 12 pairs $\{\ell_{g,k}, \bar{\ell}_{g,k}\}$ are precisely the 24 roots solving the equation

$$\prod_n (x^{L_n} - 1)^{m_n} = 0.$$

As discussed in §3 and §4, in order to preserve at least $\mathcal{N} = 2$ superconformal symmetry and hence be able to twine the graded R-sector partition function (5.1), the subgroup G must leave at least a 2-dimensional subspace in **24** pointwise invariant. In the graded partition function this corresponds to leaving the factor $\theta_i(\tau, 2z)$ in (5.2) invariant. As a result, for every conjugacy class $[g]$ of such a group G we can choose $\ell_{g,1} = \bar{\ell}_{g,1} = 1$. It is easy to see that when acting on the untwisted free fermions of the theory, contributing the terms involving θ_i with $i = 3, 4$ in

(5.2), the group element g simply replaces $\theta_i^{11}(\tau,0)$ with

$$\prod_{k=2}^{12}\theta_i(\tau,\rho_{g,k}) \tag{5.6}$$

where $\mathrm{e}(\rho_{g,k})=\ell_{g,k}$.

When trying to do the same for the contribution from the twisted fermions, contributing the term involving θ_2 in (5.2), however, we see that the above simple consideration suffers from an ambiguity. This can be seen from the fact that $\theta_2(\tau,\rho)=-\theta_2(\tau,\rho+1)$, and hence the answer cannot be determined simply by looking at the g-eigenvalues on $\mathbf{24}$. This of course is a reflection of the fact that the global symmetry group, with no superconformal structure imposed, is Spin(24), which is a 2-fold cover of $SO(24)$. As a result, to specify the twining of the twisted fermion contribution, we also need to know the action of G on the faithful 2^{12}-dimensional representation of Spin(24) spanned by Ramond sector ground states in the free fermion theory (cf. §2), henceforth denoted $\mathbf{4096}$, which decomposes as $\mathbf{4096}=\mathbf{1}+\mathbf{276}+\mathbf{1771}+\mathbf{24}+\mathbf{2024}$ in terms of the irreducible representations of Co_0.

Note that, according to the orbifold construction, just "half" of the Ramond sector ground states in the free fermion theory will contribute to the Ramond sector V of the orbifold theory under consideration. In terms of the Co_0 action, the two "halves" are $\mathbf{24}+\mathbf{2024}$, where Co_0 acts faithfully, and $\mathbf{1}+\mathbf{276}+\mathbf{1771}$, where the action factors through $Co_1=Co_0/2$. In practice, both choices give rise to equivalent theories (i.e., isomorphic super vertex operator algebras, cf. [37,38]), but they are inequivalent as Co_0-modules. For us, the ground states represented by $\mathbf{24}+\mathbf{2024}$ lie in the R sector, V, and the $\mathbf{1}$ in $\mathbf{1}+\mathbf{276}+\mathbf{1771}$ represents the Co_0-invariant $\mathcal{N}=1$ supercurrent in the NS sector of our orbifold theory.

The above discussion serves to remind us that there is, really, a vanishing term

$$0=\frac{1}{2}\frac{1}{\eta^{12}(\tau)}\theta_1(\tau,2z)\theta_1^{11}(\tau,0) \tag{5.7}$$

in (5.2), which, for certain $g\in Co_0$, will make a non-vanishing contribution to the g-twined version of (5.1). It vanishes when $g=e$ is the identity because the Ramond sector ground states in the free fermion theory come in pairs with opposite eigenvalues for $(-1)^F$. Moreover, exchanging the pair corresponds to complex conjugation $\psi_a\leftrightarrow\bar\psi_a$, for $a=1,\dots,12$, of the complex fermions. Recall that one of the complex fermions, denoted ψ_1 in (3.19), was used to construct the $U(1)$ charge operator J_0, and we are interested in the graded partition function where we introduce a chemical potential z for this operator. Because exchanging $\psi_1\leftrightarrow\bar\psi_1$ also induces a flip of $U(1)$ charges, captured by $z\leftrightarrow -z$, the contribution of the first complex fermion

does not vanish, corresponding to the fact that the identity

$$\theta_1(\tau, z) = \theta_1(\tau, z+2) = -\theta_1(\tau, -z) \tag{5.8}$$

only forces $\theta_1(\tau, z)$ to vanish at $z \in \mathbb{Z}$. Consequently, the g-twining of (5.7) makes a non-zero contribution to the g-twining of (5.2) if and only if $\rho_{g,k} \notin \mathbb{Z}$ for all $k = 2, \ldots, 12$. In other words, it is non-zero only when the cyclic group generated by g fixes nothing but a 2-plane.

By inspection we find that, among the groups we consider, such group elements must be in the conjugacy classes 23AB $\subset M_{23}$, 6AB, 12AB, 12DE, 18AB $\subset U_6(2)$, 15AB, 30AB $\subset McL$, or 20AB $\subset HS$. The pairs of these conjugacy classes corresponding to the letters A and B (or D and E) are mutually inverse, and so their respective traces, on any representation, are related by complex conjugation. In terms of our construction, choosing one over the other is the same as choosing what one labels ψ_1 and $\bar{\psi}_1$, and the same as choosing an orientation on the 2-plane fixed by the group element in **24**. As a result, from (5.8) we see that the θ_1 term in the partition functions twined by these conjugate A (D) and B (E) classes come with an opposite sign.

Let us work with the principal branch of the logarithm, and choose $\rho_{g,k} \in [0, 1/2]$ in (5.6). Then, by direct computation—we must compute directly, for the choice of labels for mutually inverse conjugacy classes is not natural—we find that the signs in (5.10) are

$$\epsilon_{g,1} = 1 \text{ for } g \text{ in } \begin{cases} 23A \subset M_{23}, \\ 20A \subset HS, \\ 15A \cup 30A \subset McL, \\ 12A \cup 12D \cup 6B \cup 18B \subset U_6(2), \end{cases} \tag{5.9}$$

and $\epsilon_{g,1} = -1$ for the inverse classes, $23B \subset M_{23}$, $20B \subset HS$, &c.

Putting these different contributions together, we conclude that for every $[g] \subset G$ where G is a subgroup of Co_0 preserving (at least) a 2-plane in **24**, the corresponding g-twined $U(1)$-graded R sector partition function reads

$$Z_g(\tau, z) = \mathrm{Tr}_V\, g(-1)^F q^{L_0 - c/24} y^{J_0} \tag{5.10}$$

$$= \frac{1}{2} \frac{1}{\eta(\tau)^{12}} \sum_{i=1}^{4} (-1)^{i+1} \epsilon_{g,i}\, \theta_i(\tau, 2z) \prod_{k=2}^{12} \theta_i(\tau, \rho_{g,k}), \tag{5.11}$$

where

$$\epsilon_{g,2} = \begin{cases} \frac{\mathrm{Tr}_{\mathbf{4096}}g}{2^{12}\prod_{k=1}^{12}\cos(\pi\rho_{g,k})} \in \{-1,1\} & \text{when} \quad \prod_{\mathrm{k}=1}^{12}\cos(\pi\rho_{\mathrm{g,k}}) \neq 0 \\ 0 & \text{when} \quad \prod_{\mathrm{k}=1}^{12}\cos(\pi\rho_{\mathrm{g,k}}) = 0 \end{cases} \tag{5.12}$$

$$\epsilon_{g,3} = \epsilon_{g,4} = 1, \tag{5.13}$$

and where the $\epsilon_{g,1}$ are as determined in the preceding paragraph.

In this section we have introduced the g-twined $U(1)$-graded Ramond sector partition function, or g-twined elliptic genus of our theory, Z_g, for any $g \in Co_0$ fixing a 2-plane in $\mathbf{24}$. We have also derived an explicit formula (5.10) for Z_g, in terms of the Frame shapes Π_g and values $\mathrm{Tr}_{\mathbf{4096}}g$. This Frame shape and trace value data is collected, for $g \in G$, for various $G \subset \mathrm{Co}_0$, in Appendix B. In §6 and §7 we will see how the above twining leads to the mock modular forms playing the role of the McKay–Thompson series in these new examples of mock modular moonshine.

6 The $\mathcal{N} = 4$ Decompositions

From the discussion in §3 it is clear that the orbifold theory discussed in §2 can be equipped with $\mathcal{N} = 4$ superconformal structure. In this section we will study the decomposition of the Ramond sector V into irreducible representations of the $\mathcal{N} = 4$ SCA and see how the decomposition leads to mock modular forms relevant for the M_{22} moonshine which we will discuss in §8.

Recall (cf. [46]) that the $\mathcal{N} = 4$ superconformal algebra contains subalgebras isomorphic to the affine $SU(2)$ and Virasoro Lie algebras. In a unitary representation the former of these acts with level $m-1$, for some integer $m > 1$, and the latter with central charge $c = 6(m-1)$.

The unitary irreducible highest weight representations $v^{\mathcal{N}=4}_{m;h,j}$ are labeled by the eigenvalues of L_0 and $\frac{1}{2}J^3_0$ acting on the highest weight state, which we denote by h and j, respectively. Cf. [47, 50]. The superconformal algebra has two types of highest weight Ramond sector representations: the *massless* (or *BPS*) representations with $h = \frac{c}{24} = \frac{m-1}{4}$ and $j \in \{0, \frac{1}{2}, \cdots, \frac{m-1}{2}\}$, and the *massive* (or *non-BPS*) representations with $h > \frac{m-1}{4}$ and $j \in \{\frac{1}{2}, 1, \cdots, \frac{m-1}{2}\}$. Their graded characters, defined as

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z) = \mathrm{tr}_{v^{\mathcal{N}=4}_{m;h,j}}\left((-1)^{J^3_0} y^{J^3_0} q^{L_0 - c/24}\right), \tag{6.1}$$

are given by

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z) = (\Psi_{1,1}(\tau,z))^{-1}\mu_{m;j}(\tau,z) \tag{6.2}$$

and

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z) = (\Psi_{1,1}(\tau,z))^{-1}\, q^{h-\frac{c}{24}-\frac{j^2}{m}}\left(\theta_{m,2j}(\tau,z) - \theta_{m,-2j}(\tau,z)\right) \tag{6.3}$$

in the massless and massive cases, respectively, [50]. In the above formulas, the function $\mu_{m;j}(\tau,z)$ is defined by setting

$$\mu_{m;j}(\tau,z) = (-1)^{1+2j} \sum_{k\in\mathbb{Z}} q^{mk^2} y^{2mk} \frac{(yq^k)^{-2j} + (yq^k)^{-2j+1} + \cdots + (yq^k)^{1+2j}}{1-yq^k}, \tag{6.4}$$

and $\Psi_{1,1}$ is a meromorphic Jacobi form (cf. §8 of [51] for more on meromorphic Jacobi forms) of weight 1 and index 1 given by

$$\Psi_{1,1}(\tau,z) = -i\,\frac{\theta_1(\tau,2z)\,\eta(\tau)^3}{(\theta_1(\tau,z))^2} = \frac{y+1}{y-1} - (y^2-y^{-2})q + \cdots . \tag{6.5}$$

Finally, we have used the theta functions

$$\theta_{m,r}(\tau,z) = \sum_{k=r \pmod{2m}} \mathrm{e}(\tfrac{k}{2})\, q^{k^2/4m} y^k, \tag{6.6}$$

defined for all $2m \in \mathbb{Z}_{>0}$ and $r-m \in \mathbb{Z}$, and satisfying

$$\theta_{m,r}(\tau,z) = \theta_{m,r+2m}(\tau,z) = \mathrm{e}(m)\,\theta_{m,-r}(\tau,-z).$$

Note that the vector-valued theta function $\theta_m = (\theta_{m,r})$, $r-m \in \mathbb{Z}/2m\mathbb{Z}$, is a vector-valued Jacobi form of weight 1/2 and index m satisfying

$$\begin{aligned}\theta_m(\tau,z) &= \sqrt{\frac{1}{2m}}\sqrt{\frac{i}{\tau}}\;\mathrm{e}(-\tfrac{m}{\tau}z^2)\,\mathcal{S}_\theta.\theta_m(-\tfrac{1}{\tau},\tfrac{z}{\tau}) \\ &= \mathcal{T}_\theta.\theta_m(\tau+1,z) \\ &= \theta_m(\tau,z+1) = \mathrm{e}(m(\tau+2z+1))\theta_m(\tau,z+\tau),\end{aligned} \tag{6.7}$$

where the $\mathcal{S}_\theta$ and $\mathcal{T}_\theta$ matrices are $2m\times 2m$ matrices with entries

$$(\mathcal{S}_\theta)_{r,r'} = \mathrm{e}(\tfrac{rr'}{2m})\,\mathrm{e}(\tfrac{-r+r'}{2}) \quad , \quad (\mathcal{T}_\theta)_{r,r'} = \mathrm{e}(-\tfrac{r^2}{4m})\,\delta_{r,r'}. \tag{6.8}$$

We will take $m\in\mathbb{Z}$ for the rest of this section. When we consider $\mathcal{N}=2$ decompositions in the next section, we will use the theta function with half-integral indices.

From the above discussion, it is clear that the graded partition function of a module for the $c=6(m-1)$ $\mathcal{N}=4$ SCA admits the following decomposition

$$\mathcal{Z}^{\mathcal{N}=4,m} = \sum_{\substack{n\geq 0, 0\leq r\leq m-1\\ r\neq 0 \text{ when } n>0}} c'_r(n-\tfrac{r^2}{4m})\,\mathrm{ch}^{\mathcal{N}=4}_{m;\frac{m-1}{4}+n,\frac{r}{2}}(\tau,z)\,. \tag{6.9}$$

Furthermore, from the identity

$$\mu_{m;\frac{r}{2}} = (-1)^r (r+1)\mu_{m;0} + (-1)^{n-1} \sum_{n=1}^{r} n\, q^{-\frac{(r-n+1)^2}{4m}} (\theta_{m,r-n+1} - \theta_{m,-(r-n+1)})$$

we arrive at

$$\mathcal{Z}^{\mathcal{N}=4,m} = (\Psi_{1,1}(\tau,z))^{-1} \left(c_0\, \mu_{m;0}(\tau,z) + \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} F_r^{(m)}(\tau)\, \theta_{m,r}(\tau,z) \right), \tag{6.10}$$

where

$$F_r^{(m)}(\tau) = \sum_{n=0}^{\infty} c_r(n - \tfrac{r^2}{4m})\, q^{n-\frac{r^2}{4m}} \quad , \quad 1 \le r \le m-1, \tag{6.11}$$

$$c_0 = \sum_{r=0}^{m-1} (-1)^r\, (r+1)\, c_r'(-\tfrac{r^2}{4m}), \tag{6.12}$$

$$c_r(n - \tfrac{r^2}{4m}) = \begin{cases} \sum_{r'=r}^{m-1} (-1)^{r'-r} (r'+1-r)\, c_{r'}'(-\frac{r'^2}{4m}) & , \ n=0 \\ c_r'(n - \frac{r^2}{4m}) & , \ n>0 \end{cases} . \tag{6.13}$$

The rest of the components of $F^{(m)} = (F_r^{(m)})$, $r \in \mathbb{Z}/2m\mathbb{Z}$, are defined by setting

$$F_r^{(m)}(\tau) = -F_{-r}^{(m)}(\tau) = F_{r+2m}^{(m)}(\tau). \tag{6.14}$$

Recall that $\mu_{m;0}(\tau,z) = -f_0^{(m)}(\tau,z) + f_0^{(m)}(\tau,-z)$, a specialisation of the Appell–Lerch sum

$$f_u^{(m)}(\tau,z) = \sum_{k\in\mathbb{Z}} \frac{q^{mk^2} y^{2mk}}{1 - yq^k\, \mathrm{e}(-u)} \tag{6.15}$$

studied in [52], has the following relation to the modular group $SL_2(\mathbb{Z})$: let the (non-holomorphic) completion of $\mu_{m;0}(\tau,z)$ be

$$\hat{\mu}_{m;0}(\tau,\bar{\tau},z) = \mu_{m;0}(\tau,z) - \mathrm{e}(-\tfrac{1}{8}) \frac{1}{\sqrt{2m}} \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \theta_{m,r}(\tau,z) \int_{-\bar{\tau}}^{i\infty} (\tau'+\tau)^{-1/2} \overline{S_{m,r}(-\bar{\tau}')}\, \mathrm{d}\tau' \ . \tag{6.16}$$

Then $\hat{\mu}_{m;0}$ transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Here $S_m = (S_{m,r})$ is the vector-valued cusp form for $SL_2(\mathbb{Z})$ whose components are given by the unary theta functions

$$S_{m,r}(\tau) = \sum_{k=r \pmod{2m}} \mathrm{e}(\tfrac{k}{2})\, k\, q^{k^2/4m} = \frac{1}{2\pi i} \frac{\partial}{\partial z} \theta_{m,r}(\tau,z)|_{z=0}.$$

For later use, note that the theta series $S_{m,r}(\tau)$ is defined for all $2m \in \mathbb{Z}$ and $r - m \in \mathbb{Z}/2m\mathbb{Z}$.

The way in which the functions $\mathcal{Z}^{(m)}$ and $\hat{\mu}_{m;0}$ transform under the Jacobi group shows that the non-holomorphic function $\sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \hat{F}_r^{(m)}(\tau)\,\theta_{m,r}(\tau,z)$ transforms as a Jacobi form of weight 1 and index m under $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, where

$$\hat{F}_r^{(m)}(\tau) = F_r^{(m)}(\tau) + c_0\,\mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2m}} \int_{-\bar{\tau}}^{i\infty} (\tau'+\tau)^{-1/2}\overline{S_{m,r}(-\bar{\tau}')}\,\mathrm{d}\tau'.$$

In other words, $F^{(m)} = (F_r^{(m)})$, $r \in \mathbb{Z}/2m\mathbb{Z}$ is a vector-valued mock modular form with a vector-valued shadow $c_0\,S_m$, whose r-th component is given by $S_{m,r}(\tau)$, with the multiplier for $SL_2(\mathbb{Z})$ given by the inverse of the multiplier system of S_m (cf. (6.8)).

Now we are ready to apply the above discussion to the $U(1)$-graded Ramond sector partition function of the theory, discussed in §5. Recall that in this case we have $c = 12$, so $m = 3$ in (6.2) and (6.3). The $\mathcal{N} = 4$ decomposition of (5.1) gives

$$\begin{aligned} Z(\tau,z) &= 21\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{1}{2},0} + \mathrm{ch}^{\mathcal{N}=4}_{3;\frac{1}{2},1} + \big(560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{3}{2},\frac{1}{2}} + 8470\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{5}{2},\frac{1}{2}} + 70576\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{7}{2},\frac{1}{2}} + \dots\big) \\ &\quad + \big(210\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{3}{2},1} + 4444\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{5}{2},1} + 42560\,\mathrm{ch}^{\mathcal{N}=4}_{3;\frac{7}{2},1} + \dots\big) \qquad (6.17) \\ &= (\Psi_{1,1}(\tau,z))^{-1}\left(24\,\mu_{3;0}(\tau,z) + \sum_{r\in\mathbb{Z}/6\mathbb{Z}} h_r(\tau)\theta_{3,r}(\tau,z)\right) \qquad (6.18) \end{aligned}$$

where $\dots$ stand for terms with expansion $\Psi_{1,1}^{-1}q^\alpha y^\beta$ with $\alpha - \beta^2/12 > 3$. More Fourier coefficients of the functions $h_r(\tau)$ are recorded in Appendix C, where $h = h_g$ for $[g] = 1A$. Note that all the graded multiplicities $c'_r(n - \frac{r^2}{12})$ appear to be non-negative. Of course, this is guaranteed by the fact that V is a module for the $\mathcal{N} = 4$ SCA as shown in §3. In particular, the Fourier coefficients of $h_r(\tau)$ appear to be all non-negative apart from that of the polar term $-2q^{-1/12}$ in h_1.

From the above discussion we see that $h = (h_r)$, for $r \in \mathbb{Z}/6\mathbb{Z}$, is a weight $1/2$ vector-valued mock modular form for $SL_2(\mathbb{Z})$ with 6 components (but just 2 linearly independent components, since $h_0 = h_3 = 0$, $h_{-1} = -h_1$, and $h_{-2} = -h_2$), with shadow given by $24\,S_3$, and multiplier system inverse to that of S_3.

This is to be contrasted with the elliptic genus of a generic non-chiral super conformal field theory. For example, the sigma model of a K3 surface has $c = 6$, and the elliptic genus is given

by

$$\begin{aligned}\mathbf{EG}(\tau,z;K3) &= \mathrm{Tr}_{\mathcal{H}_{RR}}(-1)^{F_L+F_R}y^{J_0}q^{L_0-c/24}\bar{q}^{\tilde{L}_0-\tilde{c}/24} \\ &= 20\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{1}{4},0} - 2\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{1}{4},\frac{1}{2}} + \left(90\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{5}{4},\frac{1}{2}} + 462\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{9}{4},\frac{1}{2}} + 1540\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{13}{4},\frac{1}{2}} + \dots\right) \quad (6.19) \\ &= (\Psi_{1,1}(\tau,z))^{-1}\Big\{24\,\mu_{2;0}(\tau,z) + (\theta_{2,1}(\tau,z) - \theta_{2,-1}(\tau,z)) \\ &\quad \times(-2q^{-1/8} + 90q^{7/8} + 462q^{15/8} + 1540q^{23/8} + \dots)\Big\}, \quad (6.20)\end{aligned}$$

where $\dots$ stand for terms with expansion $\Psi_{1,1}^{-1}q^{\alpha}y^{\beta}$ with $\alpha - \beta^2/8 > 3$. In this case, the coefficient multiplying the massless character $\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{1}{4},\frac{1}{2}}$ is negative, arising from the Witten index of the right-moving massless multiplets paired with the representation $v^{\mathcal{N}=4}_{2;\frac{1}{4},\frac{1}{2}}$ of the left-moving $\mathcal{N}=4$ SCA.

In §3 we have shown that the theory under consideration, as a module for the $\mathcal{N}=4$ SCA, admits a faithful action via automorphisms by a group G, as long as G is a subgroup of Co_0 fixing at least a 3-plane. For any such $g \in G$, the g-twined graded partition function $Z_g(\tau,z)$ is given by (5.10), and from the fact that the action of g commutes with the $\mathcal{N}=4$ SCA, we expect $Z_g(\tau,z)$ to admit a decomposition

$$Z_g(\tau,z) = (\Psi_{1,1}(\tau,z))^{-1}\left((\mathrm{Tr}_{\mathbf{24}}g)\,\mu_{3;0}(\tau,z) + \sum_{r\in\mathbb{Z}/6\mathbb{Z}} h_{g,r}(\tau)\theta_{3,r}(\tau,z)\right). \quad (6.21)$$

Moreover, the coefficients of

$$h_{g,r}(\tau) = a_r q^{-r^2/12} + \sum_{n=1}^{\infty}(\mathrm{Tr}_{V^G_{r,n}}g)\,q^{n-r^2/12} \quad (6.22)$$

must be characters of the G-module

$$V^G = \bigoplus_{r=1,2}\bigoplus_{n=1}^{\infty} V^G_{r,n} \quad (6.23)$$

arising from the orbifold theory discussed in §2.

Indeed, the multiplicities of the $\mathcal{N}=4$ multiplets in the decomposition (6.17) are suggestive of the following group theoretic interpretation[2]: the 21 $h=1/2$, $j=0$ massless representations transform as the 21-dimensional irreducible representation of M_{22}, and similarly, the 560 $h=3/2$, $j=1/2$ massive representations transform as $\chi_{10}+\chi_{11}$ (see Appendix B), or "$280+\overline{280}$", under M_{22}, etc.

[2]The observation that the decomposition into $\mathcal{N}=4$ characters of (a multiple of) the function $Z(\tau,z)$ returns positive integers that are suggestive of representations of the Mathieu group M_{22} was first communicated privately by Jeff Harvey to J.D. in 2010.

We have explicitly computed the first 30 or so coefficients of each q-series $h_{g,r}(\tau)$ for all conjugacy classes $[g]$ of G, for $G = M_{22}$ and $G = U_4(3)$. These can be found in the tables in Appendix C. Subsequently, we compute the first 30 or so G-modules $V^G_{r,n}$ in terms of their decompositions into irreducible representations. They can be found in the tables in Appendix D.

Finally we would like to discuss the mock modular property of the functions $h_g = (h_{g,r})$. Recall that the *Hecke congruence subgroups* of $SL_2(\mathbb{Z})$ are defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c = 0 \bmod N \right\}. \tag{6.24}$$

We expect Z_g to be a weak Jacobi form of weight zero and index 2 (possibly with multiplier) for the group $\Gamma_0(\mathrm{o}_g) \ltimes \mathbb{Z}^2$, where o_g is the order of the group element $g \in G$. This can be verified explicitly from the expression (5.10). Repeating the similar arguments as above, we conclude that each vector-valued function h_g is a vector-valued mock modular form of weight 1/2 with shadow $(\mathrm{Tr}_{\mathbf{24}} g)S_3$ for the congruence subgroup $\Gamma_0(\mathrm{o}_g)$. Note that $(\mathrm{Tr}_{\mathbf{24}} g) \neq 0$ for all $g \in M_{22}$ which are the cases of our main interest. For these cases the multiplier of h_g is again given by the inverse of the multiplier system of S_3, now restricted to $\Gamma_0(o_g)$.

In this section we have analyzed the decomposition of the Ramond sector of our orbifold theory into irreducible modules for the $\mathcal{N} = 4$ SCA, and we have demonstrated that the generating functions of irreducible $\mathcal{N} = 4$ SCA module multiplicities furnish a vector-valued mock modular form. We have also demonstrated that these multiplicities are dimensions of modules for subgroups $G < Co_0$ that point-wise fix a 3-plane in **24**, and we have analyzed the modularity of the resulting, g-twined multiplicity generating functions, for $g \in G$. We have verified that each such g-twining results in a vector-valued mock modular form with a specified shadow function. In the next section we will present directly analogous considerations for $\mathcal{N} = 2$ superconformal structures arising from 2-planes in **24**.

7 The $\mathcal{N} = 2$ Decompositions

As discussed in §4, the theory presented in §2 can be regarded as a module for an $\mathcal{N} = 2$ SCA as well as for an $\mathcal{N} = 4$ SCA. Moreover, for every subgroup $G < Co_0$ fixing a 2-plane there is an $\mathcal{N} = 2$ SCA commuting with the action of G on the theory. As a result, and as we will now demonstrate, the decomposition of the partition function (5.10) twined by elements of G into $\mathcal{N} = 2$ characters leads to sets of vector-valued mock modular forms, now of weight 1/2 and index 3/2, which are the graded characters of an infinite-dimensional G-module inherited from the Co_0-module structure on V (cf. §5).

To see what these (vector-valued) mock modular forms $\tilde{h}_g = (\tilde{h}_{g,j})$ are, let us start by recalling the characters of the irreducible representations of the $\mathcal{N} = 2$ SCA. For the SCA with central charge $c = 3(2\ell + 1) = 3\hat{c}$, the unitary irreducible highest weight representations $v^{\mathcal{N}=2}_{\ell;h,Q}$ are labeled by the two quantum numbers h and Q which are the eigenvalues of L_0 and J_0, respectively, when acting on the highest weight state [53,54]. Just as in the $\mathcal{N} = 4$ case, there are two types of Ramond sector highest weight representations: the *massless* (or *BPS*) representations with $h = \frac{c}{24} = \frac{\hat{c}}{8}$ and $Q \in \{-\frac{\hat{c}}{2} + 1, -\frac{\hat{c}}{2} + 2, \ldots, \frac{\hat{c}}{2} - 1, \frac{\hat{c}}{2}\}$, and the *massive* (or *non-BPS*) representations with $h > \frac{\hat{c}}{8}$ and $Q \in \{-\frac{\hat{c}}{2} + 1, -\frac{\hat{c}}{2} + 2, \ldots, \frac{\hat{c}}{2} - 2, \frac{\hat{c}}{2} - 1, \frac{\hat{c}}{2}\}$, $Q \neq 0$. From now on we will concentrate on the case when ℓ is half-integral, and hence $\hat{c}$ and c are even.

The graded characters, defined as

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;h,Q}(\tau, z) = \mathrm{tr}_{v^{\mathcal{N}=2}_{\ell;h,Q}} \left((-1)^{J^3_0} y^{\tilde{J}^3_0} q^{L_0 - c/24} \right), \tag{7.1}$$

are given by

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;h,Q}(\tau, z) = \mathrm{e}(\tfrac{\ell}{2})(\Psi_{1,-\frac{1}{2}}(\tau, z))^{-1} q^{h - \frac{c}{24} - \frac{j^2}{4\ell}} \theta_{\ell,j}(\tau, z) \quad , \quad j = \mathrm{sgn}(Q)\,(|Q| - 1/2)\,, \tag{7.2}$$

for the massive representations, and

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau, z) = \mathrm{e}(\tfrac{\ell+Q+1/2}{2})(\Psi_{1,-\frac{1}{2}}(\tau, z))^{-1}\, y^{Q+\frac{1}{2}}\, f^{(\ell)}_u(\tau, z + u) \quad , \quad u = \tfrac{1}{2} + \tfrac{(1+2Q)\tau}{4\ell}\,, \tag{7.3}$$

for the massless representations (with $Q \neq \frac{\hat{c}}{2}$). The character $\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau, z)$ for $Q = \frac{\hat{c}}{2}$ is given in (7.5). In the above formula, we have used the Appell–Lerch sum (6.15) and defined

$$\Psi_{1,-\frac{1}{2}} = -i\,\frac{\eta(\tau)^3}{\theta_1(\tau, z)} = \frac{1}{y^{1/2} - y^{-1/2}} + q\,(y^{1/2} - y^{-1/2}) + O(q^2).$$

Note that the above characters transform according to the rule

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau, z) = \mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,-Q}(\tau, -z)$$

under charge conjugation.

From the relation between the massless and massive characters

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}+\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,-Q}=q^{-n}\sum_{k=0}^{|Q|-1}(-1)^k\big(\mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,Q-k}+\mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,k-Q}\big)$$
$$+2(-1)^Q\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,0}\,,\quad n>0,|Q|\leq\ell\,, \tag{7.4}$$

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,\frac{\hat{c}}{2}}=q^{-n}\Big(\mathrm{ch}^{\mathcal{N}=2}_{\ell;n+c/24,\frac{\hat{c}}{2}}+\sum_{k=1}^{\frac{\hat{c}}{2}-1}(-1)^k\big(\mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,\frac{\hat{c}}{2}-k}+\mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,k-\frac{\hat{c}}{2}}\big)\Big)$$
$$+(-1)^{\frac{\hat{c}}{2}}\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,0}\,, \tag{7.5}$$

as well as the charge conjugation symmetry of the theory, we expect the $U(1)$-graded Ramond sector partition function of a theory that is invariant under charge conjugation to admit a decomposition

$$\mathcal{Z}^{\mathcal{N}=2,\ell}=C_0'\,\mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24},0}(\tau,z)+\sum_{n\geq0}C_\ell'(n-\tfrac{\ell}{4})\,\mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24}+n,\ell+\frac{1}{2}}(\tau,z)$$
$$+\sum_{n\geq0,j\in\{\frac{1}{2},\frac{3}{2},\ldots,\ell-1\}}C_j'(n-\tfrac{j^2}{4\ell})\left(\mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24}+n,j+\frac{1}{2}}(\tau,z)+\mathrm{ch}^{\mathcal{N}=2}_{\ell;\frac{c}{24}+n,-(j+\frac{1}{2})}(\tau,z)\right) \tag{7.6}$$
$$=\mathrm{e}(\tfrac{\ell}{2})(\Psi_{1,-\frac{1}{2}})^{-1}\left(C_0\,\tilde{\mu}_{\ell;0}(\tau,z)+\sum_{j-\ell\in\mathbb{Z}/2\ell\mathbb{Z}}\tilde{F}^{(\ell)}_j(\tau)\theta_{\ell,j}(\tau,z)\right) \tag{7.7}$$

when the $\mathcal{N}=2$ SCA has even central charge, $c=3(2\ell+1)$. In the last equation, we have defined

$$\tilde{\mu}_{\ell;0}=\mathrm{e}(\tfrac{1}{4})\,y^{1/2}f^{(\ell)}_u(\tau,u+z)\quad,\quad u=\frac{1}{2}+\frac{\tau}{4\ell}\,,$$

and

$$\tilde{F}^{(\ell)}_j(\tau)=\tilde{F}^{(\ell)}_{-j}(\tau)=\tilde{F}^{(\ell)}_{j+2\ell}(\tau)=\sum_{n\geq0}C_j(n-\tfrac{j^2}{4\ell})\,q^{n-\frac{j^2}{4\ell}}\,, \tag{7.8}$$

$$C_0=C_0'+2\sum_{j\in\{\frac{1}{2},\frac{3}{2},\ldots,\ell\}}(-1)^{j+\frac{1}{2}}C_j'(-\tfrac{j^2}{4\ell})\,, \tag{7.9}$$

$$C_j(n-\tfrac{j^2}{4\ell})=\begin{cases}\sum_{k=0}^{\ell-j}(-1)^kC'_{j+k}(-\frac{(j+k)^2}{4\ell}) & n=0\\ C_j'(n-\frac{j^2}{4\ell}) & n>0\end{cases}\,. \tag{7.10}$$

Similar to the case of massless $\mathcal{N}=4$ characters, through its relation to the Appell–Lerch sum, $\tilde{\mu}_{\ell;0}$ admits a completion which transforms as a weight one, half-integral index Jacobi form under the Jacobi group. More precisely, define $\widehat{\tilde{\mu}_{\ell;0}}$ by replacing $\mu_{m;0}$ with $\tilde{\mu}_{\ell;0}$ and the integer m with the half-integral ℓ in (6.16). Then $\widehat{\tilde{\mu}_{\ell;0}}$ transforms like a Jacobi form of weight 1 and index ℓ under the group $SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$. Following the same computation as in the previous

section, we hence conclude that $\tilde{F}^{(\ell)} = (\tilde{F}_j^{(\ell)})$, where $j - 1/2 \in \mathbb{Z}/2\ell\mathbb{Z}$, is a vector-valued mock modular form with a vector-valued shadow $C_0\, S_\ell = C_0(S_{\ell,j}(\tau))$.

Now we are ready to apply the above discussion to the $U(1)$-graded R sector partition function of the orbifold theory of §2. The $\mathcal{N}=2$ decomposition gives

$$\begin{aligned} Z(\tau,z) &= 23\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},0} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},2} + \Big(770\,(\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},1} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},-1}) + 13915\,(\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},1} + \mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},-1}) + \dots\Big) \\ &\quad + \Big(231\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},2} + 5796\,\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},2} + \dots\Big) \qquad (7.11) \\ &= \mathrm{e}(\tfrac{3}{4})\Psi^{-1}_{1,-\frac{1}{2}}\Big(24\,\tilde{\mu}_{\frac{3}{2};0} + (-q^{-\frac{1}{24}} + 770\,q^{\frac{23}{24}} + 13915\,q^{\frac{47}{24}} + \dots)\,(\theta_{\frac{3}{2},\frac{1}{2}} + \theta_{\frac{3}{2},-\frac{1}{2}}) \\ &\quad + (q^{-\frac{3}{8}} + 231\,q^{\frac{5}{8}} + 5796 q^{\frac{13}{8}} + \dots)\,\theta_{\frac{3}{2},\frac{3}{2}}\Big) \qquad (7.12) \end{aligned}$$

where $\dots$ denote the terms with expansion $\Psi^{-1}_{1,-\frac{1}{2}} q^\alpha y^\beta$ with $\alpha - \beta^2/6 > 2$. Again, we observe that all the multiplicities of the representations with characters $\mathrm{ch}^{\mathcal{N}=2}_{\frac{3}{2};h,Q}$ appear to be non-negative, consistent with our construction of V as an $\mathcal{N}=2$ SCA module.

In general, from the previous sections we have seen that the graded partition function twined by any element g of a subgroup G of Co_0 should admit a decomposition into $\mathcal{N}=2$ characters. We write

$$Z_g(\tau,z) = \mathrm{e}(\tfrac{3}{4})\Psi^{-1}_{1,-\frac{1}{2}}\left((\mathrm{Tr}_{\mathbf{24}} g)\,\tilde{\mu}_{\frac{3}{2};0}(\tau,z) + \sum_{j\in\{1/2,-1/2,3/2\}} \tilde{h}_{g,j}(\tau)\theta_{\frac{3}{2},j}\right). \qquad (7.13)$$

Moreover, from the discussion in §5 we have seen that

$$\tilde{h}_{g,1/2}(\tau) = \overline{\tilde{h}_{g,-1/2}(-\bar{\tau})}, \qquad (7.14)$$

and the coefficients of these functions

$$\tilde{h}_{g,j}(\tau) = a_j q^{-j^2/6} + \sum_{n=1}^{\infty} (\mathrm{Tr}_{\tilde{V}^G_{r,n}} g)\; q^{n-j^2/6} \qquad (7.15)$$

are given by characters of a G-module

$$\tilde{V}^G = \bigoplus_{j=-1/2,1/2,3/2} \bigoplus_{n=1}^{\infty} \tilde{V}^G_{j,n} \qquad (7.16)$$

which descends from the orbifold theory in §2. In particular, for any n, the G-module $\tilde{V}^G_{-1/2,n}$ is the dual of $\tilde{V}^G_{1/2,n}$. From the above discussion, we conclude that $\tilde{h}_g = (\tilde{h}_{g,j})$ is a vector-valued mock modular form for $\Gamma_0(o_g)$ with shadow $(\mathrm{Tr}_{\mathbf{24}} g) S_{3/2}$. Recall that $(\mathrm{Tr}_{\mathbf{24}} g) \neq 0$ for all $g \in M_{22}$ which are the cases of our main interest. For these case the multiplier of $\tilde{h}_g$ is given by the inverse of the multiplier system of $S_{3/2}$, restricted to $\Gamma_0(o_g)$.

To summarize, we have analyzed the decomposition of the Ramond sector of our orbifold theory into irreducible modules for the $\mathcal{N}=2$ SCA in this section, and we have demonstrated that the generating functions of irreducible $\mathcal{N}=2$ SCA module multiplicities also furnish a vector-valued mock modular form. We have demonstrated that these multiplicities are dimensions of modules for subgroups $G < Co_0$ that point-wise fix a 2-plane in **24**, and we have observed that the resulting, g-twined multiplicity generating functions, for $g \in G$, are vector-valued mock modular forms with a certain extra symmetry, relating the components labelled by $\pm 1/2$ by complex conjugation.

8 Mathieu Moonshine

In the previous sections we have seen that the orbifold theory described in §2 leads to infinite-dimensional G-modules underlying a set of vector-valued mock modular forms from its $\mathcal{N}=4$ ($\mathcal{N}=2$) structures for any subgroup G of Co_0 fixing at least a 3-plane (2-plane) in **24**. In this section we will discuss a natural property of the vector-valued mock modular forms that distinguishes subgroups of M_{24} from other 3-plane (2-plane) fixing subgroups. These considerations lead to mock modular Mathieu moonshine involving distinguished vector-valued mock modular forms of weight 1/2.

Recall the celebrated genus zero condition of monstrous moonshine, which states that the monstrous McKay–Thompson series are Hauptmoduls with only a polar term q^{-1} at the cusp represented by $i\infty$, and no poles at any other cusps. To be more precise, denote by $T_g(\tau) = \sum_{n\geq -1} q^n \operatorname{tr}_{V_n^\natural} g$ the graded character of the moonshine module $V^\natural = \bigoplus_{n\geq -1} V_n^\natural$ of Frenkel–Lepowsky–Meurman [3]. Then $T_g(\tau)$ is a function invariant under the action of a particular $\Gamma_g < SL_2(\mathbb{R})$ (specified in [1]), such that

$$\begin{aligned} &\text{(i)} \quad qT_g(\tau) = O(1) \text{ as } \tau \to i\infty, \text{ and} \\ &\text{(ii)} \quad T_g(\gamma\tau) = O(1) \text{ as } \tau \to i\infty \text{ whenever } \gamma \in SL_2(\mathbb{Z}) \text{ and } \gamma\infty \notin \Gamma_g\infty. \end{aligned} \tag{8.1}$$

Similarly, in Mathieu moonshine [11], it follows from the results of [20] that if $g \in M_{24}$ and Z_g denotes the g-twined K3 elliptic genus then

$$\begin{aligned} &\text{(i)} \quad Z_g(\tau, z) = c_g + y + y^{-1} \text{ as } \tau \to i\infty, \text{ and} \\ &\text{(ii)} \quad Z_g|_{0,1}\gamma(\tau, z) = c_{g,\gamma} \text{ as } \tau \to i\infty \text{ whenever } \gamma \in SL_2(\mathbb{Z}) \text{ and } \gamma\infty \notin \Gamma_g\infty, \end{aligned} \tag{8.2}$$

(cf. (5.4)) for some $c_g, c_{g,\gamma} \in \mathbb{C}$. In other words, the $\tau \to i\infty$ limit of $Z_g|_{0,1}\gamma$ is a z-independent constant whenever γ is not in the invariance group Γ_g. (Note that Γ_g is always a subgroup of $SL_2(\mathbb{Z})$ for $g \in M_{24}$).

A natural question is therefore: among the subgroups of Co_0 fixing 2- or 3-planes for which

we have constructed a module in this work, for which of these do the associated modular objects satisfy a condition analogous to the preceding cases of moonshine described above? We will see presently that the Mathieu groups M_{23}, M_{22}, M_{12} and M_{11} are distinguished in our setting, in that the graded characters of their respective modules yield weight zero weak Jacobi forms satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad Z_g(\tau,z) = c_g + y^2 + y^{-2} \text{ as } \tau \to i\infty, \text{ and} \\ &\text{(ii)} \quad Z_g|_{0,2}\gamma(\tau,z) = c_{g,\gamma} \text{ as } \tau \to i\infty \text{ whenever } \gamma \in SL_2(\mathbb{Z}) \text{ and } \gamma\infty \notin \Gamma_g\infty, \end{aligned} \tag{8.3}$$

for some $c_g, c_{g,\gamma} \in \mathbb{C}$. On the other hand, all the other groups mentioned in §4 contain elements g for which the condition (ii) in (8.3) is not satisfied. Thus the conditions (8.3) single out the Mathieu groups as the sporadic simple subgroups of Co_0 with this moonshine property. The constructions we have presented in this paper provide concrete realizations of the underlying mock modular Mathieu moonshine modules.

Note that the conditions (8.3) impose restrictions on the degrees of the poles (if any) of the mock modular forms h_g, $\tilde{h}_g$ (cf. §§6,7) at all cusps. In fact, for the case that G is a copy of M_{22} or M_{11} preserving $\mathcal{N}=4$ supersymmetry, the corresponding mock modular forms h_g only have poles at the infinite cusp. This property also holds for the mock modular forms attached to M_{24} via $\mathcal{N}=4$ decomposition of the twined K3 elliptic genera of Mathieu moonshine, satisfying (8.2), as was demonstrated in [20]. In more physical terms, (8.2) and (8.3) can be interpreted as the condition that the elliptic genus of any cyclic orbifold of the theory receives no contributions from $U(1)$-charged ground states in twisted sectors.

To investigate the behaviour of Z_g at cusps other than $i\infty$, we first note that for any positve integer N,

$$Z_g|_{0,2}\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)(\tau,z) = \mathrm{e}\left(N\sum_{\ell=1}^{12}\frac{\rho_\ell^2}{2}\right)\sum_{i=1}^{4}\epsilon_{i,N}(-1)^{i+1}\Theta_{i,N}(\tau,z),$$

where

$$\Theta_{i,N}(\tau,z) = \frac{\theta_i(\tau,2z)}{\eta^{12}(\tau)}\prod_{k=2}^{12}\mathrm{e}\left(\frac{\rho_k^2N^2}{2}\tau\right)\,\theta_i(\tau,\rho_k+N\rho_k\tau), \tag{8.4}$$

and $\epsilon_{i,N} = \epsilon_i$ when $2|N$, and

$$\begin{cases} \epsilon_{1,N} &= -\epsilon_1 \\ \epsilon_{2,N} &= -\epsilon_3 = -1 \\ \epsilon_{3,N} &= -\epsilon_2 \\ \epsilon_{4,N} &= -\epsilon_4 = -1 \end{cases} \tag{8.5}$$

otherwise. The above expressions can be derived from the transformation laws of Jacobi theta

functions under $SL_2(\mathbb{Z})$.

Near the infinite cusp, $\tau \to i\infty$, the different contributions have the following leading behaviour:

$$
\begin{aligned}
\Theta_{1,N}(\tau,z) &= \mathrm{e}\left(\tfrac{1}{2}+\sum_{k=1}^{12}(\tfrac{1}{2}-\rho_k)\lfloor N\rho_k\rfloor-\tfrac{\rho_k}{2}\right) q^{f_{N,1}(\Pi_g)/2}(y-y^{-1})\,[1+O(q^{1/o_g})]\\
\Theta_{2,N}(\tau,z) &= \mathrm{e}\left(\sum_{k=1}^{12}-\rho_k\lfloor N\rho_k\rfloor-\tfrac{\rho_k}{2}\right) q^{f_{N,1}(\Pi_g)/2}(y+y^{-1})\,[1+O(q^{1/o_g})]\\
\Theta_{3,N}(\tau,z) &= \mathrm{e}\left(-\sum_{k=1}^{12}\rho_k\lfloor\tfrac{1}{2}+N\rho_k\rfloor\right) q^{f_{N,2}(\Pi_g)/2}\Big(1+q^{1/2}(y^2+y^{-2})\Big)\\
&\quad\times\left(\prod_{k=2}^{12}\left(1+\mathrm{e}(-\rho_k)\,q^{\lfloor\frac{1}{2}+N\rho_k\rfloor+\frac{1}{2}-N\rho_k}\right)\prod_{n=1}^{\lfloor\frac{1}{2}+N\rho_k\rfloor}\left(1+\mathrm{e}(\rho_k)q^{\frac{1}{2}+N\rho_k-n}\right)\right)\times[1+O(q^{1/o_g})]\\
\Theta_{4,N}(\tau,z) &= \mathrm{e}\left(-\sum_{k=1}^{12}(\tfrac{1}{2}+\rho_k)\lfloor\tfrac{1}{2}+N\rho_k\rfloor\right) q^{f_{N,2}(\Pi_g)/2}\Big(1-q^{1/2}(y^2+y^{-2})\Big)\\
&\quad\times\left(\prod_{k=2}^{12}\left(1-\mathrm{e}(-\rho_k)\,q^{\lfloor\frac{1}{2}+N\rho_k\rfloor+\frac{1}{2}-N\rho_k}\right)\prod_{n=1}^{\lfloor\frac{1}{2}+N\rho_k\rfloor}\left(1-\mathrm{e}(\rho_k)\,q^{\frac{1}{2}+N\rho_k-n}\right)\right)\times[1+O(q^{1/o_g})]
\end{aligned}
\tag{8.6}
$$

with

$$f_{N,1}(\Pi_g) = -1+\sum_{k=1}^{12}(N\rho_k-1/2)^2+\lfloor N\rho_k\rfloor(1+\lfloor N\rho_k\rfloor-2N\rho_k)\,, \tag{8.7}$$

$$f_{N,2}(\Pi_g) = -1+\sum_{k=1}^{12}N^2\rho_k^2-\lfloor\tfrac{1}{2}+N\rho_k\rfloor(2N\rho_k-\lfloor\tfrac{1}{2}+N\rho_k\rfloor)\,, \tag{8.8}$$

where $\lfloor x\rfloor$ denotes the largest integer that is not greater than x.

Comparing with the second condition in (8.3), we see that it is satisfied for $\gamma=\left(\begin{smallmatrix}1&0\\N&1\end{smallmatrix}\right)$ if and only if the Π_g satisfies

$$f_{N,1}(\Pi_g) > 0, \tag{8.9}$$

together with

$$f_{N,2}(\Pi_g) > -1, \quad \frac{f_{N,2}(\Pi_g)}{2}+\left|\frac{1}{2}+\lfloor N\rho_k\rfloor-N\rho_k\right| \geq 0 \tag{8.10}$$

for the case $\epsilon_{3,N}\epsilon_{4,N}\,\mathrm{e}(\frac{1}{2}\sum_{k=1}^{12}\lfloor\frac{1}{2}N\rho_k\rfloor)=1$, and

$$f_{N,2}(\Pi_g) \geq 0 \tag{8.11}$$

for the case $\epsilon_{3,N}\epsilon_{4,N}\,\mathrm{e}(\frac{1}{2}\sum_{k=1}^{12}\lfloor\frac{1}{2}N\rho_k\rfloor)=-1$.

Now we are left to check explicitly the condition (ii) in (8.3) for the various 2- and 3-plane preserving subgroups discussed in §4. First, recall that, for n a positive integer, each cusp of $\Gamma_0(n)$ is represented by a rational number of the form u/v where u and v are coprime positive integers, v a divisor of n, and u/v is equivalent u'/v' if and only if $v = v'$, and $u = u'$ mod $(v, n/v)$. (note that the infinite cusp is also represented by $1/n$.) Via direct computation using the data of the eigenvalues of $g \in Co_0$ acting on $\mathbf{24}$, we note that among all the groups we have considered in §4, the groups $U_4(3)$, $U_6(2)$ and McL all contain a conjugacy classe with Frame shape $\Pi_g = 3^9/1^3$, and HS has a conjugacy class with Frame shape $\Pi_g = 5^5/1^1$. One can explicitly check that $f_{1,1}(\Pi_g) = 0$ for these classes and hence the corresponding twining $Z_g|_{0,2}\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)(\tau, z)$ has a non-vanishing coefficient for $q^0 y^1$ as $\tau \to i\infty$. (Note that the cusp at $\tau = 1$ is not equivalent to the cusp at $i\infty$ in these cases.) This excludes the groups $U_4(3)$, $U_6(2)$, McL and HS as candidates for moonshine satisfying (8.3).

For the subgroups of M_{24}, a simple analysis of the cusp representatives of $\Gamma_g = \Gamma(o_g)$ shows that it is sufficient to verify (8.3) for $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$ for all $N|n$ and $N < n$. For these cases, we explicitly verify that (8.9)-(8.11) are satisfied and hence the moonshine condition (8.3) is met.

We therefore conclude that we have established mock modular moonshine for all but the largest of the sporadic simple Mathieu groups, together with explicit constructions of the corresponding modules. Moreoever, the corresponding twined graded characters are mock modular forms arising from Jacobi forms satisfying the distinguishing conditions (8.3), which we may recognise as furnishing a natural analogue of the powerful principal modulus property (a.k.a. genus zero property) of monstrous moonshine.

The reader will note that many of the numbers which occur as dimensions of irreducible representations of M_{23} also occur as dimensions of irreducible representations for M_{24}. Indeed, looking at the tables in C, one is tempted to guess that there is an alternative construction, or hidden symmetry in our model, which yields an M_{24}-module with the same graded dimensions. In fact, the procedure we have explained for computing twinings can be carried out for any element of M_{24}, regarded as a subgroup of Co_0, for any such element fixes a 2-space in $\mathbf{24}$. However, there is no 2-space that is fixed by every element of a given copy of M_{24}, and explicit computations reveal that any M_{24}-module structure on the module we have constructed would have to involve virtual representations. This indicates that there is no direct extension to M_{24} of the Mathieu moonshine modules we have considered here, despite the prominent role M_{24} plays in incorporating the different groups of moonshine in the current setting. Nevertheless, there is a certain modification of our method for which M_{24} is now known to play a leading role. We refer the reader to the next, and final section for a description of this.

9 Discussion

In this paper we have demonstrated that, starting with the free field Co_0 module of [37], one can construct explicit examples of modules for various subgroups $G \subset \mathrm{Co}_0$ which underlie certain mock modular forms. In particular, subgroups which preserve a 3-plane (respectively 2-plane) in the **24** give rise to $\mathcal{N} = 4$ ($\mathcal{N} = 2$) modules with G symmetry. This gives completely explicit examples of mock modular moonshine for the smaller Mathieu groups, where the modules are known, and where the twining functions are distinguished in a manner directly similar to Mathieu moonshine. Other examples, including modules for the sporadic groups *McL* and *HS*, are also described.

There are several future directions. We considered here the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ extended chiral algebras, and the subgroups of Co_0 that they preserve. Other extended chiral algebras may also yield interesting results. For instance, supersymmetric sigma models with target a Spin(7) manifold give rise to an extended chiral algebra [55], whose representations were studied in [56]. It is an extension of the $\mathcal{N} = 1$ superconformal algebra where instead of adding a $U(1)$ current (which extends the theory to an $\mathcal{N} = 2$ superconformal theory), one chooses an additional Ising factor. Conjectural characters for the unitary, irreducible representations of this algebra were worked out in [61], where it was shown that there is a suggestive relation between the decomposition of the elliptic genus of a Spin(7) manifold into these characters and irreducible representations of finite groups. This connection was made precise in [62], where it was shown that the same $c = 12$ theory can be viewed as an SCFT with extended $\mathcal{N} = 1$ symmetry, and thus yields theories with global symmetry groups M_{24}, Co_2, and Co_3. The partition function twined by these symmetries, when decomposed into characters of the Spin(7) algebra, gives rise to two-component vector-valued mock modular forms encoding infinite-dimensional modules for the corresponding sporadic groups.

The motivation that led, eventually, to the present study was actually to find connections between geometrical target manifolds associated to $c = 12$ conformal field theories, and sporadic groups. The elliptic genera of Calabi-Yau fourfolds were computed in [57], for instance; their structure is reminiscent of some of the modules we have seen here, and we intend to further explore and describe some of these connections in a future publication. Likewise, hyperkähler fourfolds, as well as the Spin(7) manifolds mentioned above, provide a wide class of geometries where an analogue of the connections between M_{24} and K3 may be sought.

Last but not least, there are suggestive connections between the trace functions in moonshine modules, and certain special properties of the underlying conformal field theory. Both the CFT appearing in monstrous moonshine and the Co_0 module that played a starring role in this paper appear to play special roles also in AdS_3 quantum gravity, where they are candidates for CFT duals to pure (super)gravity [58]. The genus zero property of the twining functions in monstrous

moonshine can be reformulated as a condition that these class functions should be expressed as Rademacher sums based on a fixed polar part [20, 43]; this latter description then applies uniformly to monstrous moonshine and Mathieu moonshine.

In this paper we demonstrate that a similar criterion also applies to our mock modular Mathieu moonshines. In particular, we have shown that the Jacobi forms relevant for Mathieu moonshine display a specific asymptotic behaviour near non-infinite cusps, which, in physical terms, can be interpreted as a condition on orbifolds of the theory. Pursuing a deeper understanding of this property constitutes an enticing direction for the future.

Acknowledgements

We are grateful to Jeff Harvey, Sander Mack-Crane and Daniel Whalen for conversations about related subjects. We are grateful to Daniel Baumann, and the anonymous referees, for very helpful comments on earlier drafts. We note that the appearance of dimensions of M_{22} representations in an $\mathcal{N} = 4$ decomposition of the partition function of the model studied in this paper was described in correspondence from J. Harvey to J. Duncan in 2010. The expression (5.2) for $Z(\tau, z)$ first came to our attention during the course of conversations with S. Mack-Crane. We thank the Simons Center for Geometry and Physics, and in particular the organizers of the workshop on "Mock Modular Forms, Moonshine, and String Theory," for hospitality when this work was initiated. S.K. is grateful to the Aspen Center for Physics for providing the rockies during the completion of this work. X.D., S.H. and S.K. are supported by the U.S. National Science Foundation grant PHY-0756174, the Department of Energy under contract DE-AC02-76SF00515, and the John Templeton Foundation. J.D. is supported in part by the Simons Foundation (#316779) and by the U.S. National Science Foundation (DMS 1203162). T.W. is supported by a Research Fellowship (Grant number WR 166/1-1) of the German Research Foundation (DFG).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Each author contributing equally to all aspects of the production of this article.

A Jacobi Theta Functions

We define the *Jacobi theta functions* $\theta_i(\tau, z)$ as follows for $q = e(\tau)$ and $y = e(z)$:

$$\theta_1(\tau, z) = -iq^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1})\,, \tag{A.1}$$

$$\theta_2(\tau, z) = q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^{n-1})\,, \tag{A.2}$$

$$\theta_3(\tau, z) = \prod_{n=1}^{\infty}(1-q^n)(1+y\,q^{n-1/2})(1+y^{-1}q^{n-1/2})\,, \tag{A.3}$$

$$\theta_4(\tau, z) = \prod_{n=1}^{\infty}(1-q^n)(1-y\,q^{n-1/2})(1-y^{-1}q^{n-1/2})\,. \tag{A.4}$$

They transform in the following way under the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

$$\begin{aligned}
\theta_1(\tau, z) &= i\,\alpha^{-1}(\tau, z)\theta_1(-\frac{1}{\tau}, \frac{z}{\tau}) = e(-1/8)\,\theta_1(\tau+1, z)\\
&= (-1)^{\lambda+\mu}e(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z))\theta_1(\tau, z+\lambda\tau+\mu)\,,\\
\theta_2(\tau, z) &= \alpha^{-1}(\tau, z)\theta_4(-\frac{1}{\tau}, \frac{z}{\tau}) = e(-1/8)\,\theta_2(\tau+1, z)\\
&= (-1)^{\mu}e(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z))\theta_2(\tau, z+\lambda\tau+\mu)\,,\\
\theta_3(\tau, z) &= \alpha^{-1}(\tau, z)\theta_3(-\frac{1}{\tau}, \frac{z}{\tau}) = \theta_4(\tau+1, z)\\
&= e(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z))\theta_3(\tau, z+\lambda\tau+\mu)\,,\\
\theta_4(\tau, z) &= \alpha^{-1}(\tau, z)\theta_2(-\frac{1}{\tau}, \frac{z}{\tau}) = \theta_3(\tau+1, z)\\
&= (-1)^{\lambda}e(\tfrac{1}{2}(\lambda^2\tau + 2\lambda z))\theta_4(\tau, z+\lambda\tau+\mu)\,.
\end{aligned} \tag{A.5}$$

Here $\alpha(\tau, z) = \sqrt{-i\tau}\,\mathrm{e}(\frac{z^2}{2\tau})$, and $\lambda, \mu \in \mathbb{Z}$.

The weight four Eisenstein series E_4 can be written in terms of the Jacobi theta functions as

$$E_4(\tau) = \frac{1}{2}\left(\theta_2(\tau, 0)^8 + \theta_3(\tau, 0)^8 + \theta_4(\tau, 0)^8\right)\,. \tag{A.6}$$

B Character Tables

B1. Frame Shapes and Spinor Representations

Table 1: Frame Shapes and Spinor Characters for M_{22}.

$[g]$	1A	2A	3A	4A	4B	5A	6A	7A	7B	8A	11A	11B
Π_g	1^{24}	$1^8 2^8$	$1^6 3^6$	$1^4 2^4 4^4$	$1^4 2^4 4^4$	$1^4 5^4$	$1^2 2^2 3^3 6^2$	$1^3 7^3$	$1^3 7^3$	$1^2 2.4.8^2$	$1^2 11^2$	$1^2 11^2$
$\mathrm{Tr}_{\mathbf{4096}} g$	2048	0	64	0	0	0	0	8	8	0	4	4

Table 2: Frame Shapes and Spinor Characters for $U_4(3)$.

$[g]$	1A	2A	3A	3BCD	4AB	5A	6A	6BC	7AB	8A	9ABCD	12A
Π_g	1^{24}	$1^8 2^8$	$\frac{3^9}{1^3}$	$1^6 3^6$	$1^4 2^2 4^4$	$1^4 5^4$	$\frac{1^5 3.6^4}{2^4}$	$1^2 2^2 3^3 6^2$	$1^3 7^3$	$1^2 2.4.8^2$	$\frac{1^3 9^3}{3^2}$	$\frac{1.2^2 3.12^2}{4^2}$
R_g	2048	0	-8	64	0	0	72	0	8	0	4	0

Table 3: Frame Shapes and Spinor Characters for M_{23}.

$[g]$	1A	2A	3A	4A	5A	6A	7AB	8A	11AB	14AB	15AB	23AB
Π_g	1^{24}	$1^8 2^8$	$1^6 3^6$	$1^4 2^2 4^4$	$1^4 5^4$	$1^2 2^2 3^3 6^2$	$1^3 7^3$	$1^2 2.4.8^2$	$1^2 11^2$	1.2.7.14	1.3.5.15	1.23
$\mathrm{Tr}_{\mathbf{4096}} g$	2048	0	64	0	0	0	8	0	4	0	4	2

Table 4: Frame Shapes and Spinor Characters for McL.

$[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7AB
Π_g	1^{24}	$1^8 2^8$	$\frac{3^9}{1^3}$	$1^6 3^6$	$1^4 2^2 4^4$	$\frac{5^5}{1}$	$1^4 5^4$	$\frac{1^5 3.6^4}{2^4}$	$1^2 2^2 3^3 6^2$	$1^3 7^3$
$\mathrm{Tr}_{\mathbf{4096}} g$	2048	0	-8	64	0	-4	0	72	0	8

$[g]$	8A	9AB	10A	11AB	12A	14AB	15AB	30AB
Π_g	$1^2 2.4.8^2$	$\frac{1^3 9^3}{3^2}$	$\frac{1^3 5.10^2}{2^2}$	$1^2 11^2$	$\frac{1.2^2 3.12^2}{4^2}$	1.2.7.14	$\frac{1^2 15^2}{3.5}$	$\frac{2.3.5.30}{6.10}$
$\mathrm{Tr}_{\mathbf{4096}} g$	0	4	20	4	0	0	2	2

Table 5: Frame Shapes and Spinor Characters for *HS*.

$[g]$	1A	2A	2B	3A	4A	4BC	5A	5BC	6A	6B
Π_g	1^{24}	$1^8 2^8$	2^{12}	$1^6 3^6$	$\frac{2^6 4^4}{1^4}$	$1^4 2^2 4^4$	$\frac{5^5}{1}$	$1^4 5^4$	$2^3 6^3$	$1^2 2^2 3^3 6^2$
$\mathrm{Tr}_{\mathbf{4096}} g$	2048	0	0	64	0	0	-4	0	0	0

$[g]$	7A	8ABC	10A	10B	11AB	12A	15A	20AB
Π_g	$1^3 7^3$	$1^2 2.4.8^2$	$\frac{1^3 5.10^2}{2^2}$	$2^2 10^2$	$1^2 11^2$	$\frac{1^2 4.6^2 12}{3^2}$	1.3.5.15	$\frac{1.2.10.20}{4.5}$
$\mathrm{Tr}_{\mathbf{4096}} g$	8	0	20	0	4	0	4	0

Table 6: Frame Shapes and Spinor Characters for $U_6(2)$.

$[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4CDE	4F	4G
Π_g	1^{24}	$\frac{2^{16}}{1^8}$	$1^8 2^8$	2^{12}	$1^6 3^6$	$\frac{3^9}{1^3}$	$1^6 3^6$	$\frac{1^8 4^8}{2^8}$	$\frac{4^8}{2^4}$	$1^4 2^2 4^4$	$2^4 4^4$	$1^4 2^2 4^4$
$\mathrm{Tr}_{\mathbf{4096}} g$	2048	0	0	0	64	-8	64	256	0	0	0	0

$[g]$	5A	6AB	6C	6D	6E	6F	6G	6H	7A	8A	8BCD
Π_g	$1^4 5^4$	$\frac{1.6^6}{2^2 3^3}$	$\frac{2^4 6^4}{1^2 3^2}$	$\frac{1^5 3.6^4}{2^4}$	$1^2 2^2 3^2 6^2$	$\frac{1^4 2.6^5}{3^4}$	$1^2 2^2 3^2 6^2$	$2^3 6^3$	$1^3 7^3$	$\frac{1^4 8^4}{2^2 4^2}$	$1^2 2.4.8^2$
$\mathrm{Tr}_{\mathbf{4096}} g$	0	0	0	72	0	0	0	0	8	32	0

$[g]$	9ABC	10A	11AB	12AB	12C	12DE	12FGH	12I	15A	18AB
Π_g	$\frac{1^3 9^3}{3^2}$	$\frac{1^2 2.10^3}{5^2}$	$1^2 11^2$	$\frac{2.3^3 12^3}{1.4.6^3}$	$\frac{1^2 3^2 4^2 12^2}{2^2 6^2}$	$\frac{1^3 12^3}{2.3.4.6}$	$\frac{1.2^2 3.12^2}{4^2}$	2.4.6.12	1.3.5.15	$\frac{1.2.18^2}{6.9}$
$\mathrm{Tr}_{\mathbf{4096}} g$	4	0	4	4	16	12	0	0	4	0

B2. Irreducible Representations

Table 7: Character table of M_{22}. $b_p = (-1 + i\sqrt{p})/2$.

$[g]$	1A	2A	3A	4A	4B	5A	6A	7A	7B	8A	11A	11B
$[g^2]$	1A	1A	3A	2A	2A	5A	3A	7A	7B	4A	11B	11A
$[g^3]$	1A	2A	1A	4A	4B	5A	2A	7B	7A	8A	11A	11B
$[g^5]$	1A	2A	3A	4A	4B	1A	6A	7B	7A	8A	11A	11B
$[g^7]$	1A	2A	3A	4A	4B	5A	6A	1A	1A	8A	11B	11A
$[g^{11}]$	1A	2A	3A	4A	4B	5A	6A	7A	7B	8A	1A	1A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	21	5	3	1	1	1	-1	0	0	-1	-1	-1
χ_3	45	-3	0	1	1	0	0	b_7	$\overline{b_7}$	-1	1	1
χ_4	45	-3	0	1	1	0	0	$\overline{b_7}$	b_7	-1	1	1
χ_5	55	7	1	3	-1	0	1	-1	-1	1	0	0
χ_6	99	3	0	3	-1	-1	0	1	1	-1	0	0
χ_7	154	10	1	-2	2	-1	1	0	0	0	0	0
χ_8	210	2	3	-2	-2	0	-1	0	0	0	1	1
χ_9	231	7	-3	-1	-1	1	1	0	0	-1	0	0
χ_{10}	280	-8	1	0	0	0	1	0	0	0	b_{11}	$\overline{b_{11}}$
χ_{11}	280	-8	1	0	0	0	1	0	0	0	$\overline{b_{11}}$	b_{11}
χ_{12}	385	1	-2	1	1	0	-2	0	0	1	0	0

Table 8: Character table of $U_4(3)$. $b_p = (-1 + i\sqrt{p})/2$.

$[g]$	1A	2A	3A	3B	3C	3D	4A	4B	5A	6A	6B	6C	7A	7B	8A	9A	9B	9C	9D	12A
$[g^2]$	1A	1A	3A	3B	3C	3D	2A	2A	5A	3A	3B	3C	7A	7B	4A	9B	9A	9D	9C	6A
$[g^3]$	1A	2A	1A	1A	1A	1A	4A	4B	5A	2A	2A	2A	7B	7A	8A	3A	3A	3A	3A	4A
$[g^5]$	1A	2A	3A	3B	3C	3D	4A	4B	1A	6A	6B	6C	7B	7A	8A	9B	9A	9D	9C	12A
$[g^7]$	1A	2A	3A	3B	3C	3D	4A	4B	5A	6A	6B	6C	1A	1A	8A	9A	9B	9C	9D	12A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	21	5	-6	3	3	3	1	1	1	2	-1	-1	0	0	-1	0	0	0	0	-2
χ_3	35	3	8	8	-1	-1	3	-1	0	0	0	3	0	0	-1	2	2	-1	-1	0
χ_4	35	3	8	-1	8	-1	3	-1	0	0	3	0	0	0	-1	-1	-1	2	2	0
χ_5	90	10	9	9	9	0	-2	2	0	1	1	1	-1	-1	0	0	0	0	0	1
χ_6	140	12	5	-4	-4	5	4	0	0	-3	0	0	0	0	0	-1	-1	-1	-1	1
χ_7	189	-3	27	0	0	0	5	1	-1	3	0	0	0	0	1	0	0	0	0	-1
χ_8	210	2	21	3	3	3	-2	-2	0	5	-1	-1	0	0	0	0	0	0	0	1
χ_9	280	-8	10	10	1	1	0	0	0	-2	-2	1	0	0	0	$1+3b_3$	$1+3\overline{b_3}$	1	1	0
χ_{10}	280	-8	10	10	1	1	0	0	0	-2	-2	1	0	0	0	$1+3\overline{b_3}$	$1+3b_3$	1	1	0
χ_{11}	280	-8	10	1	10	1	0	0	0	-2	1	-2	0	0	0	1	1	$1+3b_3$	$1+3\overline{b_3}$	0
χ_{12}	280	-8	10	1	10	1	0	0	0	-2	1	-2	0	0	0	1	1	$1+3\overline{b_3}$	$1+3b_3$	0
χ_{13}	315	11	-9	18	-9	0	-1	-1	0	-1	2	-1	0	0	1	0	0	0	0	-1
χ_{14}	315	11	-9	-9	18	0	-1	-1	0	-1	-1	2	0	0	1	0	0	0	0	-1
χ_{15}	420	4	-39	6	6	-3	4	0	0	1	-2	-2	0	0	0	0	0	0	0	1
χ_{16}	560	-16	-34	2	2	2	0	0	0	2	2	2	0	0	0	-1	-1	-1	-1	0
χ_{17}	640	0	-8	-8	-8	1	0	0	0	0	0	0	b_7	$\overline{b_7}$	0	1	1	1	1	0
χ_{18}	640	0	-8	-8	-8	1	0	0	0	0	0	0	$\overline{b_7}$	b_7	0	1	1	1	1	0
χ_{19}	729	9	0	0	0	0	-3	1	-1	0	0	0	1	1	-1	0	0	0	0	0
χ_{20}	896	0	32	-4	-4	-4	0	0	1	0	0	0	0	0	0	-1	-1	-1	-1	0

Table 9: Character table of M_{23}. $b_p = (-1 + i\sqrt{p})/2$.

$[g]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	11A	11B	14A	14B	15A	15B	23A	23B
$[g^2]$	1A	1A	3A	2A	5A	3A	7A	7B	4A	11B	11A	7A	7B	15A	15B	23A	23B
$[g^3]$	1A	2A	1A	4A	5A	2A	7B	7A	8A	11A	11B	14B	14A	5A	5A	23A	23B
$[g^5]$	1A	2A	3A	4A	1A	6A	7B	7A	8A	11A	11B	14B	14A	3A	3A	23B	23A
$[g^7]$	1A	2A	3A	4A	5A	6A	1A	1A	8A	11B	11A	2A	2A	15B	15A	23B	23A
$[g^{11}]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	1A	1A	14A	14B	15B	15A	23B	23A
$[g^{23}]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	11A	11B	14A	14B	15A	15B	1A	1A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	4	2	2	0	1	1	0	0	0	-1	-1	-1	-1	-1	-1
χ_3	45	-3	0	1	0	0	b_7	$\overline{b_7}$	-1	1	1	$-b_7$	$-\overline{b_7}$	0	0	-1	-1
χ_4	45	-3	0	1	0	0	$\overline{b_7}$	b_7	-1	1	1	$-\overline{b_7}$	$-b_7$	0	0	-1	-1
χ_5	230	22	5	2	0	1	-1	-1	0	-1	-1	1	1	0	0	0	0
χ_6	231	7	6	-1	1	-2	0	0	-1	0	0	0	0	1	1	1	1
χ_7	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	b_{15}	$\overline{b_{15}}$	1	1
χ_8	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	$\overline{b_{15}}$	b_{15}	1	1
χ_9	253	13	1	1	-2	1	1	1	-1	0	0	-1	-1	1	1	0	0
χ_{10}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	896	0	-4	0	1	0	0	0	0	b_{11}	$\overline{b_{11}}$	0	0	1	1	-1	-1
χ_{13}	896	0	-4	0	1	0	0	0	0	$\overline{b_{11}}$	b_{11}	0	0	1	1	-1	-1
χ_{14}	990	-18	0	2	0	0	b_7	$\overline{b_7}$	0	0	0	b_7	$\overline{b_7}$	0	0	1	1
χ_{15}	990	-18	0	2	0	0	$\overline{b_7}$	b_7	0	0	0	$\overline{b_7}$	b_7	0	0	1	1
χ_{16}	1035	27	0	-1	0	0	-1	-1	1	1	1	-1	-1	0	0	0	0
χ_{17}	2024	8	-1	0	-1	-1	1	1	0	0	0	1	1	-1	-1	0	0

Table 10: Character table of *McL*. $b_p = (-1 + i\sqrt{p})/2$, $a = 1 + 3b_3$.

$[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7A	7B	8A	9A	9B	10A	11A	11B	12A	14A	14B	15A	15B	30A	30B
$[g^2]$	1A	1A	3A	3B	2A	5A	5B	3A	3B	7A	7B	4A	9B	9A	5A	11B	11A	6A	7A	7B	15A	15B	15A	15B
$[g^3]$	1A	2A	1A	1A	4A	5A	5B	2A	2A	7B	7A	8A	3A	3A	10A	11A	11B	4A	14B	14A	5A	5A	10A	10A
$[g^5]$	1A	2A	3A	3B	4A	1A	1A	6A	6B	7B	7A	8A	9B	9A	2A	11A	11B	12A	14B	14A	3A	3A	6A	6A
$[g^7]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	1A	1A	8A	9A	9B	10A	11B	11A	12A	2A	2A	15B	15A	30B	30A
$[g^{11}]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7A	7B	8A	9B	9A	10A	1A	1A	12A	14A	14B	15B	15A	30B	30A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	-5	4	2	-3	2	3	0	1	1	0	1	1	1	0	0	-1	-1	-1	0	0	-2	-2
χ_3	231	7	15	6	-1	6	1	7	-2	0	0	-1	0	0	2	0	0	-1	0	0	0	0	2	2
χ_4	252	28	9	9	4	2	2	1	1	0	0	0	0	0	-2	-1	-1	1	0	0	-1	-1	1	1
χ_5	770	-14	-13	5	-2	-5	0	7	1	0	0	0	-1	-1	1	0	0	1	0	0	b_{15}	$\overline{b_{15}}$	b_{15}	$\overline{b_{15}}$
χ_6	770	-14	-13	5	-2	-5	0	7	1	0	0	0	-1	-1	1	0	0	1	0	0	$\overline{b_{15}}$	b_{15}	$\overline{b_{15}}$	b_{15}
χ_7	896	0	32	-4	0	-4	1	0	0	0	0	0	-1	-1	0	b_{11}	$\overline{b_{11}}$	0	0	0	2	2	0	0
χ_8	896	0	32	-4	0	-4	1	0	0	0	0	0	-1	-1	0	$\overline{b_{11}}$	b_{11}	0	0	0	2	2	0	0
χ_9	1750	70	-5	13	2	0	0	-5	1	0	0	0	-2	-2	0	1	1	-1	0	0	0	0	0	0
χ_{10}	3520	64	-44	10	0	-5	0	4	-2	-1	-1	0	1	1	-1	0	0	0	1	1	1	1	-1	-1
χ_{11}	3520	-64	-44	10	0	-5	0	-4	2	-1	-1	0	1	1	1	0	0	0	-1	-1	1	1	1	1
χ_{12}	4500	20	45	-9	4	0	0	5	-1	-1	-1	0	0	0	0	1	1	1	-1	-1	0	0	0	0
χ_{13}	4752	-48	54	0	0	2	2	-6	0	-1	-1	0	0	0	2	0	0	0	1	1	-1	-1	-1	-1
χ_{14}	5103	63	0	0	3	3	-2	0	0	0	0	1	0	0	3	-1	-1	0	0	0	0	0	0	0
χ_{15}	5544	-56	36	9	0	19	-1	4	1	0	0	0	0	0	-1	0	0	0	0	0	1	1	-1	-1
χ_{16}	8019	-45	0	0	3	-6	-1	0	0	$-b_7$	$-\overline{b_7}$	-1	0	0	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0	0	0
χ_{17}	8019	-45	0	0	3	-6	-1	0	0	$-\overline{b_7}$	$-b_7$	-1	0	0	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0	0	0
χ_{18}	8250	10	15	6	-2	0	0	-5	-2	$-b_7$	$-\overline{b_7}$	0	0	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	0	0
χ_{19}	8250	10	15	6	-2	0	0	-5	-2	$-\overline{b_7}$	$-b_7$	0	0	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	0	0
χ_{20}	9625	105	40	-5	-3	0	0	0	3	0	0	-1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{21}	9856	0	-80	-8	0	6	1	0	0	0	0	0	a	$\overline{a}$	0	0	0	0	0	0	0	0	0	0
χ_{22}	9856	0	-80	-8	0	6	1	0	0	0	0	0	$\overline{a}$	a	0	0	0	0	0	0	0	0	0	0
χ_{23}	10395	-21	27	0	-1	-5	0	3	0	0	0	1	0	0	-1	0	0	-1	0	0	b_{15}	$\overline{b_{15}}$	$-b_{15}$	$-\overline{b_{15}}$
χ_{24}	10395	-21	27	0	-1	-5	0	3	0	0	0	1	0	0	-1	0	0	-1	0	0	$\overline{b_{15}}$	b_{15}	$-\overline{b_{15}}$	$-b_{15}$

Table 11: Character table of *HS*. $b_p = (-1 + i\sqrt{p})/2$, $a_p = i\sqrt{p}$.

$[g]$	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	7A	8A	8B	8C	10A	10B	11A	11B	12A	15A	20A	20B
$[g^2]$	1A	1A	1A	3A	2A	2A	2A	5A	5B	5C	3A	3A	7A	4B	4C	4C	5A	5B	11B	11A	6B	15A	10A	10A
$[g^3]$	1A	2A	2B	1A	4A	4B	4C	5A	5B	5C	2B	2A	7A	8A	8B	8C	10A	10B	11A	11B	4A	5B	20A	20B
$[g^5]$	1A	2A	2B	3A	4A	4B	4C	1A	1A	1A	6A	6B	7A	8A	8B	8C	2A	2B	11A	11B	12A	3A	4A	4A
$[g^7]$	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	1A	8A	8B	8C	10A	10B	11B	11A	12A	15A	20A	20B
$[g^{11}]$	1A	2A	2B	3A	4A	4B	4C	5A	5B	5C	6A	6B	7A	8A	8B	8C	10A	10B	1A	1A	12A	15A	20B	20A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	-2	4	-6	2	2	-3	2	2	-2	0	1	0	0	0	1	-2	0	0	0	-1	-1	-1
χ_3	77	13	1	5	5	5	1	2	-3	2	1	1	0	1	-1	-1	-2	1	0	0	-1	0	0	0
χ_4	154	10	10	1	-2	6	-2	4	4	-1	1	1	0	0	0	0	0	0	0	0	1	1	-2	-2
χ_5	154	10	-10	1	-10	-2	2	4	4	-1	-1	1	0	0	2	-2	0	0	0	0	-1	1	0	0
χ_6	154	10	-10	1	-10	-2	2	4	4	-1	-1	1	0	0	-2	2	0	0	0	0	-1	1	0	0
χ_7	175	15	11	4	15	-1	3	0	5	0	2	0	0	-1	1	1	0	1	-1	-1	0	-1	0	0
χ_8	231	7	-9	6	15	-1	-1	6	1	1	0	-2	0	-1	-1	-1	2	1	0	0	0	1	0	0
χ_9	693	21	9	0	21	5	1	-7	3	-2	0	0	0	1	-1	-1	1	-1	0	0	0	0	1	1
χ_{10}	770	34	-10	5	-14	2	-2	-5	0	0	-1	1	0	-2	0	0	-1	0	0	0	1	0	1	1
χ_{11}	770	-14	10	5	-10	-2	-2	-5	0	0	1	1	0	0	0	0	1	0	0	0	-1	0	a_5	$\overline{a_5}$
χ_{12}	770	-14	10	5	-10	-2	-2	-5	0	0	1	1	0	0	0	0	1	0	0	0	-1	0	$\overline{a_5}$	a_5
χ_{13}	825	25	9	6	-15	1	1	0	-5	0	0	-2	-1	1	1	1	0	-1	0	0	0	1	0	0
χ_{14}	896	0	16	-4	0	0	0	-4	1	1	-2	0	0	0	0	0	0	1	b_{11}	$\overline{b_{11}}$	0	1	0	0
χ_{15}	896	0	16	-4	0	0	0	-4	1	1	-2	0	0	0	0	0	0	1	$\overline{b_{11}}$	b_{11}	0	1	0	0
χ_{16}	1056	32	0	-6	0	0	0	6	-4	1	0	2	-1	0	0	0	2	0	0	0	0	-1	0	0
χ_{17}	1386	-6	18	0	6	-2	-2	11	6	1	0	0	0	0	0	0	-1	-2	0	0	0	0	1	1
χ_{18}	1408	0	16	4	0	0	0	8	-7	-2	-2	0	1	0	0	0	0	1	0	0	0	-1	0	0
χ_{19}	1750	-10	10	-5	-10	6	2	0	0	0	1	-1	0	-2	0	0	0	0	1	1	-1	0	0	0
χ_{20}	1925	5	-19	-1	5	5	-3	0	5	0	-1	-1	0	1	1	1	0	1	0	0	-1	-1	0	0
χ_{21}	1925	5	1	-1	-35	-3	1	0	5	0	1	-1	0	1	-1	-1	0	1	0	0	1	-1	0	0
χ_{22}	2520	24	0	0	24	-8	0	-5	0	0	0	0	0	0	0	0	-1	0	1	1	0	0	-1	-1
χ_{23}	2750	-50	-10	5	10	2	2	0	0	0	-1	1	-1	0	0	0	0	0	0	0	1	0	0	0
χ_{24}	3200	0	-16	-4	0	0	0	0	-5	0	2	0	1	0	0	0	0	-1	-1	-1	0	1	0	0

Table 12: Character table of $U_6(2)$ - Part I.

$[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	5A
$[g^2]$	1A	1A	1A	1A	3A	3B	3C	2A	2A	2B	2B	2B	2B	2B	5A
$[g^3]$	1A	2A	2B	2C	1A	1A	1A	4A	4B	4C	4D	4E	4F	4G	5A
$[g^5]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	1A
$[g^7]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	5A
$[g^{11}]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4C	4D	4E	4F	4G	5A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	-10	6	-2	4	-5	4	6	-2	2	2	2	-2	2	2
χ_3	231	39	7	-9	6	15	6	23	7	-1	-1	-1	-1	-1	1
χ_4	252	60	28	12	9	9	9	12	-4	4	4	4	4	4	2
χ_5	385	-95	17	-7	25	7	-2	1	9	5	5	5	-7	-3	0
χ_6	440	120	24	8	35	8	-1	24	8	0	0	0	8	0	0
χ_7	560	-80	-16	16	20	20	2	16	-16	0	0	0	0	0	0
χ_8	616	-24	40	8	-14	-5	13	8	8	8	8	8	0	0	1
χ_9	770	-30	-14	10	5	-13	5	34	-6	-2	-2	-2	2	-2	0
χ_{10}	770	-30	-14	10	5	-13	5	34	-6	-2	-2	-2	2	-2	0
χ_{11}	1155	195	35	19	30	-6	3	-13	3	11	-5	-5	3	3	0
χ_{12}	1155	195	35	19	30	-6	3	-13	3	-5	11	-5	3	3	0
χ_{13}	1155	195	35	19	30	-6	3	-13	3	-5	-5	11	3	3	0
χ_{14}	1386	-246	58	-30	36	9	9	-6	2	-2	-2	-2	-6	6	1
χ_{15}	1540	260	4	-28	55	1	1	-12	-12	4	4	4	4	-4	0
χ_{16}	3080	-440	40	-8	65	2	-7	40	-8	0	0	0	-8	0	0
χ_{17}	3080	-440	40	-8	65	2	-7	40	-8	0	0	0	-8	0	0
χ_{18}	3520	-320	64	0	10	-44	10	64	0	0	0	0	0	0	0
χ_{19}	4620	-180	44	-36	-15	57	12	28	-20	4	4	4	-4	-4	0
χ_{20}	4928	320	64	64	-4	68	5	0	0	0	0	0	0	0	-2
χ_{21}	5544	-24	-56	24	9	36	9	72	24	0	0	0	-8	0	-1
χ_{22}	6160	400	-48	-16	40	-50	4	48	16	0	0	0	0	0	0
χ_{23}	6160	400	-48	-16	40	-50	4	48	16	0	0	0	0	0	0
χ_{24}	6930	690	98	42	45	-36	-9	18	-6	6	6	6	10	-2	0
χ_{25}	8064	384	128	0	-36	-36	18	0	0	0	0	0	0	0	-1
χ_{26}	9240	-360	88	-8	-30	33	-3	-8	-8	24	-8	-8	0	0	0
χ_{27}	9240	-360	88	-8	-30	33	-3	-8	-8	-8	24	-8	0	0	0
χ_{28}	9240	-360	88	-8	-30	33	-3	-8	-8	-8	-8	24	0	0	0
χ_{29}	10395	315	-21	-45	0	27	0	75	-13	-1	-1	-1	3	-1	0
χ_{30}	10395	315	-21	-45	0	27	0	75	-13	-1	-1	-1	3	-1	0
χ_{31}	10395	315	-21	-45	0	27	0	-21	19	15	-1	-1	3	-1	0
χ_{32}	10395	315	-21	-45	0	27	0	-21	19	-1	15	-1	3	-1	0
χ_{33}	10395	315	-21	-45	0	27	0	-21	19	-1	-1	15	3	-1	0
χ_{34}	11264	1024	0	0	104	32	-4	0	0	0	0	0	0	0	-1
χ_{35}	13860	420	100	36	-45	9	-18	84	20	4	4	4	-4	4	0
χ_{36}	14784	-1344	64	0	114	-12	6	-64	0	0	0	0	0	0	-1
χ_{37}	18711	-1161	-57	63	81	0	0	-9	15	3	3	3	-1	-5	1
χ_{38}	18711	1431	87	-9	81	0	0	-9	-9	-9	-9	-9	-1	-1	1
χ_{39}	20790	-810	6	-18	0	54	0	6	14	-6	-6	-6	6	2	0
χ_{40}	20790	-810	6	-18	0	54	0	6	14	-6	-6	-6	6	2	0
χ_{41}	24640	-960	64	-64	-20	-92	-11	0	0	0	0	0	0	0	0
χ_{42}	25515	-405	-117	27	0	0	0	27	-21	3	3	3	3	3	0
χ_{43}	25515	-405	-117	27	0	0	0	27	-21	3	3	3	3	3	0
χ_{44}	32768	0	0	0	-64	-64	8	0	0	0	0	0	0	0	-2
χ_{45}	37422	270	30	54	-81	0	0	-18	6	-6	-6	-6	-2	-6	2
χ_{46}	40095	1215	-81	-9	0	0	0	-81	-9	3	3	3	-9	3	0

Table 13: Character table of $U_6(2)$ - Part II. $a_q = q + 6i\sqrt{3}$, $c_{-3} = (1 - 3i\sqrt{3})/2$.

$[g]$	6A	6B	6C	6D	6E	6F	6G	6H	7A	8A	8B	8C	8D	9A	9B	9C
$[g^2]$	3B	3B	3A	3B	3A	3C	3C	3C	7A	4B	4C	4D	4E	9B	9A	9C
$[g^3]$	2A	2A	2A	2B	2B	2A	2B	2C	7A	8A	8B	8C	8D	3B	3B	3B
$[g^5]$	6B	6A	6C	6D	6E	6F	6G	6H	7A	8A	8B	8C	8D	9B	9A	9C
$[g^7]$	6A	6B	6C	6D	6E	6F	6G	6H	1A	8A	8B	8C	8D	9A	9B	9C
$[g^{11}]$	6B	6A	6C	6D	6E	6F	6G	6H	7A	8A	8B	8C	8D	9B	9A	9C
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	-4	3	0	2	0	-2	1	2	0	0	0	1	1	1
χ_3	3	3	6	7	-2	0	-2	0	0	3	-1	-1	-1	0	0	0
χ_4	-3	-3	9	1	1	3	1	3	0	0	0	0	0	0	0	0
χ_5	-5	-5	1	-1	5	-2	2	2	0	1	-1	-1	-1	1	1	1
χ_6	12	12	3	0	3	3	3	-1	-1	0	0	0	0	2	2	-1
χ_7	-8	-8	4	-4	-4	-2	2	-2	0	0	0	0	0	-1	-1	2
χ_8	3	3	-6	-5	-2	3	1	-1	0	0	0	0	0	-2	-2	-2
χ_9	a_{-3}	$\overline{a}_{-3}$	-3	7	1	-3	1	1	0	2	0	0	0	-1	-1	-1
χ_{10}	$\overline{a}_{-3}$	a_{-3}	-3	7	1	-3	1	1	0	2	0	0	0	-1	-1	-1
χ_{11}	6	6	6	2	2	-3	-1	1	0	-1	3	-1	-1	0	0	0
χ_{12}	6	6	6	2	2	-3	-1	1	0	-1	-1	3	-1	0	0	0
χ_{13}	6	6	6	2	2	-3	-1	1	0	-1	-1	-1	3	0	0	0
χ_{14}	-3	-3	-12	1	4	-3	1	-3	0	-2	0	0	0	0	0	0
χ_{15}	17	17	-1	1	-5	-1	1	-1	0	0	0	0	0	1	1	1
χ_{16}	a_{-8}	$\overline{a}_{-8}$	1	-2	1	1	1	1	0	0	0	0	0	c_{-3}	$\bar{c}_{-3}$	-1
χ_{17}	$\overline{a}_{-8}$	a_{-8}	1	-2	1	1	1	1	0	0	0	0	0	$\bar{c}_{-3}$	c_{-3}	-1
χ_{18}	4	4	-14	4	-2	4	-2	0	-1	0	0	0	0	1	1	1
χ_{19}	9	9	9	-7	5	0	-4	0	0	0	0	0	0	0	0	0
χ_{20}	-4	-4	-4	4	4	5	1	1	0	0	0	0	0	-1	-1	2
χ_{21}	12	12	9	4	1	-3	1	-3	0	0	0	0	0	0	0	0
χ_{22}	a_4	$\overline{a}_4$	-8	-6	0	-2	0	2	0	0	0	0	0	$-c_{-3}$	$-\bar{c}_{-3}$	1
χ_{23}	$\overline{a}_4$	a_4	-8	-6	0	-2	0	2	0	0	0	0	0	$-\bar{c}_{-3}$	$-c_{-3}$	1
χ_{24}	-12	-12	-3	-4	5	-3	-1	-3	0	2	0	0	0	0	0	0
χ_{25}	-12	-12	12	-4	-4	0	2	0	0	0	0	0	0	0	0	0
χ_{26}	9	9	-6	1	-2	-3	1	1	0	0	0	0	0	0	0	0
χ_{27}	9	9	-6	1	-2	-3	1	1	0	0	0	0	0	0	0	0
χ_{28}	9	9	-6	1	-2	-3	1	1	0	0	0	0	0	0	0	0
χ_{29}	-9	-9	0	3	0	0	0	0	0	-1	1	1	1	0	0	0
χ_{30}	-9	-9	0	3	0	0	0	0	0	-1	1	1	1	0	0	0
χ_{31}	-9	-9	0	3	0	0	0	0	0	-1	-3	1	1	0	0	0
χ_{32}	-9	-9	0	3	0	0	0	0	0	-1	1	-3	1	0	0	0
χ_{33}	-9	-9	0	3	0	0	0	0	0	-1	1	1	-3	0	0	0
χ_{34}	16	16	-8	0	0	4	0	0	1	0	0	0	0	-1	-1	-1
χ_{35}	-3	-3	3	1	-5	0	-2	0	0	0	0	0	0	0	0	0
χ_{36}	-12	-12	-6	4	-2	0	-2	0	0	0	0	0	0	0	0	0
χ_{37}	0	0	9	0	-3	0	0	0	0	-1	1	1	1	0	0	0
χ_{38}	0	0	9	0	-3	0	0	0	0	-1	-1	-1	-1	0	0	0
χ_{39}	$18i\sqrt{3}$	$-18i\sqrt{3}$	0	-6	0	0	0	0	0	2	0	0	0	0	0	0
χ_{40}	$-18i\sqrt{3}$	$18i\sqrt{3}$	0	-6	0	0	0	0	0	2	0	0	0	0	0	0
χ_{41}	12	12	12	4	4	3	1	-1	0	0	0	0	0	-2	-2	1
χ_{42}	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	1	0	0	0	0	2	2	-1
χ_{45}	0	0	-9	0	3	0	0	0	0	-2	0	0	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	-1	3	1	1	1	0	0	0

Table 14: Character table of $U_6(2)$ - Part III. $a_q = q + 6i\sqrt{3}$, $b_p = (-1 + i\sqrt{p})/2$, $d^n_m = m + in\sqrt{3}$.

$[g]$	10A	11A	11B	12A	12B	12C	12D	12E	12F	12G	12H	12I	15A	18A	18B
$[g^2]$	5A	11B	11A	6B	6A	6C	6B	6A	6D	6D	6D	6E	15A	9B	9A
$[g^3]$	10A	11A	11B	4A	4A	4A	4B	4B	4C	4D	4E	4F	5A	6A	6B
$[g^5]$	2A	11A	11B	12B	12A	12C	12E	12D	12F	12G	12H	12I	3A	18B	18A
$[g^7]$	10A	11B	11A	12A	12B	12C	12D	12E	12F	12G	12H	12I	15A	18A	18B
$[g^{11}]$	10A	1A	1A	12B	12A	12C	12E	12D	12F	12G	12H	12I	15A	18B	18A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	0	0	0	-3	-3	0	1	1	-1	-1	-1	-2	-1	-1	-1
χ_3	-1	0	0	5	5	2	1	1	-1	-1	-1	2	1	0	0
χ_4	0	-1	-1	3	3	-3	-1	-1	1	1	1	1	-1	0	0
χ_5	0	0	0	1	1	1	-3	-3	-1	-1	-1	-1	0	1	1
χ_6	0	0	0	6	6	3	2	2	0	0	0	-1	0	0	0
χ_7	0	-1	-1	-2	-2	4	2	2	0	0	0	0	0	1	1
χ_8	1	0	0	-1	-1	2	-1	-1	-1	-1	-1	0	1	0	0
χ_9	0	0	0	d^{-3}_{-2}	$\bar{d}^{-3}_{-2}$	1	d^{-1}_0	d^1_0	1	1	1	-1	0	d^1_0	d^{-1}_0
χ_{10}	0	0	0	$\bar{d}^{-3}_{-2}$	d^{-3}_{-2}	1	d^1_0	d^{-1}_0	1	1	1	-1	0	d^{-1}_0	d^1_0
χ_{11}	0	0	0	-4	-4	2	0	0	2	-2	-2	0	0	0	0
χ_{12}	0	0	0	-4	-4	2	0	0	-2	2	-2	0	0	0	0
χ_{13}	0	0	0	-4	-4	2	0	0	-2	-2	2	0	0	0	0
χ_{14}	-1	0	0	3	3	0	-1	-1	1	1	1	0	1	0	0
χ_{15}	0	0	0	-3	-3	3	-3	-3	1	1	1	1	0	-1	-1
χ_{16}	0	0	0	d^{-3}_{-5}	$\bar{d}^{-3}_{-5}$	1	d^{-1}_1	$\bar{d}^{-1}_1$	0	0	0	1	0	$d^{-1/2}_{-1/2}$	$\bar{d}^{-1/2}_{-1/2}$
χ_{17}	0	0	0	$\bar{d}^{-3}_{-5}$	d^{-3}_{-5}	1	$\bar{d}^{-1}_1$	d^{-1}_1	0	0	0	1	0	$\bar{d}^{-1/2}_{-1/2}$	$d^{-1/2}_{-1/2}$
χ_{18}	0	0	0	-8	-8	-2	0	0	0	0	0	0	0	1	1
χ_{19}	0	0	0	1	1	1	1	1	1	1	1	-1	0	0	0
χ_{20}	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	-1
χ_{21}	1	0	0	0	0	-3	0	0	0	0	0	1	-1	0	0
χ_{22}	0	0	0	$-6b_3$	$-6\bar{b}_3$	0	d^{-1}_1	$\bar{d}^{-1}_1$	0	0	0	0	0	$d^{-1/2}_{-1/2}$	$\bar{d}^{-1/2}_{-1/2}$
χ_{23}	0	0	0	$-6\bar{b}_3$	$-6b_3$	0	$\bar{d}^{-1}_1$	d^{-1}_1	0	0	0	0	0	$\bar{d}^{-1/2}_{-1/2}$	$d^{-1/2}_{-1/2}$
χ_{24}	0	0	0	0	0	-3	0	0	0	0	0	1	0	0	0
χ_{25}	-1	1	1	0	0	0	0	0	0	0	0	0	-1	0	0
χ_{26}	0	0	0	1	1	-2	1	1	-3	1	1	0	0	0	0
χ_{27}	0	0	0	1	1	-2	1	1	1	-3	1	0	0	0	0
χ_{28}	0	0	0	1	1	-2	1	1	1	1	-3	0	0	0	0
χ_{29}	0	0	0	$-\bar{a}_{-3}$	$-a_{-3}$	0	d^{-2}_{-1}	$\bar{d}^{-2}_{-1}$	-1	-1	-1	0	0	0	0
χ_{30}	0	0	0	$-a_{-3}$	$-\bar{a}_{-3}$	0	$\bar{d}^{-2}_{-1}$	d^{-2}_{-1}	-1	-1	-1	0	0	0	0
χ_{31}	0	0	0	-3	-3	0	1	1	3	-1	-1	0	0	0	0
χ_{32}	0	0	0	-3	-3	0	1	1	-1	3	-1	0	0	0	0
χ_{33}	0	0	0	-3	-3	0	1	1	-1	-1	3	0	0	0	0
χ_{34}	-1	0	0	0	0	0	0	0	0	0	0	0	-1	1	1
χ_{35}	0	0	0	3	3	3	-1	-1	1	1	1	-1	0	0	0
χ_{36}	1	0	0	8	8	2	0	0	0	0	0	0	-1	0	0
χ_{37}	-1	0	0	0	0	-3	0	0	0	0	0	-1	1	0	0
χ_{38}	1	0	0	0	0	-3	0	0	0	0	0	-1	1	0	0
χ_{39}	0	0	0	$6b_3$	$6\bar{b}_3$	0	d^1_{-1}	$\bar{d}^1_{-1}$	0	0	0	0	0	0	0
χ_{40}	0	0	0	$6\bar{b}_3$	$6b_3$	0	$\bar{d}^1_{-1}$	d^1_{-1}	0	0	0	0	0	0	0
χ_{41}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{42}	0	$-b_{11}$	$-\bar{b}_{11}$	0	0	0	0	0	0	0	0	0	0	0	0
χ_{43}	0	$-\bar{b}_{11}$	$-b_{11}$	0	0	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	-1	-1	0	0	0	0	0	0	0	0	0	1	0	0
χ_{45}	0	0	0	0	0	3	0	0	0	0	0	1	-1	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

C Coefficient Tables

Table 15: The twined series for M_{22}. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,1}(\tau)$.

$-D$ \ $[g]$	1A	2A	3A	4AB	5A	6A	7AB	8A	11AB
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2
11	560	-16	2	0	0	2	0	0	-1
23	8470	54	-8	-2	0	0	0	2	0
35	70576	-144	16	0	-4	0	2	0	0
47	435820	332	4	4	0	-4	0	0	0
59	2187328	-704	-50	0	8	-2	-4	0	0
71	9493330	1394	58	-6	0	2	0	-2	0
83	36792560	-2640	38	0	0	6	0	0	2
95	130399766	4822	-172	6	-14	4	0	-2	2
107	429229920	-8480	174	0	0	-2	0	0	-2
119	1327987562	14506	104	-6	22	-8	6	2	0
131	3895785632	-24288	-502	0	-8	-6	4	0	2
143	10912966810	39770	466	10	0	2	-10	2	-2
155	29351354032	-63888	316	0	-28	12	0	0	0
167	76141761850	101018	-1232	-14	0	8	0	-2	0
179	191223891936	-157344	1098	0	56	-6	0	0	0
191	466389602756	241764	710	12	-14	-18	0	0	0
203	1107626293840	-366960	-2876	0	0	-12	10	0	0
215	2567229121428	550676	2472	-12	-62	8	6	4	4
227	5818567673360	-817840	1658	0	0	26	-20	0	2
239	12917927405852	1203068	-6148	20	112	20	0	0	-4
251	28135083779792	-1753840	5174	0	-48	-10	-4	0	0
263	60194978116000	2535488	3382	-24	0	-34	0	-4	2
275	126660856741328	-3637232	-12634	0	-112	-26	0	0	-3
287	262393981258310	5179526	10430	22	0	14	20	-2	0

Table 16: The twined series for M_{22}. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,2}(\tau)$.

$-D$ \ $[g]$	1A	2A	3A	4AB	5A	6A	7AB	8A	11AB
-4	1	1	1	1	1	1	1	1	1
8	210	2	3	-2	0	-1	0	0	1
20	4444	-4	7	4	-1	-1	-1	0	0
32	42560	16	-19	4	0	1	0	-2	1
44	281512	-24	19	-8	7	3	0	0	0
56	1481964	12	24	-12	-6	0	1	0	0
68	6649200	-16	-81	16	0	-1	5	0	-3
80	26455264	64	70	24	-6	-2	-4	4	0
92	95731405	-83	70	-27	0	-2	0	1	0
104	320626372	36	-248	-36	22	0	-4	0	0
116	1006567156	-44	217	44	-14	1	0	0	1
128	2990338680	168	183	52	0	3	0	-6	-1
140	8469129448	-216	-656	-72	-17	0	3	0	-1
152	23000871960	88	558	-88	0	-2	10	0	0
164	60186506768	-112	431	112	48	-1	-11	0	3
176	152335803872	416	-1582	144	-28	2	0	8	-1
188	374172530930	-494	1340	-166	0	4	-5	-2	0
200	894352929498	202	981	-202	-42	1	0	0	-5
212	2085157528300	-244	-3545	244	0	-1	0	0	0
224	4751675601024	864	2946	280	104	-6	9	-12	0
236	10602363945184	-1056	2077	-352	-51	-3	18	0	0
248	23199658816580	420	-7480	-420	0	0	-20	0	4
260	49851590654096	-496	6146	496	-84	2	0	0	0
272	105323108387200	1792	4228	608	0	4	-10	16	0
284	219021850730991	-2097	-15099	-697	196	-3	0	3	0

Table 17: The twined series for $U_4(3)$. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,1}(\tau)$.

$-D$ \ $[g]$	1A	2A	3A	3BCD	4AB	5A	6A	6BC	7AB	8A	9ABCD	12A
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
11	560	-16	-34	2	0	0	2	2	0	0	-1	0
23	8470	54	-116	-8	-2	0	0	0	0	2	1	-2
35	70576	-144	-272	16	0	-4	0	0	2	0	1	0
47	435820	332	-662	4	4	0	2	-4	0	0	-2	-2
59	2187328	-704	-1454	-50	0	8	-2	-2	-4	0	4	0
71	9493330	1394	-2732	58	-6	0	-4	2	0	-2	1	0
83	36792560	-2640	-5254	38	0	0	6	6	0	0	-7	0
95	130399766	4822	-9802	-172	6	-14	10	4	0	-2	2	0
107	429229920	-8480	-16782	174	0	0	-2	-2	0	0	3	0
119	1327987562	14506	-28930	104	-6	22	-14	-8	6	2	-7	0
131	3895785632	-24288	-49066	-502	0	-8	-6	-6	4	0	5	0
143	10912966810	39770	-79058	466	10	0	14	2	-10	2	4	-2
155	29351354032	-63888	-127484	316	0	-28	12	12	0	0	-11	0
167	76141761850	101018	-203264	-1232	-14	0	-4	8	0	-2	13	-2
179	191223891936	-157344	-313578	1098	0	56	-6	-6	0	0	3	0
191	466389602756	241764	-482842	710	12	-14	-6	-18	0	0	-22	0
203	1107626293840	-366960	-736772	-2876	0	0	-12	-12	10	0	10	0
215	2567229121428	550676	-1098876	2472	-12	-62	-4	8	6	4	9	0
227	5818567673360	-817840	-1634074	1658	0	0	26	26	-20	0	-22	0
239	12917927405852	1203068	-2412226	-6148	20	112	38	20	0	0	17	2
251	28135083779792	-1753840	-3502486	5174	0	-48	-10	-10	-4	0	14	0
263	60194978116000	2535488	-5067686	3382	-24	0	-58	-34	0	-4	-32	0
275	126660856741328	-3637232	-7287046	-12634	0	-112	-26	-26	0	0	32	0
287	262393981258310	5179526	-10348570	10430	22	0	38	14	20	-2	11	-2

Table 18: The twined series for $U_4(3)$. The table shows the Fourier coefficients multiplying $q^{-D/12}$ in the q-expansion of the function $h_{g,2}(\tau)$.

$-D$ \ $[g]$	1A	2A	3A	3BCD	4AB	5A	6A	6BC	7AB	8A	9ABCD	12A
-4	1	1	1	1	1	1	1	1	1	1	1	1
8	210	2	21	3	-2	0	5	-1	0	0	0	1
20	4444	-4	97	7	4	-1	-7	-1	-1	0	1	1
32	42560	16	197	-19	4	0	13	1	0	-2	-1	1
44	281512	-24	577	19	-8	7	-15	3	0	0	-2	1
56	1481964	12	1176	24	-12	-6	24	0	1	0	3	0
68	6649200	-16	2313	-81	16	0	-31	-1	5	0	-3	1
80	26455264	64	4552	70	24	-6	40	-2	-4	4	-2	0
92	95731405	-83	8440	70	-27	0	-56	-2	0	1	7	0
104	320626372	36	14440	-248	-36	22	72	0	-4	0	-5	0
116	1006567156	-44	25759	217	44	-14	-89	1	0	0	1	-1
128	2990338680	168	42861	183	52	0	117	3	0	-6	9	1
140	8469129448	-216	69904	-656	-72	-17	-144	0	3	0	-5	0
152	23000871960	88	114066	558	-88	0	178	-2	10	0	-9	2
164	60186506768	-112	181097	431	112	48	-223	-1	-11	0	11	1
176	152335803872	416	280208	-1582	144	-28	272	2	0	8	-10	0
188	374172530930	-494	436490	1340	-166	0	-326	4	-5	-2	-4	2
200	894352929498	202	662499	981	-202	-42	403	1	0	0	24	-1
212	2085157528300	-244	993025	-3545	244	0	-487	-1	0	0	-17	1
224	4751675601024	864	1485462	2946	280	104	582	-6	9	-12	-6	-2
236	10602363945184	-1056	2189455	2077	-352	-51	-705	-3	18	0	25	-1
248	23199658816580	420	3186440	-7480	-420	0	840	0	-20	0	-16	0
260	49851590654096	-496	4635278	6146	496	-84	-994	2	0	0	-16	-2
272	105323108387200	1792	6654706	4228	608	0	1186	4	-10	16	34	2
284	219021850730991	-2097	9475383	-15099	-697	196	-1401	-3	0	3	-30	-1

Table 19: The twined series for M_{23}. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau).b_p = (-1+i\sqrt{p})/2$.

$-D$ \ $[g]$	1A	2A	3A	4A	5A	6A	7AB	8A	11AB	14AB	15AB	23A	23B
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
23	770	-14	5	-2	0	1	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
47	13915	43	10	-1	0	-2	-1	1	0	1	0	0	0
71	132825	-119	21	5	-5	1	0	-1	0	0	1	0	0
95	915124	308	31	4	4	-1	0	0	1	0	1	0	0
119	5069867	-693	59	-13	2	3	-2	-1	0	0	-1	0	0
143	24053215	1407	85	-9	0	-3	4	-1	-1	0	0	-1	-1
167	101268540	-2772	135	24	0	3	2	2	-1	0	0	0	0
191	387746282	5306	200	14	-18	-4	0	0	0	0	0	0	0
215	1372935090	-9710	300	-38	15	4	-5	2	-1	-1	0	0	0
239	4552039296	17136	414	-20	11	-6	0	2	2	0	-1	0	0
263	14265412315	-29589	610	63	0	6	0	-3	0	0	0	1	1
287	42568680715	50155	835	35	0	-5	-6	-1	0	0	0	0	0
311	121665949240	-83160	1165	-108	-45	9	10	-2	0	0	0	0	0
335	334658246604	135148	1581	-60	34	-11	3	-4	0	-1	1	0	0
359	889413095662	-216482	2158	170	22	10	0	4	3	0	-2	0	0
383	2291482148835	342259	2865	87	0	-11	-9	1	0	1	0	0	0
407	5739333227670	-533610	3855	-250	0	15	0	2	-2	0	0	0	0
431	14008317423968	821296	5051	-124	-107	-17	0	6	-1	0	1	-1	-1
455	33388385201699	-1250717	6656	371	79	16	-11	-5	0	1	1	0	0
479	77853744768906	1886234	8649	190	56	-19	22	0	-4	0	-1	1	1
503	177881794535250	-2816798	11250	-554	0	22	8	-4	3	2	0	0	0
527	398808419854845	4167709	14430	-283	0	-26	0	-7	0	0	0	1	1
551	878461575586727	-6116665	18581	799	-218	29	-22	7	0	-2	1	0	0
575	1903241478167799	8909383	23631	395	154	-29	0	1	0	0	1	$\overline{b_{23}}$	b_{23}

Table 20: The twined series for M_{23}. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$.

$-D$ \ $[g]$	1A	2A	3A	4A	5A	6A	7AB	8A	11AB	14AB	15AB	23AB
-9	1	1	1	1	1	1	1	1	1	1	1	1
15	231	7	6	-1	1	-2	0	-1	0	0	1	1
39	5796	-28	18	4	1	2	0	0	-1	0	-2	0
63	65505	97	-15	1	0	1	-1	1	0	-1	0	1
87	494385	-239	60	-7	0	4	3	1	1	-1	0	0
111	2922381	525	90	-3	-9	-6	0	1	0	0	0	1
135	14525511	-1113	-75	15	6	-3	0	-1	0	0	0	-1
159	63447087	2255	228	7	7	-4	-3	-1	0	1	-2	0
183	250188435	-4333	360	-29	0	8	0	-1	2	0	0	1
207	907876585	7945	-260	-15	0	4	0	-3	0	0	0	0
231	3073155810	-14174	762	50	-25	10	-3	2	-1	1	2	0
255	9804660777	24809	1062	25	17	-10	9	1	-1	1	2	0
279	29717775186	-42286	-759	-78	16	-7	1	2	0	1	1	0
303	86116649220	70308	2070	-36	0	-18	0	4	-3	0	0	0
327	239806592730	-115046	2880	122	0	16	-9	-2	1	-1	0	-1
351	644418434331	185563	-1926	59	-64	10	0	-1	1	0	-1	0
375	1676994065901	-294483	5256	-195	46	24	0	-3	0	0	1	-1
399	4238788584987	460571	6948	-101	37	-28	-9	-5	0	-1	-2	0
423	10432762525295	-711985	-4645	295	0	-13	19	3	0	-1	0	0
447	25058448433770	1088842	12120	146	0	-32	3	2	4	-1	0	0
471	58848028309224	-1647128	15912	-424	-136	40	0	4	0	0	2	0
495	135349351964727	2466359	-10281	-201	92	23	-16	7	-1	0	-1	0
519	305326880332593	-3660335	26646	617	68	46	0	-7	-2	0	-4	0
543	676433083819185	5388145	34110	305	0	-50	0	-3	0	0	0	0
567	1473468429847035	-7867397	-21840	-901	0	-32	-15	-5	-5	1	0	1

Table 21: The twined series for *McL*. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$. $b_p = (-1+i\sqrt{p})/2$.

$-D$ \ $[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7AB	8A	9AB	10A	11AB	12A	14AB	$15A, \overline{15B}$	$30A, \overline{30B}$
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
23	770	-14	-13	5	-2	-5	0	7	1	0	0	-1	1	0	1	0	b_{15}	b_{15}
47	13915	43	-17	10	-1	-10	0	7	-2	-1	1	1	-2	0	-1	1	$-b_{15}$	b_{15}
71	132825	-119	-42	21	5	-25	-5	22	1	0	-1	0	1	0	2	0	$-b_{15}$	b_{15}
95	915124	308	-68	31	4	-26	4	32	-1	0	0	-2	-2	1	-2	0	b_{15}	b_{15}
119	5069867	-693	-112	59	-13	-58	2	60	3	-2	-1	2	2	0	2	0	$-b_{15}$	$(b_{15}-2)$
143	24053215	1407	-167	85	-9	-85	0	81	-3	4	-1	1	-3	-1	-3	0	$-\overline{b_{15}}$	$(b_{15}-1)$
167	101268540	-2772	-279	135	24	-135	0	141	3	2	2	-3	3	-1	3	0	$(b_{15}-1)$	$(b_{15}-1)$
191	387746282	5306	-394	200	14	-218	-18	194	-4	0	0	2	-4	0	-4	0	$-2b_{15}$	$2b_{15}$
215	1372935090	-9710	-600	300	-38	-285	15	304	4	-5	2	0	5	-1	4	-1	0	$2b_{15}$
239	4552039296	17136	-837	414	-20	-404	11	411	-6	0	2	-3	-4	2	-5	0	$(2b_{15}-1)$	$(2b_{15}-3)$
263	14265412315	-29589	-1208	610	63	-610	0	612	6	0	-3	4	6	0	6	0	2	$(2b_{15}-2)$
287	42568680715	50155	-1667	835	35	-835	0	829	-5	-6	-1	1	-5	0	-7	0	$-b_{15}$	$(3b_{15}-2)$
311	121665949240	-83160	-2345	1165	-108	-1210	-45	1179	9	10	-2	-5	10	0	9	0	$(b_{15}-2)$	$(3b_{15}-2)$
335	334658246604	135148	-3153	1581	-60	-1546	34	1567	-11	3	-4	3	-12	0	-9	-1	$(1-2b_{15})$	$(4b_{15}-1)$
359	889413095662	-216482	-4313	2158	170	-2138	22	2167	10	0	4	1	8	3	11	0	$\overline{b_{15}}$	$(5b_{15}-3)$
383	2291482148835	342259	-5748	2865	87	-2865	0	2860	-11	-9	1	-6	-11	0	-12	1	$(4b_{15}-1)$	$(4b_{15}-3)$
407	5739333227670	-533610	-7692	3855	-250	-3855	0	3864	15	0	2	6	15	-2	14	0	$(2-2b_{15})$	$(4b_{15}-4)$
431	14008317423968	821296	-10096	5051	-124	-5157	-107	5032	-17	0	6	2	-19	-1	-16	0	$(1-b_{15})$	$(5b_{15}-3)$
455	33388385201699	-1250717	-13333	6656	371	-6576	79	6679	16	-11	-5	-7	18	0	17	1	$(4b_{15}-1)$	$(6b_{15}-3)$
479	77853744768906	1886234	-17280	8649	190	-8594	56	8624	-19	22	0	6	-16	-4	-20	0	$(1-3b_{15})$	$(7b_{15}-5)$
503	177881794535250	-2816798	-22500	11250	-554	-11250	0	11272	22	8	-4	0	22	3	22	2	0	$(8b_{15}-4)$
527	398808419854845	4167709	-28887	14430	-283	-14430	0	14413	-26	0	-7	-9	-26	0	-25	0	$(3-3b_{15})$	$(9b_{15}-5)$
551	878461575586727	-6116665	-37138	18581	799	-18798	-218	18602	29	-22	7	8	30	0	28	-2	$(2-5b_{15})$	$(9b_{15}-6)$
575	1903241478167799	8909383	-47253	23631	395	-23476	154	23599	-29	0	1	3	-32	0	-31	0	2	$(10b_{15}-6)$

Table 22: The twined series for McL. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$.

$-D$ \ $[g]$	1A	2A	3A	3B	4A	5A	5B	6A	6B	7AB	8A	9AB	10A	11AB	12A	14AB	15AB	30AB
-9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	231	7	15	6	-1	6	1	7	-2	0	-1	0	2	0	-1	0	0	2
39	5796	-28	45	18	4	21	1	5	2	0	0	0	-3	-1	1	0	0	0
63	65505	97	30	-15	1	30	0	22	1	-1	1	0	2	0	-2	-1	0	2
87	494385	-239	150	60	-7	60	0	22	4	3	1	0	-4	1	2	-1	0	2
111	2922381	525	225	90	-3	81	-9	57	-6	0	1	0	5	0	-3	0	0	2
135	14525511	-1113	159	-75	15	161	6	63	-3	0	-1	3	-3	0	3	0	-1	3
159	63447087	2255	570	228	7	237	7	122	-4	-3	-1	0	5	0	-2	1	0	2
183	250188435	-4333	900	360	-29	360	0	164	8	0	-1	0	-8	2	4	0	0	4
207	907876585	7945	505	-260	-15	510	0	289	4	0	-3	-5	10	0	-3	0	0	4
231	3073155810	-14174	1905	762	50	735	-25	361	10	-3	2	0	-9	-1	5	1	0	6
255	9804660777	24809	2655	1062	25	1077	17	551	-10	9	1	0	9	-1	-5	1	0	6
279	29717775186	-42286	1518	-759	-78	1536	16	710	-7	1	2	0	-16	0	6	1	3	5
303	86116649220	70308	5175	2070	-36	2070	0	1071	-18	0	4	0	18	-3	-9	0	0	6
327	239806592730	-115046	7200	2880	122	2880	0	1408	16	-9	-2	0	-16	1	8	-1	0	8
351	644418434331	185563	3879	-1926	59	3806	-64	1999	10	0	-1	9	18	1	-13	0	-1	9
375	1676994065901	-294483	13140	5256	-195	5301	46	2580	24	0	-3	0	-23	0	12	0	0	10
399	4238788584987	460571	17370	6948	-101	6987	37	3530	-28	-9	-5	0	31	0	-14	-1	0	10
423	10432762525295	-711985	9260	-4645	295	9270	0	4532	-13	19	3	-10	-30	0	16	-1	0	12
447	25058448433770	1088842	30300	12120	146	12120	0	6124	-32	3	2	0	32	4	-16	-1	0	14
471	58848028309224	-1647128	39780	15912	-424	15774	-136	7876	40	0	4	0	-38	0	20	0	0	16
495	135349351964727	2466359	20562	-10281	-201	20652	92	10442	23	-16	7	0	44	-1	-18	0	-3	17
519	305326880332593	-3660335	66615	26646	617	26718	68	13231	46	0	-7	0	-50	-2	23	0	0	16
543	676433083819185	5388145	85275	34110	305	34110	0	17155	-50	0	-3	0	50	0	-25	0	0	20
567	1473468429847035	-7867397	43725	-21840	-901	43710	0	21637	-32	-15	-5	15	-62	-5	29	1	0	22

Table 23: The twined series for HS. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$. $a_p = i\sqrt{p}$.

$-D$ \ $[g]$	1A	2A	2B	3A	4A	4BC	5A	5BC	6A	6B	7A	8ABC	10A	10B	11AB	12A	15A	$20A, \overline{20B}$
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
23	770	-14	10	5	-10	-2	-5	0	1	1	0	0	1	0	0	-1	0	a_5
47	13915	43	-45	10	-25	-1	-10	0	0	-2	-1	1	-2	0	0	2	0	0
71	132825	-119	129	21	-35	5	-25	-5	3	1	0	-1	1	-1	0	1	1	$\overline{a_5}$
95	915124	308	-300	31	-60	4	-26	4	-3	-1	0	0	-2	0	1	-3	1	0
119	5069867	-693	667	59	-125	-13	-58	2	1	3	-2	-1	2	2	0	1	-1	0
143	24053215	1407	-1425	85	-185	-9	-85	0	-3	-3	4	-1	-3	0	-1	1	0	a_5
167	101268540	-2772	2820	135	-240	24	-135	0	3	3	2	2	3	0	-1	-3	0	$\overline{a_5}$
191	387746282	5306	-5278	200	-394	14	-218	-18	-4	-4	0	0	-4	2	0	2	0	$(\overline{a_5}+1)$
215	1372935090	-9710	9634	300	-630	-38	-285	15	4	4	-5	2	5	-1	-1	0	0	a_5
239	4552039296	17136	-17176	414	-860	-20	-404	11	-4	-6	0	2	-4	-1	2	-2	-1	0
263	14265412315	-29589	29715	610	-1145	63	-610	0	6	6	0	-3	6	0	0	4	0	0
287	42568680715	50155	-50085	835	-1645	35	-835	0	-9	-5	-6	-1	-5	0	0	-1	0	$\overline{a_5}$
311	121665949240	-83160	82944	1165	-2420	-108	-1210	-45	9	9	10	-2	10	-1	0	-5	0	0
335	334658246604	135148	-135268	1581	-3244	-60	-1546	34	-7	-11	3	-4	-12	2	0	5	1	(a_5+1)
359	889413095662	-216482	216822	2158	-4126	170	-2138	22	12	10	0	4	8	2	3	2	-2	$(\overline{a_5}-1)$
383	2291482148835	342259	-342085	2865	-5665	87	-2865	0	-13	-11	-9	1	-11	0	0	-7	0	a_5
407	5739333227670	-533610	533110	3855	-7930	-250	-3855	0	13	15	0	2	15	0	-2	5	0	a_5
431	14008317423968	821296	-821544	5051	-10260	-124	-5157	-107	-15	-17	0	6	-19	1	-1	3	1	$\overline{a_5}$
455	33388385201699	-1250717	1251459	6656	-12909	371	-6576	79	18	16	-11	-5	18	-1	0	-6	1	$(\overline{a_5}+1)$
479	77853744768906	1886234	-1885854	8649	-17146	190	-8594	56	-21	-19	22	0	-16	-4	-4	5	-1	$(\overline{a_5}-1)$
503	177881794535250	-2816798	2815690	11250	-23010	-554	-11250	0	22	22	8	-4	22	0	3	0	0	$2a_5$
527	398808419854845	4167709	-4168275	14430	-29195	-283	-14430	0	-24	-26	0	-7	-26	0	0	-8	0	0
551	878461575586727	-6116665	6118263	18581	-36305	799	-18798	-218	27	29	-22	7	30	-2	0	7	1	$2\overline{a_5}$
575	1903241478167799	8909383	-8908593	23631	-46925	395	-23476	154	-33	-29	0	1	-32	2	0	1	1	0

Table 24: The twined series for *HS*. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$.

$-D$ \ $[g]$	1A	2A	2B	3A	4A	4BC	5A	5BC	6A	6B	7A	8ABC	10A	10B	11AB	12A	15A	20AB
-9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	231	7	-9	6	15	-1	6	1	0	-2	0	-1	2	1	0	0	1	0
39	5796	-28	36	18	36	4	21	1	0	2	0	0	-3	1	-1	0	-2	1
63	65505	97	-95	-15	65	1	30	0	-5	1	-1	1	2	0	0	-1	0	0
87	494385	-239	225	60	105	-7	60	0	0	4	3	1	-4	0	1	0	0	0
111	2922381	525	-531	90	189	-3	81	-9	0	-6	0	1	5	-1	0	0	0	-1
135	14525511	-1113	1143	-75	319	15	161	6	9	-3	0	-1	-3	-2	0	1	0	-1
159	63447087	2255	-2241	228	471	7	237	7	0	-4	-3	-1	5	-1	0	0	-2	1
183	250188435	-4333	4275	360	675	-29	360	0	0	8	0	-1	-8	0	2	0	0	0
207	907876585	7945	-7975	-260	1025	-15	510	0	-10	4	0	-3	10	0	0	2	0	0
231	3073155810	-14174	14274	762	1554	50	735	-25	0	10	-3	2	-9	-1	-1	0	2	-1
255	9804660777	24809	-24759	1062	2169	25	1077	17	0	-10	9	1	9	1	-1	0	2	-1
279	29717775186	-42286	42130	-759	2930	-78	1536	16	19	-7	1	2	-16	0	0	-1	1	0
303	86116649220	70308	-70380	2070	4140	-36	2070	0	0	-18	0	4	18	0	-3	0	0	0
327	239806592730	-115046	115290	2880	5850	122	2880	0	0	16	-9	-2	-16	0	1	0	0	0
351	644418434331	185563	-185445	-1926	7835	59	3806	-64	-36	10	0	-1	18	0	1	-4	-1	0
375	1676994065901	-294483	294093	5256	10269	-195	5301	46	0	24	0	-3	-23	-2	0	0	1	-1
399	4238788584987	460571	-460773	6948	13851	-101	6987	37	0	-28	-9	-5	31	-3	0	0	-2	1
423	10432762525295	-711985	712575	-4645	18775	295	9270	0	45	-13	19	3	-30	0	0	1	0	0
447	25058448433770	1088842	-1088550	12120	24450	146	12120	0	0	-32	3	2	32	0	4	0	0	0
471	58848028309224	-1647128	1646280	15912	31320	-424	15774	-136	0	40	0	4	-38	0	0	0	2	0
495	135349351964727	2466359	-2466761	-10281	41015	-201	20652	92	-59	23	-16	7	44	4	-1	5	-1	0
519	305326880332593	-3660335	3661569	26646	53817	617	26718	68	0	46	0	-7	-50	4	-2	0	-4	2
543	676433083819185	5388145	-5387535	34110	68625	305	34110	0	0	-50	0	-3	50	0	0	0	0	0
567	1473468429847035	-7867397	7865595	-21840	86395	-901	43710	0	90	-32	-15	-5	-62	0	-5	-2	0	0

Table 25: The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ - part I. $a_p = i\sqrt{p}$.

$-D$ \ $[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4CDE	4F	4G	5A	$6A, \overline{6B}$
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
23	770	-30	-14	10	5	-13	5	34	-6	-2	2	-2	0	-3- $6a_3$
47	13915	-5	43	-45	10	-17	10	139	-13	-1	3	-1	0	-5
71	132825	-199	-119	129	21	-42	21	505	-15	5	-7	5	-5	-10- $12a_3$
95	915124	180	308	-300	31	-68	31	1412	-28	4	-4	4	4	-18- $18a_3$
119	5069867	-917	-693	667	59	-112	59	3627	-69	-13	11	-13	2	-26- $30a_3$
143	24053215	1055	1407	-1425	85	-167	85	8559	-97	-9	7	-9	0	-43-$36a_3$
167	101268540	-3300	-2772	2820	135	-279	135	19068	-108	24	-20	24	0	-69- $78a_3$
191	387746282	4490	5306	-5278	200	-394	200	40154	-190	14	-14	14	-18	-100- $90a_3$
215	1372935090	-10894	-9710	9634	300	-600	300	81250	-334	-38	42	-38	15	-148- $156a_3$
239	4552039296	15456	17136	-17176	414	-837	414	158736	-440	-20	24	-20	11	-213-$204a_3$
263	14265412315	-32005	-29589	29715	610	-1208	610	300987	-541	63	-69	63	0	-298- $306a_3$
287	42568680715	46795	50155	-50085	835	-1667	835	555307	-805	35	-37	35	0	-419- $408a_3$
311	121665949240	-87784	-83160	82944	1165	-2345	1165	1001016	-1264	-108	104	-108	-45	-583- $606a_3$
335	334658246604	128780	135148	-135268	1581	-3153	1581	1767388	-1652	-60	52	-60	34	-793- $768a_3$
359	889413095662	-225074	-216482	216822	2158	-4313	2158	3062830	-1978	170	-162	170	22	-1073- $1092a_3$
383	2291482148835	330755	342259	-342085	2865	-5748	2865	5217427	-2789	87	-85	87	0	-1444-$1428a_3$
407	5739333227670	-548970	-533610	533110	3855	-7692	3855	8751462	-4090	-250	254	-250	0	-1914- $1938a_3$
431	14008317423968	801024	821296	-821544	5051	-10096	5051	14473168	-5192	-124	136	-124	-107	-2532- $2496a_3$
455	33388385201699	-1277277	-1250717	1251459	6656	-13333	6656	23625763	-6269	371	-381	371	79	-3327- $3366a_3$
479	77853744768906	1851562	1886234	-1885854	8649	-17280	8649	38101658	-8478	190	-190	190	56	-4328- $4284a_3$
503	177881794535250	-2861710	-2816798	2815690	11250	-22500	11250	60762514	-11782	-554	546	-554	0	-5614- $5658a_3$
527	398808419854845	4109885	4167709	-4168275	14430	-28887	14430	95895325	-14739	-283	269	-283	0	-7237- $7194a_3$
551	878461575586727	-6190873	-6116665	6118263	18581	-37138	18581	149872439	-17753	799	-785	799	-218	-9268- $9318a_3$
575	1903241478167799	8814743	8909383	-8908593	23631	-47253	23631	232094167	-23265	395	-393	395	154	-11827- $11766a_3$

Table 26: The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ - part II. $a_p = i\sqrt{p}$.

$-D$ \ $[g]$	6C	6D	6E	6F	6G	6H	7A	8A	8BCD	9ABC	10A	11AB	$12A, \overline{12B}$	12C	$12D, \overline{12E}$	12FGH	12I	15A	$18A, \overline{18B}$
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
23	-3	7	1	-3	1	1	0	2	0	-1	0	0	-2+ $3a_3$	1	a_3	1	-1	0	$-a_3$
47	-14	7	-2	4	-2	0	-1	-1	1	1	0	0	-5+ $6a_3$	-2	-1-$2a_3$	-1	0	0	1
71	-19	22	1	-1	1	3	0	1	-1	0	1	0	-8+ $12a_3$	1	0	2	-1	1	-1+a_3
95	-33	32	-1	-3	-1	-3	0	0	0	-2	0	1	-19+ $15a_3$	-1	-1+a_3	-2	-1	1	0
119	-53	60	3	1	3	1	-2	3	-1	2	-2	0	-27+ $27a_3$	3	3+a_3	2	-1	-1	1+a_3
143	-91	81	-3	5	-3	-3	4	-5	-1	1	0	-1	-45+ $42a_3$	-3	-1-$2a_3$	-3	1	0	-1
167	-129	141	3	-9	3	3	2	4	2	-3	0	-1	-66+ $69a_3$	3	$3a_3$	3	1	0	$-a_3$
191	-208	194	-4	8	-4	-4	0	-2	0	2	0	0	-103+ $99a_3$	-4	-1-$3a_3$	-4	-2	0	2
215	-292	304	4	-4	4	4	-5	6	2	0	1	-1	-146+ $150a_3$	4	2+$2a_3$	4	0	0	-1+a_3
239	-426	411	-6	0	-6	-4	0	-4	2	-3	1	2	-213+ $210a_3$	-6	-5-$2a_3$	-5	0	-1	$-a_3$
263	-598	612	6	2	6	6	0	3	-3	4	0	0	-297+ $303a_3$	6	5+a_3	6	0	0	2
287	-845	829	-5	7	-5	-9	-6	-5	-1	1	0	0	-425+ $414a_3$	-5	-1-$2a_3$	-7	-1	0	-2+a_3
311	-1147	1179	9	-19	9	9	10	8	-2	-5	1	0	-576+ $585a_3$	9	2+$7a_3$	9	-1	0	-1-$2a_3$
335	-1603	1567	-11	17	-11	-7	3	-16	-4	3	0	0	-797+ $792a_3$	-11	-5-$8a_3$	-9	1	1	2+a_3
359	-2138	2167	10	-8	10	12	0	14	4	1	-4	3	-1067+ $1080a_3$	10	5+$4a_3$	11	0	-2	-2+a_3
383	-2887	2860	-11	-1	-11	-13	-9	-9	1	-6	0	0	-1448+ $1434a_3$	-11	-8-$2a_3$	-12	-1	0	-1-a_3
407	-3825	3864	15	-3	15	13	0	18	2	6	0	-2	-1911+ $1923a_3$	15	11+$5a_3$	14	-1	0	3+a_3
431	-5085	5032	-17	21	-17	-15	0	-12	6	2	-1	-1	-2540+ $2526a_3$	-17	-8-$10a_3$	-16	1	1	-3+a_3
455	-6624	6679	16	-30	16	18	-11	11	-5	-7	3	0	-3314+ $3333a_3$	16	4+$11a_3$	17	0	1	-$3a_3$
479	-8687	8624	-19	31	-19	-21	22	-18	0	6	2	-4	-4342+ $4320a_3$	-19	-6-$12a_3$	-20	-1	-1	4
503	-11206	11272	22	-22	22	22	8	18	-4	0	0	3	-5603+ $5625a_3$	22	11+$11a_3$	22	0	0	-4+$2a_3$
527	-14482	14413	-26	8	-26	-24	0	-35	-7	-9	0	0	-7244+ $7221a_3$	-26	-18-$9a_3$	-25	2	0	-1-$2a_3$
551	-18523	18602	29	-13	29	27	-22	35	7	8	2	0	-9259+ $9285a_3$	29	19+$11a_3$	28	1	1	5+a_3
575	-23689	23599	-29	35	-29	-33	0	-25	1	3	-2	0	-11846+ $11811a_3$	-29	-12-$15a_3$	-31	-3	1	-4+a_3

Table 27: The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ - part I.

$-D$ \ $[g]$	1A	2A	2B	2C	3A	3B	3C	4A	4B	4CDE	4F	4G	5A	6AB	6C	6D
-9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	231	39	7	-9	6	15	6	23	7	-1	-1	-1	1	3	6	7
39	5796	36	-28	36	18	45	18	84	20	4	-4	4	1	9	18	5
63	65505	225	97	-95	-15	30	-15	353	33	1	1	1	0	18	33	22
87	494385	-15	-239	225	60	150	60	993	49	-7	9	-7	0	30	60	22
111	2922381	909	525	-531	90	225	90	2685	93	-3	5	-3	-9	45	90	57
135	14525511	-505	-1113	1143	-75	159	-75	6487	167	15	-17	15	6	71	149	63
159	63447087	3183	2255	-2241	228	570	228	14719	239	7	-9	7	7	114	228	122
183	250188435	-2925	-4333	4275	360	900	360	31395	323	-29	27	-29	0	180	360	164
207	907876585	10025	7945	-7975	-260	505	-260	64553	505	-15	9	-15	0	269	524	289
231	3073155810	-11166	-14174	14274	762	1905	762	127490	802	50	-46	50	-25	381	762	361
255	9804660777	29097	24809	-24759	1062	2655	1062	243945	1097	25	-23	25	17	531	1062	551
279	29717775186	-36270	-42286	42130	-759	1518	-759	453842	1426	-78	82	-78	16	738	1497	710
303	86116649220	78660	70308	-70380	2070	5175	2070	824596	2052	-36	44	-36	0	1035	2070	1071
327	239806592730	-103590	-115046	115290	2880	7200	2880	1465578	2986	122	-126	122	0	1440	2880	1408
351	644418434331	201115	185563	-185445	-1926	3879	-1926	2555051	3947	59	-61	59	-64	1963	3898	1999
375	1676994065901	-273555	-294483	294093	5256	13140	5256	4376077	5037	-195	189	-195	46	2628	5256	2580
399	4238788584987	488475	460571	-460773	6948	17370	6948	7377627	6875	-101	91	-101	37	3474	6948	3530
423	10432762525295	-675025	-711985	712575	-4645	9260	-4645	12257551	9535	295	-289	295	0	4592	9227	4532
447	25058448433770	1137450	1088842	-1088550	12120	30300	12120	20093306	12298	146	-142	146	0	6060	12120	6124
471	58848028309224	-1583640	-1647128	1646280	15912	39780	15912	32531448	15448	-424	432	-424	-136	7956	15912	7876
495	135349351964727	2548791	2466359	-2466761	-10281	20562	-10281	52070711	20407	-201	215	-201	92	10350	20631	10442
519	305326880332593	-3553935	-3660335	3661569	26646	66615	26646	82458145	27217	617	-631	617	68	13323	26646	13231
543	676433083819185	5524785	5388145	-5387535	34110	85275	34110	129280929	34465	305	-311	305	0	17055	34110	17155
567	1473468429847035	-7692805	-7867397	7865595	-21840	43725	-21840	200804347	42747	-901	891	-901	0	21761	43616	21637

Table 28: The twined series for $U_6(2)$. The table shows the Fourier coefficients multiplying $q^{-D/24}$ in the q-expansion of the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ - part II.

$-D$ \ $[g]$	6E	6F	6G	6H	7A	8A	8BCD	9ABC	10A	11AB	12AB	12C	12DE	12FGH	12I	15A	18AB
-9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	-2	0	-2	0	0	3	-1	0	-1	0	5	2	1	-1	2	1	0
39	2	0	2	0	0	0	0	0	1	-1	3	-6	-1	1	2	-2	0
63	1	3	1	-5	-1	1	1	0	0	0	20	5	0	-2	1	0	0
87	4	0	4	0	3	-3	1	0	0	1	30	0	-2	2	0	0	0
111	-6	0	-6	0	0	9	1	0	-1	0	39	-6	3	-3	2	0	0
135	-3	-7	-3	9	0	-5	-1	3	0	0	79	1	-1	3	1	0	-1
159	-4	0	-4	0	-3	3	-1	0	3	0	130	16	2	-2	0	-2	0
183	8	0	8	0	0	-9	-1	0	0	2	156	-24	-4	4	0	0	0
207	4	14	4	-10	0	9	-3	-5	0	0	275	20	7	-3	0	0	-1
231	10	0	10	0	-3	-6	2	0	-1	-1	383	2	-5	5	2	2	0
255	-10	0	-10	0	9	9	1	0	-3	-1	513	-18	5	-5	-2	2	0
279	-7	-21	-7	19	1	-14	2	0	0	0	764	5	-8	6	1	1	0
303	-18	0	-18	0	0	24	4	0	0	-3	1069	34	9	-9	2	0	0
327	16	0	16	0	-9	-18	-2	0	0	1	1368	-72	-8	8	0	0	0
351	10	28	10	-36	0	15	-1	9	0	1	1985	50	5	-13	2	-1	1
375	24	0	24	0	0	-27	-3	0	0	0	2644	16	-12	12	0	1	0
399	-28	0	-28	0	-9	27	-5	0	5	0	3414	-60	14	-14	4	-2	0
423	-13	-43	-13	45	19	-25	3	-10	0	0	4654	19	-14	16	-1	0	2
447	-32	0	-32	0	3	30	2	0	0	4	6152	92	16	-16	-4	0	0
471	40	0	40	0	0	-36	4	0	0	0	7788	-168	-20	20	0	2	0
495	23	69	23	-59	-16	55	7	0	-4	-1	10400	119	28	-18	-1	-1	0
519	46	0	46	0	0	-51	-7	0	0	-2	13357	34	-23	23	2	-4	0
543	-50	0	-50	0	0	45	-3	0	0	0	16917	-138	25	-25	-2	0	0
567	-32	-94	-32	90	-15	-69	-5	15	0	-5	21907	52	-33	29	0	0	-1

D Decomposition Tables

Table 29: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of M_{22}.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
-1	-2	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	1	1	0
23	0	0	0	0	2	2	4	4	6	4	4	8
35	0	2	8	8	6	16	22	34	32	48	48	60
47	2	26	42	42	60	100	160	208	232	268	268	380
59	2	92	228	228	258	480	738	1028	1132	1394	1394	1900
71	26	472	952	952	1204	2130	3334	4508	4966	5966	5966	8240
83	78	1710	3754	3754	4516	8192	12708	17410	19112	23284	23284	31928
95	300	6220	13194	13194	16254	29148	45402	61752	68016	82218	82218	113216
107	950	20228	43616	43616	53078	95744	148822	203204	223388	271160	271160	372564
119	3040	63080	134624	134624	164946	296536	461482	628868	691920	838076	838076	1152800
131	8706	184100	395460	395460	482650	869408	1352058	1844428	2028652	2459952	2459952	3381724
143	24720	517270	1106932	1106932	1354036	2436238	3790274	5167368	5684522	6888684	6888684	9473140
155	66016	1388930	2978518	2978518	3638646	6551148	10189794	13897114	15285968	18531236	18531236	25478420
167	171904	3606408	7724632	7724632	9443990	16996718	26440708	36052396	39659114	48067156	48067156	66095608
179	430782	9052216	19403004	19403004	23710442	42682664	66393106	90540888	99592828	120725462	120725462	165992492
191	1052210	22086048	47318486	47318486	57840438	104106714	161947152	220829728	242915594	294432862	294432862	404852680
203	2496320	52439410	112383780	112383780	137347678	247235148	384582826	524442118	576882244	699266588	699266588	961480364
215	5789798	121561770	260469430	260469430	318366892	573046530	891413422	1215547168	1337108322	1620714052	1620714052	2228498796
227	13116978	275489850	590364314	590364314	721533716	1298781032	2020314748	2754999510	3030488952	3673355264	3673355264	5050837768
239	29128906	611659466	1310656938	1310656938	1601947284	2883475316	4485422600	6116448648	6728109656	8155231968	8155231968	11213482124

Table 30: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,2}(\tau)$ into irreducible representations χ_n of M_{22}.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
-4	1	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0
20	0	1	1	1	1	1	2	2	1	3	3	4
32	0	1	5	5	5	10	15	18	24	26	26	38
44	2	15	28	28	35	61	96	136	147	179	179	242
56	2	70	149	149	184	332	516	706	770	936	936	1284
68	15	310	675	675	821	1486	2307	3138	3468	4196	4196	5778
80	62	1260	2686	2686	3286	5906	9192	12528	13770	16702	16702	22964
92	215	4535	9711	9711	11871	21367	33238	45338	49855	60440	60440	83094
104	716	15162	32528	32528	39754	71562	111316	151798	167022	202408	202408	278332
116	2277	47678	102132	102132	124829	224684	349513	476604	524228	635466	635466	873748
128	6752	141611	303407	303407	370835	667490	1038325	1415886	1557452	1887842	1887842	2595774
140	19067	400933	859281	859281	1050214	1890431	2940645	4009962	4411059	5346659	5346659	7351704
152	51870	1089096	2333682	2333682	2852302	5134124	7986426	10890628	11979584	14520766	14520766	19965996
164	135731	2849794	6106599	6106599	7463621	13434478	20898099	28497414	31347100	37996546	37996546	52245218
176	343436	7212750	15456158	15456158	18890818	34003540	52894358	72128538	79341684	96171541	96171541	132236044
188	843662	17716592	37963923	37963923	46400388	83520644	129921032	177165164	194881422	236220080	236220080	324802452
200	2016488	42346314	90741956	90741956	110906886	199632354	310539240	423462681	465808750	564616772	564616772	776347956
212	4701307	98728774	211562274	211562274	258575987	465436954	724012941	987290154	1086019814	1316387235	1316387235	1810032796
224	10713686	224984946	482109963	482109963	589245588	1060641874	1649887462	2249846600	2474830808	2999795266	2999795266	4124718268
236	23905058	502005985	1075726834	1075726834	1314777297	2366599091	3681376388	5020058920	5522064385	6693411661	6693411661	9203440774
248	52307782	1098468036	2353861448	2353861448	2876941610	5178495256	8055436866	10984686398	12083156304	14646249036	14646249036	20138593100
260	112400070	2360397820	5057994232	5057994232	6181993066	11127587228	17309580294	23603973264	25964369548	31471963978	31471963978	43273949924

Table 31: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
-1	-2	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	2	4	4	7	7
47	0	6	4	4	14	22	22	26	34	34
59	0	12	20	20	52	82	120	132	189	189
71	4	72	102	102	270	422	532	606	801	801
83	8	232	380	380	984	1548	2112	2344	3164	3164
95	38	860	1398	1398	3618	5622	7492	8350	11125	11125
107	124	2748	4554	4554	11732	18306	24784	27532	36834	36834
119	420	8640	14228	14228	36676	57060	76674	85316	113706	113706
131	1156	24972	41622	41622	107060	166678	225288	250260	334074	334074
143	3368	70446	116944	116944	300968	468176	631070	701512	935210	935210
155	8908	188592	314208	314208	808100	1257446	1698156	1886748	2516637	2516637
167	23350	490188	815984	815984	2098802	3264774	4405146	4895338	6526850	6526850
179	58388	1229276	2048450	2048450	5267788	8195334	11065144	12294420	16394948	16394948
191	142936	3000480	4998134	4998134	12853814	19994842	26987278	29987754	39982901	39982901
203	338664	7121124	11868186	11868186	30518832	47475942	64096288	71217412	94962320	94962320
215	786382	16511002	27512230	27512230	70749080	110054184	148559990	165070996	220092869	220092869
227	1780628	37411832	62351670	62351670	160334544	249414368	336717232	374129064	498851309	498851309
239	3955942	83070266	138437866	138437866	355990060	553762164	747550924	830621184	1107491427	1107491427
251	8612692	180905952	301506846	301506846	775306952	1206043980	1628176080	1809082032	2412136251	2412136251
263	18432618	387072550	645093274	645093274	1658826474	2580396830	3483475608	3870548166	5160722937	5160722937
275	38777988	814425664	1357371466	1357371466	3490390608	5429518660	7329886448	8144312112	10859139417	10859139417
287	80345934	1687237678	2812006070	2812006070	7230903886	11248072954	15184775484	16872013154	22496001167	22496001167

Table 32: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part II.

-D	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}
-1	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	1	0	0	0	0
23	0	0	2	2	2	1	2	2	2	2
35	7	7	6	6	10	16	14	14	16	16
47	34	34	46	46	62	74	86	86	100	116
59	189	189	206	206	290	391	432	432	480	596
71	801	801	934	934	1242	1626	1864	1864	2130	2586
83	3164	3164	3532	3532	4756	6379	7216	7216	8192	10064
95	11125	11125	12636	12636	16856	22345	25568	25568	29148	35728
107	36834	36834	41346	41346	55278	73819	84136	84136	95744	117656
119	113706	113706	128270	128270	171022	227677	260274	260274	296536	364172
131	334074	334074	375590	375590	501258	668609	763500	763500	869408	1068560
143	935210	935210	1053068	1053068	1404072	1871144	2138648	2138648	2436238	2993494
155	2516637	2516637	2830546	2830546	3775230	5034454	5751956	5751956	6551148	8051776
167	6526850	6526850	7345188	7345188	9793632	13055603	14921266	14921266	16996718	20888326
179	16394948	16394948	18442654	18442654	24593074	32792777	37473264	37473264	42682664	52460200
191	39982901	39982901	44986624	44986624	59982140	79970261	91395836	91395836	104106714	127950562
203	94962320	94962320	106828846	106828846	142445378	189931510	217054970	217054970	247235148	303871648
215	220092869	220092869	247617812	247617812	330156976	440195866	503083826	503083826	573046530	704309074
227	498851309	498851309	561199172	561199172	748280636	997717729	1140227136	1140227136	1298781032	1596305804
239	1107491427	1107491427	1245957224	1245957224	1661276556	2215005306	2531441198	2531441198	2883475316	3544000144
251	2412136251	2412136251	2713634904	2713634904	3618212136	4824304831	5513444680	5513444680	6280138936	7718796324
263	5160722937	5160722937	5805874860	5805874860	7741166340	10321492731	11796004996	11796004996	13436398852	16514369292
275	10859139417	10859139417	12216494216	12216494216	16288726840	21718346527	24820871652	24820871652	28272484272	34749166956
287	22496001167	22496001167	25308126986	25308126986	33744168882	44992097964	51419567602	51419567602	58570125574	71987317446

Table 33: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,2}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
-4	1	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0
20	0	0	0	0	1	1	1	0	1	1
32	0	0	1	1	2	1	4	3	3	3
44	1	2	4	4	8	14	16	21	26	26
56	0	8	18	18	42	64	92	104	129	129
68	3	37	74	74	185	285	396	428	573	573
80	10	170	288	288	742	1142	1558	1720	2276	2276
92	30	606	1038	1038	2648	4114	5574	6189	8226	8226
104	98	2040	3458	3458	8852	13740	18616	20668	27505	27505
116	315	6453	10820	10820	27785	43187	58374	64812	86349	86349
128	932	19200	32105	32105	82484	128231	173260	192443	256447	256447
140	2585	54343	90862	90862	233470	363072	490425	544792	726201	726201
152	7068	147802	246650	246650	634020	986096	1331594	1479426	1972157	1972157
164	18479	386845	645246	645246	1658877	2580201	3483848	3870656	5160351	5160351
176	46678	979182	1632930	1632930	4198392	6530326	8817060	9796194	13060772	13060772
188	114644	2405530	4010504	4010504	10311866	16040080	21655492	24061078	32080056	32080056
200	273958	5750076	9585456	9585456	24646996	38338768	51759604	57509747	76677496	76677496
212	638587	13406477	22347452	22347452	57462835	89385017	120673444	134079840	178770283	178770283
224	1455274	30552132	50924466	50924466	130945830	203691226	274988136	305540174	407382190	407382190
236	3246731	68171414	113625730	113625730	292176084	454492898	613572808	681744339	908985651	908985651
248	7103972	149171154	248629024	248629024	639325160	994500692	1342587586	1491758880	1989002072	1989002072
260	15265066	320543266	534252608	534252608	1373783498	2136989558	2884951508	3205494608	4273978516	4273978516
272	32250408	677225340	1128728962	1128728962	2902432966	4514885396	6095118216	6772343354	9029770416	9029770416
284	67064204	1408309053	2347212278	2347212278	6035669290	9388804084	12674919317	14083228602	18777609455	18777609455

Table 34: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/12}$ in the function $h_{g,1}(\tau)$ into irreducible representations χ_n of $U_4(3)$ - part II.

-D	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}
-4	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
20	1	1	0	0	0	0	1	1	1	1
32	3	3	3	3	5	6	8	8	10	14
44	26	26	27	27	31	46	54	54	61	81
56	129	129	142	142	182	248	289	289	332	410
68	573	573	636	636	842	1124	1299	1299	1486	1842
80	2276	2276	2544	2544	3372	4512	5178	5178	5906	7278
92	8226	8226	9223	9223	12251	16366	18749	18749	21367	26305
104	27505	27505	30902	30902	41142	54888	62812	62812	71562	88062
116	86349	86349	97044	97044	129268	172448	197214	197214	224684	276276
128	256447	256447	288351	288351	384273	512502	585938	585938	667490	820618
140	726201	726201	816744	816744	1088680	1451756	1659543	1659543	1890431	2323911
152	1972157	1972157	2218282	2218282	2957154	3943264	4507162	4507162	5134124	6310848
164	5160351	5160351	5804744	5804744	7738814	10318996	11794091	11794091	13434478	16513050
176	13060772	13060772	14692426	14692426	19588686	26119004	29851826	29851826	34003540	41794712
188	32080056	32080056	36088522	36088522	48115930	64156020	73323427	73323427	83520644	102655962
200	76677496	76677496	86259890	86259890	115010066	153348868	175259316	175259316	199632354	245367882
212	178770283	178770283	201113152	201113152	268146424	357531400	408612706	408612706	465436954	572065334
224	407382190	407382190	458299758	458299758	611059410	814750620	931151309	931151309	1060641874	1303622708
236	908985651	908985651	1022601213	1022601213	1363458009	1817950926	2077669720	2077669720	2366599091	2908753703
248	1989002072	1989002072	2237616450	2237616450	2983474090	3977974852	4546274088	4546274088	5178495256	6364807716
260	4273978516	4273978516	4808209568	4808209568	6410924444	8547913920	9769068796	9769068796	11127587228	13676730320
272	9029770416	9029770416	10158468546	10158468546	13544593866	18059479228	20639439888	20639439888	23509622436	28895264904
284	18777609455	18777609455	21124777990	21124777990	28166327222	37555131344	42920200803	42920200803	48888805889	60088352145

Table 35: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
-1	-1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	1	1	0	0	0
71	0	0	1	1	3	3	2	2	4
95	1	4	4	4	24	23	21	21	24
119	0	10	23	23	110	115	113	113	122
143	3	57	104	104	555	552	547	547	604
167	11	217	450	450	2266	2289	2283	2283	2500
191	40	851	1705	1705	8792	8800	8789	8789	9646
215	132	2945	6066	6066	30883	31076	31062	31062	34002
239	455	9865	20061	20061	102784	103141	103119	103119	112980
263	1393	30717	62965	62965	321420	322980	322951	322951	353668
287	4196	91938	187731	187731	960228	964123	964080	964080	1056018
311	11898	262221	536800	536800	2742534	2754938	2754882	2754882	3017118
335	32865	722082	1476142	1476142	7546650	7578723	7578641	7578641	8300712
359	87129	1917740	3923759	3923759	20051805	20140197	20140092	20140092	22057824
383	224778	4942779	10108150	10108150	51668700	51891433	51891287	51891287	56834066
407	562443	12376668	25318792	25318792	129399884	129965529	129965340	129965340	142342008
431	1373557	30213098	61794663	61794663	315850951	317219639	317219382	317219382	347432516
455	3272651	72004775	147289246	147289246	752794475	756074520	756074191	756074191	828078940
479	7632762	167908464	343437978	343437978	1755376093	1762997610	1762997173	1762997173	1930905618
503	17436751	383624438	784701896	784701896	4010659821	4028113401	4028112844	4028112844	4411737282
527	39096804	860103679	1759278672	1759278672	8991927075	9030999117	9030998389	9030998389	9891102068
551	86113417	1894529513	3875207902	3875207902	19806533653	19892683368	19892682446	19892682446	21787212032
575	186578208	4104664777	8395853450	8395853450	42912264213	43098789349	43098788160	43098788160	47203452886

Table 36: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part II.

-D	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
-1	0	0	0	0	0	0	0	0
23	1	0	0	0	0	0	0	0
47	1	1	1	1	1	1	2	3
71	11	11	11	11	14	14	12	26
95	68	68	80	80	87	87	96	182
119	389	389	444	444	496	496	508	1003
143	1811	1811	2111	2111	2324	2324	2454	4777
167	7661	7661	8892	8892	9848	9848	10246	20084
191	29245	29245	34052	34052	37596	37596	39394	76950
215	103695	103695	120586	120586	133306	133306	139204	272375
239	343525	343525	399819	399819	441658	441658	462028	903231
263	1076965	1076965	1252987	1252987	1384656	1384656	1447084	2830347
287	3212975	3212975	3738996	3738996	4130944	4130944	4319566	8446311
311	9184195	9184195	10686484	10686484	11808192	11808192	12343520	24139819
335	25260382	25260382	29394631	29394631	32477570	32477570	33956132	66400838
359	67136828	67136828	78121437	78121437	86318725	86318725	90238598	176470194
383	172966448	172966448	201271974	201271974	222385345	222385345	232499596	454660159
407	433225520	433225520	504113505	504113505	557004065	557004065	582313360	1138754982
431	1057386979	1057386979	1230418627	1230418627	1359497348	1359497348	1421306694	2779430485
455	2520265222	2520265222	2932664346	2932664346	3240340814	3240340814	3387607678	6624675836
479	5876631765	5876631765	6838273397	6838273397	7555669131	7555669131	7899140768	15447177148
503	13427082844	13427082844	15624224127	15624224127	17263391744	17263391744	18048043408	35293998406
527	30103271271	30103271271	35029285569	35029285569	38704205367	38704205367	40463558496	79128667059
551	66309028068	66309028068	77159558254	77159558254	85254464108	85254464108	89129562686	174297913365
575	143662505416	143662505417	167170968143	167170968143	184708934751	184708934751	193104947248	377627303791

Table 37: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9
-9	1	0	0	0	0	0	0	0	0
15	0	0	0	0	0	1	0	0	0
39	0	1	0	0	1	0	0	0	0
63	0	0	0	0	2	1	2	2	2
87	1	2	2	2	10	12	10	10	12
111	0	8	12	12	72	71	65	65	76
135	1	29	66	66	317	324	327	327	354
159	8	148	278	278	1456	1451	1439	1439	1584
183	25	536	1108	1108	5614	5666	5650	5650	6186
207	90	1969	3996	3996	20527	20571	20585	20585	22554
231	301	6612	13574	13574	69200	69575	69539	69539	76160
255	979	21229	43224	43224	221297	222128	222072	222072	243296
279	2892	63977	131140	131140	669671	672826	672862	672862	736834
303	8477	185925	379810	379810	1942293	1950361	1950253	1950253	2136178
327	23485	516995	1058004	1058004	5406043	5430193	5430053	5430053	5947048
351	63229	1390158	2842550	2842550	14531106	14593220	14593319	14593319	15983498
375	164314	3616159	7398130	7398130	37808739	37974823	37974566	37974566	41590710
399	415733	9142805	18698262	18698262	95575487	95988502	95988148	95988148	105130940
423	1022444	22498248	46023364	46023364	235220486	236247135	236247364	236247364	258745612
447	2456955	54045257	110540364	110540364	564999523	567449989	567449375	567449375	621494632
471	5768329	126911807	259601046	259601046	1326827403	1332605575	1332604789	1332604789	1459516642
495	13269156	291907776	597070540	597070540	3051727864	3064982355	3064982875	3064982875	3356890620
519	29930003	658478668	1346907692	1346907692	6884144714	6914096414	6914095094	6914095094	7572573738
543	66312927	1458848820	2983976756	2983976756	15251512500	15317793334	15317791616	15317791616	16776640436
567	144441012	3177751973	6499993004	6499993004	33222076673	33366564639	33366565723	33366565723	36544317696

Table 38: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of M_{23} - part II.

-D	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
-9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	1	1	0	0	1	1	0	1
63	4	4	6	6	6	6	8	13
87	41	41	42	42	49	49	48	97
111	220	220	254	254	280	280	302	582
135	1099	1099	1278	1278	1418	1418	1462	2879
159	4783	4783	5568	5568	6143	6143	6460	12594
183	18920	18920	21967	21967	24308	24308	25342	49625
207	68482	68482	79749	79749	88055	88055	92194	180159
231	232064	232064	269912	269912	298347	298347	311662	609707
255	739984	739984	861169	861169	951374	951374	995038	1945438
279	2243393	2243393	2610275	2610275	2884379	2884379	3014774	5896262
303	6500043	6500043	7563999	7563999	8357121	8357121	8738192	17086836
327	18102016	18102016	21063317	21063317	23273934	23273934	24329868	47580312
351	48641673	48641673	56602448	56602448	62539370	62539370	65385226	127861367
375	126586391	126586391	147298443	147298443	162753713	162753713	170146608	332736007
399	319954666	319954666	372313294	372313294	411370026	411370026	430076524	841030812
423	787500731	787500731	916360448	916360448	1012501148	1012501148	1058511762	2069990475
447	1891483817	1891483817	2201005302	2201005302	2431907356	2431907356	2542467146	4971917548
471	4442040173	4442040173	5168908580	5168908580	5711193875	5711193875	5970765562	11676191108
495	10216574441	10216574441	11888393029	11888393029	13135596047	13135596047	13732710656	26855037539
519	23047036703	23047036703	26818346389	26818346389	29631903464	29631903464	30978745714	60580719175
543	51059233415	51059233415	59414411523	59414411523	65647584650	65647584650	68631657584	134212929307
567	111221993159	111221993159	129421909124	129421909124	142999706146	142999706146	149499558772	292354823899

Table 39: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
-1	-1	0	0	0	0	0	0	0
23	0	0	0	0	1	0	0	0
47	0	0	0	0	0	0	0	0
71	0	0	0	0	1	1	0	0
95	0	1	1	1	2	1	1	1
119	0	1	2	1	7	7	5	5
143	0	3	8	8	23	22	24	24
167	1	5	27	29	93	92	101	101
191	0	13	103	113	336	336	386	386
215	2	37	357	381	1192	1191	1370	1370
239	6	122	1181	1292	3911	3909	4541	4541
263	18	357	3673	3990	12260	12259	14231	14231
287	49	1065	10971	11987	36504	36503	42467	42467
311	135	2983	31296	34082	104381	104379	121375	121375
335	379	8247	86127	94007	286919	286918	333865	333865
359	995	21796	228754	249432	762672	762670	887304	887304
383	2564	56231	589479	643218	1964549	1964545	2286050	2286050
407	6386	140568	1476138	1610016	4920870	4920869	5725731	5725731
431	15622	343337	3603184	3931104	12009753	12009751	13975118	13975118
455	37167	817799	8587440	9367446	28625805	28625800	33309272	33309272
479	86756	1907498	20024583	21845882	66746654	66746652	77669274	77669274
503	198014	4357044	45751192	49908836	152506325	152506321	177460315	177460315
527	444173	9769828	102575149	111902115	341912961	341912958	397863480	397863480
551	977997	21517663	225940987	246477854	753141922	753141920	876380147	876380147
575	2119404	46622397	489518783	534024793	1631720458	1631720453	1898731972	1898731972

Table 40: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part II.

-D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
-1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	1	0	0	0	0	0	0
71	0	1	1	1	0	1	1	2
95	3	5	3	5	4	5	5	8
119	8	21	22	25	27	27	32	46
143	49	99	92	120	123	138	146	214
167	193	396	402	507	536	572	628	909
191	765	1535	1513	1946	2039	2212	2385	3459
215	2657	5375	5400	6874	7271	7783	8486	12269
239	8900	17881	17816	22816	24056	25887	28072	40626
263	27745	55882	55963	71465	75500	81007	88095	127410
287	83033	166945	166766	213315	225153	241943	262694	380029
311	236919	476743	476982	609553	643803	691150	751132	1086405
335	652314	1311878	1311412	1676845	1770483	1901666	2065592	2987879
359	1732646	3485567	3486207	4456239	4706100	5053141	5490489	7941451
383	4465544	8981556	8980403	11481470	12123772	13020310	14144455	20459305
407	11182141	22493230	22494840	28756211	30367390	32608993	35428696	51244665
431	27296592	54903746	54901023	70187986	74116978	79593979	86469919	125073387
455	65055014	130856090	130859914	167289297	176659468	189704760	206102830	298112039
479	151701090	305132577	305126393	390081042	411922132	442353736	480575987	695120690
503	346597295	697161444	697170138	891261397	941176611	1010687552	1098039598	1588233526
527	777084367	1563042221	1563028664	1998200428	2110092165	2265963373	2461774474	3560784919
551	1711669416	3442912979	3442931988	4401459954	4647951685	4991249366	5422610680	7843413933
575	3708476391	7459319124	7459290309	9536046829	10070049592	10813885859	11748391635	16993217734

Table 41: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of McL - part III.

-D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
-1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	1
71	2	1	1	1	1	1	1	2
95	8	8	8	10	10	10	11	11
119	46	46	46	53	53	55	59	60
143	214	221	221	261	261	263	279	279
167	909	928	928	1076	1076	1110	1175	1175
191	3459	3561	3561	4167	4167	4252	4487	4489
215	12269	12607	12607	14687	14687	15064	15899	15900
239	40626	41815	41815	48824	48824	49950	52683	52683
263	127410	131025	131025	152796	152796	156541	165130	165131
287	380029	391030	391030	456320	456320	467137	492677	492679
311	1086405	1117568	1117568	1303640	1303640	1335140	1408229	1408230
335	2987879	3074116	3074116	3586785	3586785	3672503	3873309	3873312
359	7941451	8169871	8169871	9531019	9531019	9760341	10294270	10294273
383	20459305	21049092	21049092	24558072	24558072	25146550	26521644	26521644
407	51244665	52720052	52720052	61505504	61505504	62983039	66427796	66427799
431	125073387	128677387	128677387	150125530	150125530	153726332	162132945	162132948
455	298112039	306697753	306697753	357811165	357811165	366401965	386440375	386440376
479	695120690	715147222	715147222	834342808	834342808	854361992	901084137	901084142
503	1588233526	1633981014	1633981014	1906304694	1906304694	1952063468	2058818682	2058818686
527	3560784919	3663364558	3663364558	4273934999	4273934999	4376498280	4615836213	4615836219
551	7843413933	8069346991	8069346991	9414224038	9414224038	9640181689	10167382764	10167382771
575	16993217734	17482746319	17482746319	20396558046	20396558046	20886051640	22028253515	22028253520

Table 42: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
-9	1	0	0	0	0	0	0	0
15	0	0	1	0	0	0	0	0
39	0	0	0	1	0	0	0	0
63	0	0	1	0	0	0	0	0
87	1	0	1	1	1	1	0	0
111	0	0	4	1	3	3	2	2
135	1	0	6	5	12	12	14	14
159	1	3	24	23	55	55	62	62
183	2	5	72	72	219	219	247	247
207	2	22	245	259	777	777	905	905
231	6	73	804	865	2644	2644	3060	3060
255	18	251	2556	2785	8404	8404	9775	9775
279	36	716	7659	8308	25493	25493	29645	29645
303	105	2128	22218	24238	73822	73822	85899	85899
327	276	5867	61728	67254	205653	205653	239219	239219
351	730	15803	165837	180937	552420	552420	642880	642880
375	1881	41051	431408	470406	1437895	1437895	1672987	1672987
399	4756	103927	1090489	1189748	3633969	3633969	4228700	4228700
423	11625	255433	2683334	2926781	8944652	8944652	10408024	10408024
447	27977	614030	6445575	7031919	21483285	21483285	24998994	24998994
471	65536	1441297	15135916	16510850	50453393	50453393	58708484	58708484
495	150811	3315763	34812830	37978515	116039405	116039405	135028617	135028617
519	339981	7478672	78530602	85667592	261770033	261770033	304603271	304603271
543	753426	16570437	173981507	189800269	579930706	579930706	674830133	674830133
567	1640491	36091901	378977806	413425592	1263264431	1263264431	1469977138	1469977138

Table 43: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part II.

-D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
-9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	1	0
63	0	0	0	1	0	1	1	0
87	1	1	2	2	2	2	6	4
111	7	13	10	15	14	19	21	25
135	24	54	56	75	78	81	95	130
159	129	254	244	320	334	365	398	562
183	482	975	986	1251	1328	1414	1566	2235
207	1776	3566	3536	4559	4792	5173	5607	8093
231	5969	12026	12064	15394	16274	17445	19023	27451
255	19154	38471	38380	49139	51846	55752	60533	87506
279	57809	116393	116510	148894	157282	168786	183543	265369
303	167931	337637	337390	431517	455558	489419	531575	768802
327	467083	939698	940034	1201486	1268952	1362371	1480572	2141239
351	1255917	2525910	2525280	3228953	3409380	3661782	3977741	5753503
375	3267135	6572140	6573020	8402301	8873332	9527924	10352451	14973440
399	8260107	16613698	16612158	21238355	22426902	24084770	26165020	37845752
423	20326853	40887536	40889694	52272318	55200610	59275991	64401011	93150396
447	48828286	98212448	98208856	125553959	132583074	142379305	154680790	223734839
471	114662453	230638380	230643438	294852626	311367160	334361281	363262412	525430528
495	263731843	530473486	530465432	678158814	716131132	769034337	835486965	1208473427
519	594922426	1196650730	1196662060	1529814670	1615490472	1734806216	1884740065	2726136749
543	1318037615	2651128016	2651110530	3389217725	3579004576	3843378364	4175506851	6039574829
567	2871035001	5774897738	5774922228	7382694796	7796138458	8371967563	9095496261	13155976526

Table 44: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *McL* - part III.

-D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
-9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0
63	0	0	0	1	1	1	1	1
87	4	5	5	5	5	5	6	6
111	25	27	27	32	32	31	34	34
135	130	132	132	153	153	161	168	168
159	562	583	583	685	685	695	733	733
183	2235	2298	2298	2671	2671	2743	2898	2898
207	8093	8336	8336	9753	9753	9968	10503	10503
231	27451	28225	28225	32895	32895	33718	35576	35576
255	87506	90070	90070	105135	105135	107589	113467	113467
279	265369	272959	272959	318375	318375	326136	343976	343976
303	768802	791067	791067	923068	923068	945024	996685	996685
327	2141239	2202779	2202779	2569637	2569637	2631602	2775597	2775597
351	5753503	5919487	5919487	6906560	6906560	7071840	7458445	7458445
375	14973440	15404421	15404421	17971134	17971134	18403192	19409807	19409807
399	37845752	38936669	38936669	45427164	45427164	46516151	49059813	49059813
423	93150396	95832723	95832723	111803325	111803325	114488565	120749919	120749919
447	223734839	230181494	230181494	268547565	268547565	274989746	290027769	290027769
471	525430528	540564243	540564243	630654439	630654439	645794430	681112299	681112299
495	1208473427	1243288847	1243288847	1450509683	1450509683	1485315445	1566542113	1566542113
519	2726136749	2804662530	2804662530	3272097750	3272097750	3350637733	3533877823	3533877823
543	6039574829	6213562155	6213562155	7249168236	7249168236	7423133737	7829083804	7829083804
567	13155976526	13534944287	13534944287	15790750779	15790750779	16169749982	17054036789	17054036789

Table 45: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
-1	-1	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0
71	0	0	0	1	0	0	1	0
95	0	2	3	2	5	5	3	6
119	0	2	8	19	14	14	21	24
143	0	16	44	80	90	90	92	131
167	4	50	174	357	339	339	407	515
191	7	203	681	1334	1372	1372	1521	2039
215	33	672	2372	4787	4720	4720	5435	7112
239	99	2292	7929	15766	15887	15887	17929	23777
263	330	7047	24733	49595	49392	49392	56348	74177
287	948	21207	73979	147695	148048	148048	167874	221911
311	2759	60245	211115	422628	422054	422054	480211	633338
335	7516	166233	581198	1161707	1162651	1162651	1320213	1743556
359	20108	440909	1543852	3088701	3087208	3087208	3509785	4631476
383	51591	1137213	3978753	7955779	7958171	7958171	9040892	11936173
407	129518	2846173	9963420	19929384	19925689	19925689	22646724	29890200
431	315659	6949871	24321125	48638189	48643912	48643912	55271192	72963281
455	753088	16560006	57964332	115934578	115925926	115925926	131743166	173892768
479	1754960	38620903	135165363	270321371	270334509	270334509	307184590	405495839
503	4011289	88231069	308818831	617651188	617631664	617631664	701874719	926456288
527	8990981	197828069	692381433	1384742351	1384771352	1384771352	1573573454	2077143971
551	19807924	435736187	1525098819	3050227006	3050184616	3050184616	3466163569	4575295928
575	42910275	944081846	3304250864	6608457916	6608519893	6608519893	7509616830	9912751919

Table 46: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part II.

-D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
-1	0	0	0	0	0	0	0	0
23	0	0	1	0	0	0	0	0
47	0	1	0	0	1	0	0	0
71	3	2	3	4	3	3	3	2
95	14	19	15	15	18	17	17	22
119	79	85	94	94	95	106	106	117
143	375	432	413	412	449	478	478	575
167	1585	1739	1776	1777	1887	2061	2061	2396
191	6059	6780	6708	6709	7219	7803	7803	9247
215	21454	23769	23898	23897	25544	27789	27789	32642
239	71118	79177	78950	78950	84686	91864	91864	108441
263	222914	247447	247836	247836	265365	288353	288353	339513
287	665120	739461	738796	738797	791857	859693	859693	1013725
311	1901066	2111639	2112736	2112736	2263145	2458353	2458353	2896427
335	5228981	5811159	5809370	5809369	6225096	6760018	6760018	7968570
359	13897177	15439550	15442403	15442404	16544170	17969139	17969139	21175541
383	35804287	39785466	39780944	39780943	42624415	46290660	46290660	54560442
407	89677291	99637115	99644150	99644149	106758449	115949075	115949075	136648460
431	218879656	243206606	243195756	243195757	260571613	282991729	282991729	333534637
455	521694040	579649779	579666270	579666271	621063719	674519766	674519766	794956158
479	1216464055	1351642677	1351617767	1351617768	1448172795	1572792424	1572792424	1853668197
503	2779404150	3088203718	3088240865	3088240863	3308813035	3593587213	3593587213	4235268766
527	6231379962	6923790672	6923735648	6923735648	7418312272	8056712554	8056712554	9495455515
551	13725964622	15251021256	15251101926	15251101928	16340430775	17746732420	17746732420	20915725781
575	29738144692	33042457875	33042340266	33042340266	35402559005	38449273233	38449273233	45315309735

Table 47: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part III.

-D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
-1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
47	0	0	0	1	1	1	1	1
71	4	5	6	4	6	7	9	9
95	26	27	35	42	40	53	56	67
119	162	164	202	215	221	286	317	361
143	740	756	941	1054	1046	1371	1487	1743
167	3180	3230	4009	4374	4396	5744	6289	7290
191	12073	12281	15273	16866	16834	22047	24028	28003
215	42967	43635	54213	59519	59591	77977	85159	99000
239	142122	144411	179524	197695	197583	258692	282195	328520
263	445985	453032	563011	618937	619159	810439	884610	1029079
287	1329897	1351104	1679391	1847957	1847627	2418829	2639284	3071601
311	3802615	3862871	4800963	5280034	5280632	6912578	7544029	8777736
335	10457085	10623310	13203972	14526071	14525165	19015093	20749740	24146350
359	27795624	28236528	35094660	38601412	38602966	50534149	55147824	64169990
383	71606337	72743550	90413520	99459154	99456838	130198910	142079975	165332469
407	179357824	182203997	226459775	249099125	249102903	326096807	355863082	414090303
431	437754013	444703911	552723198	608005819	608000215	795929685	868568813	1010705622
455	1043395661	1059955647	1317413812	1449139547	1449148385	1897063405	2070216248	2408967373
479	2432916045	2471536933	3071871642	3379082403	3379069491	4423514093	4827235467	5617163685
503	5558825615	5647056684	7018707863	7720543541	7720563319	10106911327	11029385777	12834168544
527	12462733462	12660561531	15735791983	17309423366	17309394714	22659582306	24727692056	28774079262
551	27451966739	27887702980	34661549588	38127628080	38127670972	49912570342	54468122422	63381031865
575	59476232959	60420314767	75096290859	82606031374	82605970008	108138748123	118008497342	137319059194

Table 48: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
-9	1	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	1
39	0	0	1	0	0	0	1	0
63	0	0	0	0	1	1	0	1
87	1	0	2	3	1	1	3	3
111	0	2	6	10	13	13	12	21
135	1	4	25	54	45	45	61	73
159	2	36	118	217	232	232	251	349
183	8	117	436	882	850	850	1000	1298
207	19	455	1585	3138	3191	3191	3567	4768
231	76	1504	5333	10710	10609	10609	12169	15972
255	224	4906	17078	34000	34168	34168	38662	51207
279	680	14653	51536	103287	102985	102985	117337	154628
303	1937	42824	149641	298883	299362	299362	339698	448887
327	5448	118790	416242	832940	832131	832131	946482	1248628
351	14492	319869	1119026	2237183	2238459	2238459	2542349	3357136
375	37900	831414	2911159	5823587	5821517	5821517	6617566	8733350
399	95506	2103267	7359765	14717082	14720250	14720250	16724285	22079147
423	235400	5173804	18111488	36226467	36221486	36221486	41165993	54334528
447	564823	12431424	43505965	87006255	87013768	87013768	98871502	130517592
471	1327268	29188063	102164805	204337166	204325655	204325655	232200431	306494073
495	3051197	67140916	234984819	469957788	469974829	469974829	534044327	704954744
519	6885123	151446502	530077046	1060170981	1060145461	1060145461	1204737801	1590230348
543	15250485	335540316	1174370946	2348714622	2348751902	2348751902	2668997408	3523111975
567	33223874	730874824	2558094287	5116227145	5116172310	5116172310	5813889540	7674283501

Table 49: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part II.

-D	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}
-9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0
63	1	1	0	0	1	1	1	3
87	7	7	10	10	9	10	10	10
111	46	54	48	48	56	54	54	71
135	227	239	254	254	268	299	299	344
159	990	1118	1089	1089	1183	1265	1265	1520
183	3908	4310	4367	4367	4655	5071	5071	5937
207	14181	15813	15709	15709	16879	18296	18296	21661
231	48025	53238	53426	53426	57171	62148	62148	73107
255	153185	170410	170086	170086	182391	197917	197917	233542
279	464346	515565	516126	516126	552755	600589	600589	707419
303	1345538	1495628	1494707	1494707	1601901	1739300	1739300	2050680
327	3747019	4162388	4163910	4163910	4460701	4845163	4845163	5709189
351	10068963	11189191	11186751	11186751	11986913	13017514	13017514	15344148
375	26203117	29112145	29116039	29116039	31194087	33880203	33880203	39927180
399	66230865	73593695	73587635	73587635	78846635	85629401	85629401	100925428
423	163012176	181118490	181127903	181127903	194061566	210766813	210766813	248396323
447	391537803	435051059	435036721	435036721	466117227	506225002	506225002	596634205
471	919501026	1021654112	1021675875	1021675875	1094643301	1188857894	1188857894	1401136588
495	2114832583	2349834128	2349801630	2349801630	2517659061	2734316595	2734316595	3222614325
519	4770733874	5300784874	5300833219	5300833219	5679443225	6168239640	6168239640	7269672253
543	10569264743	11743672359	11743601341	11743601341	12582461650	13665284242	13665284242	16105571907
567	23022947110	25580985954	25581089850	25581089850	27408265179	29767082573	29767082573	35082549764

Table 50: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of *HS* - part III.

-D	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}
-9	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
39	1	1	0	0	0	0	1	0
63	2	2	2	4	2	4	4	5
87	19	19	19	20	20	28	32	34
111	89	93	111	131	125	168	181	213
135	466	468	580	622	628	822	904	1043
159	1973	2007	2489	2770	2750	3612	3932	4587
183	7859	7976	9884	10828	10850	14204	15532	18023
207	28328	28783	35784	39464	39398	51610	56267	65550
231	96152	97662	121313	133278	133362	174568	190608	221641
255	306248	311150	386726	425729	425535	557162	607872	707534
279	929020	943669	1172777	1289537	1289797	1688384	1842742	2143915
303	2690703	2733527	3397533	3738214	3737680	4893216	5339398	6213691
327	7494835	7613625	9462567	10407432	10408176	13625020	14869371	17301389
351	20136861	20456746	25425987	27970908	27969538	36615272	39955933	46495887
375	52408277	53239679	66170567	72784072	72786002	95282852	103981276	120993550
399	132459084	134562341	167247845	183978541	183975195	240841718	262820443	305831383
423	326029062	331202866	411649683	452805738	452810502	592768390	646874829	752720667
447	783069157	795500581	988727552	1087614155	1087606345	1423778908	1553720168	1807974775
471	1839012867	1868200964	2321978347	2554155967	2554167059	3343632878	3648816574	4245880168
495	4229650434	4296791328	5340477195	5874555816	5874538248	7690311850	8392189392	9765479321
519	9541491409	9692937895	12047321308	13252008175	13252033063	17348106814	18931489848	22029335806
543	21138496831	21474037147	26690041661	29359113857	29359075739	38433714858	41941519745	48804725354
567	46045944874	46776819698	58138785609	63952566006	63952619706	83719772698	91360913541	106310810073

Table 51: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
-1	-1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	1
47	0	0	0	0	0	0	0	0	0	0
71	0	0	0	0	0	0	0	0	1	1
95	0	1	1	0	0	0	0	1	1	2
119	0	1	2	1	0	2	2	0	4	5
143	0	3	5	2	1	3	3	3	7	9
167	1	4	10	8	3	11	13	10	22	27
191	0	8	28	17	18	33	32	32	56	61
215	2	15	67	57	54	99	112	103	172	180
239	3	36	187	156	196	280	312	330	475	487
263	9	73	477	461	592	807	971	998	1394	1411
287	12	179	1318	1279	1803	2265	2723	2936	3892	3914
311	33	404	3451	3556	5080	6244	7741	8287	10843	10878
335	66	1029	9188	9534	14074	16751	20811	22680	29067	29109
359	160	2473	23558	25066	37214	43863	55156	60008	76443	76503
383	334	6120	59837	63872	96109	111842	140844	154261	194953	195033
407	795	14678	147655	159223	240240	278362	352289	385653	486133	486240
431	1763	35220	358129	386854	586899	676374	856699	940408	1181472	1181610
455	4088	82414	848112	920162	1397738	1607611	2040543	2239707	2810456	2810645
479	9121	190679	1971916	2141266	3260343	3741059	4750527	5220237	6541054	6541292
503	20500	431992	4492407	4887625	7446731	8536684	10850420	11923091	14931553	14931866
527	45009	964995	10058179	10947884	16697919	19121267	24309087	26726129	33447564	33447965
551	98354	2117328	22125220	24103804	36775178	42092946	53536679	58860381	73643280	73643797
575	210968	4579167	47903632	52199413	79680703	91155986	115951616	127512423	159487514	159488167

Table 52: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part II.

-D	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}
-1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	1	0
71	0	0	0	0	0	0	1	1	0
95	0	0	0	1	0	1	2	3	1
119	0	0	0	0	0	3	5	8	3
143	1	1	1	5	4	14	16	22	17
167	10	10	10	12	11	45	49	63	55
191	39	39	39	62	61	159	164	204	219
215	157	157	157	189	207	510	519	625	715
239	532	532	532	693	744	1642	1653	1946	2383
263	1732	1732	1732	2090	2302	4975	4992	5815	7273
287	5207	5207	5207	6424	7052	14671	14694	16983	21717
311	15071	15071	15071	18136	20097	41417	41450	47724	61506
335	41585	41585	41585	50419	55773	113397	113440	130238	169159
359	111048	111048	111048	133476	148102	299955	300016	343936	448042
383	286479	286479	286479	345208	382864	771309	771388	883312	1154153
407	718884	718884	718884	863390	958742	1928130	1928238	2206704	2886846
431	1755665	1755665	1755665	2110561	2343230	4702139	4702278	5378796	7045333
455	4187951	4187951	4187951	5027797	5584747	11198147	11198333	12806125	16782848
479	9767990	9767990	9767990	11731080	13029827	26101380	26101618	29842923	39131147
503	22326103	22326103	22326103	26797674	29770742	59614785	59615100	68152090	89385449
527	50061479	50061479	50061479	60096749	66762700	133631223	133631622	152753296	200393822
551	110289605	110289605	110289605	132364423	147060137	294300418	294300936	336394392	441360427
575	238965109	238965109	238965109	286811937	318652800	637563387	637564041	728720690	956216022

Table 53: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part III.

-D	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}	χ_{27}	χ_{28}
-1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0
71	1	1	0	1	0	0	0	0	0
95	0	2	1	2	1	1	1	1	1
119	4	8	5	7	5	4	4	4	4
143	12	24	22	24	20	22	24	24	24
167	59	89	81	85	83	86	97	97	97
191	204	281	291	296	302	345	388	388	388
215	750	942	982	991	1064	1194	1360	1360	1360
239	2420	2932	3180	3191	3473	4006	4564	4564	4564
263	7688	9022	9799	9816	10856	12481	14266	14266	14266
287	22754	26332	28983	29006	32232	37369	42728	42728	42728
311	65313	74730	82323	82356	92034	106605	122027	122027	122027
335	179162	203895	225671	225714	252683	293554	336080	336080	336080
359	476877	540363	598307	598368	671268	779662	892976	892976	892976
383	1227450	1387781	1539339	1539418	1728219	2009529	2301754	2301754	2301754
407	3076064	3471701	3851620	3851728	4327763	5031900	5764588	5764588	5764588
431	7505177	8462435	9395281	9395420	10559896	12283559	14072625	14072625	14072625
455	17892382	20159114	22383595	22383781	25167016	29274597	33540710	33540710	33540710
479	41714567	46979165	52179270	52179508	58675953	68265693	78215160	78215160	78215160
503	95319311	107312059	119196613	119196928	134058491	155968449	178705755	178705755	178705755
527	213691022	240528777	267203769	267204168	300539689	349688430	400669808	400669808	400669808
551	470720019	529752466	588518852	588519370	661990039	770250507	882558528	882558528	882558528
575	1019815583	1147598073	1274985629	1274986283	1434203044	1668815367	1912148601	1912148601	1912148601

Table 54: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part IV.

-D	χ_{29}	χ_{30}	χ_{31}	χ_{32}	χ_{33}	χ_{34}	χ_{35}	χ_{36}	χ_{37}
-1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	1	0	0	0	0	0	0	0	0
71	1	0	0	0	0	0	1	0	0
95	4	3	1	1	1	1	3	0	1
119	11	9	4	4	4	6	12	4	11
143	43	39	26	26	26	30	47	30	44
167	140	135	105	105	105	124	181	141	208
191	504	495	430	430	430	476	641	581	769
215	1659	1647	1512	1512	1512	1679	2189	2112	2800
239	5391	5373	5106	5106	5106	5576	7089	7143	9196
263	16513	16488	15981	15981	15981	17469	21956	22579	29034
287	48944	48909	47964	47964	47964	52139	64970	67811	86401
311	138812	138765	137052	137052	137052	149008	184876	194405	247540
335	380846	380778	377745	377745	377745	409888	506964	535979	680273
359	1009253	1009164	1003898	1003898	1003898	1089307	1345063	1426142	1809476
383	2597544	2597424	2588431	2588431	2588431	2806561	3461147	3677682	4660418
407	6498438	6498279	6483169	6483169	6483169	7029317	8662775	9215765	11676421
431	15853998	15853785	15828769	15828769	15828769	17157026	21132786	22502085	28495813
455	37769072	37768797	37727937	37727937	37727937	40892977	50353789	53644469	67927686
479	88050558	88050195	87984244	87984244	87984244	95353051	117386017	125107300	158383477
503	201135952	201135486	201030242	201030242	201030242	217864013	268167991	285875651	361897629
527	450900478	450899874	450733710	450733710	450733710	488448796	601164912	640979164	811353013
551	993105129	993104358	992844595	992844595	992844595	1075912674	1324106543	1411960695	1787223481
575	2151522265	2151521277	2151118874	2151118874	2151118874	2331033239	2868612560	3059214800	3872097331

Table 55: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{1}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part V.

-D	χ_{38}	χ_{39}	χ_{40}	χ_{41}	χ_{42}	χ_{43}	χ_{44}	χ_{45}	χ_{46}
-1	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0
71	0	0	0	0	1	1	1	0	0
95	2	2	2	3	3	3	3	3	2
119	9	11	11	12	17	17	18	21	17
143	49	55	55	67	69	69	87	94	91
167	199	228	228	265	295	295	362	409	415
191	786	881	881	1049	1088	1088	1384	1557	1631
215	2771	3105	3105	3657	3864	3864	4892	5575	5869
239	9251	10311	10311	12230	12681	12681	16220	18452	19609
263	28943	32259	32259	38155	39773	39773	50831	57985	61760
287	86560	96303	96303	114151	118296	118296	151675	172971	184763
311	247286	275091	275091	325787	338162	338162	433503	494841	528970
335	680702	756753	756753	896888	929109	929109	1192384	1360994	1456375
359	1808802	2010778	2010778	2382438	2469363	2469363	3168960	3618306	3873101
383	4661495	5180729	5180729	6140008	6359343	6359343	8164490	9321948	9982360
407	11674769	12974774	12974774	15375595	15927891	15927891	20449076	23351239	25008800
431	28498396	31668560	31668560	37532560	38869434	38869434	49911204	56994271	61050043
455	67923794	75478392	75478392	89450970	92643752	92643752	118961732	135851564	145527486
479	158389396	175998083	175998083	208588414	216007314	216007314	277390339	316772980	339359186
503	361888871	402117862	402117862	476572073	493536047	493536047	633786948	723786642	775414671
527	811366088	901543092	901543092	1068489749	1106464669	1106464669	1420941137	1622719280	1738525374
551	1787204411	1985829461	1985829461	2353547126	2437220934	2437220934	3129929312	3574428126	3829570352
575	3872125240	4302423193	4302423193	5099152355	5280310504	5280310504	6781185810	7744222864	8297132508

Table 56: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part I.

-D	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
-9	1	0	0	0	0	0	0	0	0	0
15	0	0	1	0	0	0	0	0	0	0
39	0	0	0	1	0	0	0	0	0	0
63	0	0	1	0	0	1	0	0	0	0
87	1	0	1	1	0	1	0	0	1	1
111	0	0	4	1	0	2	0	0	2	2
135	1	0	5	4	1	5	2	0	5	5
159	1	2	14	8	4	13	6	7	14	14
183	2	2	24	19	10	30	24	20	43	43
207	2	8	62	40	42	79	66	66	113	113
231	5	16	140	124	128	213	228	223	342	342
255	10	54	379	334	423	585	648	693	964	964
279	17	107	946	928	1242	1639	1952	2037	2782	2782
303	30	299	2574	2556	3642	4502	5434	5889	7697	7697
327	64	716	6655	6947	10035	12149	15072	16267	21026	21026
351	122	1817	17366	18206	27087	31983	39848	43501	55450	55450
375	287	4447	43965	47011	70199	82214	103514	112936	143220	143220
399	608	10973	109822	117781	177721	206063	259946	284997	359275	359275
423	1400	26176	267074	288683	436768	504591	639120	700427	881263	881263
447	3111	62208	638341	691001	1049704	1207687	1530976	1681362	2109987	2109987
471	7070	144100	1491474	1619959	2463620	2829836	3593518	3946210	4947675	4947675
495	15666	329562	3422339	3719775	5667740	6498220	8255052	9073094	11363145	11363145
519	34864	738943	7702776	8384938	12782104	14643912	18617246	20462399	25615443	25615443
543	75899	1632596	17046286	18562586	28320987	32418588	41222644	45326175	56711601	56711601
567	164118	3545538	37091514	40418996	61684063	70582578	89782368	98720482	123491729	123491729

Table 57: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part II.

-D	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}
-9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0
63	0	0	0	0	0	0	0	0	0
87	0	0	0	0	0	0	0	0	1
111	0	0	0	0	0	2	2	4	4
135	1	1	1	0	0	8	8	10	9
159	5	5	5	11	9	30	30	40	44
183	26	26	26	30	32	100	100	127	139
207	97	97	97	136	143	350	350	416	497
231	363	363	363	436	475	1106	1106	1310	1596
255	1172	1172	1172	1484	1607	3458	3458	4034	5088
279	3638	3638	3638	4370	4842	10248	10248	11856	15102
303	10611	10611	10611	12975	14297	29442	29442	33922	43783
327	29814	29814	29814	35856	39741	81284	81284	93406	121084
351	80286	80286	80286	97039	107493	217714	217714	249613	325236
375	209675	209675	209675	251918	279626	564586	564586	646742	844320
399	530506	530506	530506	638563	708547	1425020	1425020	1631017	2133713
423	1307537	1307537	1307537	1570043	1743695	3502378	3502378	4006776	5246147
447	3142048	3142048	3142048	3775558	4192520	8406902	8406902	9614460	12599677
471	7383374	7383374	7383374	8863180	9845654	19730666	19730666	22560343	29576649
495	16985215	16985215	16985215	20394871	22654755	45366836	45366836	51864617	68021751
519	38326695	38326695	38326695	46000805	51105811	102311008	102311008	116954636	153417371
543	84919350	84919350	84919350	101933701	113244297	226631616	226631616	259049862	339876627
567	185003006	185003006	185003006	222026326	246680793	493602894	493602894	564184563	740284029

Table 58: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part III.

-D	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}	χ_{27}	χ_{28}
-9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	1	0	0	0	0	0	0	0
63	0	1	0	0	0	0	0	0	0
87	1	3	1	1	0	1	0	0	0
111	2	7	4	4	2	4	3	3	3
135	9	20	15	15	12	11	13	13	13
159	32	60	55	55	48	59	63	63	63
183	142	204	193	193	196	219	243	243	243
207	477	630	662	662	698	801	909	909	909
231	1672	2051	2165	2165	2352	2689	3057	3057	3057
255	5228	6225	6779	6779	7442	8627	9834	9834	9834
279	15977	18566	20286	20286	22548	26013	29759	29759	29759
303	46068	52957	58400	58400	65100	75588	86454	86454	86454
327	128664	146652	161869	161869	181184	210189	240627	240627	240627
351	345074	391657	433851	433851	486308	565195	647221	647221	647221
375	899003	1017030	1126986	1126986	1265124	1470206	1683993	1683993	1683993
399	2270753	2564491	2845570	2845570	3196124	3717138	4257972	4257972	4257972
423	5591173	6305860	6998318	6998318	7865522	9147020	10479406	10479406	10479406
447	13425927	15130943	16801605	16801605	18887958	21972919	25174005	25174005	25174005
471	31535156	35518958	39444133	39444133	44354258	51598009	59118234	59118234	59118234
495	72522016	81656325	90701880	90701880	102004268	118679661	135979088	135979088	135979088
519	163610220	184167767	204577702	204577702	230098050	267714806	306745221	306745221	306745221
543	362451055	407928442	453185010	453185010	509745480	593117732	679593708	679593708	679593708
567	789548632	888501021	987096186	987096186	1110356978	1291964951	1480349568	1480349568	1480349568

Table 59: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part IV.

-D	χ_{29}	χ_{30}	χ_{31}	χ_{32}	χ_{33}	χ_{34}	χ_{35}	χ_{36}	χ_{37}
-9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0
63	1	1	0	0	0	0	1	0	0
87	2	2	0	0	0	1	2	0	2
111	8	8	3	3	3	4	8	3	5
135	24	24	13	13	13	17	35	16	32
159	95	95	69	69	69	77	118	89	123
183	322	322	267	267	267	306	424	367	517
207	1124	1124	1011	1011	1011	1113	1471	1390	1822
231	3633	3633	3411	3411	3411	3763	4830	4796	6268
255	11444	11444	11019	11019	11019	12011	15140	15497	19864
279	34164	34164	33372	33372	33372	36393	45506	47238	60491
303	98541	98541	97104	97104	97104	105478	131014	137526	174934
327	272918	272918	270372	270372	270372	293697	363656	383784	487925
351	732055	732055	727612	727612	727612	789280	975053	1033033	1310274
375	1901107	1901107	1893498	1893498	1893498	2053906	2534016	2690675	3411830
399	4801562	4801562	4788741	4788741	4788741	5191571	6399104	6805582	8621691
423	11807813	11807813	11786521	11786521	11786521	12777678	15741552	16756387	21225130
447	28351578	28351578	28316673	28316673	28316673	30690911	37794342	40258951	50975995
471	66557162	66557162	66500652	66500652	66500652	72075158	88736216	94560774	119725056
495	153056164	153056164	152965722	152965722	152965722	165771974	204055733	217516112	275355244
519	345212969	345212969	345069777	345069777	345069777	373954834	460266342	490719721	621183917
543	764740650	764740650	764516151	764516151	764516151	828475182	1019607264	1087225658	1376174113
567	1665696713	1665696713	1665348005	1665348005	1665348005	1804658930	2220884907	2368380807	2997763988

Table 60: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/24}$ in the function $\tilde{h}_{g,\frac{3}{2}}(\tau)$ into irreducible representations χ_n of $U_6(2)$ - part V.

-D	χ_{38}	χ_{39}	χ_{40}	χ_{41}	χ_{42}	χ_{43}	χ_{44}	χ_{45}	χ_{46}
-9	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0
63	0	0	0	1	0	0	0	0	0
87	1	1	1	1	2	2	2	2	0
111	6	7	7	9	8	8	11	9	9
135	28	32	32	38	45	45	51	58	54
159	130	145	145	175	179	179	225	250	255
183	503	566	566	662	715	715	891	1012	1044
207	1846	2060	2060	2457	2536	2536	3233	3662	3861
231	6220	6951	6951	8205	8608	8608	10951	12474	13215
255	19937	22198	22198	26322	27278	27278	34934	39779	42381
279	60355	67201	67201	79550	82728	82728	105878	120825	128899
303	175147	194787	194787	230873	239200	239200	306831	350044	374214
327	487555	542191	542191	642249	666188	666188	854421	975422	1043352
351	1310841	1457089	1457089	1726958	1788772	1788772	2296034	2621064	2805657
375	3410897	3791267	3791267	4492367	4655102	4655102	5975070	6822630	7304754
399	8623112	9583033	9583033	11357471	11762638	11762638	15102657	17244669	18468738
423	21222871	23584843	23584843	27949916	28950941	28950941	37171558	42447872	45465426
447	50979357	56648802	56648802	67138492	69528362	69528362	89282411	101955129	109216491
471	119719865	133032283	133032283	157661257	163281960	163281960	209673562	239444613	256509711
495	275362888	305972205	305972205	362631248	375524639	375524639	482245140	550717850	589999521
519	621172387	690217270	690217270	818019198	847121958	847121958	1087869446	1242355809	1330999539
543	1376190840	1529134899	1529134899	1812299925	1876700029	1876700029	2410108430	2752364303	2948823657
567	2997739281	3330883649	3330883649	3947676846	4087990850	4087990850	5249918560	5995502658	6423520020

References

[1] J. H. Conway and S. P. Norton, "Monstrous moonshine," Bull. London Math. Soc. **11**, 308 (1979).

[2] I. Frenkel, J. Lepowsky and A. Meurman, *A moonshine module for the Monster*, in *Vertex operators in mathematics and physics (Berkeley, Calif., 1983)*, vol. 3 of Math. Sci. Res. Inst. Publ., Springer, 1985.

[3] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, vol. 134 of Pure and Applied Mathematics, Elsevier Science, 1989.

[4] R. E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras," Invent. Math. **109**, 405 (1992).

[5] L. J. Dixon, P. H. Ginsparg and J. A. Harvey, "Beauty and the beast: Superconformal symmetry in a Monster module," Comm. Math. Phys. **119**, 221 (1988).

[6] T. Gannon, *Moonshine beyond the Monster: The bridge connecting algebras, modular forms, and physics*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2006.

[7] J. F. R. Duncan, M. J. Griffin, and K. Ono, "Moonshine", Research in the Mathematical Sciences (2015) 2:11. arXiv:1411.6571.

[8] M. C. N. Cheng, J. F. R. Duncan and J. A. Harvey, "Umbral Moonshine," Communications in Number Theory and Physics 8 (2014), no. 2. arXiv:1204.2779 [math.RT].

[9] M. C. N. Cheng, J. F. R. Duncan and J. A. Harvey, "Umbral Moonshine and the Niemeier Lattices," Research in the Mathematical Sciences 2014 **1**:3. arXiv:1307.5793 [math.RT].

[10] M. C. N. Cheng, J. F. R. Duncan and J. A. Harvey, "Weight One Jacobi Forms and Umbral Moonshine," (to appear).

[11] T. Eguchi, H. Ooguri and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," Exper. Math. **20**, 91 (2011) [arXiv:1004.0956 [hep-th]].

[12] M. C. N. Cheng, "K3 Surfaces, N=4 Dyons, and the Mathieu Group M_{24}," Commun. Num. Theor. Phys. **4**, 623 (2010) [arXiv:1005.5415 [hep-th]].

[13] M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu twining characters for K3," JHEP **1009**, 058 (2010) [arXiv:1006.0221 [hep-th]];
M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," JHEP **1010**, 062 (2010) [arXiv:1008.3778 [hep-th]].

[14] T. Eguchi and K. Hikami, "Note on Twisted Elliptic Genus of K3 Surface," Phys. Lett. B **694**, 446 (2011) [arXiv:1008.4924 [hep-th]].

[15] T. Gannon, "Much ado about Mathieu," arXiv:1211.5531 [math.RT].

[16] A. Taormina and K. Wendland, "The Symmetries of the tetrahedral Kummer surface in the Mathieu group M_{24}," arXiv:1008.0954 [hep-th].

[17] M. R. Gaberdiel, S. Hohenegger and R. Volpato, "Symmetries of K3 sigma models," Commun. Num. Theor. Phys. **6**, 1 (2012) [arXiv:1106.4315 [hep-th]].

[18] S. Govindarajan, "Unravelling Mathieu Moonshine," Nucl. Phys. B **864**, 823 (2012) [arXiv:1106.5715 [hep-th]].

[19] A. Taormina and K. Wendland, "The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}," JHEP **1308**, 125 (2013) [arXiv:1107.3834 [hep-th]].

[20] M. C. N. Cheng and J. F. R. Duncan, "On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine," Commun. Num. Theor. Phys. **6**, 697 (2012) [arXiv:1110.3859 [math.RT]].

[21] M. Raum, "M24-twisted Product Expansions are Siegel Modular Forms," arXiv:1208.3453 [math.NT].

[22] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch and R. Volpato, "Generalised Mathieu Moonshine," Commun. Num. Theor. Phys. **7**, 145 (2013) [arXiv:1211.7074 [hep-th]].

[23] M. C. N. Cheng and J. F. R. Duncan, "On the Discrete Groups of Mathieu Moonshine," Perspectives in Representation Theory, Contemporary Mathematics 610 (2014). arXiv:1212.0906 [math.NT].

[24] M. R. Gaberdiel, D. Persson and R. Volpato, "Generalised Moonshine and Holomorphic Orbifolds," arXiv:1302.5425 [hep-th].

[25] A. Taormina and K. Wendland, "Symmetry-surfing the moduli space of Kummer K3s," arXiv:1303.2931 [hep-th].

[26] T. Creutzig and G. Hoehn, "Mathieu Moonshine and the Geometry of K3 Surfaces," arXiv:1309.2671 [math.QA].

[27] D. Persson and R. Volpato, "Second Quantized Mathieu Moonshine," arXiv:1312.0622 [hep-th].

[28] M. C. N. Cheng and S. Harrison, "Umbral Moonshine and K3 Surfaces," arXiv:1406.0619 [hep-th].

[29] S. Hohenegger and S. Stieberger, "BPS Saturated String Amplitudes: K3 Elliptic Genus and Igusa Cusp Form," Nucl. Phys. B **856**, 413 (2012) [arXiv:1108.0323 [hep-th]].

[30] M. C. N. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru and T. Wrase, "Mathieu Moonshine and N=2 String Compactifications," JHEP **1309**, 030 (2013) [arXiv:1306.4981 [hep-th]].

[31] J. A. Harvey and S. Murthy, "Moonshine in Fivebrane Spacetimes," JHEP **1401**, 146 (2014) [arXiv:1307.7717 [hep-th]].

[32] S. Harrison, S. Kachru and N. M. Paquette, "Twining Genera of (0,4) Supersymmetric Sigma Models on K3," JHEP **1404**, 048 (2014) [arXiv:1309.0510 [hep-th]].

[33] T. Wrase, "Mathieu moonshine in four dimensional $\mathcal{N} = 1$ theories," JHEP **1404**, 069 (2014) [arXiv:1402.2973 [hep-th]].

[34] K. Ono, L. Rolen, and S. Trebat-Leder, *Classical and Umbral Moonshine: Connections and p-adic Properties*, J. Ramanujan Math. Soc., to appear (ArXiv e-prints (2014)).

[35] J. F. R. Duncan and S. Mack-Crane, "Derived Equivalences of K3 Surfaces and Twined Elliptic Genera," arXiv:1506.06198.

[36] J. F. R. Duncan and J. A. Harvey, "The Umbral Mooonshine Module for the Unique Unimodular Niemeier Root System", arXiv:1412.8191.

[37] J. F. Duncan, "Super-moonshine for Conway's largest sporadic group," Duke Math. J., 139(2):255-315, 2007. arXiv:math/0502267.

[38] J. F. R. Duncan and S. Mack-Crane, "The Moonshine Module for Conway's Group," Forum of Mathematics, Sigma 3 (2015). arXiv:1409.3829.

[39] N. Benjamin, S. M. Harrison, S. Kachru, N. M. Paquette and D. Whalen, "On the elliptic genera of manifolds of Spin(7) holonomy," arXiv:1412.2804.

[40] M. C. N. Cheng, S. M. and Harrison, S. Kachru D. and Whalen, "Exceptional Algebra and Sporadic Groups at c=12," arXiv:1503.07219.

[41] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. Springer, New York, 1999.

[42] J. F. R. Duncan, M. J. Griffin, and K. Ono, "Proof of the Umbral Moonshine Conjecture," arXiv:1503.01472.

[43] J. F. Duncan and I. B. Frenkel, "Rademacher sums, Moonshine and Gravity," Commun. Num. Theor. Phys. **5**, 849 (2011) [arXiv:0907.4529 [math.RT]].

[44] C. Dong, H. Li, G. Mason, and P. S. Montague, "The radical of a vertex operator algebra," in *The Monster and Lie algebras (Columbus, OH, 1996)*, Ohio State Univ. Math. Res. Inst. Publ., **7** (1998).

[45] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.*, Oxford: Clarendon Press, 1985.

[46] T. Eguchi and A. Taormina, "Unitary representations of the $N = 4$ superconformal algebra," Phys. Lett. B **196**, 1 (1987).

[47] T. Eguchi and A. Taormina, "On the Unitary Representations of $N = 2$ and $N = 4$ Superconformal Algebras," Phys. Lett. B **210**, 125 (1988).

[48] V. Kac, "Vertex algebras for beginners," University Lecture Series, American Mathematical Society, **10** (1988).

[49] V. A. Kostelecky, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, "Conformal Techniques, Bosonization and Tree Level String Amplitudes," Nucl. Phys. B **288**, 173 (1987).

[50] T. Eguchi and A. Taormina, "Character Formulas for the $N = 4$ Superconformal Algebra," Phys. Lett. B **200**, 315 (1988).

[51] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," arXiv:1208.4074 [hep-th].

[52] S. Zwegers, "Mock Theta Functions," arXiv:0807.4834 [math.NT].

[53] V. K. Dobrev, "Characters of the Unitarizable Highest Weight Modules Over the $N = 2$ Superconformal Algebras," Phys. Lett. B **186**, 43 (1987).

[54] E. Kiritsis, "Character Formulae and the Structure of the Representations of the $N = 1$, $N = 2$ Superconformal Algebras," Int. J. Mod. Phys. A **3**, 1871 (1988).

[55] S. L. Shatashvili and C. Vafa, "Superstrings and manifold of exceptional holonomy," Selecta Math. **1**, 347 (1995) [hep-th/9407025].

[56] D. Gepner and B. Noyvert, "Unitary representations of SW(3/2,2) superconformal algebra," Nucl. Phys. B **610**, 545 (2001) [hep-th/0101116].

[57] C. D. D. Neumann, "The Elliptic genus of Calabi-Yau 3-folds and 4-folds: Product formulae and generalized Kac-Moody algebras," J. Geom. Phys. **29**, 5 (1999) [hep-th/9607029].

[58] E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th].

[59] R. Dijkgraaf, J. M. Maldacena, G. W. Moore and E. P. Verlinde, "A Black hole Farey tail," hep-th/0005003;
J. Manschot and G. W. Moore, "A Modern Farey Tail," Commun. Num. Theor. Phys. **4**, 103 (2010) [arXiv:0712.0573 [hep-th]].

[60] M. C. N. Cheng and J. F. R. Duncan, "Rademacher Sums and Rademacher Series," Conformal Field Theory, Automorphic Forms and Related Topics, Contributions in Mathematical and Computational Sciences, vol. 8, Springer, 2014, pp. 143-182. arXiv:1210.3066 [math.NT].

[61] N. Benjamin, S. M. Harrison, S. Kachru, N. M. Paquette and D. Whalen, "On the elliptic genera of manifolds of Spin(7) holonomy," arXiv:1412.2804 [hep-th].

[62] M. C. N. Cheng, S. M. Harrison, S. Kachru and D. Whalen, "Exceptional Algebra and Sporadic Groups at c=12," arXiv:1503.07219 [hep-th].

arXiv:2201.13321v2 [hep-th] 21 Feb 2022

Snowmass White Paper: Moonshine

Sarah M. Harrison[1,2], Jeffrey A. Harvey[3], and Natalie M. Paquette[4]

[1]Department of Mathematics and Statistics, McGill University, Montreal, QC, Canada
[2]Department of Physics, McGill University, Montreal, QC, Canada
[3]Enrico Fermi Institute and Department of Physics, University of Chicago ,
5640 Ellis Ave., Chicago IL 60637, USA
[4]Department of Physics, University of Washington, Seattle, WA, 98195, USA

Abstract

We present a brief overview of Moonshine with an emphasis on connections to physics. Moonshine collectively refers to a set of phenomena connecting group theory, analytic number theory, and vertex operator algebras or conformal field theories. Modern incarnations of Moonshine arise in various BPS observables in string theory and, via dualities, invariants in enumerative geometry. We survey old and new developments, and highlight some of the many open questions that remain.

Contents

1 Introduction 2

2 Connections to CFT/VOA 3

3 Connections to string theory 5

4 Mathematical connections 6

5 Open questions 8

1 Introduction

This paper contains a brief review of various types of moonshine starting with the earliest example of monstrous moonshine. Other reviews of moonshine with various points of view can be found in [2, 67, 66, 117] and the book [89] describes moonshine as of 2006.

Monstrous moonshine had its origins in observations of McKay and Thompson of connections between the representation theory of the Monster ($\mathbb{M}$), the largest sporadic finite simple group, and the elliptic modular invariant

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + \cdots . \tag{1.1}$$

Here $q = e^{2\pi i\tau}$ and τ is a complex parameter taking values in the upper half plane $\mathrm{Im}(\tau) > 0$. $J(\tau)$ is the unique function with a simple pole only at $q = 0$ with residue one and vanishing constant term that is invariant under modular transformations $\tau \to (a\tau + b)/(c\tau + d)$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The idea of a connection between modular functions and the representation theory of the Monster group is so outlandish that it was referred to as moonshine by Conway and Norton [43]. A connection between these two areas of mathematics eventually emerged with a striking connection to physics: consistent two-dimensional conformal field theories (CFTs) have partition functions which can be computed as path integrals on a two-dimensional torus and are modular invariant functions as a consequence of invariance under global diffeomorphisms acting on the torus. Thus the construction of a (holomorphic) CFT with Monster symmetry in [77, 79] (or more precisely a vertex operator algebra (VOA)) led to an explanation of this bizarre connection. A physical perspective on this construction was given in [53]. We now understand that it is easy to find more connections of this sort. One can start with a positive definite lattice L with a large automorphism group G and construct a lattice VOA whose partition function will have modular properties and have coefficients that can be written as dimensions of representations of a group closely related to G. However, the word moonshine, in the words of R. Borcherds, "...should only be applied to things that are weird and special; if there are an infinite number of examples of something, then it is not moonshine." The thing that makes Monstrous moonshine special is a connection to genus zero subgroups of $SL(2, \mathbb{R})$ as explained later in this review.

The classic era of moonshine involved the study of Monstrous moonshine and a supersymmetric variant known as Conway moonshine [78, 57, 62]. A new era of moonshine began in 2010 with the observation in [73] of a surprising connection between the elliptic genus of $K3$ surfaces and the representation theory of another sporadic group, the Mathieu group M_{24}.

This relation involves a new kind of modular object, a mock modular form. This relation was soon extended to umbral moonshine [26, 33, 36], which conjectured relationships between a set of 23 distinguished mock modular forms and 23 finite groups which arise from the automorphism groups of Niemeier lattices. The main conjecture of Mathieu and umbral moonshine was proven in [88, 58] but as we explain below, there is still a great deal of mystery surrounding the origins of Mathieu and umbral moonshine. More recently there have been additional new types of moonshine: for the O'Nan sporadic group [59] and penumbral moonshine [60] which encompasses moonshine for the Thompson group [105] as a special case, much as umbral moonshine generalizes Mathieu moonshine. Although the connections of these new types of moonshine to physics are at the moment unknown, they do deserve to be called moonshine in that they are all special and finite in number and this again arises through connections to genus zero subgroups of $SL(2,\mathbb{R})$ [33, 41, 60].

2 Connections to CFT/VOA

Monstrous moonshine had its origins in observations of J. McKay and J. Thompson of connections between the representation theory of the Monster, the largest sporadic finite simple group, and the elliptic modular invariant $J(\tau)$. For example, the coefficient $196884 = 1 + 196883$ is the sum of the dimensions of the two smallest irreducible representations of $\mathbb{M}$. These observations suggested the existence of an infinite-dimensional graded vector space, $V^\natural$, carrying representations of $\mathbb{M}$, and a generalization $T_g(\tau)$ of $J(\tau)$, known as McKay-Thompson series. These series are constructed by replacing the dimensions of representations by characters of group elements of $\mathbb{M}$. Conway and Norton organized and generalized these observations and made several conjectures, the most important of which [145] was that each of the $T_g(\tau)$ (which depend only on the conjugacy class of g) [1] were invariant under a special class of subgroups of $SL(2,\mathbb{R})$ known as genus zero subgroups because the quotient of the upper half plane by these groups can be given the structure of a genus zero Riemann surface. The proof of this genus zero conjecture involved the construction of a Vertex Operator Algebra (VOA) on $V^\natural$ by Frenkel, Lepowsky and Meurman (FLM) [77, 79] following on the definition of a Vertex Algebra by Borcherds [10] and followed by a tour de force proof by Borcherds of the genus zero property [11]. Borcherds defined and employed a new class of algebras known as Borcherds-Kac-Moody (BKM) algebras, see [90] for a discussion of his work.

Remarkably these ideas and techniques have a close relationship to physics. In particular, a VOA is, roughly speaking, equivalent to the chiral part of a two-dimensional conformal field theory (CFT) and the construction of FLM can be viewed as the first example of an orbifold CFT, an idea which was developed independently in the context of string compactifications and then eventually abstracted into a general mechanism for constructing new CFTs from old by gauging global symmetries. Recently, a physical explanation for the genus zero property of Monstrous moonshine has been proposed in the context of a family of special heterotic string compactifications including $V^\natural$ or its orbifolds on the worldsheet [135, 134]; there, the genus zero property is shown to follow from self-duality of the models under T-dualities.

[1] "Generalized moonshine" for a larger class of functions $T_{g,h}(\tau)$ labeled by commuting pairs $g, h \in \mathbb{M}$ was conjectured by Norton [129, 130] and proved by Carnahan [16, 18, 19], who constructed new BKMs to establish their genus zero properties. The natural physical origin of these functions is as twisted sector partition functions of orbifolds of $V^\natural$; see e.g. [53, 147, 148] for details.

Moonshine for another sporadic group, the Conway group Co_1 arises in a superconformal field theory with central charge $c = 12$ [78, 57, 62]. The corresponding McKay-Thompson series are also associated to genus zero subgroups of $SL(2, \mathbb{R})$ [62].

Moonshine relations between modular functions and other sporadic groups that are involved in the Monster group have also been constructed using ideas from conformal field theory starting with work of Höhn on the Baby Monster group [109]. An important ingredient in his and other constructions is the fact that certain minimal model CFTs and parafermion CFTs can be embedded in the Monster CFT and the discrete symmetries of the minimal model or parafermion CFT then act as elements of the Monster group. For example, the $\mathbb{Z}_2$ symmetry of the $c = 1/2$ Ising model acts as an element of the 2A conjugacy class of the Monster, an obervation due to Miyamoto. Many generalizations of this idea have been worked out, see [54, 5] for further examples and constructions. A similar phenomenon for the Conway CFT is explored in [25].

One of the goals of current research into moonshine is to find CFT/VOA explanations for umbral and penumbral moonshine. The Mathieu moonshine example of umbral moonshine arose through a study of the decomposition of the elliptic genus of $K3$ surfaces into representations of the $N = 4$ superconformal algebra that exists for any $K3$ sigma-model. The multiplicities of massive $N = 4$ characters are captured by a weight $1/2$ mock modular form with q expansion

$$H^{(2)}(\tau) = 2(-q^{-1/8} + 45q^{7/8} + 231q^{15/8} + \cdots). \tag{2.1}$$

Additionally, associated to each $g \in M_{24}$ is a McKay-Thompson series $H_g^{(2)}(\tau)$, also a weight $1/2$ mock modular form for a subgroup of $SL(2, \mathbb{Z})$, whose form was originally conjectured in [40, 81, 80, 72].

The occurrence of dimensions of M_{24} irreducible representations would not be surprising if there existed $K3$ sigma-models with M_{24} group actions that preserve the $N = 4$ algebra. However both classical algebro-geometric analysis [127] and quantum analysis of $K3$ sigma models [82] shows that no such $K3$ sigma-model exists[2]. The result of the classification of [82] is that all symmetry groups of K3 sigma-models are (proper) subgroups of the Conway group, the automorphism group of the Leech lattice. In particular, this implies there exist K3 sigma-models with symmetries which lie outside of M_{24}, as well as elements of M_{24} which can never appear as symmetries of K3 sigma models. This result was generalized in [31] to include singular points in the moduli space of K3 CFTs (which correspond to points of enhanced gauge symmetry in spacetimes), and where the possible twining genera of K3 sigma models were classified and shown to be related to the umbral and Conway moonshine constructions of [30] and [55], respectively.

Given that the elliptic genus only captures contributions of BPS states, which are independent of the moduli of the theory, one attempt to overcome the problem that no K3 CFT has M_{24} symmetry involves the idea of "symmetry-surfing" which involves combining symmetries that appear at different points in the moduli space of $K3$ sigma-models, first explored for Kummer surfaces in [142, 143], and moving away from the Kummer locus in [83]. A similar idea was investigated for UV Landau-Ginzburg models which flow to K3 CFTs in

[2] One may also study global symmetries that preserve only the $\mathcal{N} = (4, 1)$ superconformal symmetry required for the definition of the elliptic genus and its decomposition into $\mathcal{N} = 4$ characters, rather than the full $\mathcal{N} = (4, 4)$ algebra [101]; although the resulting symmetry groups are larger, this has not led to M_{24} symmetry.

the IR in [29]. The elliptic genus of $K3$ can also be viewed as the partition function of a "half-twisted" topological field theory and is closely related [120] to another mathematical construction known as the chiral de Rham complex. It has been suggested that this might be the correct arena to understand Mathieu moonshine, see for example [150].

The modular objects appearing in umbral moonshine are mock modular forms of a type discovered by Ramanujan [140] and put into a modern mathematical framework by Zwegers [155]. Mock modularity appears in computations of the elliptic genus of non-compact sigma models [74, 146] so it is natural to look for a CFT involving both a $K3$ surface and a non-compact sigma model in order to explain the mock modularity of the functions appearing in umbral moonshine. Such models appear in the description of fivebranes wrapping $K3$ surfaces [103] and in the description of string theory near an ADE singularity [30]. Analysis of these CFTs lead to intriguing connections to the mock modular forms of umbral moonshine but have not yet led to an explicit construction of the umbral moonshine modules. Another attempt to understand Mathieu moonshine involves exploiting connections between certain $K3$ sigma-models and the Conway moonshine CFT [55, 144].[3]

However, the existence of generalized Mathieu [84] and umbral moonshines [24] suggests that there should eventually be a CFT/VOA-related explanation for these phenomena. The case of umbral moonshine corresponding to Niemeier root system $3E_8$ was given a VOA interpretation in [56]. Umbral moonshine can be formulated in terms of a specific set of optimal mock Jacobi forms which descend from meromorphic Jacobi forms by a procedure discussed in [45]. There exist VOA constructions (based on free fields) of the (twined) meromorphic Jacobi forms of umbral moonshine corresponding to Niemeier root systems $4A_6$ and $2A_{12}$ [61] and $4D_6, 3D_8, 2D_{12}$, and D_{24} [32], thus furnishing a solution to the module problem for these cases. Additionally, in [4], the relationship between umbral moonshine, the K3 elliptic genus, and the Conway moonshine module was exploited to construct a module for the case of umbral moonshine corresponding to the Niemeier lattice with root system $6D_4$. However, many cases, including the original case of Mathieu moonshine corresponding to root system $24A_1$ are still missing module constructions. Furthermore, a uniform construction of umbral moonshine modules is still lacking, as none of the existing constructions can be generalized to all cases at once.

3 Connections to string theory

Moonshine's close historical connections to VOA (and consequently, CFT) imply connections to distinguished worldsheet string theories. Less immediate are the many beautiful intersections of moonshine with spacetime string theory, which may provide an arena for resolving the outstanding mysteries of moonshine, or provide a setting for novel applications of number theory and large sporadic groups to physics.

One theme of these explorations is the fact that BPS state counting functions are automorphic objects. For judiciously chosen compactifications, these objects may coincide with, or otherwise be related to, moonshine functions. The central role of mock modular forms in Mathieu and umbral moonshines is suggestive of noncompact systems, such as the near-horizon limit of NS5-branes and little string theories; perturbative BPS states in such systems are counted by the spacetime helicity supertrace, whose decomposition into mock modular

[3] An analogous story for T^4 sigma models was explored in [149, 3].

forms and relations to moonshine were explored in [103, 104]. By contrast, BPS counting functions for certain compact fivebranes wrapping Calabi-Yau divisors produce skew-holomorphic Jacobi forms with moonshine properties [51, 27]. The new supersymmetric index for the heterotic string on $K3 \times T^2$, and its type II duals, inherit moonshine features from the $K3$ elliptic genus [34], which persist in 4d $\mathcal{N} = 1$ compactifications with flux [152, 132] and in orbifolds [49, 21, 22, 7], although it is not yet understood whether or how M_{24} acts in these systems. Siegel forms counting 1/4-BPS dyons (black holes, at large charge) in 4d $\mathcal{N} = 4$ compactifications from type II on $K3 \times T^2$ and its CHL orbifolds may be decomposed into mock modular forms [45] and exhibit a variety of (umbral and Conway) moonshine connections [40, 133, 9, 23, 137] and novel string dualities [136]. Furthermore, there are many suggestive connections between dyon counting functions and BKM algebras [46, 38, 39, 91, 92, 93, 94]. (Recently, other number theoretic quantities have been recognized in black hole counting functions which also await explanation [95, 115]).

On another front, special low-dimensional spacetime string theory constructions [108] have been proposed to explain the genus zero properties of Monstrous [135, 134] and Conway moonshines [97], with the associated BKM algebras [98, 99] playing the role of BPS spectrum-generating algebras. BKM algebras have long been suggested to furnish an algebra of BPS states in string compactifications [107, 106], and it is a fascinating open possibility that moonshine groups may be best understood as symmetries of BPS algebras. Other low-dimensional string compactifications [4] naturally enjoy the umbral [116] and Conway [116, 102, 8] groups as subgroups of the duality group, suggesting a governing role for large sporadic groups in organizing spacetime symmetries. The role of mock modular forms in these compactifications is yet to be understood, though a proposal from a D1-NS5 system in 2d appears in [154].

Finally, in the context of holography, moonshine VOAs have been explored as instances of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [151, 86, 64], particularly as putative duals to pure 3d gravity (so-called extremal CFTs) with $l_{AdS} \sim \alpha'^{1/2}$. Many questions about such constructions remain. However, the Rademacher summability properties of moonshine functions (see below) are inspired by, and may presage further connections to, holography [42].

4 Mathematical connections

One way to construct a modular form is by a Poincaré series, roughly constructed via an infinite sum over $SL(2, \mathbb{Z})$ images of a seed function [138]. This sum may have convergence issues depending on the weight of the modular form and form of the seed function, which were resolved in weight zero by a regularization procedure due to Rademacher [139]. We refer to these regularized Poincaré series as Rademacher sums. However, after regularization, this Rademacher sum may no longer be a modular function, but have mock modular properties (see, e.g., [37]). The obstruction to modularity occurs in what is known as Eichler cohomology and work of Knopp shows that it vanishes when the invariance group imposed on the Rademacher sums has genus zero. See [124] for an overview of these mathematical developments.

The connection between Rademacher sums and moonshine phenomena began with [64]. Rademacher sums for other weights have been developed in the mathematical literature, see [37] for a review. These have played an important role in the characterization of the mock

[4] See also the closely related works [100, 118].

modular forms of Mathieu and umbral moonshine [42, 37], as well as the trace functions appearing in penumbral moonshine [60].

Rademacher sums have also been proposed to have a physical interpretation in terms of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [52, 126], in which the terms in the sum correspond to saddle points of the gravitational path integral with toroidal boundary condition. This has been developed further in [50, 126, 64, 128]. Other applications of Rademacher sums include exact asymptotic expansions for Fourier coefficients of certain modular forms, which have been applied to 2d CFT partitions functions [1], Vafa-Witten invariants [13], and the entropy of supersymmetric black holes in string theory [47, 48, 44, 75, 15].

While the monstrous moonshine conjectures concerned a relation between the monster group and classical modular functions, more recent incarnations of moonshine—including umbral and penumbral moonshine—involve connections between finite groups and more exotic automorphic forms of varying weights, including mock modular forms, (weak, mock, meromorphic and skew-holomorphic) Jacobi forms, and Siegel modular forms. For a review of the modular objects connected to Mathieu moonshine, see [35]. See [41] for the classification of "optimal" mock Jacobi forms of weight 1, including those which appear in umbral moonshine, and a connection to genus zero groups and [60] for a parallel discussion of skew-holomorphic Jacobi forms and genus zero groups in penumbral moonshine.

There have also been some tantalizing connections between automorphic forms connected with moonshine and invariants natural in enumerative geometry. In the case of K3 surfaces, enumerative invariants have been studied since the work of Yau and Zaslow [153], who conjectured that the generating function for rational curves with n double points on K3 is given by $1/\eta^{24}(\tau)$, equal to the generating function for Euler characteristics of $\mathrm{Hilb}^{[n]}(\mathrm{K3})$. Since then, further refinements of this formula have been conjectured by considering Gopakumar-Vafa invariants arising from M theory on $K3 \times T^2$ [122] as well as motivic stable pairs invariants [121], which reproduce the generating function of Hodge numbers of $\mathrm{Hilb}^{[n]}(\mathrm{K3})$. In [121] the question was posed whether these invariants are naturally related to dimensions of M_{24} representations. Though the answer to this question appears to be negative [102], in [28], inspired by [82], it was proposed that there is a natural action of subgroups of the Conway group on these invariants. In [110], the classification result of [82] was reinterpreted in terms of groups of derived equivalences of K3 surfaces. Finally, the Igusa cusp form Φ_{10} has been conjectured to reproduce the full Gromov-Witten theory of $K3 \times T^2$ [131], and been generalized to CHL orbifolds of $K3 \times T^2$ in [14, 76], where a number of the Siegel modular forms arising from second quantized Mathieu moonshine have appeared [137].

The phenomenon of O'Nan moonshine has been observed to have connections to the arithmetic of elliptic curves [65] as has moonshine for the Thompson group [123]. We are just beginning to scratch the surface of this intriguing connection between sporadic groups, automorphic forms, and arithmetic geometry.

In our brief description of developments in moonshine we have omitted one aspect of moonshine which at the moment seems far from physics. The representation theory of finite groups can be defined over fields of positive characteristic p. The study of these representations is known as modular representation theory. In the papers [141, 12] Ryba and Borcherds observed some very interesting behavior in the p-modular behavior of the module $V^\natural$ of the Monster VOA. One of their main conjectures was recently proved by Carnahan [17]. A nice overview of these developments as well as observations and speculations regarding connections

between modular moonshine and generalized moonshine can be found in [20]. New relations between penumbral moonshine and generalized moonshine have also been found [68, 69]. These developments point towards the possibility of a unified theory of moonshine that would encompass all known examples. At the moment the connection of these results to physics is unclear, but as mentioned earlier, there are reasons to expect some kind of structure related to CFT or VOAs to lie behind these new connections.

5 Open questions

Moonshine mysteries, of mathematical and physical natures, abound. Chief among these mysteries is the missing umbral moonshine modules: while we have several isolated examples of umbral moonshine modules a uniform construction still eludes us, including a module for the M_{24} moonshine that instigated these modern developments. Similarly, we do not yet have modules for the penumbral and O'Nan moonshines, and it is not yet known if and how these examples may be connected to physical ideas. More broadly, although all instances of moonshine enjoy some connection to a genus zero property, we do not yet understand the origin of these properties; one may hope that physics will ultimately provide a uniform answer to the question "what is moonshine?" Further, whether and how Mathieu and umbral moonshines are ultimately related to the geometry of K3 surfaces, or string theory on K3, is not yet understood, in spite of the context of the original observation of [73]. The symmetry surfing program has had some success, but further progress will likely require studying suitable connections on the bundle of K3 CFTs over their moduli space. It would also be interesting to develop a physical interpretation of the Rademacher summability properties of the McKay-Thompson series of umbral moonshine.

We may also ask a number of other questions raised by the aforementioned works: is there a generalization of the BKM algebras found in Monstrous and Conway moonshines in the case of the other moonshines? If so, is there a relationship to the BPS states of special string compactifications? Can we characterize the natural geometric or physical objects which have sporadic group symmetries? Are there instances of moonshine, or at least interesting sporadic group symmetries, that act on enumerative invariants of Calabi-Yau threefolds (or other spaces)? How could we understand such observations in a topological string? These questions, while already numerous, are far from exhaustive. Their eventual answers will doubtless shape our understanding of the mathematical foundations of string theory.

In spite of these mysteries, new connections between moonshine and other areas of physics are just beginning to be unearthed. We content ourselves with mentioning just a few new directions. One recent proposal [111] for understanding Mathieu moonshine takes place in the arena of algebraic topology, employing topological modular forms[5], and building on recent work which defines a new mock modular "secondary elliptic genus" as a natural invariant of 2d supersymmetric QFTs [87]. Non-invertible symmetries and topological defect lines have been studied in the Monster VOA [125] and naturally capture its self-dualities. Moonshine groups have long been understood to be automorphism groups of classical error-correcting codes, but connections between non-chiral CFTs/sigma models to quantum error-correcting

[5]Here, a natural possibility is that the M_{24} symmetries may emerge when acting on a K3 surface with 24 points removed, as in $E_8 \times E_8$ small heterotic instantons. Preliminary explorations of Mathieu moonshine in the heterotic string on K3 include [96].

codes have been recently uncovered [101] and provide useful input to the modular bootstrap program [71] [6]. We anticipate that new connections between moonshine and other parts of string theory, field theory, and mathematics will continue to emerge.

Acknowledgements

We thank S. Carnahan, J. Duncan, M. Gaberdiel, G. Moore, B. Rayhaun, A. Taormina and K. Wendland for helpful comments on a draft version of this manuscript. The work of S.M.H. is supported by the National Science and Engineering Council of Canada and the Canada Research Chairs program. The work of J.H. is supported by National Science Foundation grant PHY-1520748. N.P. acknowledges support from the University of Washington and the DOE award DE-SC0022347.

References

[1] L. F. Alday and J.-B. Bae, "Rademacher Expansions and the Spectrum of 2d CFT," *JHEP* **11** (2020) 134, `arXiv:2001.00022 [hep-th]`.

[2] V. Anagiannis and M. C. N. Cheng, "TASI Lectures on Moonshine," *PoS* **TASI2017** (2018) 010, `arXiv:1807.00723 [hep-th]`.

[3] V. Anagiannis, M. C. N. Cheng, J. Duncan, and R. Volpato, "Vertex operator superalgebra/sigma model correspondences: the four-torus case," *PTEP. Prog. Theor. Exp. Phys.* no. 8, (2021) Paper No. 08B102, 32.

[4] V. Anagiannis, M. C. N. Cheng, and S. M. Harrison, "K3 Elliptic Genus and an Umbral Moonshine Module," *Commun. Math. Phys.* **366** no. 2, (2019) 647–680, `arXiv:1709.01952 [hep-th]`.

[5] J.-B. Bae, J. A. Harvey, K. Lee, S. Lee, and B. C. Rayhaun, "Conformal Field Theories with Sporadic Group Symmetry," *Commun. Math. Phys.* **388** no. 1, (2021) 1–105, `arXiv:2002.02970 [hep-th]`.

[6] J.-B. Bae, K. Lee, and S. Lee, "Monster Anatomy," *JHEP* **07** (2019) 026, `arXiv:1811.12263 [hep-th]`.

[7] A. Banlaki, A. Chowdhury, A. Kidambi, and M. Schimpf, "On Mathieu moonshine and Gromov-Witten invariants," *JHEP* **02** (2020) 082, `arXiv:1811.11619 [hep-th]`.

[8] Z. K. Baykara and J. A. Harvey, "Conway Subgroup Symmetric Compactifications Redux," `arXiv:2108.04386 [hep-th]`.

[9] N. Benjamin and S. M. Harrison, "Symmetries of the refined D1/D5 BPS spectrum," *JHEP* **11** (2017) 091, `arXiv:1708.02244 [hep-th]`.

[10] R. E. Borcherds, "Vertex algebras, Kac-Moody algebras, and the Monster," *Proc. Nat. Acad. Sci. U.S.A.* **83** no. 10, (1986) 3068–3071.

[11] R. E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras," *Invent. Math.* **109** no. 2, (1992) 405–444.

[6] We also note that that modular representation theory has been recently employed to great effect in the modular bootstrap, providing another close connection between number theory and conformal field theory [119].

[12] R. E. Borcherds and A. J. E. Ryba, "Modular Moonshine. II," *Duke Math. J.* **83** no. 2, (1996) 435–459.

[13] K. Bringmann and J. Manschot, "From sheaves on P^2 to a generalization of the Rademacher expansion," *Am. J. Math.* **135** no. 4, (2013) 1039–1065, arXiv:1006.0915 [math.NT].

[14] J. Bryan and G. Oberdieck, "CHL Calabi–Yau threefolds: curve counting, Mathieu moonshine and Siegel modular forms," *Commun. Num. Theor. Phys.* **14** no. 4, (2020) 785–862, arXiv:1811.06102 [math.AG].

[15] G. L. Cardoso, S. Nampuri, and M. Rosselló, "Rademacher expansion of a Siegel modular form for $\mathcal{N} = 4$ counting," arXiv:2112.10023 [hep-th].

[16] S. Carnahan, "Generalized moonshine i: genus-zero functions," *Algebra & Number Theory* **4** no. 6, (2010) 649–679.

[17] S. Carnahan, "A self-dual integral form of the Moonshine module," *SIGMA Symmetry Integrability Geom. Methods Appl.* **15** (2019) Paper No. 030, 36.

[18] S. Carnahan, "Generalized moonshine II: Borcherds products," *Duke Math. J.* **161** (2012) 893–950, arXiv:0908.4223 [math.RT].

[19] S. H.-L. Carnahan, "Generalized Moonshine IV: Monstrous Lie algebras," arXiv:1208.6254 [math.RT].

[20] S. Carnahan, "Monstrous Moonshine over Z?," arXiv:1804.04161 [math.RT].

[21] A. Chattopadhyaya and J. R. David, "$\mathcal{N} = 2$ heterotic string compactifications on orbifolds of $K3 \times T^2$," *JHEP* **01** (2017) 037, arXiv:1611.01893 [hep-th].

[22] A. Chattopadhyaya and J. R. David, "Gravitational couplings in $\mathcal{N} = 2$ string compactifications and Mathieu Moonshine," *JHEP* **05** (2018) 211, arXiv:1712.08791 [hep-th].

[23] A. Chattopadhyaya and J. R. David, "Dyon degeneracies from Mathieu moonshine symmetry," *Phys. Rev. D* **96** no. 8, (2017) 086020, arXiv:1704.00434 [hep-th].

[24] M. C. N. Cheng, P. de Lange, and D. P. Z. Whalen, "Generalised Umbral Moonshine," *SIGMA* **15** (2019) 014, arXiv:1608.07835 [math.RT].

[25] M. C. N. Cheng, X. Dong, J. F. R. Duncan, S. Harrison, S. Kachru, and T. Wrase, "Mock Modular Mathieu Moonshine Modules," *Res. Math. Sci.* **2** (2015) 13, arXiv:1406.5502 [hep-th].

[26] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral Moonshine," *Commun. Num. Theor. Phys.* **08** (2014) 101–242, arXiv:1204.2779 [math.RT].

[27] M. C. N. Cheng, J. F. R. Duncan, S. M. Harrison, J. A. Harvey, S. Kachru, and B. C. Rayhaun, "Attractive Strings and Five-Branes, Skew-Holomorphic Jacobi Forms and Moonshine," *JHEP* **07** (2018) 130, arXiv:1708.07523 [hep-th].

[28] M. C. N. Cheng, J. F. R. Duncan, S. M. Harrison, and S. Kachru, "Equivariant K3 Invariants," *Commun. Num. Theor. Phys.* **11** (2017) 41–72, arXiv:1508.02047 [hep-th].

[29] M. C. N. Cheng, F. Ferrari, S. M. Harrison, and N. M. Paquette, "Landau-Ginzburg Orbifolds and Symmetries of K3 CFTs," *JHEP* **01** (2017) 046, arXiv:1512.04942 [hep-th].

[30] M. C. N. Cheng and S. Harrison, "Umbral Moonshine and K3 Surfaces," *Commun. Math. Phys.* **339** no. 1, (2015) 221–261, arXiv:1406.0619 [hep-th].

[31] M. C. N. Cheng, S. M. Harrison, R. Volpato, and M. Zimet, "K3 String Theory, Lattices and Moonshine," *Research in the Mathematical Sciences* **5** no. 3, (2018) 1–45, arXiv:1612.04404 [hep-th].

[32] M. C. N. Cheng and J. F. R. Duncan, "Meromorphic Jacobi Forms of Half-Integral Index and Umbral Moonshine Modules," *Commun. Math. Phys.* **370** no. 3, (2019) 759–780, arXiv:1707.01336 [math.RT].

[33] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Umbral moonshine and the Niemeier lattices," *Res. Math. Sci.* **1** (2014) Art. 3, 81.

[34] M. C. N. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, and T. Wrase, "Mathieu Moonshine and N=2 String Compactifications," *JHEP* **09** (2013) 030, arXiv:1306.4981 [hep-th].

[35] M. C. N. Cheng and J. F. R. Duncan, "The Largest Mathieu Group and (Mock) Automorphic Forms," *Proc. Symp. Pure Math.* **85** (2012) 53–82, arXiv:1201.4140 [math.RT].

[36] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, "Weight One Jacobi Forms and Umbral Moonshine," *J. Phys. A* **51** no. 10, (2018) 104002, arXiv:1703.03968 [math.NT].

[37] M. C. N. Cheng and J. F. R. Duncan, "Rademacher Sums and Rademacher Series," *Contrib. Math. Comput. Sci.* **8** (2014) 143–182, arXiv:1210.3066 [math.NT].

[38] M. C. N. Cheng and E. P. Verlinde, "Wall Crossing, Discrete Attractor Flow, and Borcherds Algebra," *SIGMA* **4** (2008) 068, arXiv:0806.2337 [hep-th].

[39] M. C. N. Cheng and A. Dabholkar, "Borcherds-Kac-Moody Symmetry of N=4 Dyons," *Commun. Num. Theor. Phys.* **3** (2009) 59–110, arXiv:0809.4258 [hep-th].

[40] M. C. N. Cheng, "K3 Surfaces, N=4 Dyons, and the Mathieu Group M24," *Commun. Num. Theor. Phys.* **4** (2010) 623–658, arXiv:1005.5415 [hep-th].

[41] M. C. N. Cheng and J. F. R. Duncan, "Optimal Mock Jacobi Theta Functions," *Adv. Math.* **372** (2020) 107284, arXiv:1605.04480 [math.NT].

[42] M. C. N. Cheng and J. F. R. Duncan, "On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine," *Commun. Num. Theor. Phys.* **6** (2012) 697–758, arXiv:1110.3859 [math.RT].

[43] J. H. Conway and S. P. Norton, "Monstrous moonshine," *Bull. London Math. Soc.* **11** no. 3, (1979) 308–339.

[44] A. Dabholkar, J. Gomes, and S. Murthy, "Nonperturbative black hole entropy and Kloosterman sums," *JHEP* **03** (2015) 074, arXiv:1404.0033 [hep-th].

[45] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," arXiv:1208.4074 [hep-th].

[46] A. Dabholkar and S. Nampuri, "Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift," *JHEP* **11** (2007) 077, arXiv:hep-th/0603066.

[47] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, "Exact and asymptotic degeneracies of small black holes," *JHEP* **08** (2005) 021, arXiv:hep-th/0502157.

[48] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, "Precision counting of small black holes," *JHEP* **10** (2005) 096, arXiv:hep-th/0507014.

[49] S. Datta, J. R. David, and D. Lust, "Heterotic string on the CHL orbifold of K3," *JHEP* **02** (2016) 056, arXiv:1510.05425 [hep-th].

[50] J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot, and E. Verlinde, "A Farey Tail for Attractor Black Holes," *JHEP* **11** (2006) 024, arXiv:hep-th/0608059.

[51] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos," *JHEP* **11** (2011) 129, arXiv:hep-th/0702146.

[52] R. Dijkgraaf, J. M. Maldacena, G. W. Moore, and E. P. Verlinde, "A Black hole Farey tail," arXiv:hep-th/0005003.

[53] L. J. Dixon, P. H. Ginsparg, and J. A. Harvey, "Beauty and the Beast: Superconformal Symmetry in a Monster Module," *Commun. Math. Phys.* **119** (1988) 221–241.

[54] C. Dong, R. L. Griess, Jr., and G. Höhn, "Framed vertex operator algebras, codes and the Moonshine module," *Comm. Math. Phys.* **193** no. 2, (1998) 407–448.

[55] J. F. R. Duncan and S. Mack-Crane, "Derived equivalences of K3 surfaces and twined elliptic genera," *Res. Math. Sci.* **3** (2016) Paper No. 1, 47.

[56] J. F. R. Duncan and J. A. Harvey, "The umbral moonshine module for the unique unimodular Niemeier root system," *Algebra Number Theory* **11** no. 3, (2017) 505–535.

[57] J. F. Duncan, "Super-moonshine for Conway's largest sporadic group," *Duke Math. J.* **139** no. 2, (2007) 255–315.

[58] J. F. R. Duncan, M. J. Griffin, and K. Ono, "Proof of the umbral moonshine conjecture," *Res. Math. Sci.* **2** (2015) Art. 26, 47.

[59] J. F. R. Duncan, M. H. Mertens, and K. Ono, "O'Nan moonshine and arithmetic," *Amer. J. Math.* **143** no. 4, (2021) 1115–1159.

[60] J. F. R. Duncan, J. A. Harvey, and B. C. Rayhaun, "An Overview of Penumbral Moonshine," arXiv:2109.09756 [math.RT].

[61] J. Duncan and A. O'Desky, "Super Vertex Algebras, Meromorphic Jacobi Forms and Umbral Moonshine," *J. Algebra* **515** (2018) 389–407, arXiv:1705.09333 [math.RT].

[62] J. F. R. Duncan and S. Mack-Crane, "The Moonshine Module for Conway's Group," *SIGMA* **3** (2015) e10, arXiv:1409.3829 [math.RT].

[63] J. F. R. Duncan and S. Mack-Crane, "Derived equivalences of K3 surfaces and twined elliptic genera," *Res. Math. Sci.* **3** (2016) Paper No. 1, 47.

[64] J. F. Duncan and I. B. Frenkel, "Rademacher sums, Moonshine and Gravity," *Commun. Num. Theor. Phys.* **5** (2011) 849–976, arXiv:0907.4529 [math.RT].

[65] J. F. Duncan, M. H. Mertens, and K. Ono, "O'nan moonshine and arithmetic," *American Journal of Mathematics* **143** no. 4, (2021) 1115–1159.

[66] J. F. R. Duncan, "From the monster to Thompson to O'Nan," in *Vertex operator algebras, number theory and related topics*, vol. 753 of *Contemp. Math.*, pp. 73–93. Amer. Math. Soc., [Providence], RI, [2020] ©2020. https://doi.org/10.1090/conm/753/15165.

[67] J. F. R. Duncan, M. J. Griffin, and K. Ono, "Moonshine," *Res. Math. Sci.* **2** (2015) Art. 11, 57.

[68] J. F. R. Duncan, J. A. Harvey, and B. C. Rayhaun, "Modular Products and Modules for Finite Groups," arXiv:2202.08271 [math.RT].

[69] J. F. R. Duncan, J. A. Harvey, and B. C. Rayhaun, "Two New Avatars of Moonshine for the Thompson Group," arXiv:2202.08277 [math.RT].

[70] A. Dymarsky and A. Shapere, "Quantum stabilizer codes, lattices, and CFTs," *JHEP* **21** (2020) 160, arXiv:2009.01244 [hep-th].

[71] A. Dymarsky and A. Shapere, "Solutions of modular bootstrap constraints from quantum codes," *Phys. Rev. Lett.* **126** no. 16, (2021) 161602, arXiv:2009.01236 [hep-th].

[72] T. Eguchi and K. Hikami, "Note on twisted elliptic genus of $K3$ surface," *Phys. Lett. B* **694** (2011) 446–455, arXiv:1008.4924 [hep-th].

[73] T. Eguchi, H. Ooguri, and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M_{24}," *Exper. Math.* **20** (2011) 91–96, arXiv:1004.0956 [hep-th].

[74] T. Eguchi and Y. Sugawara, "Non-holomorphic Modular Forms and SL(2,R)/U(1) Superconformal Field Theory," *JHEP* **03** (2011) 107, arXiv:1012.5721 [hep-th].

[75] F. Ferrari and V. Reys, "Mixed Rademacher and BPS Black Holes," *JHEP* **07** (2017) 094, arXiv:1702.02755 [hep-th].

[76] F. Fischbach, A. Klemm, and C. Nega, "Lost chapters in CHL black holes: untwisted quarter-BPS dyons in the $\mathbb{Z}_2$ model," *JHEP* **01** (2021) 157, arXiv:2005.07712 [hep-th].

[77] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A natural representation of the Fischer-Griess Monster with the modular function J as character," *Proc. Nat. Acad. Sci. U.S.A.* **81** no. 10, , Phys. Sci., (1984) 3256–3260.

[78] I. B. Frenkel, J. Lepowsky, and A. Meurman, "A moonshine module for the Monster," in *Vertex operators in mathematics and physics (Berkeley, Calif., 1983)*, vol. 3 of *Math. Sci. Res. Inst. Publ.*, pp. 231–273. Springer, New York, 1985. https://doi.org/10.1007/978-1-4613-9550-8_12.

[79] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, vol. 134 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[80] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu Moonshine in the elliptic genus of K3," *JHEP* **10** (2010) 062, arXiv:1008.3778 [hep-th].

[81] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Mathieu twining characters for K3," *JHEP* **09** (2010) 058, arXiv:1006.0221 [hep-th].

[82] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, "Symmetries of K3 sigma models," *Commun. Num. Theor. Phys.* **6** (2012) 1–50, arXiv:1106.4315 [hep-th].

[83] M. R. Gaberdiel, C. A. Keller, and H. Paul, "Mathieu Moonshine and Symmetry Surfing," *J. Phys. A* **50** no. 47, (2017) 474002, arXiv:1609.09302 [hep-th].

[84] M. R. Gaberdiel, D. Persson, H. Ronellenfitsch, and R. Volpato, "Generalized Mathieu Moonshine," *Commun. Num. Theor Phys.* **07** (2013) 145–223, arXiv:1211.7074 [hep-th].

[85] M. R. Gaberdiel, A. Taormina, R. Volpato, and K. Wendland, "A K3 sigma model with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry," *JHEP* **02** (2014) 022, arXiv:1309.4127 [hep-th].

[86] D. Gaiotto, "Monster symmetry and Extremal CFTs," *JHEP* **11** (2012) 149, arXiv:0801.0988 [hep-th].

[87] D. Gaiotto and T. Johnson-Freyd, "Mock modularity and a secondary elliptic genus," arXiv:1904.05788 [hep-th].

[88] T. Gannon, "Much ado about Mathieu," *Adv. Math.* **301** (2016) 322–358.

[89] T. Gannon, *Moonshine beyond the Monster*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2006. https://doi.org/10.1017/CBO9780511535116. The bridge connecting algebra, modular forms and physics.

[90] P. Goddard, "The work of Richard Ewen Borcherds," in *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, no. Extra Vol. I, pp. 99–108. 1998.

[91] S. Govindarajan and K. Gopala Krishna, "Generalized Kac-Moody Algebras from CHL dyons," *JHEP* **04** (2009) 032, arXiv:0807.4451 [hep-th].

[92] S. Govindarajan, D. P. Jatkar, and K. Gopala Krishna, "BKM superalgebras from counting dyons in N=4 supersymmetric type II compactifications," *Nucl. Phys. B* **859** (2012) 143–158, arXiv:1106.1318 [hep-th].

[93] S. Govindarajan, "BKM Lie superalgebras from counting twisted CHL dyons," *JHEP* **05** (2011) 089, arXiv:1006.3472 [hep-th].

[94] S. Govindarajan and K. Gopala Krishna, "BKM Lie superalgebras from dyon spectra in Z(N) CHL orbifolds for composite N," *JHEP* **05** (2010) 014, arXiv:0907.1410 [hep-th].

[95] M. Gunaydin, S. Kachru, and A. Tripathy, "Black holes and Bhargava's invariant theory," *J. Phys. A* **53** no. 44, (2020) 444001, arXiv:1903.02323 [hep-th].

[96] S. Harrison, S. Kachru, and N. M. Paquette, "Twining Genera of (0,4) Supersymmetric Sigma Models on K3," *JHEP* **04** (2014) 048, arXiv:1309.0510 [hep-th].

[97] S. M. Harrison, N. M. Paquette, D. Persson, and R. Volpato, "BPS Algebras in 2D String Theory," arXiv:2107.03507 [hep-th].

[98] S. M. Harrison, N. M. Paquette, D. Persson, and R. Volpato, "Fun with F_{24}," *JHEP* **02** (2021) 039, arXiv:2009.14710 [hep-th].

[99] S. M. Harrison, N. M. Paquette, and R. Volpato, "A Borcherds–Kac–Moody Superalgebra with Conway Symmetry," *Commun. Math. Phys.* **370** no. 2, (2019) 539–590, arXiv:1803.10798 [hep-th].

[100] S. M. Harrison, S. Kachru, N. M. Paquette, R. Volpato, and M. Zimet, "Heterotic sigma models on T^8 and the Borcherds automorphic form Φ_{12}," *JHEP* **10** (2017) 121, arXiv:1610.00707 [hep-th].

[101] J. A. Harvey and G. W. Moore, "Moonshine, superconformal symmetry, and quantum error correction," *JHEP* **05** (2020) 146, arXiv:2003.13700 [hep-th].

[102] J. A. Harvey and G. W. Moore, "Conway Subgroup Symmetric Compactifications of Heterotic String," *J. Phys. A* **51** no. 35, (2018) 354001, arXiv:1712.07986 [hep-th].

[103] J. A. Harvey and S. Murthy, "Moonshine in Fivebrane Spacetimes," *JHEP* **01** (2014) 146, arXiv:1307.7717 [hep-th].

[104] J. A. Harvey, S. Murthy, and C. Nazaroglu, "ADE Double Scaled Little String Theories, Mock Modular Forms and Umbral Moonshine," *JHEP* **05** (2015) 126, arXiv:1410.6174 [hep-th].

[105] J. A. Harvey and B. C. Rayhaun, "Traces of Singular Moduli and Moonshine for the Thompson Group," *Commun. Num. Theor. Phys.* **10** (2016) 23–62, arXiv:1504.08179 [math.RT].

[106] J. A. Harvey and G. W. Moore, "On the algebras of BPS states," *Commun. Math. Phys.* **197** (1998) 489–519, arXiv:hep-th/9609017.

[107] J. A. Harvey and G. W. Moore, "Algebras, BPS states, and strings," *Nucl. Phys. B* **463** (1996) 315–368, arXiv:hep-th/9510182.

[108] J. A. Harvey, G. W. Moore, and C. Vafa, "Quasicrystalline Compactification," *Nucl. Phys. B* **304** (1988) 269–290.

[109] G. Höhn, *Selbstduale Vertexoperatorsuperalgebren und das Babymonster*, vol. 286 of *Bonner Mathematische Schriften [Bonn Mathematical Publications]*. Universität Bonn, Mathematisches Institut, Bonn, 1996. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1995.

[110] D. Huybrechts, "On derived categories of K3 surfaces and Mathieu groups," arXiv:1309.6528 [math.AG].

[111] T. Johnson-Freyd, "Topological Mathieu Moonshine," arXiv:2006.02922 [math.AT].

[112] T. Johnson-Freyd, "Supersymmetry and the Suzuki chain," *Tunis. J. Math.* **3** no. 2, (2021) 309–359.

[113] T. Johnson-Freyd, "Correction to: The Moonshine anomaly," *Comm. Math. Phys.* **375** no. 2, (2020) 1579.

[114] T. Johnson-Freyd, "The Moonshine anomaly," *Comm. Math. Phys.* **365** no. 3, (2019) 943–970.

[115] S. Kachru and A. Tripathy, "Black Holes and Hurwitz Class Numbers," *Int. J. Mod. Phys. D* **26** no. 12, (2017) 1742003, arXiv:1705.06295 [hep-th].

[116] S. Kachru, N. M. Paquette, and R. Volpato, "3D String Theory and Umbral Moonshine," *J. Phys. A* **50** no. 40, (2017) 404003, arXiv:1603.07330 [hep-th].

[117] S. Kachru, "Elementary introduction to Moonshine," arXiv:1605.00697 [hep-th].

[118] S. Kachru and A. Tripathy, "The hidden symmetry of the heterotic string," *Adv. Theor. Math. Phys.* **21** (2017) 1729–1745, arXiv:1702.02572 [hep-th].

[119] J. Kaidi, Y.-H. Lin, and J. Parra-Martinez, "Holomorphic modular bootstrap revisited," *JHEP* **12** (2021) 151, arXiv:2107.13557 [hep-th].

[120] A. Kapustin, "Chiral de Rham complex and the half-twisted sigma-model," arXiv:hep-th/0504074.

[121] S. Katz, A. Klemm, and R. Pandharipande, "On the motivic stable pairs invariants of K3 surfaces," arXiv:1407.3181 [math.AG].

[122] S. H. Katz, A. Klemm, and C. Vafa, "M theory, topological strings and spinning black holes," *Adv. Theor. Math. Phys.* **3** (1999) 1445–1537, arXiv:hep-th/9910181.

[123] M. Khaqan, "Elliptic curves and Thompson's sporadic simple group," *J. Number Theory* **224** (2021) 274–306.

[124] M. I. Knopp, "Rademacher on $J(\tau)$, Poincaré series of nonpositive weights and the Eichler cohomology," *Notices Amer. Math. Soc.* **37** no. 4, (1990) 385–393.

[125] Y.-H. Lin and S.-H. Shao, "Duality Defect of the Monster CFT," *J. Phys. A* **54** no. 6, (2021) 065201, arXiv:1911.00042 [hep-th].

[126] J. Manschot and G. W. Moore, "A Modern Farey Tail," *Commun. Num. Theor. Phys.* **4** (2010) 103–159, arXiv:0712.0573 [hep-th].

[127] S. Mukai, "Finite groups of automorphisms of $K3$ surfaces and the Mathieu group," *Invent. Math.* **94** no. 1, (1988) 183–221.

[128] S. Murthy and B. Pioline, "A Farey tale for N=4 dyons," *JHEP* **09** (2009) 022, arXiv:0904.4253 [hep-th].

[129] S. Norton, "Generalized moonshine," *Proc. Sympos. Pure Math., The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)* **47** (1987) 209–210.

[130] S. Norton, "From moonshine to the monster," *Proceedings on Moonshine and Related Topics (Montréal, QC, 1999), CRM Proc. Lecture Notes* **30** (2001) 163–171.

[131] G. Oberdieck and R. Pandharipande, "Curve counting on $K3 \times E$, the Igusa cusp form χ_{10}, and descendent integration," *Progr. Math.* **315** (2016) 245–278.

[132] N. M. Paquette and T. Wrase, "Comments on M_{24} representations and CY_3 geometries," *JHEP* **11** (2014) 155, arXiv:1409.1540 [hep-th].

[133] N. M. Paquette, R. Volpato, and M. Zimet, "No More Walls! A Tale of Modularity, Symmetry, and Wall Crossing for 1/4 BPS Dyons," *JHEP* **05** (2017) 047, arXiv:1702.05095 [hep-th].

[134] N. M. Paquette, D. Persson, and R. Volpato, "BPS Algebras, Genus Zero, and the Heterotic Monster," *J. Phys. A* **50** no. 41, (2017) 414001, arXiv:1701.05169 [hep-th].

[135] N. M. Paquette, D. Persson, and R. Volpato, "Monstrous BPS-Algebras and the Superstring Origin of Moonshine," *Commun. Num. Theor. Phys.* **10** (2016) 433–526, arXiv:1601.05412 [hep-th].

[136] D. Persson and R. Volpato, "Fricke S-duality in CHL models," *JHEP* **12** (2015) 156, arXiv:1504.07260 [hep-th].

[137] D. Persson and R. Volpato, "Second Quantized Mathieu Moonshine," *Commun. Num. Theor. Phys.* **08** (2014) 403–509, arXiv:1312.0622 [hep-th].

[138] H. Poincaré, "Fonctions modulaires et fonctions fuchsiennes," *Annales de la Faculté des Sciences de Toulouse* **3** 125–149.

[139] H. Rademacher, "The fourier series and the functional equation of the absolute modular invariant $j(\tau)$," *American Journal of Mathematics* **61** no. 1, (1939) 237–248.

[140] S. Ramanujan, "Last letter to G. H. Hardy,".

[141] A. J. E. Ryba, "Modular Moonshine?," in *Moonshine, the Monster, and related topics (South Hadley, MA, 1994)*, vol. 193 of *Contemp. Math.*, pp. 307–336. Amer. Math. Soc., Providence, RI, 1996. https://doi.org/10.1090/conm/193/02378.

[142] A. Taormina and K. Wendland, "The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}," *JHEP* **08** (2013) 125, arXiv:1107.3834 [hep-th].

[143] A. Taormina and K. Wendland, "Symmetry-surfing the moduli space of Kummer K3s," *Proc. Symp. Pure Math.* **90** (2015) 129–154, arXiv:1303.2931 [hep-th].

[144] A. Taormina and K. Wendland, "The Conway Moonshine Module is a reflected K3 theory," *Adv. Theor. Math. Phys.* **24** no. 5, (2020) 1247–1323, arXiv:1704.03813 [hep-th].

[145] J. G. Thompson, "Some numerology between the Fischer-Griess Monster and the elliptic modular function," *Bull. London Math. Soc.* **11** no. 3, (1979) 352–353.

[146] J. Troost, "The non-compact elliptic genus: mock or modular," *JHEP* **06** (2010) 104, arXiv:1004.3649 [hep-th].

[147] M. P. Tuite, "Monstrous moonshine from orbifolds," *Communications in mathematical physics* **146** no. 2, (1992) 277–309.

[148] M. P. Tuite, "Monstrous and generalized moonshine and permutation orbifolds," in *Moonshine: the first quarter century and beyond*, vol. 372, pp. 378–392. Cambridge Univ. Press Cambridge, 2010.

[149] R. Volpato, "On symmetries of $\mathcal{N} = (4,4)$ sigma models on T^4," *JHEP* **08** (2014) 094, arXiv:1403.2410 [hep-th].

[150] K. Wendland, "Hodge-Elliptic Genera and How They Govern K_3 Theories," *Commun. Math. Phys.* **368** no. 1, (2019) 187–221, arXiv:1705.09904 [hep-th].

[151] E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th].

[152] T. Wrase, "Mathieu moonshine in four dimensional $\mathcal{N} = 1$ theories," *JHEP* **04** (2014) 069, arXiv:1402.2973 [hep-th].

[153] S.-T. Yau and E. Zaslow, "BPS states, string duality, and nodal curves on K3," *Nucl. Phys. B* **471** (1996) 503–512, arXiv:hep-th/9512121.

[154] M. Zimet, "Umbral Moonshine and String Duality," arXiv:1803.07567 [hep-th].

[155] S. Zwegers, "Ph.D. Thesis, Utrecht, 2002,".

Mathieu Moonshine and the Geometry of K3 Surfaces

Thomas Creutzig* and Gerald Höhn†

Abstract

We compare the moonshine observation of Eguchi, Ooguri and Tachikawa relating the Mathieu group M_{24} and the complex elliptic genus of a K3 surface with the symmetries of geometric structures on K3 surfaces.

Two main results are that the complex elliptic genus of a K3 surface is a virtual module for the Mathieu group M_{24} and also for a certain vertex operator superalgebra V^G where G is the holonomy group.

1 Introduction

Moonshine refers to an unexpected relation between finite simple groups, automorphic forms and physics, with the most prominent example of monstrous moonshine [CN, Bo]. A few years ago, a similar observation was made concerning the Mathieu group M_{24} and the complex elliptic genus of a K3 surface.

1.1 Mathieu Moonshine

Eguchi, Ooguri and Tachikawa [EOT] observed that the sum decomposition of the Jacobi form $2\,\phi_{0,1}(z;\tau)$ of weight 0 and index 1 into the (expected) characters of the $\mathcal{N}=4$ super Virasoro algebra at central charge $c=6$ has multiplicities which are simple sums of dimensions of irreducible representations of the largest Mathieu group M_{24}. The Jacobi form $2\,\phi_{0,1}(z;\tau)$ has several interpretations, it is the complex elliptic genus $\chi_{-y}(q,\mathcal{L}X)$ of a K3 surface X and in string theory it has the meaning of the partition function of physical states in the Ramond-sector of a sigma model on X.

The observation of Eguchi, Ooguri and Tachikawa suggests the existence of a graded M_{24}-module $K=\bigoplus_{n=0}^{\infty} K_n\, q^{n-1/8}$ whose graded character is given by the

*Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada. email: creutzig@ualberta.ca

†Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506-2602, USA. email: gerald@math.ksu.edu

sum decomposition of the elliptic genus into characters of the $\mathcal{N}=4$ super Virasoro algebra. Subsequently the analogues to McKay-Thompson series in monstrous moonshine where proposed in several works [C, CD, EH, GHV, GHV2]. The McKay-Thompson series for g in M_{24} are of the form

$$\Sigma_g(q) \;=\; q^{-1/8}\sum_{n=0}^{\infty}\mathrm{Tr}(g|K_n)\,q^n \;=\; \frac{e(g)}{24}\,\Sigma(q)-\frac{f_g(q)}{\eta(q)^3} \tag{1}$$

where $\Sigma=\Sigma_e$ is the graded dimension of K (an explicit mock modular form), $e(g)$ is the character of the 24-dimensional permutation representation of M_{24}, the series f_g is a certain explicit modular form of weight 2 for some subgroup $\Gamma_0(N_g)$ of $\mathrm{SL}(2,\mathbb{Z})$ and η is the Dedekind eta function. Terry Gannon [G] has proven that these McKay-Thompson series indeed determine a M_{24}-module:

Theorem 1.1. *The McKay-Thompson series as in [GHV, EH] determine a virtual graded M_{24}-module $K=\bigoplus_{n=0}^{\infty}K_n\,q^{n-1/8}$. For $n\geq 1$ the K_n are honest (and not only virtual) M_{24}-representations.*

The proof uses techniques of modular forms and group theory. It does not explain any connection to geometry or physics. Since the complex elliptic genus is both an invariant of K3 surfaces and a string theory object, it is believed that there is a geometric and/or physics explanation of the Mathieu moonshine observation. In this work, we investigate implications of the geometry to the moonshine.

1.2 The complex elliptic genus

The complex elliptic genus appears originally in the physical literature. A rigorous mathematical definition and a first investigation of its basic topological properties, like rigidity have been first given by Krichever [Kr] and the second author [Ho] following similar work by Ochanine, Witten, Hirzebruch and Bott/Taubes for other elliptic genera. The complex elliptic genus equals the graded dimension of the cohomology of the chiral de Rham complex of a complex manifold [MSV, BL].

We recall that a complex genus in the sense of Hirzebruch [Hi] is a graded ring homomorphism from the complex bordism ring into some other graded ring R. The complex elliptic genus can be formally defined as the S^1-equivariant χ_y-genus of the loop space of a manifold

$$\chi_y(q,\mathcal{L}X):=(-y)^{-d/2}\chi_y\big(X,\bigotimes_{n=1}^{\infty}\Lambda_{yq^n}T^*\otimes\bigotimes_{n=1}^{\infty}\Lambda_{y^{-1}q^n}T\otimes\bigotimes_{n=1}^{\infty}S_{q^n}(T^*\oplus T)\big)$$

having values in $\mathbb{Q}[y^{1/2},y^{-1/2}][[q]]$. Up to normalization, this leads to a four-variable genus with values in a polynomial ring $\mathbb{Q}[A,B,C,D]$. For manifolds X with vanishing first Chern class it takes values in $\mathbb{Q}[B,C,D]$ only and it is a Jacobi form of weight 0 and index half the complex dimension of X (see [Ho]). For automorphisms g of X one has the corresponding equivariant elliptic genus $\chi_y(g;q,\mathcal{L}X)$.

The Jacobi form $2\,\phi_{0,1}(z;\tau)$ equals the complex elliptic genus of K3 surfaces (cf. for example [Ho], p. 62).

1.3 Results and Organization

By combining Mukai's [Mu] classification of groups acting symplectically on K3 surfaces and information from rational character tables we prove in Section 2 (see Corollary 2.7):

Theorem. *The complex elliptic genus of a K3 surface can be given the structure of a virtual M_{24}-module which is compatible with the H-module structure for all possible groups H of symplectic automorphisms of K3 surfaces under restriction.*

We remark that all such groups H of symplectic automorphisms of K3 surfaces are subgroups of M_{24}.

We then explain in Section 3 how to construct bundles over a K3 surface that at each fiber have the structure of a (possibly twisted) module M for a vertex operator superalgebra V and that the corresponding twisted Todd genus is a character of a virtual module for a vertex operator subalgebra V^G where G is the structure group of the bundle (see Theorem 3.1). For the case of the complex elliptic genus of a K3 surface we have (Corollary 3.12):

Theorem. *The complex elliptic genus of a K3 surface is the graded character of a natural virtual vertex operator algebra module for V^G and its $\mathcal{N}=4$ super Virasoro vertex subalgebra.*

We also obtain explicit formulas for the decomposition of M into characters of the $\mathcal{N}=4$ super Virasoro algebra.

The equivariant complex elliptic genus of a K3 surface can be computed by using the holomorphic Lefschetz fixed-point formula. On the other hand, the known Mathieu moonshine uses the McKay-Thompson series, i.e. equivariant elliptic genera in the sense of string theory and super conformal field theory on K3 surfaces. A priori these two could be different, but fortunately we find (Theorem 4.3):

Theorem. *For a finite symplectic automorphism g acting on a K3 surface X the equivariant elliptic genus and the $\mathcal{N} = 4$ character determined by the McKay-Thompson series of Mathieu moonshine agree.*

We also compare in this final section the geometric structures for K3 surfaces which we have considered with the proposed Mathieu moonshine.

2 Symplectic group actions on K3 surfaces and the Mathieu group M_{24}

A K3 surface is a simply connected complex surface with trivial canonical line bundle. An action of a group H on X by complex automorphisms is called symplectic if the

Table 1: Fixed point sets of non-trivial symplectic automorphisms g

order of g	# of fixed points	eigenvalues of g at the fixed points
2	8	$8 \times (-1, -1)$
3	6	$6 \times (\zeta_3, \zeta_3^{-1})$
4	4	$4 \times (i, -i)$
5	4	$2 \times (\zeta_5, \zeta_5^{-1})$ and $2 \times (\zeta_5^2, \zeta_5^{-2})$
6	2	$2 \times (\zeta_6, \zeta_6^{-1})$
7	3	(ζ_7, ζ_7^{-1}), $(\zeta_7^2, \zeta_7^{-2})$ and $(\zeta_7^3, \zeta_7^{-3})$
8	2	(ζ_8, ζ_8^{-1}) and $(\zeta_8^3, \zeta_8^{-3})$

induced action on the space $H^0(X, \Omega^2)$ of holomorphic 2-forms is trivial. For a symplectic action one can find a hyperkähler metric g on X for which H acts by isometries and which is a Kähler metric for the given complex structure. Recall that a metric for a four dimensional manifold is called hyperkähler if the holonomy group is contained in $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. Conversely, given a hyperkähler metric g on X there is a family of complex structures on the underlying differentiable manifold parameterized by a 2-sphere and the action of a group H by isometries is symplectic for all of them.

Let g be a non-trivial symplectic automorphism of finite order acting on a K3 surface X. By [Ni] such automorphisms have only isolated fixed points. The holomorphic Lefschetz fixed point formula [AB] shows that only seven cases are possible [Ni] (see also [Mu]). We list them together with the eigenvalues of g acting on the tangent spaces in Table 1. Here, ζ_n stands for a primitive n-th root of unity.

The finite symplectic automorphism groups of K3 surfaces have been classified by Mukai. They are all isomorphic to subgroups of the Mathieu group M_{23} of a particular type. We recall that M_{23} is isomorphic to a one-point stabilizer for the permutation action of M_{24} on 24 elements.

Theorem 2.1 (Mukai [Mu])**.** *A finite group H acting symplectically on a K3 surface is isomorphic to a subgroup of M_{23} with at least five orbits on the regular permutation representation of the Mathieu group $M_{24} \supset M_{23}$ on 24 elements.*

Remark 2.2. 1. Every such group H is a subgroup of 11 maximal groups explicitily given in [Mu]. They are of the following type (cf. [Mu] for details):

no.	1	2	3	4	5	6	7	8	9	10	11
structure	$L_2(7)$	A_6	S_5	2^4A_5	2^4S_4	$A_{4,4}$	T_{192}	2^4D_{12}	3^2D_8	3^2Q_8	$SL_2(3)$
order	168	360	120	960	384	288	192	192	72	72	48

2. The non-trivial minimal subgroups correspond to the seven possibilities of finite symplectic groups actions as listed in Table 1. Below, we list the ATLAS

names [AT] of the corresponding generators (as well as the trivial element) as elements in the two Mathieu groups M_{23}, M_{24} and the Conway group Co_0 which contains M_{24} as a subgroup.

order	1	2	3	4	5	6	7	8
M_{23}	$1A$	$2A$	$3A$	$4A$	$5A$	$6A$	$7A$, $7B$	$8A$
M_{24}	$1A$	$2A$	$3A$	$4B$	$5A$	$6A$	$7A$, $7B$	$8A$
Co_0	$1A+$	$2A+$	$3B+$	$4C+$	$5B+$	$6E+$	$7B+$	$8E+$

3. There are in total 80 abstract isomorphism types of symplectic automorphism groups H (including the trivial one) acting on K3 surfaces. Considered as a subgroup of $O(L)$, where $L = H^2(X, \mathbb{Z})$ is the intersection lattice of X, the group is unique up to conjugacy in 75 of those cases. For five cases, there exist two possibilities, each; see [Ha].

Given a holomorphic vector bundle E on a K3 surface X associated to the tangent bundle, and a finite automorphism group H of X, the induced action of H on the cohomology groups $H^q(X, E)$ makes the holomorphic Euler characteristic $\chi(X, E) = \sum_{q=0}^{2}(-1)^q \dim H^q(X, E)$ into a virtual H-module. For H symplectic, we know from the holomorphic Lefschetz formula and Table 1 that the value of

$$\chi(g; X, E) = \mathrm{tr}(g|H^0(X, E)) - \mathrm{tr}(g|H^1(X, E)) + \mathrm{tr}(g|H^2(X, E))$$

depends only on the order of $g \in H$ and is rational. In particular, we see that the structure of the virtual H-module $\chi(X, E)$ for all eleven maximal Mukai groups H is completely determined by the value of $\chi(g; X, E)$ for elements g of order 1, 2, ..., 8. We axiomatize this situation in the following definition.

Definition 2.3. A *Mukai* (resp. *virtual Mukai*) *representation* ρ is a family $(\rho_i)_{i=1,\ldots,11}$ of rational characters (resp. generalized rational characters) of the eleven maximal symplectic groups H_i, $i = 1, \ldots, 11$, such that $\rho_i(g)$ for $g \in H_i$ depends only on the order of g and this value is independent of i.

Theorem 2.4. *Let $\rho = (\rho_i)_{i=1,\ldots,11}$ be a virtual Mukai representation. Then there exists a virtual rational representation μ of M_{24} such that $\mu|_{H_i} = \rho_i$ for each of the eleven Mukai groups H_i.*

Remark 2.5.

1. The embedding of H_i into M_{24} may not be unique but the value of the characters is, i.e. $\mu|_{H_i}$ is always well-defined.

2. The representation μ of M_{24} is not unique. In fact, it can be changed by any virtual rational representation of M_{24} with the property that the character for the eight conjugacy classes of elements belonging to symplectic automorphism is zero.

Proof: Let L_i be the $\mathbb{Z}$-module generated by the rational characters of the group H_i. We consider the submodule L_i^0 of L_i defined by the property that the values for those conjugacy classes of H_i for which the elements have the same order are equal. Let M be the $\mathbb{Z}$-module of $\mathbb{Z}$-valued functions on the set $\{1, 2, \ldots, 8\}$. We let N_i be the submodule of M consisting of those functions on $\{1, 2, \ldots, 8\}$ for which the restriction to the subset which occur as orders of elements of H_i belongs to L_i^0. Set $N = \bigcap_{i=1}^{11} N_i$.

It follows from the definition that for a Mukai representation $\rho = (\rho_i)$ the individual characters ρ_i belong to L_i^0. Since the value of the character is independent of i they define together an element of N.

Let K be the submodule of M generated by the restriction of the generalized rational characters of M_{24} to those conjugacy classes of order 1, ..., 8 which are listed in the table under Remark 2.2.2. above. A computer calculation with Magma [Ma] shows $K = N$. Since the Schur indices of the M_{24}-representations are all 1 [Be], a generalized rational character for M_{24} actually defines a virtual rational M_{24}-module. This proves the theorem. □

Remark 2.6. One has $M/N \cong 2 \times 4 \times 24 \times 40320$. We also list for each i the possible orders of elements of H_i and the structure of the quotient modules N_i/N in the following table.

no.	1	2	3	4	5	6
orders	$1-4,7$	$1-5$	$1-6$	$1-5$	$1-4,8$	$1-4,6$
N_i/N	$4 \times 12 \times 480$	$4 \times 4 \times 672$	$2 \times 4 \times 672$	12×168	3×420	2×280

no.	7	8	9	10	11
orders	$1-4,6$	$1-4,6$	$1-4,6$	$1-4$	$1-4,6,8$
N_i/N	2×840	2×840	$2 \times 4 \times 1120$	$4 \times 4 \times 3360$	2×1680

The calculation shows that it is enough to use only the groups H_1, H_5, H_6 and in addition one of the three groups H_2, H_3 or H_4. It is not enough to use only three of the groups H_i.

Let K' and K'' be the submodules of M generated by the restriction of the rational characters of M_{23} respectively Co_0 to those conjugacy classes of order 1, ..., 8 which are listed in the table in Remark 2.2.2. Since $K \subset K'$ and $K' \subset N$ one has $K' = N$. A calculation shows that the lattice K'' for the Conway group Co_0 has index 2 in $K = K' = N$.

Corollary 2.7. *The complex elliptic genus of a K3 surface can be given the structure of a virtual M_{24}-module which is compatible with the H_i-module structure for each of the eleven Mukai groups H_i under restriction.*

Proof: The coefficients of the elliptic genus of a K3 surface X are of the form $\chi(X,E)$ where E is a vector bundle associated to the tangent bundle of X. Thus $\chi(g;X,E)$ defines a generalized rational H_i-character for each of the H_i. The result follows now from Theorem 2.4. $\square$

It was shown by Gannon [G], that the M_{24}-modules of Mathieu moonshine can be extended to virtual Co_0-modules. Thus in these cases it follows from Theorem 4.3 that the above mentioned $\mathbb{Z}_2$-obstructions in K''/N vanish.

3 The elliptic genus of a K3 surface

We discuss properties of the elliptic genus related to the hyperkähler geometry of K3 surfaces using vertex algebras.

3.1 The complex elliptic genus and vertex algebras

Let V be a vertex operator superalgebra, $M = \bigoplus M_n$ be a (twisted) V-module and G be a subgroup of the automorphism group of M such that

$$M = \bigoplus_{\lambda \in \mathrm{Irr}(G)} \lambda \otimes M_\lambda \tag{2}$$

as $G \times V^G$-module (i.e. the M_λ are trivial G-modules) with finite dimensional λ.

For a G-principal bundle P over a closed manifold X we have the associated bundle $\underline{M} = P \times_G M$ over X which at every fiber has the structure of a vertex operator superalgebra module isomorphic to M. The grading of M provides the decomposition $\underline{M} = \bigoplus_n \underline{M_n}\, q^n$ into finite dimensional vector bundles over X, where we introduced the formal variable q. Let $\underline{W_\lambda}$ be the complex vector bundle over X associated to a finite dimensional representation λ of G. Then the decomposition (2) of M leads to a decomposition

$$\underline{M} = \bigoplus_{\lambda \in \mathrm{Irr}(G)} \underline{W_\lambda} \otimes \underline{M_\lambda}$$

where we consider $\underline{M_\lambda}$ as the trivial bundle $X \times M_\lambda$.

On a d-dimensional complex manifold X, one can consider the Dolbeault operator twisted by a holomorphic vector bundle E. The holomorphic Euler characteristic $\chi(X,E) = \sum_{q=0}^{d} (-1)^q \dim H^q(X,E)$ for the sheaf of holomorphic sections in E equals the index of the Dolbeault operator and can be computed by the twisted Todd genus

$$\chi(X,E) = \mathrm{Td}(X)\mathrm{ch}(E)[X]$$

where $\mathrm{Td}(X)$ is the total Todd class of X and $\mathrm{ch}(E)$ is the Chern character of E; cf. [Hi]. In the following we assume that the bundles $\underline{W_\lambda}$ and hence $\underline{M}$ are holomorphic bundles.

We are interested in the twisted Todd genus

$$\chi(X, \underline{M}) = \sum_n \chi(X, \underline{M_n})\, q^n \in \mathbb{Z}[[q]].$$

Theorem 3.1. *The Todd genus $\chi(X, \underline{M})$ is the graded character of a natural virtual vertex operator superalgebra module for V^G.*

Proof: One has

$$\chi(X, \underline{M}) \;=\; \bigoplus_{\lambda \in \mathrm{Irr}(G)} \chi(X, \underline{W_\lambda} \otimes \underline{M_\lambda}) \;=\; \bigoplus_{\lambda \in \mathrm{Irr}(G)} \chi(X, \underline{W_\lambda}) \otimes M_\lambda$$

where $\chi(X, E)$ is here identified with the virtual complex vector space of dimension $\chi(X, E)$. The first equality is trivial. The last equality needs some justification. We have the diagram

$$\begin{array}{ccc}
\Gamma(X, A^q \otimes \underline{W_\lambda}) \otimes M_\lambda & \longrightarrow & \Gamma(X, A^q \otimes \underline{W_\lambda} \otimes \underline{M_\lambda}) \\
\downarrow \partial^q_{\underline{W_\lambda}} \otimes \mathrm{id} & & \downarrow \partial^q_{\underline{W_\lambda} \otimes \underline{M_\lambda}} \\
\Gamma(X, A^{q+1} \otimes \underline{W_\lambda}) \otimes M_\lambda & \longrightarrow & \Gamma(X, A^{q+1} \otimes \underline{W_\lambda} \otimes \underline{M_\lambda})
\end{array}$$

where A^q is the exterior bundle $\Lambda^q T^*_{\mathbb{C}} X$, ∂^q_E denotes the twisted Dolbeault operator with coefficients in a holomorphic bundle E, and the horizontal arrows are isomorphisms induced by the constant sections in the trivial bundle $\underline{M_\lambda}$. A V^G-module structure on all spaces is obtained by using constant sections in the bundle $\underline{V^G}$ and the fiberwise action on the last tensor factor.

For a local holomorphic frame (Φ_j) of a bundle E and a form α, the operator ∂^q_E is defined by $\partial^q_E(\alpha \otimes \Phi_j) = (\partial^q \alpha) \otimes \Phi_j$. Thus we see that the above diagram is commutative in the sense of V^G-modules and we have the induced V^G-module maps on the level of cohomology. $\square$

For the complex elliptic genus we will take

$$V = \left(\bigotimes_{n \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (\Lambda_{yq^n}(\mathbb{C}^*) \otimes \Lambda_{y^{-1}q^n}(\mathbb{C})) \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(\mathbb{C} \oplus \mathbb{C}) \right)^{\otimes d}$$

and

$$M = \left(\Lambda_y(\mathbf{C}^*) \otimes \bigotimes_{n \in \mathbb{Z}_{>0}} (\Lambda_{yq^n}(\mathbb{C}^*) \otimes \Lambda_{y^{-1}q^n}(\mathbb{C})) \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(\mathbb{C} \oplus \mathbb{C}^*) \right)^{\otimes d}$$

as vertex operator superalgebra and module, resp., for G we take $GL(d, \mathbb{C}) < \mathrm{Aut}(V)$ and for P we take the holomorphic $GL(d, \mathbb{C})$-principal bundle associated to the tangent bundle of the complex manifold X.

Thus $\chi_y(q, \mathcal{L}X) := (-y)^{-d/2}\chi(X, \underline{M})$ has the structure of a virtual $V^{GL(d,\mathbb{C})}$-module if we verify our assumptions on M. If the structure group of P is reduced to a smaller subgroup G of $GL(d, \mathbb{C})$ — for example if X is a complex manifold with trivial canonical line bundle — $\chi_y(q, \mathcal{L}X)$ will be a virtual module for the larger vertex operator superalgebra $V^G \supset V^{GL(d,\mathbb{C})}$.

Remark 3.2. 1. In a differential geometric setting it was shown by H. Tamanoi [Ta1, Ta2] that the level two elliptic genus of spin manifolds is a vertex operator algebra module over the vertex operator superalgebra formed by the parallel sections in a certain vertex superalgebra bundle $\underline{V}$ which can be identified with V^G where G is the holonomy group.

2. In [BL] it was observed that the complex elliptic genus of complex manifolds is the graded dimension of a vertex algebra defined as the cohomology of the chiral de Rham complex [MSV]. It was shown in [MSV] that a global holomorphic volume form defines an $\mathcal{N}=2$ superconformal structure on the space of sections of the chiral de Rham complex and its cohomology. In [BHS] the authors considered hyperkähler manifolds and showed that the chiral de Rham complex has an $\mathcal{N}=4$ superconformal structure of central charge $6d$ where d is the complex dimension of the manifold. For the elliptic genus, this has also been investigated in the preprint [Zh]. In [He] it was shown that this superconformal structure can be decomposed into two commuting $\mathcal{N} = 4$ superconformal structures of central charge $3d$. In both cases the superconformal structures depend on the hyperkähler metric. It is not clear if these superconformal structures can be extended to a vertex algebra isomorphic to V^G. For the character considerations in the present paper, it is sufficient to consider only the above established V^G-module structure for the elliptic genus.

Finally, assume that a compact group H acts on X by automorphisms compatible with the reduction of the structure group to G. Then $\chi_y(q, \mathcal{L}X)$ is also a virtual H-module. More precisely, one has

$$\chi(X, \underline{M}) = \bigoplus_{\lambda \in \mathrm{Irr}(G)} \chi(X, \underline{W_\lambda}) \otimes \underline{M_\lambda}$$

as $H \times V^G$-module. Thus for the computation of the equivariant elliptic genus $\chi(g; X, \underline{M})$ for $g \in H$ it is enough to know $\chi(g; X, \underline{W_\lambda})$ for all $\lambda \in \mathrm{Irr}(G)$.

3.2 The algebra V^G

We restrict now to the case of a K3 surface so that the complex dimension is $d = 2$ and the structure group can be reduced to $G = \mathrm{SL}(2, \mathbb{C})$. We will describe the vertex algebra V and the fixed point subalgebra $V^G = V^{\mathrm{SU}(2)}$ in more detail.

The vertex algebra V is generated by fermionic fields $b_1(z)$, $c_1(z)$, $b_2(z)$, $c_2(z)$ and bosonic fields $x_1(z)$, $y_1(z)$, $x_2(z)$, $y_2(z)$ with non-regular operator products

$$b_i(z)c_j(w) \ \sim \ \frac{\delta_{i,j}}{(z-w)}, \qquad x_i(z)y_j(w) \ \sim \ \frac{\delta_{i,j}}{(z-w)^2}.$$

The vectors $b(z) = (b_1(z), b_2(z))^t$, $x(z) = (x_1(z), x_2(z))^t$ carry the standard representation of SU(2), while $c(z) = (c_1(z), c_2(z))$, $y(z) = (y_1(z), y_2(z))$ carry its conjugate representation. Define the basic invariant fields

$$\begin{aligned} \mathrm{tr}_{n,m}(v,w) &=: \partial^n v_1(z)\partial^m w_1(z) : + : \partial^n v_2(z)\partial^m w_2(z) :, \\ \det{}_{n,m}(v,w) &=: \partial^n v_1(z)\partial^m w_2(z) : - : \partial^n v_2(z)\partial^m w_1(z) : . \end{aligned}$$

Lemma 3.3. *V^G is strongly generated by the basic invariant fields, i.e.* $\{\det_{n,m}(b,b), \det_{n,m}(b,x), \det_{n,m}(x,x), \det_{n,m}(c,c), \det_{n,m}(c,y), \det_{n,m}(y,y),$ $tr_{n,m}(b,c), tr_{n,m}(b,y), tr_{n,m}(x,c), tr_{n,m}(x,y) \mid n,m \in \mathbb{Z}_{\geq 0}\}$.

Proof: Weyl's fundamental theorem of invariant theory for the fundamental representation of SU(N) [We] states that all invariants can be expressed in terms of the basic invariants, i.e. traces and determinants. Hence especially every element of V^G can be expressed as a normally ordered polynomial in above generators. □

Lemma 3.4. *V^G contains the following subalgebras:*

1. *The simple affine vertex algebra $L_1(sl(2))$ of $sl(2)$ of level one, generated by $tr_{0,0}(b,c)$, $\det_{0,0}(b,b)$, $\det_{0,0}(c,c)$.*
2. *The super Virasoro algebra Vir_6 of central charge 6, generated by $tr_{0,0}(x,y) + (tr_{1,0}(b,c) - tr_{0,1}(b,c))/2$.*
3. *The $\mathcal{N}=4$ super Virasoro algebra of central charge 6, generated by $L_1(sl(2))$, Vir_6 and $tr_{0,0}(x,c)$, $tr_{0,0}(b,y)$, $\det_{0,0}(b,x)$, $\det_{0,0}(c,y)$.*

Proof: These are well-known statements: $\mathrm{tr}_{0,0}(b,c)$, $\det_{0,0}(b,b)$, $\det_{0,0}(c,c)$ generate a rank one lattice vertex algebra isomorphic to the simple affine vertex algebra $L_1(sl(2))$ of $sl(2)$ of level one. The realization of the $\mathcal{N}=4$ super Virasoro algebra in terms of these free fields appeared in [ABD]. □

Remark 3.5. The universal $\mathcal{N}=4$ super Virasoro algebra at central charge 6 is the quantum Hamiltonian reduction for a non-principal nilpotent element of the universal affine vertex superalgebra $V_{-2}(psl(2|2))$ of $psl(2|2)$ at level minus two. This is a non-generic position and hence the reduced algebra is not simple [HS]. To our knowledge it is not proven whether the free field realization describes the simple quotient of the $\mathcal{N}=4$ super Virasoro algebra at central charge 6.

Remark 3.6. The conformal dimension Δ with respect to the Virasoro field $\mathrm{tr}_{0,0}(x,y) + (\mathrm{tr}_{1,0}(b,c) - \mathrm{tr}_{0,1}(b,c))/2$ are $\Delta(b_i) = \Delta(c_i) = 1/2$ and $\Delta(x_i) = \Delta(y_i) = 1$. So that the basic invariants have conformal dimension $\Delta(\mathrm{tr}_{n,m}(v,w)) = \Delta(\det_{n,m}(v,w)) = n + m + \Delta(v) + \Delta(w)$.

We define the set

$$\begin{aligned} S = \{&\det_{0,0}(b,b), \det_{0,0}(b,x), \det_{0,0}(c,c), \det_{0,0}(c,y), \mathrm{tr}_{0,0}(b,c), \mathrm{tr}_{0,0}(b,y), \\ &\mathrm{tr}_{0,0}(x,c), \mathrm{tr}_{0,0}(x,y), \det_{1,0}(b,x), \det_{1,0}(x,x), \det_{1,0}(c,y), \det_{1,0}(y,y), \\ &\mathrm{tr}_{1,0}(b,c), \mathrm{tr}_{1,0}(b,y), \mathrm{tr}_{1,0}(x,c), \mathrm{tr}_{0,0}(x,y)\} \end{aligned}$$

of 16 fields containing the eight strong generators of the $\mathcal{N}=4$ super Virasoro algebra and eight additional fields of conformal dimension 2, 5/2, 5/2, 5/2, 5/2, 3, 3, 3.

Lemma 3.7. *Every basic invariant is a normally ordered polynomial of the fields in S and their derivatives.*

Proof: The idea of the proof is as for Theorem 7.1 of [CL], which uses [FKRW]. It is well-known (see e.g. Section 7 of [CL]), that $\mathrm{tr}_{0,0}(b,c)$ and $\mathrm{tr}_{0,1}(b,c)$ strongly generate the simple $\mathcal{W}_{1+\infty}$ algebra of central charge two. Especially, every $\mathrm{tr}_{n,m}(b,c)$ is expressible as a normally ordered polynomial in $\mathrm{tr}_{0,0}(b,c)$ and $\mathrm{tr}_{0,1}(b,c)$ and their derivatives:

$$\mathrm{tr}_{n,m}(b,c) = P_{n,m}(\mathrm{tr}_{0,0}(b,c), \mathrm{tr}_{0,1}(b,c)).$$

Let $\circ_m$ denote the m-th order pole contribution in the operator product algebra. Then

$$\begin{aligned} \mathrm{tr}_{0,0}(x,c) \circ_1 \mathrm{tr}_{0,n}(b,c) &= \mathrm{tr}_{0,n}(x,c), & \det_{0,0}(b,x) \circ_1 \mathrm{tr}_{0,n}(x,c) &= \det_{0,n}(x,x), \\ \mathrm{tr}_{0,0}(y,b) \circ_1 \mathrm{tr}_{n,0}(b,c) &= -\mathrm{tr}_{0,n}(y,b), & \det_{0,0}(c,y) \circ_1 \mathrm{tr}_{0,n}(y,b) &= \det_{0,n}(y,y), \\ \det_{0,0}(b,b) \circ_1 \mathrm{tr}_{n,0}(b,c) &= -2\det_{0,n}(b,b), & \det_{0,0}(c,c) \circ_1 \mathrm{tr}_{0,n}(b,c) &= 2\det_{0,n}(c,c), \\ \det_{0,0}(b,b) \circ_1 \mathrm{tr}_{0,n}(x,c) &= -2\det_{0,n}(x,b), & \det_{0,0}(c,c) \circ_1 \mathrm{tr}_{0,n}(y,b) &= -2\det_{0,n}(y,c), \\ \mathrm{tr}_{0,0}(x,c) \circ_1 \mathrm{tr}_{0,n}(y,b) &= \mathrm{tr}_{0,n}(y,x) + \mathrm{tr}_{1,n}(c,b). \end{aligned}$$

We consider the example of $\mathrm{tr}_{m,n}(x,c)$ and $\det_{m,n}(x,x)$. It follows from above the equation that

$$\begin{aligned} \mathrm{tr}_{0,n}(x,c) &= \mathrm{tr}_{0,0}(x,c) \circ_1 \mathrm{tr}_{0,n}(b,c) = \mathrm{tr}_{0,0}(x,c) \circ_1 P_{0,n}(\mathrm{tr}_{0,0}(b,c), \mathrm{tr}_{0,1}(b,c)) \\ &= Q_{0,n}(\mathrm{tr}_{0,0}(b,c), \mathrm{tr}_{0,0}(x,c), \mathrm{tr}_{0,1}(b,c), \mathrm{tr}_{0,1}(x,c)) \end{aligned}$$

for some polynomial $Q_{0,n}$ in $tr_{0,0}(b,c)$, $\mathrm{tr}_{0,0}(x,c)$, $\mathrm{tr}_{0,1}(b,c)$, $\mathrm{tr}_{0,1}(x,c)$ and their derivatives. From this we get

$$\begin{aligned} \det_{0,n}(x,x) &= \det_{0,0}(b,x) \circ_1 \mathrm{tr}_{0,n}(x,c) \\ &= \det_{0,0}(b,x) \circ_1 Q_{0,n}(\mathrm{tr}_{0,0}(b,c), \mathrm{tr}_{0,0}(x,c), \mathrm{tr}_{0,1}(b,c), \mathrm{tr}_{0,1}(x,c)) \\ &= R_{0,n}(\mathrm{tr}_{0,0}(b,c), \mathrm{tr}_{0,0}(x,c), \det_{0,0}(b,x), \mathrm{tr}_{0,1}(b,c), \mathrm{tr}_{0,1}(x,c), \det_{0,1}(b,x)) \end{aligned}$$

for some polynomial $R_{0,n}$ in $\mathrm{tr}_{0,0}(b,c)$, $\mathrm{tr}_{0,0}(x,c)$, $\det_{0,0}(b,x)$, $\mathrm{tr}_{0,1}(b,c)$, $\mathrm{tr}_{0,1}(x,c)$, $\det_{0,1}(b,x)$ and their derivatives. The claim for $\mathrm{tr}_{m,n}(x,c)$ and $\det_{m,n}(x,x)$ follows

by induction to $N = n + m$, since every $\mathrm{tr}_{n,m}(v, w)$ (respectively $\det_{n,m}(v, w)$) is a linear combination of $\mathrm{tr}_{0,n+m}(v, w)$ (respectively $\det_{0,n+m}(v, w)$) and the derivatives of the $\mathrm{tr}_{n',m'}(v, w)$ (respectively $\det_{n',m'}(v, w)$) with $n' + m' < n + m$.

The cases of the seven remaining types of basic invariants are proven in complete analogy. □

We can thus characterize V^G

Theorem 3.8. *V^G is strongly generated by the 16 fields of the set S.*

Proof: Lemma 3.3 says that every field of V^G is a normally ordered polynomial of the basic invariants and their derivatives, which in turn are by Lemma 3.7 themselves normally ordered polynomials in the fields of the set S and their derivatives. □

Going back to general dimension d and the corresponding V as at the end of the last subsection one has:

Remark 3.9.

- For $G = \mathrm{SL}(d, \mathbb{C})$ the vertex superalgebra V^G contains the $\mathcal{N}=2$ super Virasoro algebra of central charge $3d$ as subalgebra (special Kähler manifolds).
- For $G = \mathrm{Sp}(d', \mathbb{C})$ the vertex algebra V^G contains the $\mathcal{N}=4$ super Virasoro algebra of central charge $6d' = 3d$ as subalgebra (hyperkähler manifolds).

3.3 Characters for V^G-modules

In this section, we consider characters for the fixed point algebra V^G for the case $G = \mathrm{SL}(2, \mathbb{C})$.

We have the decompositions

$$V = \bigoplus_{N=0}^{\infty} \lambda_N \otimes V_{\lambda_N} \quad \text{and} \quad M = \bigoplus_{N=0}^{\infty} \lambda_N \otimes M_{\lambda_N},$$

where λ_N is the $N + 1$-dimensional irreducible representation of G. The V_{λ_N} are modules in the Neveu-Schwarz-sector of the vertex superalgebra V^G, while the M_{λ_N} are in the so-called Ramond-sector. The two are related by an automorphism of the $\mathcal{N}=2$ super Virasoro algebra that is called spectral flow in physics. A character of a module W is defined as

$$\mathrm{ch}_W(y; q) = \mathrm{tr}_W(q^{L_0 - \frac{1}{4}} y^{J_0}),$$

where L_0 is the zero-mode of the Virasoro subalgebra and J_0 the zero-mode of the Heisenberg subalgebra. Between the characters of V and M one has the identity $\mathrm{ch}_M(y; q) = q^{1/4} y \, \mathrm{ch}_V(yq^{1/2}; q)$. In the case of the $\mathcal{N} = 4$ super Virasoro algebra,

it is believed [ET] that modules fall into a generic family of irreducible modules parameterized by conformal weight h, called typical or massive, with character

$$\mathrm{ch}_h(y;q) = q^{h-3/8}\frac{\vartheta_3(y;q)^2}{\eta(q)^3}.$$

For $h = 1/4$ the module is not irreducible and the character is given by the one of three atypical or massless modules

$$\mathrm{ch}_{1/4}(y;q) = 2\,\mathrm{ch}_{1/4,0}(y;q) + \mathrm{ch}_{1/4,1}(y;q)$$

with

$$\mathrm{ch}_{1/4,0}(y;q) = \frac{\vartheta_3(y;q)}{\eta(q)^3}\sum_{\alpha\in\mathbb{Z}+\frac{1}{2}}\frac{y^{\alpha+1/2}q^{\alpha(\alpha-1)/2}}{1+yq^\alpha}.$$

Theorem 3.10. *The character* $\mathrm{ch}_{V_N}(y;q)$ *decomposes into a sum of atypical (massless) characters* $\mathrm{ch}^a_N(y;q)$ *and a sum of typical (massive characters)* $\mathrm{ch}^t_N(y;q)$ *of the* $\mathcal{N}=4$ *super Virasoro algebra of central charge six, where*

$$\begin{aligned}
\mathrm{ch}_{V_N}(y;q) &= \mathrm{ch}^a_N(y;q) + \mathrm{ch}^t_N(y;q)\\
\mathrm{ch}^a_N(y;q) &= \frac{\vartheta_3(y;q)}{\eta(q)^3}\sum_{\alpha\in\mathbb{Z}+\frac{1}{2}}\frac{y^{\alpha+1/2}q^{\alpha(\alpha-1)/2}}{1+yq^\alpha}\times\begin{cases}-2 & \text{if } N=0,\\ 1 & \text{if } N=1,\\ 0 & \text{else}\end{cases}\\
\mathrm{ch}^t_N(y;q) &= \frac{\vartheta_3(y;q)^2}{\eta(q)^3}(h_N(q) - 2h_{N+1}(q) + 2h_{N+3}(q) - h_{N+4}(q))\\
h_{N+1}(q) &= \frac{1}{\eta(q)^3}\sum_{\substack{m,r,s\in\mathbb{Z}+\frac{1}{2}\\ r,s>0}}(-1)^{r+s+1}q^{r|m|+s|N-m|+(\mathrm{sgn}(m)r+\mathrm{sgn}(m-N)s)^2/2-N/2}.
\end{aligned}$$

Before proving the theorem, we remark:

Remark 3.11. The Ramond-characters are immediately obtained via $\mathrm{ch}_{M_N}(y;q) = q^{1/4}y\,\mathrm{ch}_{V_N}(yq^{1/2};q)$. Let $\mathrm{ch}^R_h(y;q) = q^{h-3/8}\frac{\vartheta_2(y;q)}{\eta(q)^3}$ be the Ramond-character of conformal weight h of the $\mathcal{N}=4$ super Virasoro algebra. Expanding the $h_N(q)$ one can explicitily compute the multiplicities of the modules, and the result is of the form

$$\begin{aligned}
\mathrm{ch}_{V_N}(y;q) &= \mathrm{ch}^a_N(y;q) + \sum_{h\in\mathbb{Z}/4}\mathrm{mult}(N,h)\,\mathrm{ch}_h(y;q),\\
\mathrm{ch}_{M_N}(y;q) &= \mathrm{ch}^{R,a}_N(y;q) + \sum_{h\in\mathbb{Z}/4}\mathrm{mult}(N,h)\,\mathrm{ch}^R_h(y;q),
\end{aligned}$$

with the same multiplicities $\mathrm{mult}(N,h)$ in the Ramond- and NS-sector. On the other hand, from the explicit form of M, it is clear that only states with conformal dimension in $\mathbb{Z}+1/4$ appear implying $\mathrm{mult}(N,h)\neq 0$ only if $h\in\mathbb{Z}+1/4$.

Proof: The proof of this theorem uses the ideas of chapter 8 of [DMZ], where meromorphic Jacobi forms have been decomposed into Appell-Lerch sums and holomorphic Jacobi forms, see also [BF].

The graded character of V, graded by the natural Virasoro zero-mode L_0, by the charge f of the group $\mathrm{SL}(2,\mathbb{C})$ and by the zero-mode J_0 of the level one affine $sl(2)$ subalgebra is for $0<|q|<1$ and $|q|<|z|<|q|^{-1}$

$$\begin{aligned}\mathrm{ch}_V(z,y;q) &= \mathrm{tr}_V(q^{L_0-\frac{1}{4}}z^f y^{J_0})\\ &= q^{-\frac{1}{4}}\prod_{n=1}^{\infty}\frac{(1+zyq^{n-\frac{1}{2}})(1+zy^{-1}q^{n-\frac{1}{2}})(1+z^{-1}yq^{n-\frac{1}{2}})(1+z^{-1}y^{-1}q^{n-\frac{1}{2}})}{(1+zq^n)^2(1+z^{-1}q^n)^2}\\ &= \frac{\vartheta_3(y;q)^2}{\eta(q)^6}(1-z^{-1})^2\sum_{m,m'\in\mathbb{Z}}\frac{z^{m+m'}}{(1+yq^{m-\frac{1}{2}})(1+y^{-1}q^{m'-\frac{1}{2}})}.\end{aligned}$$

Here, the last identity follows directly from the denominator identity of the affine Lie superalgebra of $sl(2|1)$ [KW]. By defining

$$g_N=\frac{\vartheta_3(y;q)}{\eta(q)^3}\sum_{m\in\mathbb{Z}}\frac{1}{(1+yq^{m-\frac{1}{2}})(1+y^{-1}q^{N-m-\frac{1}{2}})},$$

the character decomposes as

$$\mathrm{ch}_V(z,y;q)=\frac{\vartheta_3(y;q)}{\eta(q)^3}\sum_{N\in\mathbb{Z}}z^N(g_N-2g_{N+1}+g_{N+2}).$$

We need to compute the g_N in order to determine the characters of V^G. For this consider the decomposition

$$\mathrm{ch}_V(z,y;q)=\sum_{N=0}^{\infty}\chi_N\cdot\mathrm{ch}_{V_N}$$

of the character of V where $\chi_N=\sum_{i=0}^{N}z^{-N+2i}$ is the character of λ_N and ch_{V_N} the character of V_N. Since the coefficient of z^N, $N\geq 0$, contains exactly the contributions for the representations λ_N, λ_{N+2}, ..., the character ch_N equals the difference of the coefficients of z^N and z^{N+2}, i.e. one has

$$\mathrm{ch}_{V_N}=\frac{\vartheta_3(y;q)}{\eta(q)^3}(g_N-2g_{N+1}+2g_{N+3}-g_{N+4}).$$

In order to compute g_N explicitily, we follow [DMZ]. The elliptic transformation of $g_N(u;\tau)$ for $y=e^{2\pi iu}$, $q=e^{2\pi i\tau}$ is

$$g_N(z+m\tau+n;\tau)=y^{-m}q^{-m^2/2}g_N(z;\tau)$$

meaning that the g_N have elliptic transformation properties as a Jacobi form of index 1/2. We define the Fourier part of g_N as follows: let

$$g_{N,\ell}^{u_0}=q^{-\ell^2/2}\int_{u_0}^{u_0+1}g_N(u;\tau)y^{-\ell}du.$$

Then the elliptic transformation of g_N implies

$$g_{N,\ell}^{u_0+\tau} = g_{N,\ell+1}^{u_0}$$

and hence $h_N = g_{N,\ell}^{-\ell\tau}$, $\ell \in \mathbb{Z}$, is independent of the choice of the integer ℓ. Expanding g_N as a power series in q and y allows to compute the integral for h_N. The result is

$$h_{N+1}(q) = \frac{1}{\eta(q)^3} \sum_{\substack{m,r,s\in\mathbb{Z}+\frac{1}{2}\\ r,s>0}} (-1)^{r+s+1} q^{r|m|+s|N-m|+(\mathrm{sgn}(m)r+\mathrm{sgn}(m-N)s)^2/2-N/2}.$$

The Fourier part of g_N is now

$$g_N^F = \vartheta_3(y;\tau)h_N(\tau).$$

The advantage is that this immediately provides a decomposition of characters if g_N is holomorphic, as in that case $g_N^F = g_N$. Looking at the definition of g_N, we see that g_1 is meromorphic but not holomorphic, while for all $N \neq 1$ the g_N are holomorphic. In summary, we have shown

$$\mathrm{ch}_N = \frac{\vartheta_3(y;q)^2}{\eta(q)^3}(h_N - 2h_{N+1} + 2h_{N+3} - h_{N+4})$$

for $N \geq 2$.

It remains to compute g_1. The poles of $g_1(y;q)$ are at the points $(-q^{\alpha-1/2};q)$ for integers α. Its (weighted) residues are related using the elliptic transformations as follows. Let C_α be a small contour including only the pole at $y = -q^{\alpha-1/2}$, then we have

$$\begin{aligned}\mathrm{Res}_{y=-q^{\alpha-\frac{1}{2}}}(g_1(y;q)y^{-\ell}) &= \frac{1}{2\pi i}\oint_{C_\alpha} y^{-\ell} g_1(y;q)du = \frac{1}{2\pi i}\oint_{C_0} (yq^\alpha)^{-\ell} g_1(yq^\alpha;q)du \\ &= (-1)^{\ell+\alpha} q^{\frac{1}{2}(\alpha+\ell-2\alpha\ell-\alpha^2)} \mathrm{Res}_{y=-q^{-\frac{1}{2}}}(g_1(y;q)) \\ &= \frac{(-1)^{\ell+\alpha}}{2\pi i} q^{\frac{1}{2}(\alpha+\ell-2\alpha\ell-\alpha^2-1/4)}.\end{aligned}$$

We define the polar part of g_N as

$$g_N^P = g_N - g_N^F.$$

Of course $g_N^P = 0$ if $N \neq 1$. In order to compute g_1^P we proceed as in the proof of Theorem 8.1 of [DMZ]. Let $\tau = \tau_1 + i\tau_2$ for τ_1, $\tau_2 \in \mathbb{R}$ and define A by $\mathrm{Im}(u) = A\tau_2$ and choose a contour as described in Theorem 8.1 of [DMZ]. Further fix a point

Table 2: Decomposition multiplicities of V^G-module characters into characters for the $\mathcal{N}=4$ super Virasoro algebra

	atypical	typical $\mathrm{ch}_h(y;q)$ for $h=$											
	$\mathrm{ch}_{1/4,0}(y;q)$	0	1	2	3	4	5	6	7	8	9	10	11
V_{λ_0}	-2	1	0	1	0	1	3	2	6	11	13	24	43
V_{λ_1}	1	0	0	0	2	2	2	8	10	16	30	46	68
V_{λ_2}	0	0	1	0	1	3	5	7	14	22	39	60	97
V_{λ_3}	0	0	0	2	0	2	6	8	14	28	38	70	112
V_{λ_4}	0	0	0	0	3	1	3	9	15	22	45	67	112
V_{λ_5}	0	0	0	0	0	4	2	6	12	22	36	66	102
V_{λ_6}	0	0	0	0	0	0	5	3	9	18	30	50	95
V_{λ_7}	0	0	0	0	0	0	0	6	4	12	24	42	66
V_{λ_8}	0	0	0	0	0	0	0	0	7	5	15	30	54
V_{λ_9}	0	0	0	0	0	0	0	0	0	8	6	18	36
$V_{\lambda_{10}}$	0	0	0	0	0	0	0	0	0	0	9	7	21

$Q = A\tau + B$ for A, $B \in \mathbb{R}$ so that $g_1(e^{2\pi i Q}, q)$ is finite. Then

$$\begin{aligned}
g_1^P(y;q) &= g_1(y;q) - g_1^F(y;q) \\
&= \sum_{\ell\in\mathbb{Z}} (g_{1,\ell}^Q(q) - h_1(q)) q^{\ell^2/2} y^\ell \\
&= 2\pi i \sum_{\ell\in\mathbb{Z},\, \alpha\in\mathbb{Z}+\frac{1}{2}} \frac{\mathrm{sgn}(\alpha - A) - \mathrm{sgn}(\alpha+\ell)}{2} \mathrm{Res}_{y=-q^\alpha}(g_1(y;q) y^{-\ell}) y^\ell \\
&= \sum_{\ell\in\mathbb{Z},\, \alpha\in\mathbb{Z}+\frac{1}{2}} \frac{\mathrm{sgn}(\alpha - A) - \mathrm{sgn}(\alpha+\ell)}{2} y^\ell (-1)^{\ell+\alpha+1/2} q^{-\alpha^2/2-\alpha\ell} \\
&= \sum_{\alpha\in\mathbb{Z}+\frac{1}{2}} \frac{(yq^{-\alpha})^{-\alpha+1/2}}{1+yq^{-\alpha}} q^{-\alpha^2/2} = \sum_{\alpha\in\mathbb{Z}+\frac{1}{2}} \frac{y^{\alpha+1/2} q^{\alpha(\alpha-1)/2}}{1+yq^\alpha}.
\end{aligned}$$

□

We finally list explicit decompositions of the first few V^G-module characters into the first few $\mathcal{N}=4$ characters in Table 2.

3.4 Implications for the elliptic genus of a K3 surface

We can apply now the discussion of the previous subsections to the case of K3 surfaces.

Let V be the vertex algebra studied in Subsection 3.2 and let $G = \mathrm{SL}(2,\mathbb{C})$.

Corollary 3.12. *The elliptic genus of a K3 surface is the graded character of a natural virtual vertex operator algebra module for V^G and its $\mathcal{N}=4$ super Virasoro vertex subalgebra.*

Proof: A K3 surface has a trivial canonical line bundle. This allows to reduce the structure group of the holomorphic tangent bundle from $\mathrm{GL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$. As discussed in Subsection 3.2, the vertex algebra V and the V-module M can be decomposed into direct sums of V^G-modules. Theorem 3.1 and Lemma 3.4 part 3. together with the description of the elliptic genus as twisted Todd genus give now the stated virtual vertex algebra module structures. □

For the explicit decomposition of the elliptic genus into characters of V^G and the $\mathcal{N}=4$ super Virasoro vertex algebra one has to compute the holomorphic Euler characteristics $\chi(X,\underline{\lambda_n})$ where λ_n is the $(n+1)$-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. This will be done in the final section.

4 The elliptic genus and Mathieu Moonshine

In Section 2, we have seen that the elliptic genus of a K3 surface can be given the structure of a virtual M_{24}-module extending the virtual H-module structures for symplectic automorphism groups H.

In Section 3, it was shown that the Calabi-Yau structure of K3 surfaces leads to a $V^{SU(2)}$-module structure for its elliptic genus. This implies in particular that there is a decomposition of the elliptic genus into characters of the $\mathcal{N}=4$ super Virasoro vertex operator algebra compatible with the above virtual M_{24}-module structures.

Somewhat disappointingly, we will show in this section that the resulting virtual M_{24}-module structures cannot be made compatible with the predicted McKay-Thompson series of Mathieu moonshine.

4.1 Rational Mathieu Moonshine

The coefficients of the complex elliptic genus of a complex manifold are the holomorphic Euler characteristics of certain vector bundles associated to the tangent bundle of the manifold. In the case of a K3 surface X, the structure group of the principal bundle corresponding to the tangent bundle can be reduced to SU(2). Every complex vector bundle associated to the tangent bundle of X is as a differentiable vector bundle isomorphic to a direct sum of vector bundles associated to irreducible representations of SU(2). The irreducible representations of SU(2) are exactly the symmetric powers of the two-dimensional defining representation of SU(2).

We collect all the symmetric powers of the tangent bundle T of X in the gener-

ating series $S_tT = \bigoplus_{n=0}^{\infty} S^nT \cdot t^n$ and investigate the expression

$$\chi(X, S_tT) = \sum_{n=0}^{\infty} \chi(X, S^nT) \cdot t^n \in \mathbb{Z}[[t]].$$

Let E be a holomorphic vector bundle on a complex manifold X. Recall from Section 3.1 that by the Riemann-Roch theorem one has

$$\chi(X, E) = \mathrm{Td}(X)\mathrm{ch}(E)[X].$$

For the symmetric powers of the tangent bundle we get $\mathrm{ch}(S_tT) = \prod_{i=1}^{d}(1 - e^{x_i}t)^{-1}$ where the x_i are the formal roots of the total Chern class of T and d is the complex dimension of X. Thus $\chi(X, S_tT)$ is the complex genus for the characteristic power series

$$Q(x) = \frac{x}{1 - e^{-x}} \cdot \frac{1}{1 - e^x t} = \frac{1}{1 - t}\left(1 - \frac{1 + t}{2(t - 1)}\, x + \frac{1 + 10t + t^2}{12(t - 1)^2}\, x^2 + \cdots\right)$$

in the sense of [Hi].

For X a K3 surface, the only non-vanishing Chern number is $c_2[X] = 24$ which equals the Euler characteristic of X. Using the methods developed in [Hi], we easily obtain

$$\begin{aligned} r_{1A} := \chi(X, S_tT) &= \frac{2 - 28t + 2\,t^2}{(t - 1)^4} \qquad (3) \\ &= 2 - 20\,t - 90\,t^2 - 232\,t^3 - 470\,t^4 - 828\,t^5 - 1330\,t^6 + \cdots . \end{aligned}$$

Given a non-trivial symplectic automorphism g of X, we can compute the equivariant genus $r_g(t) = \chi(g; X, S_tT)$ by using the information from Table 1 and applying the equivariant Riemann-Roch [AB] formula:

$$\chi(g; X, S_tT) = \sum_{p_i} \frac{1}{(1 - \lambda_i)(1 - \lambda_i^{-1})} \cdot \frac{1}{(1 - \lambda_i^{-1}\, t)(1 - \lambda_i\, t)}.$$

Here, $(\lambda_i, \lambda_i^{-1})$ are the eigenvalues of g acting on the tangent space of the isolated fixed point p_i.

We recall from Section 2 that a finite symplectic automorphism g of a K3 surface can be considered as an element of M_{23} or M_{24} via Mukai's theorem. Up to taking inverses, the associated conjugacy class in M_{23} or M_{24} depends only on the order of g. For the so arising non-trivial conjugacy classes of M_{23} one obtains the following

seven rational functions:

$$\begin{aligned}
r_{2A} &= \frac{2}{\Phi_2(t)^2} \\
r_{3A} &= \frac{2}{\Phi_3(t)} \\
r_{4A} &= \frac{2}{\Phi_4(t)} \\
r_{5A} &= \frac{2+2t+2t^2}{\Phi_5(t)} \\
r_{6A} &= \frac{2}{\Phi_6(t)} \\
r_{7AB} &= \frac{2(1+t^4)+3(t+t^3)+4t^2}{\Phi_7(t)} \\
r_{8A} &= \frac{2+2t+2t^2}{\Phi_8(t)}.
\end{aligned}$$

Here, $\Phi_n(t)$ denotes the n-th cyclotomic polynomial of degree $\varphi(n)$.

We like to make $\chi(X, S_tT)$ into a graded virtual rational M_{23}- or M_{24}-module with characters r_{1A}, r_{2A}, $\ldots$, r_{7AB}, r_{8A} as above. By Theorem 2.4, we know that this is always possible. For a rational M_{23}-module structure, we have to provide formulas for $r_g = \mathrm{tr}(g|\chi(X, S_tT))$ for g belonging to a conjugacy class of M_{23} of the four remaining types $11AB$, $14AB$, $15AB$ and $23AB$. We note that for the other eight types of conjugacy classes $[g]$ the rational function $r_g(t)$ has the form $\frac{P(t)}{Q(t)}$ where $Q(t)$ is a power of the n-th cyclotomic polynomial with n the order of g and $P(t)$ is a palindromic polynomial of degree two less than the degree of $Q(t)$. By requiring that the resulting rational functions have the same structure as for the previous eight classes, we found the following (non-uniquely determined) expressions:

$$\begin{aligned}
r_{11AB} &= \frac{2(1+t^8)+4(t+t^7)+2(t^2+t^6)+(t^3+t^5)+4t^4}{\Phi_{11}(t)} \\
r_{14AB} &= \frac{2(1+t^4)+(t+t^3)-2t^2}{\Phi_{14}(t)} \\
r_{15AB} &= \frac{2(1+t^6)+(t+t^5)-3(t^2+t^4)-6t^3}{\Phi_{15}(t)} \\
r_{23AB} &= \frac{2(1+t^{20})+5(t+t^{19})+7(t^2+t^{18})+5(t^3+t^{17})+5(t^4+t^{16})}{\Phi_{23}(t)} \\
&\quad + \frac{-5(t^5+t^{15})-(t^6+t^{14})+13(t^8+t^{12})+6(t^9+t^{11})+15t^{10}}{\Phi_{23}(t)}.
\end{aligned}$$

The general formula

$$r_{NA} = m(N) \sum_{n\in(\mathbb{Z}/N\mathbb{Z})^*} \frac{1}{(1-e^{2\pi in/N})(1-e^{-2\pi in/N})} \cdot \frac{1}{(1-e^{-2\pi in/N}\,t)(1-e^{2\pi in/N}\,t)}$$

for $N \leq 8$ where the constant $m(N)$ is chosen such that $r_{NA} = 2 + O(t)$ cannot be used for $N > 8$ although it has the correct shape since the t-expansion contains non-integral coefficients.

The character table of M_{23} allows to compute the corresponding multiplicities $m_\chi(t)$ of the irreducible M_{23}-modules in $\chi(X, S_tT)$. The resulting rational functions in t have integral coefficients, i.e. $\chi(X, S_tT)$ becomes by taking the above characters r_{11AB}, r_{14AB}, r_{15AB} and r_{23AB} indeed a virtual M_{23}-module. For example, for the multiplicity of the trivial character one has $m_{\chi_1}(t) = \frac{P(t)}{Q(t)}$ with $P(t)$ a polynomial of degree 70 and $Q(t)$ a product of cyclotomic polynomials of total degree 72.

The partial fraction decompositions of the multiplicity functions $m_\chi(t)$ contain for all M_{23}-characters χ a dominating term $\frac{c_\chi}{(t-1)^4}$ for $t \to 1$ with $c_\chi < 0$; for example, $m_{\chi_1} = -\frac{1}{425040}$. Thus for large enough n, the virtual M_{23}-modules $-\chi(X, S^nT)$ are actual all honest M_{23}-modules. We have not been able to make the $-\chi(X, S^nT)$ simultaneously for all n into honest M_{23}-modules under the assumptions made. More precisely, by taking the characters for elements of order 1 to 8 to zero but by allowing arbitrary values for the elements of order 11, 14, 15 and 23, one is able to add integral linear combinations of the following four virtual rational M_{23}-modules:

irred.	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
11	2	0	2	2	−2	0	0	0	0	0	0	−1	−1	0	0	2	0
14	2	−2	1	1	2	0	0	0	−2	0	0	0	0	−1	−1	−2	2
15	2	−2	0	0	0	2	−1	−1	2	0	0	2	2	0	0	0	−2
23	2	−2	−2	−2	0	2	2	2	0	−1	−1	−2	−2	2	2	0	0

We list the decomposition of $-\chi(X, S_tT)$ into irreducible virtual M_{23}-modules for the first 21 terms in Table 3.

We expect that our rational M_{23} moonshine extends to a rational moonshine for M_{24} or even Co_0. However, because of the non-uniqueness in choosing additional characters and the incompatibility with Mathieu moonshine discussed in the next subsection, we do not investigate this further.

4.2 Comparing rational and Mathieu Moonshine

We will compare the virtual M_{23}-modules structures for the holomorphic Euler characteristics $\chi(X, S^nT)$ as in the previous subsection with Mathieu moonshine.

First, we have to verify that for symplectic automorphisms on a K3 surface the equivariant elliptic genus coincides with the prediction of Mathieu moonshine as in [GHV2, EH].

For a compact Lie group H acting on a complex manifold X by automorphisms, the equivariant elliptic genus

$$\chi_y(g; q, \mathcal{L}X) := (-y)^{-d/2}\chi_y\big(g; X, \bigotimes_{n=1}^{\infty} \Lambda_{yq^n}T^* \otimes \bigotimes_{n=1}^{\infty} \Lambda_{y^{-1}q^n}T \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(T^* \oplus T)\big)$$

Table 3: Decomposition of $-\chi(X, S_t T)$ into irreducible M_{23}-modules

n	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
0	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	−1	0	0	0	1	0	0	1	0	0	1	1	0	0	0	−1
4	2	−2	−2	−2	0	2	0	0	2	0	0	1	1	1	1	0	−2
5	2	−1	0	0	−1	1	−1	−1	2	0	0	1	1	0	0	1	−1
6	0	0	0	0	0	0	−1	−1	0	0	0	1	1	0	0	0	0
7	−1	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	0	1
8	−2	2	2	2	0	−2	0	0	−2	1	1	0	0	−1	−1	0	2
9	−1	1	0	0	0	−1	2	2	−1	0	0	−2	−2	1	1	1	2
10	−2	2	1	1	0	−2	0	0	0	1	1	0	0	0	0	0	2
11	1	−1	−1	−1	1	1	1	1	0	0	0	0	0	1	1	0	2
12	0	0	0	0	0	0	−1	−1	0	1	1	2	2	0	0	0	2
13	2	−1	−1	−1	0	2	1	1	0	0	0	0	0	2	2	2	2
14	0	0	0	0	0	0	0	0	0	1	1	1	1	2	2	2	2
15	−1	0	−1	−1	1	0	1	1	1	1	1	1	1	2	2	2	3
16	0	0	1	1	0	0	1	1	0	2	2	1	1	2	2	2	4
17	−1	1	1	1	1	0	0	0	1	2	2	3	3	1	1	2	5
18	0	0	0	0	0	2	0	0	2	2	2	4	4	3	3	2	4
19	2	−1	0	0	0	3	0	0	3	2	2	4	4	3	3	4	5
20	2	−2	0	0	0	2	0	0	2	3	3	4	4	4	4	4	6

is an element in $R(H)[y,y^{-1}][[q]]$ and can also be interpreted for fixed $g \in H$ of finite order as a Jacobi form for a certain congruence subgroup. The holomorphic Lefschetz fixed point formula [AB] allows to determine the equivariant elliptic genus from information on the fixed point set and the action on its normal bundle.

For a K3 surface, we can use Table 1 to compute the equivariant elliptic genus for finite symplectic automorphisms similar as done done for $\chi(g;X,S_tT)$ in the previous subsection.

Let $q = e^{2\pi i\tau}$, $y = e^{2\pi iu}$ for τ in the upper half of the complex plane, and $u \in \mathbb{C}$, and consider the following quotient of a standard Jacobi theta function and Dedekind's eta function

$$\varphi(u;\,\tau) := \vartheta_1(u;\,\tau)\eta(\tau)^{-3} = -i(y^{1/2} - y^{-1/2})\prod_{n=1}^{\infty}(1-yq^n)(1-y^{-1}q^n)(1-q^n)^{-2}.$$

It has the following modular and elliptic transformation properties

$$\begin{aligned}\varphi\Big(\frac{u}{ct+d};\,\frac{a\tau+b}{c\tau+d}\Big) &= (c\tau+d)^{-1}e^{\frac{\pi icu^2}{c\tau+d}}\varphi(u;\,\tau), \qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\\ \varphi(u+\lambda\tau+\mu;\,\tau) &= e^{-\pi i(\lambda^2\tau+2\lambda u)}(-1)^{\lambda+\mu}\varphi(u;\,\tau), \qquad \lambda,\,\mu\in\mathbb{Z}.\end{aligned}$$

Definition 4.1. A weak Jacobi form J of weight k and index m for some subgroup of $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function on $\mathbb{H}\times\mathbb{C}$ with no poles at the cusp $q=0$, satisfying

$$\begin{aligned}J\Big(\frac{u}{ct+d};\,\frac{a\tau+b}{c\tau+d}\Big) &= (c\tau+d)^{k}e^{\frac{2\pi imcu^2}{c\tau+d}}J(u;\,\tau), \qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma \subset \mathrm{SL}(2,\mathbb{Z})\\ J(u+\lambda\tau+\mu;\,\tau) &= e^{-2\pi im(\lambda^2\tau+2\lambda u)}J(u;\,\tau), \qquad \lambda,\,\mu\in\mathbb{Z}.\end{aligned}$$

Lemma 4.2. *For a non-trivial finite symplectic automorphism g of a K3 surface X the equivariant elliptic genus $\chi_{-y}(g;q,\mathcal{L}X)$ is a weak Jacobi form for $\Gamma_0(\mathrm{ord}(g))$ of weight zero and index one.*

Proof: Let $(\lambda_i,\lambda_i^{-1})$ be the eigenvalues of g acting on the tangent space of a fixed point p_i. The fixed point formula gives

$$\begin{aligned}\chi_y(g;q,\mathcal{L}X) &= \sum_{p_i}(-y)^{-1}\frac{(1+y\lambda_i)(1+y\lambda_i^{-1})}{(1-\lambda_i)(1-\lambda_i^{-1})}\times\\ &\quad\times\prod_{n=1}^{\infty}\frac{(1+yq^n\lambda_i)(1+y^{-1}q^n\lambda_i^{-1})(1+yq^n\lambda_i^{-1})(1+y^{-1}q^n\lambda_i)}{\big((1-q^n\lambda_i)(1-q^n\lambda_i^{-1})\big)^2}.\end{aligned}$$

Let $N=\mathrm{ord}(g)$ and define multiplicities $m(N)$ by $m(2)=8$, $m(3)=3$, $m(4)=2$, $m(5)=m(6)=1$ and $m(7)=m(8)=1/2$. Then comparing with Table 1, we see that

$$\chi_{-y}(g;q,\mathcal{L}X) = m(N)\sum_{n\in(\mathbb{Z}/N\mathbb{Z})^*}\frac{\varphi\big(u+\frac{n}{N};\,\tau\big)\varphi\big(u-\frac{n}{N};\,\tau\big)}{\varphi\big(\frac{n}{N};\,\tau\big)\varphi\big(-\frac{n}{N};\,\tau\big)}.$$

Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be in $\Gamma_0(N)$, that means $c \equiv 0 \bmod N$, and it follows that $d \pmod N$ is a unit in $\mathbb{Z}/N\mathbb{Z}$. Define $u_\pm = u \pm (c\tau + d)n/N$ and $v_\pm = \pm(c\tau + d)n/N$. Then

$$\frac{\varphi\big(\frac{u}{c\tau+d} + \frac{n}{N};\, \frac{a\tau+b}{c\tau+d}\big)\varphi\big(\frac{u}{c\tau+d} - \frac{n}{N};\, \frac{a\tau+b}{c\tau+d}\big)}{\varphi\big(\frac{n}{N};\, \frac{a\tau+b}{c\tau+d}\big)\varphi\big(-\frac{n}{N};\, \frac{a\tau+b}{c\tau+d}\big)} = \frac{\varphi\big(\frac{u_+}{c\tau+d};\, \frac{a\tau+b}{c\tau+d}\big)\varphi\big(\frac{u_-}{c\tau+d};\, \frac{a\tau+b}{c\tau+d}\big)}{\varphi\big(\frac{v_+}{c\tau+d};\, \frac{a\tau+b}{c\tau+d}\big)\varphi\big(\frac{v_-}{c\tau+d};\, \frac{a\tau+b}{c\tau+d}\big)}$$
$$= e^{\frac{2\pi i u^2}{c\tau+d}}\, \frac{\varphi(u_+;\,\tau)\varphi(u_-;\,\tau)}{\varphi(v_+;\,\tau)\varphi(v_-;\,\tau)} = e^{\frac{2\pi i u^2}{c\tau+d}}\, \frac{\varphi\big(u + \frac{dn}{N};\,\tau\big)\varphi\big(u - \frac{dn}{N};\,\tau\big)}{\varphi\big(\frac{dn}{N};\,\tau\big)\varphi\big(-\frac{dn}{N};\,\tau\big)}$$

where the second equality follows from the modular properties of φ, and the third one from the elliptic ones. Note that contributions from numerator and denominator lead to various cancellations. Inserting this identity in the expression for the equivariant elliptic genus proves the modular transformation property. Its elliptic transformation property follows immediately from the elliptic properties of φ. □

Now we can easily show:

Theorem 4.3. *For a non-trivial finite symplectic automorphism g acting on a K3 surface X the equivariant elliptic genus and the twining character determined by the McKay-Thompson series of Mathieu moonshine agree, i.e. one has*

$$\chi_{-y}(g; q, \mathcal{L}X) = \frac{e(g;X)}{12}\,\phi_{0,1} + f_g\,\phi_{-2,1},$$

where f_g is the modular form as in equation (1).

Proof: By Lemma 4.2, the g-equivariant elliptic genus is a weak Jacobi form of weight zero and index one for $\Gamma_0(N)$ with $N = \mathrm{ord}(g)$. By a result of [AI] every such Jacobi form is a linear combination of $\phi_{0,1}$ and $h_N\,\phi_{-2,1}$, where the $\phi_{m,1}$ are weak Jacobi forms of weight m and index 1, and h_N is a modular form of weight 2 for $\Gamma_0(N)$. For $y = -1$, the equivariant elliptic genus specializes to the equivariant Euler characteristic of X (cf. [Ho]) whereas $\phi_{0,1}$ and $\phi_{-2,1}$ have the values 12 and 0, respectively. The space of modular forms of weight 2 for $\Gamma_0(N)$ is a finite dimensional vector space of dimension at most 3. By considering a base one sees that comparing the first three coefficients of h_N with f_g is completing the proof. □

Remark 4.4. It was observed by several authors (e.g., [C]) that the McKay-Thompson series of the Mathieu moonshine observation for those elements $g \in M_{24}$ which can be realized by symplectic group actions on a K3 surface are identical to the equivariant elliptic genus. However, the elliptic genus there is understood as the elliptic genus of an $\mathcal{N} = (4,4)$ super conformal field theory and not in the sense of the topological defined elliptic genus of a manifold as in the present paper. Without rigorous definition of $\mathcal{N} = (4,4)$ super conformal field theories it is not obvious that both definitions are the same.

The sum decompositions of the complex genus of a K3 surface into characters of

modules of the holonomy subalgebra and the $\mathcal{N}=4$ super Virasoro algebra

$$\begin{aligned}
\chi_{-y}(q,\mathcal{L}X) &= (-y)^{-d/2}\chi_y\big(X,\bigotimes_{n=1}^{\infty}\Lambda_{yq^n}T^*\otimes\bigotimes_{n=1}^{\infty}\Lambda_{y^{-1}q^n}T\otimes\bigotimes_{n=1}^{\infty}S_{q^n}(T^*\oplus T)\big)\\
&= \sum_{n=0}^{\infty}\chi(X,S^nT)\cdot\mathrm{ch}_{M_n}(y;\,q)\\
&= 24\cdot\mathrm{ch}_{h=1/4,l=0}(y;\,q)+\sum_{n=0}^{\infty}A_n\cdot\mathrm{ch}_{h=1/4+n}(y;\,q) \qquad (4)
\end{aligned}$$

as discussed in Section 3 allows to express the coefficients A_n in terms of Todd genera twisted by symmetric powers. This is easily achieved for small n with the help of Theorem 3.10. Alternatively, one can use the identities $S^nT=S^n(T^*)$, $\Lambda_tT=S^0T\oplus S^1T\,t\oplus S^0T\,t^2$, the Clebsch-Gordan rule $S^iT\otimes S^jT=S^{|i-j|}T\oplus S^{|i-j|+2}T\oplus\cdots\oplus S^{i+j}T$ and the explicit formulas for the $\mathcal{N}=4$ characters $\mathrm{ch}_{h=1/4,l=0}$ and $\mathrm{ch}_{h=1/4+n}$.

The first few coefficients are given by

$$\begin{aligned}
\sum_{n=0}^{\infty}A_nq^n &= -\chi(X,\mathbb{C})-\chi(X,S^2T)\,q-\chi(X,\mathbb{C}\oplus 2S^3T)\,q^2\\
&\quad-\chi(X,2T\oplus S^2T\oplus 3S^4T)\,q^3\\
&\quad-\chi(X,\mathbb{C}\oplus 2T\oplus 3S^2T\oplus 2S^3T\oplus S^4T\oplus 4S^5T)\,q^4+\cdots\,.
\end{aligned}$$

Here we wrote shortly nF for the n-fold direct sum vector bundle $F^{\oplus n}$.

According to Mathieu moonshine, the A_n can be considered as the dimensions of virtual M_{24}-modules K_n (actually honest ones for $n>0$) with explicit given McKay-Thompson series $\Sigma_g(q)=q^{-1/8}\sum_{n=0}\mathrm{Tr}(g|K_n)\,q^n$.

We like to investigate if Mathieu moonshine which uses the characters for the $\mathcal{N}=4$ super Virasoro algebra can be extended to a rational M_{24}-moonshine for the symmetric powers by using the characters for the holonomy invariant subalgebra. In this case the series $r_g(t)=\chi(g;X,S_tT)$ can be computed recursively by the above formula given Σ_g and $\chi(g;X,T)$. The coefficient 24 of $\mathrm{ch}_{h=1/4,l=0}$ in equation (4) is the Euler characteristic of the K3 surface. As M_{24}-character it is interpreted in Mathieu moonshine as the permutation character $\underline{1}+\underline{23}$ of M_{24} for its natural action on 24 elements. This implies that $\chi(g;X,T)$ equals the virtual M_{24}-character $3\cdot\underline{1}-\underline{23}$.

From Theorem 4.3 it follows that for the M_{23} conjugacy classes $[g]\in\{1A,2A,3A,4A,5A,6A,7AB,8A\}$ we get the rational functions already listed in Section 4.1. We checked this for the first ten coefficients.

For the remaining classes $\{11AB, 14AB, 15AB, 23AB\}$ we get:

$$\begin{aligned}
r_{11AB} &= 2 + 2\,t - 2\,t^2 - t^3 - \frac{2}{3}\,t^4 - \cdots \\
r_{14AB} &= 2 + 3\,t - t^2 - t^3 - \frac{5}{3}\,t^4 - \cdots \\
r_{15AB} &= 2 + 3\,t - \frac{1}{2}\,t^3 - 2\,t^4 - \cdots \\
r_{23AB} &= 2 + 3\,t + 2\,t^2 - 2\,t^3 - \frac{7}{3}\,t^4 - \cdots .
\end{aligned}$$

Since in each case either the coefficient of t^3 or of t^4 is not integral, we conclude that $\chi(X, S_tT)$ cannot have the structure of a virtual M_{23}- or M_{24}-module compatible with the prediction of Mathieu moonshine.

This conclusion even holds if we do not assume that $\chi(g; X, T) = \alpha$ is the virtual M_{24}-character $3 \cdot \underline{1} - \underline{23}$. For example one has

$$r_{15AB} = 2 + \alpha\,t - \frac{1}{2}\,t^3 - \frac{2\alpha}{3}\,t^4 - \frac{3 + 4\alpha}{12}\,t^5 + \cdots$$

and so α and the coefficient of t^5 cannot simultaneously be integral.

We also could not find simple rational expressions for those r_g. However, for the two M_{24} conjugacy classes $2B$ and $4A$ we found the following formulas (verified up to order 20):

$$\begin{aligned}
r_{2B(M_{24})} &= \frac{2 + 8t + 2t^2}{\Phi_2(t)^2\Phi_4(t)}, \\
r_{4A(M_{24})} &= \frac{2(1 + t^5) + 6(t + t^4) + 12(t^2 + t^3)}{\Phi_2(t)\Phi_4(t)\Phi_8(t)}.
\end{aligned}$$

For the remaining conjugacy classes of M_{24} not contained in M_{23} we could again not find such expressions. Also, the r_g contain always non-integral coefficients in those cases.

Combining this with the results of Section 3, we conclude that although the vertex algebra structure arising from the hyperkähler geometry of K3 surfaces can explain the $\mathcal{N}=4$ super Virasoro module structure of the elliptic genus, it cannot simultaneously explain the virtual M_{24}-module structure. There is no M_{24}-moonshine for the decomposition of $\chi_y(q, \mathcal{L}X)$ into characters of V^G-modules compatible with the prediction of Mathieu moonshine.

This leads to the question if the McKay-Thompson series of Mathieu moonshine can be modified to make them compatible with the V^G-module structure of the elliptic genus. Our r_g for g of type $11AB$, $14AB$, $15AB$, $23AB$ cannot be described by weight 2 modular forms h_g for $\Gamma_0(n)$, $n = \mathrm{ord}(g)$, in the form explained in the proof of Theorem 4.3. It is however at least sometimes possible to obtain rational expressions for r_g for certain modular forms $h_g \in M_2(\Gamma_0(n))$, for example one can realize

$$\tilde{r}_{11AB} = \frac{2(1 + t^8) + 4(t + t^7) + \frac{32}{5}(t^2 + t^6) + \frac{38}{5}(t^3 + t^5) + \frac{42}{5}t^4}{\Phi_{11}(t)}.$$

This gives however not a M_{23}- or M_{24}-character.

The McKay-Thompson series for a set of modular forms $f_g \in M_2(\Gamma_0(N))$ for g a conjugacy class in M_{24} describe a graded virtual M_{24}-modules if the coefficients of the f_g satisfy certain divisibility conditions which in turn can easily be described by the integral structure of $M_2(\Gamma_0(N))$; cf. [G]. In particular, the condition to define a graded virtual M_{24}-module alone does not determine the f_g uniquely.

We recall some arguments why the predicted McKay-Thompson series of Mathieu moonshine as in [GHV2, EH] should be the correct choice: The first is that the K_n for $n > 0$ are indeed honest M_{24}-modules which decompose for the first few n into irreducibles with very small multiplicity (in fact only one ot two irreducibles are present for $1 \leq n \leq 5$). Secondly, Gaberdiel, Hohenegger, and Volpato determined in [GHV2] the expected structure of symplectic like symmetry groups of $\mathcal{N} = (4,4)$ super conformal field theories. This allows to realize additional (but not all!) conjugacy classes of M_{24} as symmetries of such conformal field theories. For some of those they showed that the equivariant elliptic genus in the sense of conformal field theory equals the one predicted by Mathieu moonshine. Finally, Cheng and Duncan gave in [CD] a description of the corresponding Mock modular forms in terms of Rademacher sums (thought as an analogy of the genus zero property of the McKay-Thompson series of Monstrous moonshine) distinguishing the chosen f_g of Mathieu moonshine.

We conclude with the remark that a refined geometric approach should consider group actions on K3 surfaces with generalized Calabi-Yau structures (cf. [Hu]), the corresponding generalization of the chiral De Rham complex [HZ1] and the additional super Virasoro module structures coming from generalized Calabi-Yau metric structures [HZ2]. However, it may be too much to expect that this approach alone can explain Mathieu moonshine.

References

[AB] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes I,* Ann. of Math. (2) **86** (1967) 374–407.

[ABD] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice and P. Di Vecchia *et al.*, *Dual String with U(1) Color Symmetry*, Nucl. Phys. B **111** (1976) 77.

[AI] H. Aoki and T. Ibukiyama, *Simple graded rings of Siegel modular orms, differential operators and Borcherds products*, Int. J. Math. **16** (2005), 249–279.

[AT] Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; Wilson, R. A. *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups,* With computational assistance from J. G. Thackray. Oxford University Press, Eynsham, 1985. xxxiv+252 pp.

[Be] M. Benard *Schur indexes of sporadic simple groups,* J. Algebra **58** (1979), 508–522.

[Bo] R. E. Borcherds, *Monstrous Moonshine and Monstrous Lie Superalgebras*, Invent. Math. **109** (1992), 405–444.

[BF] K. Bringmann, A. Folsom, *On the asymptotic behavior of Kac-Wakimoto characters*, Proc. Amer. Math. Soc. **141** (2013) 1567–1576.

[BHS] D. Ben-Zvi, R. Heluani and M. Szczesny, *Supersymmetry of the chiral de Rham complex*, Compos. Math. **144** (2008), 503–521.

[BL] L. A. Borisov and A. Libgober, *Elliptic genera of singular varieties, orbifold elliptic genus and chiral de Rham complex*, Mirror symmetry, IV (Montreal, QC, 2000), 325–342, AMS/IP Stud. Adv. Math., 33, Amer. Math. Soc., Providence, RI, 2002.

[C] M. C. N. Cheng, *K3 Surfaces, $N = 4$ Dyons, and the Mathieu Group M_{24}*, Comm. Num. Theor. Phys. **4** (2010) 623–657.

[CD] M. C. N. Cheng and J. F. R. Duncan *On Rademacher Sums, the Largest Mathieu Group, and the Holographic Modularity of Moonshine,* preprint (2011), arXiv:1110.3859.

[CL] T. Creutzig, and A. R. Linshaw, *The super $\mathcal{W}_{1+\infty}$ algebra with positive integral central charge*, arXiv:1209.6032.

[CN] J. H. Conway, and S. P. Norton, *Monstrous moonshine*, *Bull. London Math. Soc.*, **11** (1979), 308–339.

[DMZ] A. Dabholkar, S. Murthy, and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms*, arXiv:1208.4074.

[EH] T. Eguchi and K. Hikami, *Note on twisted elliptic genus of $K3$ surface,* Phys. Lett. B **694** (2011), 446–455, arXiv:1008.4924.

[EOT] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group M_{24}*, Exper. Math. **20** (2011) 91–96.

[ET] T. Eguchi, and A. Taormina, *On the unitary representations of $N = 2$ and $N = 4$ superconformal algebras*, Phys. Lett. B **210** (1988), 125–132.

[FKRW] E. Frenkel, V. Kac, A. Radul, and W. Wang, *$\mathcal{W}_{1+\infty}$ and $\mathcal{W}(\mathfrak{gl}_N)$ with central charge N*, Commun. Math. Phys. **170** (1995), 337–357.

[G] T. Gannon, *Much ado about Mathieu*, arXiv:1211.5531 [math.RT].

[GHV] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *M*athieu twining characters for $K3$, J. High Energy Phys. (2010), no. 9, **058**, 20 pp, arXiv:1006.0221.

[GHV2] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu Moonshine in the elliptic genus of K3*, JHEP **1010** (2010) 062, 24 pp.

[GHV2] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Symmetries of K3 sigma models*, Commun. Num. Theor. Phys. **6** (2012) 1–50.

[Ha] K. Hashimoto, *Finite symplectic actions on the K3 lattice,* Nagoya Math. J. **206** (2012), 99–153.

[He] R. Heluani, *Supersymmetry of the chiral de Rham complex: 2 Commuting sectors*, Int. Math. Res. Not. IMRN 2009, No. 6, 953–987.

[HZ1] R. Heluani, and M. Zabzine, Maxim, *Generalized Calabi-Yau manifolds and the chiral de Rham complex*, Adv. Math. **223** (2010), 1815–1844.

[HZ2] R. Heluani, amd M. Zabzine, *Superconformal structures on generalized Calabi-Yau metric manifolds*, Comm. Math. Phys. **306** (2011), 333–364.

[Hi] F. Hirzebruch, *Topological methods in algebraic geometry.* Third enlarged edition. New appendix and translation from the second German edition by R. L. E. Schwarzenberger, with an additional section by A. Borel. Die Grundlehren der Mathematischen Wissenschaften, Band 131 Springer-Verlag New York, Inc., New York 1966 x+232 pp.

[Ho] G. Höhn, *Komplexe elliptische Geschlechter und S^1-äquivariante Kobordismustheorie,* Diploma thesis, Bonn 1991, arXiv:math/0405232.

[HS] C. Hoyt, and S. Reif, *Simplicity of vacuum modules over affine Lie superalgebras*, J. Algebra **321** (2009), 2861–2874.

[Hu] D. Huybrechts, *Generalized Calabi-Yau structures, K3 surfaces, and B-fields*, Internat. J. Math. **16** (2005), 13–36.

[Kr] I. M. Krichever, *Generalized elliptic genera and Baker-Akhiezer functions,* (Russian) Mat. Zametki **47** (1990), 34–45; translation in Math. Notes **47** (1990), 132–142.

[KW] V. Kac, and M. Wakimoto, *Integrable Highest Weight Modules over Affine Superalgebras and Number Theory*, Progr. Math. **123** (1994), 415–456.

[Ma] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language,* J. Symbolic Comput. **24** (1997), 235–265.

[Mu] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group,* Invent. Math. **94** (1988), 183–221.

[MSV] F. Malikov, V. Schechtman, and A. Vaintrob, *Chiral de Rham complex*, Comm. Math. Phys. **204** (1999), 439–473.

[Ni] Nikulin, V. V. *Finite groups of automorphisms of Kählerian K3 surfaces.* (Russian) Trudy Moskov. Mat. Obshch. **38** (1979), 75–137.

[Ta1] H. Tamanoi, *Elliptic genera and vertex operator super-algebras*, Lecture Notes in Mathematics, 1704, Springer-Verlag, 1999.

[Ta2] H. Tamanoi, *Elliptic genera and vertex operator super algebras*, Proc. Japan Acad. Ser. A Math. Sci., **71** (1995), 177–181.

[We] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1946.

[Zh] J. Zhou, *Superconformal Vertex Algebras in Differential Geometry. I*, preprint 2000, arXiv:math/0006201.